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A New Reverse Extended Hardy–Hilbert's Inequality with Two Partial Sums and Parameters

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Abstract: By using the methods of real analysis and the mid-value theorem, we introduce some lemmas and obtain a new reverse extended Hardy–Hilbert's inequality with two partial sums and multi-parameters. We also give a few equivalent conditions of the best possible constant factor related to several parameters in the new inequality. Some particular inequalities are deduced.

Keywords: constant factor; mid-value theorem; Bernoulli function; parameter; reverse

MSC: 26D15

1. Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m,\ b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then we have the following well-known Hardy–Hilbert's inequality (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is best possible.

In 2006, Krnic et al. [2] obtained the following inequality, which is a generalization of (1): for $\lambda_i \in (0,2](i = 1,2)$, $\lambda_1 + \lambda_2 = \lambda \in (0,4]$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{2}$$

with the best possible constant factor $B(\lambda_1,\lambda_2)$, in which

$$B(u,v) = \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}\,dt(u,v>0) \tag{3}$$

is the beta function. In particular, for $p = q = 2$, $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$, (2) deduces to Yang's inequality in [3]. In 2019, following the way of (2), by using Abel's summation by parts formula, Adiyasuren et al. [4] provided the following extension of (2) involving two partial sums and some parameters, for

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty}m^{-p\lambda_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-q\lambda_2-1}B_n^q\right)^{\frac{1}{q}}, \tag{4}$$

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where $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ is the best possible constant factor, and $A_m := \sum_{i=1}^{m} a_i$ and $B_n := \sum_{k=1}^{n} b_k$ $(m,n \in \mathrm{N} = \{1,2,\cdots\})$, satisfy the following inequalities:

$$0 < \sum_{m=1}^{\infty} m^{-p\lambda_1-1} A_m^p < \infty \text{ and } 0 < \sum_{n=1}^{\infty} n^{-q\lambda_2-1} B_n^q < \infty.$$

Both (1) and (2) with their integral analogues played an important role in real analysis, in which some generalizations of (1) are given and a relation between (1) and the other Hilbert-type inequality is obtained (cf. [5–16]). In 2021, Gu et. al. [17] provided a generalization of (4) with $\frac{1}{(m^\alpha+n^\beta)^\lambda}(\alpha,\beta \in (0,1])$ as the kernel of inequality. In 2016, by using the weight coefficients and the techniques of real analysis, Hong et al. [18] considered a few equivalent statements of the generalization of (1) with the best possible constant factor related to multi-parameters. The other further results were obtained by [19–29].

In this article, based on the idea of [17,18], by using the techniques of real analysis and the mid-value theorem, we introduce some preserving lemmas and then give a reverse of (2) with two partial sums and multi-parameters, which is a new reverse version of the inequality in [16]. We also consider a few equivalent conditions of the best possible constant factor in the reverse inequality related to multi-parameters. Furthermore, several inequalities are deduced by setting some particular parameters.

2. Some Lemmas

In what follows, we assume that $0 < p < 1$ $(q < 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in (0,6]$, $\alpha,\beta \in (0,1]$, $\lambda_1 \in (0,\frac{2}{\alpha}] \cap (0,\lambda)$, $\lambda_2 \in (0,\frac{2}{\beta}] \cap (0,\lambda)$, $\hat{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$. We also assume that $a_m, b_n \geq 0$, $A_m := \sum_{j=1}^{m} a_j$, $B_n := \sum_{k=1}^{n} b_k$ $(m,n \in \mathrm{N})$, satisfy $A_m = o(e^{tm^\alpha})$, $B_n = o(e^{tn^\beta})$ $(t > 0; m,n \to \infty)$, and the following inequalities:

$$0 < \sum_{m=1}^{\infty} m^{p(1-\alpha\hat{\lambda}_1)-1} a_m^p < \infty, 0 < \sum_{n=1}^{\infty} n^{q(1-\beta\hat{\lambda}_2)-1} b_n^q < \infty. \tag{5}$$

Lemma 1. *For $t > 0$, the following inequalities on the partial sums are valid:*

$$\sum_{m=1}^{\infty} e^{-tm^\alpha} m^{\alpha-1} A_m \geq \frac{1}{t\alpha} \sum_{m=1}^{\infty} e^{-tm^\alpha} a_m, \tag{6}$$

$$\sum_{n=1}^{\infty} e^{-tn^\beta} n^{\beta-1} B_n \geq \frac{1}{t\beta} \sum_{n=1}^{\infty} e^{-tn^\beta} b_n. \tag{7}$$

Proof. Since $A_m e^{-tm^\alpha} = o(1)(m \to \infty)$, by Abel's summation by parts formula, it follows that

$$\begin{aligned}\sum_{m=1}^{\infty} e^{-tm^\alpha} a_m &= \lim_{m\to\infty} A_m e^{-tm^\alpha} + \sum_{m=1}^{\infty} A_m \left[e^{-tm^\alpha} - e^{-t(m+1)^\alpha}\right] \\ &= \sum_{m=1}^{\infty} A_m \left[e^{-tm^\alpha} - e^{-t(m+1)^\alpha}\right].\end{aligned}$$

We set function $g(x) = e^{-tx^\alpha}, x \in [m, m+1]$. Then, we find $g'(x) = -t\alpha x^{\alpha-1}e^{-tx^\alpha}$, and for $\alpha \in (0,1]$, $h(x) := x^{\alpha-1}e^{-tx^\alpha}$ is decreasing in $[m, m+1]$. By the mid-value theorem, we have

$$\begin{aligned}\sum_{m=1}^{\infty} e^{-tm^\alpha} a_m &= -\sum_{m=1}^{\infty} A_m(g(m+1) - g(m)) \\ &= -\sum_{m=1}^{\infty} A_m g\prime(m+\theta) = t\alpha \sum_{m=1}^{\infty} (m+\theta)^{\alpha-1} e^{-t(m+\theta)^\alpha} A_m \\ &\leq t\alpha \sum_{m=1}^{\infty} m^{\alpha-1} e^{-tm^\alpha} A_m (\theta \in (0,1)).\end{aligned}$$

Hence, we have (6). In the same way, inequality (7) follows.
The lemma is proved. □

In the following lemma, for estimating the weight coefficient in Lemma 3, we introduce some results related to the Bernoulli functions and the related formulas.

Lemma 2. *(Ref. [5]'s section 2.2.3, [30]) (i) If* $(-1)^i \frac{d^i}{dt^i} g(t) > 0$, $t \in [m, \infty) (m \in N)$ *with* $g^{(i)}(\infty) = 0$ $(i = 0,1,2,3)$, $P_i(t)$, B_i $(i \in N)$ *are Bernoulli functions and Bernoulli numbers of i-order, then*

$$\int_m^\infty P_{2k-1}(t) g(t) dt = -\varepsilon_k \frac{B_{2k}}{2k} g(m) (0 < \varepsilon_k < 1; k = 1, 2, \cdots). \tag{8}$$

In particular, for $k = 1$, *since* $B_2 = \frac{1}{6}$, *we find*

$$-\frac{1}{12} g(m) < \int_m^\infty P_1(t) g(t) dt < 0; \tag{9}$$

For $k = 2$, *based on* $B_4 = -\frac{1}{30}$, *it follows that*

$$0 < \int_m^\infty P_3(t) g(t) dt < \frac{1}{120} g(m). \tag{10}$$

(ii) (Ref. [5]'s section 2.2.3, [30]) Suppose that $f(t) (> 0) \in C^3[m, \infty)$, $f^{(i)}(\infty) = 0$ $(i = 0,1,2,3)$. *We have the following Euler–Maclaurin summation formula:*

$$\sum_{k=m}^{\infty} f(k) = \int_m^\infty f(t) dt + \frac{1}{2} f(m) + \int_m^\infty P_1(t) f'(t) dt, \tag{11}$$

$$\int_m^\infty P_1(t) f'(t) dt = -\frac{1}{12} f'(m) + \frac{1}{6} \int_m^\infty P_3(t) f'''(t) dt. \tag{12}$$

Lemma 3. *For* $s \in (0, 6]$, $s_2 \in (0, \frac{2}{\beta}] \cap (0, s)$, $k_s(s_2) := B(s_2, s - s_2)$, *the weight coefficient is defined as follows:*

$$\varpi_s(s_2, m) := m^{\alpha(s-s_2)} \sum_{n=1}^{\infty} \frac{\beta n^{\beta s_2 - 1}}{(m^\alpha + n^\beta)^s} (m \in \mathrm{N}). \tag{13}$$

Then, the following inequalities are valid:

$$0 < k_s(s_2)(1 - O(\frac{1}{m^{\alpha s_2}})) < \varpi_s(s_2, m) < k_s(s_2) (m \in \mathrm{N}). \tag{14}$$

where we indicate that $O(\frac{1}{m^{\alpha s_2}}) := \frac{1}{k_s(s_2)} \int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du > 0$.

Proof. For any $m \in \mathrm{N}$, the function $g(m,t)$ is defined by: $g(m,t) := \frac{\beta t^{\beta s_2 - 1}}{(m^\alpha + t^\beta)^s} (t > 0)$. In view of (11), we have

$$\begin{aligned} \sum_{n=1}^{\infty} g(m,n) &= \int_1^\infty g(m,t)dt + \tfrac{1}{2} g(m,1) + \int_1^\infty P_1(t) g'(m,t)dt \\ &= \int_0^\infty g(m,t)dt - h(m), \end{aligned}$$

where we set $h(m) := \int_0^1 g(m,t)dt - \frac{1}{2} g(m,1) - \int_1^\infty P_1(t) g'(m,t)dt$.

We obtain $-\frac{1}{2} g(m,1) = \frac{-\beta}{2(m^\alpha + 1)^s}$. By integration by parts, it follows that

$$\begin{aligned} &\int_0^1 g(m,t)dt = \beta \int_0^1 \frac{t^{\beta s_2 - 1}}{(m^\alpha + t^\beta)^s} dt \overset{u=t^\beta}{=} \int_0^1 \frac{u^{s_2 - 1}}{(m^\alpha + u)^s} du \\ &= \frac{1}{s_2} \int_0^1 \frac{du^{s_2}}{(m^\alpha + u)^s} = \frac{1}{s_2} \frac{u^{s_2}}{(m^\alpha + u)^s} \Big|_0^1 + \frac{s}{s_2} \int_0^1 \frac{u^{s_2}}{(m^\alpha + u)^{s+1}} du \\ &= \frac{1}{s_2} \frac{1}{(m^\alpha + 1)^s} + \frac{s}{s_2(s_2+1)} \int_0^1 \frac{du^{s_2+1}}{(m^\alpha + u)^{s+1}} \\ &> \frac{1}{s_2} \frac{1}{(m^\alpha + 1)^s} + \frac{s}{s_2(s_2+1)} \left[\frac{u^{s_2+1}}{(m^\alpha + u)^{s+1}} \right]_0^1 + \frac{s(s+1)}{s_2(s_2+1)(m^\alpha+1)^{s+2}} \int_0^1 u^{s_2+1} du \\ &= \frac{1}{s_2} \frac{1}{(m^\alpha + 1)^s} + \frac{\lambda}{s_2(s_2+1)} \frac{1}{(m^\alpha + 1)^{s+1}} + \frac{s(s+1)}{s_2(s_2+1)(s_2+2)} \frac{1}{(m^\alpha + 1)^{s+2}}, \end{aligned}$$

$$\begin{aligned} &-g'(m,t) = -\frac{\beta(\beta s_2 - 1) t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} + \frac{\beta^2 s t^{\beta + \beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} \\ &= -\frac{\beta(\beta s_2 - 1) t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} + \frac{\beta^2 s (m^\alpha + t^\beta - m^\alpha) t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} = \frac{\beta(\beta s - \beta s_2 + 1) t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} - \frac{\beta^2 s m^\alpha t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}}, \end{aligned}$$

and for $0 < s_2 \leq \frac{2}{\beta},\ 0 < \beta \leq 1,\ s_2 < s \leq 6$, it follows that

$$(-1)^i \frac{d^i}{dt^i} [\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s}] > 0, (-1)^i \frac{d^i}{dt^i} [\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}}] > 0 \ (i = 0, 1, 2, 3).$$

By (9), (10), (11) and (12), we obtain

$$\beta(\beta s - \beta s_2 + 1) \int_1^\infty P_1(t) \frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} dt > -\frac{\beta(\beta s - \beta s_2 + 1)}{12(m^\alpha + 1)^s},$$

$$\begin{aligned} &-\beta^2 m^\alpha s \int_1^\infty P_1(t) \frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} dt \\ &= \frac{\beta^2 m^\alpha s}{12(m^\alpha+1)^{s+1}} - \frac{\beta^2 m^\alpha s}{6} \int_1^\infty P_3(t) \left[\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} \right]'' dt \\ &> \frac{\beta^2 m^\alpha s}{12(m^\alpha+1)^{s+1}} - \frac{\beta^2 m^\alpha s}{720} \left[\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}} \right]''_{t=1} \\ &> \frac{\beta^2 (m^\alpha + 1 - 1) s}{12(m^\alpha+1)^{s+1}} - \frac{\beta^2 (m^\alpha + 1) s}{720} \left[\frac{(s+1)(s+2)\beta^2}{(m^\alpha+1)^{s+3}} + \frac{\beta(s+1)(5 - \beta - 2\beta s_2)}{(m^\alpha+1)^{s+2}} + \frac{(2 - \beta s_2)(3 - \beta s_2)}{(m^\alpha+1)^{s+1}} \right] \\ &= \frac{\beta^2 s}{12(m^\alpha+1)^s} - \frac{\beta^2 s}{12(m^\alpha+1)^{s+1}} \\ &\quad - \frac{\beta^2 s}{720} [\frac{(s+1)(s+2)\beta^2}{(m^\alpha+1)^{s+2}} + \frac{\beta(s+1)(5 - \beta - 2\beta s_2)}{(m^\alpha+1)^{s+1}} + \frac{(2 - \beta s_2)(3 - \beta s_2)}{(m^\alpha+1)^s}]. \end{aligned}$$

Hence, we have $h(m) > \frac{1}{(m^\alpha+1)^s} h_1 + \frac{\lambda}{(m^\alpha+1)^{s+1}} h_2 + \frac{s(s+1)}{(m^\alpha+1)^{s+2}} h_3$, where

$$h_1 := \frac{1}{s_2} - \frac{\beta}{2} - \frac{\beta - \beta^2 s_2}{12} - \frac{\beta^2 s (2 - \beta s_2)(3 - \beta s_2)}{720},$$

$$h_2 := \frac{1}{s_2(s_2+1)} - \frac{\beta^2}{12} - \frac{\beta^3 (s+1)(5 - \beta - 2\beta s_2)}{720},$$

and $h_3 := \frac{1}{s_2(s_2+1)(s_2+2)} - \frac{\beta^4(s+2)}{720}$. We can find

$$h_1 \geq \frac{1}{s_2} - \frac{\beta}{2} - \frac{\beta - \beta^2 s_2}{12} - \frac{s\beta^2(2-\beta s_2)(3-\beta s_2)}{720} = \frac{g(s_2)}{720 s_2},$$

where the function $g(\sigma)(\sigma \in (0, \frac{2}{\beta}])$ is indicated by

$$g(\sigma) := 720 - (420\beta + 6s\beta^2)\sigma + (60\beta^2 + 5s\beta^3)\sigma^2 - s\beta^4\sigma^3.$$

For $\beta \in (0,1],\ s \in (0,6]$, we obtain

$$\begin{aligned} g'(\sigma) &= -(420\beta + 6s\beta^2) + 2(60\beta^2 + 5s\beta^3)\sigma - 3\beta^4\sigma^2 \\ &\leq -420\beta - 6s\beta^2 + 2(60\beta^2 + 5s\beta^3)\frac{2}{\beta} = (14s\beta - 180)\beta < 0, \end{aligned}$$

and then it follows that $h_1 \geq \frac{g(s_2)}{720\ s_2} \geq \frac{g(2/\beta)}{720\ s_2} = \frac{1}{6\ s_2} > 0$. For $s_2 \in (0, \frac{2}{\beta}]$, we still find

$$h_2 > \frac{\beta^2}{6} - \frac{\beta^2}{12} - \frac{5(s+1)\beta^2}{720} = (\frac{1}{12} - \frac{s+1}{140})\beta^2 > 0,$$

and $h_3 \geq (\frac{1}{24} - \frac{s+2}{720})\beta^3 > 0 (s \in (0,6])$.

Hence, it follows that $h(m) > 0$. Setting $t = m^{\frac{\alpha}{\beta}} u^{\frac{1}{\beta}}$, we have

$$\begin{aligned} \varpi_s(s_2, m) &= m^{\alpha(s-s_2)} \sum_{n=1}^{\infty} g(m,n) < m^{\alpha(s-s_2)} \int_0^{\infty} g(m,t)dt \\ &= \beta m^{\alpha(s-s_2)} \int_0^{\infty} \frac{t^{\beta s_2 - 1}dt}{(m^\alpha + t^\beta)^s} = \int_0^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} = B(s_2, s - s_2) = k_s(s_2). \end{aligned}$$

On the other hand, in view of (11), we find

$$\begin{aligned} \sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)dt + \frac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt \\ &= \int_1^{\infty} g(m,t)dt + H(m), \end{aligned}$$

where we set $H(m) := \frac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt$.

We find $\frac{1}{2}g(m,1) = \frac{\beta}{2(m^\alpha+1)^s}$, and

$$g'(m,t) = -\frac{\beta(\beta s - \beta s_2 + 1)t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} + \frac{\beta^2 s m^\alpha t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}}.$$

For $s_2 \in (0, \frac{2}{\beta}] \cap (0, s),\ 0 < s \leq 6$, by (7), we find

$$-\beta(\beta s - \beta s_2 + 1)\int_1^{\infty} P_1(t)\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s}dt > 0,$$

and

$$\beta^2 m^\alpha s \int_1^{\infty} P_1(t)\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}}dt > -\frac{\beta^2 m^\alpha s}{12(m^\alpha + 1)^{s+1}} > -\frac{\beta^2 s}{12(m^\alpha + 1)^s}.$$

Hence, it follows that

$$H(m) > \frac{\beta}{2(m^\alpha + 1)^s} - \frac{\beta^2 s}{12(m^\alpha + 1)^s} \geq \frac{\beta}{2(m^\alpha + 1)^s} - \frac{6\beta}{12(m^\alpha + 1)^s} = 0.$$

Then, we have

$$\begin{aligned}\varpi_s(\lambda_2, m) &= m^{\alpha(s-s_2)} \sum_{n=1}^{\infty} g(m,n) > m^{\alpha(s-s_2)} \int_1^{\infty} g(m,t)dt \\ &= m^{\alpha(s-s_2)} \int_0^{\infty} g(m,t)dt - m^{\alpha(s-s_2)} \int_0^1 g(m,t)dt \\ &= k_s(s_2)\left[1 - \frac{1}{k_s(s_2)} \int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du\right] > 0,\end{aligned}$$

where we indicate that $O(\frac{1}{m^{\alpha s_2}}) = \frac{1}{k_s(s_2)} \int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du$, satisfying

$$0 < \int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du < \int_0^{\frac{1}{m^\alpha}} u^{s_2-1} du = \frac{1}{s_2 m^{\alpha s_2}}.$$

Therefore, inequalities (14) are valid.

The lemma is proved. □

In view of Lemma 3, the key inequality is obtained as follows:

Lemma 4. *We have the reverse inequality, as follows:*

$$\begin{aligned}I_\lambda &:= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\beta)^\lambda} > \left(\frac{1}{\beta} k_\lambda(\lambda_2)\right)^{\frac{1}{p}} \left(\frac{1}{\alpha} k_\lambda(\lambda_1)\right)^{\frac{1}{q}} \\ &\times \left[\sum_{m=1}^{\infty} \left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right) m^{p(1-\alpha\hat{\lambda}_1)-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\beta\hat{\lambda}_2)-1} b_n^q\right]^{\frac{1}{q}}.\end{aligned} \quad (15)$$

Proof. By the symmetry, for $s_1 \in (0, \frac{2}{\alpha}] \cap (0, s), s \in (0, 6]$, we obtain the inequalities of the next weight coefficient, as follows:

$$\begin{aligned}&0 < k_s(s_1)\left(1 - O\left(\frac{1}{n^{\beta s_1}}\right)\right) \\ &< \omega_s(s_1, n) := n^{\beta(s-s_1)} \sum_{m=1}^{\infty} \frac{\alpha m^{\alpha s_1 - 1}}{(m^\alpha + n^\beta)^s} < k_s(s_1) (n \in \mathrm{N}),\end{aligned} \quad (16)$$

where we indicate $O(\frac{1}{n^{\beta s_1}}) := \frac{1}{k_s(s_1)} \int_0^{\frac{1}{n^\beta}} \frac{u^{s_1-1}}{(1+u)^s} du > 0$.

Using the reverse Hölder's inequality (cf. [31]), it follows that

$$\begin{aligned}I_\lambda &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m^\alpha + n^\beta)^\lambda} \left[\frac{m^{\alpha(1-\lambda_1)/q} (\beta n^{\beta-1})^{1/p}}{n^{\beta(1-\lambda_2)/p} (\alpha m^{\alpha-1})^{1/q}} a_m\right] \left[\frac{n^{\beta(1-\lambda_2)/p} (\alpha m^{\alpha-1})^{1/q}}{m^{\alpha(1-\lambda_1)/q} (\beta n^{\beta-1})^{1/p}} b_n\right] \\ &\geq \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\beta}{(m^\alpha + n^\beta)^\lambda} \frac{m^{\alpha(1-\lambda_1)(p-1)} n^{\beta-1} a_m^p}{n^{\beta(1-\lambda_2)} (\alpha m^{\alpha-1})^{p-1}}\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\alpha}{(m^\alpha + n^\beta)^\lambda} \frac{n^{\beta(1-\lambda_2)(q-1)} m^{\alpha-1} b_n^q}{m^{\alpha(1-\lambda_1)} (\beta n^{\beta-1})^{q-1}}\right]^{\frac{1}{q}} \\ &= \left(\frac{1}{\alpha}\right)^{\frac{1}{q}} \left(\frac{1}{\beta}\right)^{\frac{1}{p}} \left[\sum_{m=1}^{\infty} \varpi_\lambda(\lambda_2, m) m^{p(1-\alpha\hat{\lambda}_1)-1} a_m^p\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{n=1}^{\infty} \omega_\lambda(\lambda_1, n) n^{q(1-\beta\hat{\lambda}_2)-1} b_n^q\right]^{\frac{1}{q}}.\end{aligned}$$

By (14), (16) and (5) (for $s = \lambda, s_i = \lambda_i (i = 1, 2)$), since $0 < p < 1 (q < 0)$ and the assumptions, we obtain (15).

The lemma is proved. □

3. Main Results

By Lemma 1 and Lemma 4, the following theorem follows:

Theorem 1. *The following reverse inequality with two partial sums and parameters is valid:*

$$\begin{aligned} I := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{\alpha-1}n^{\beta-1}}{(m^{\alpha}+n^{\beta})^{\lambda+2}}A_m B_n > \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)\alpha\beta}\left(\frac{1}{\beta}k_{\lambda}(\lambda_2)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_{\lambda}(\lambda_1)\right)^{\frac{1}{q}} \\ \times\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right)m^{p(1-\alpha\hat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\hat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{17}$$

In particular, for $\lambda_1+\lambda_2=\lambda$, we have

$$0<\sum_{m=1}^{\infty}m^{p(1-\alpha\lambda_1)-1}a_m^p<\infty,\ 0<\sum_{n=1}^{\infty}n^{q(1-\beta\lambda_2)-1}b_n^q<\infty,$$

as well as:

$$\begin{aligned} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{\alpha-1}n^{\beta-1}A_m B_n}{(m^{\alpha}+n^{\beta})^{\lambda+2}} > \left(\frac{1}{\alpha}\right)^{1+\frac{1}{q}}\left(\frac{1}{\beta}\right)^{1+\frac{1}{p}}\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}B(\lambda_1,\lambda_2) \\ \times\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right)m^{p(1-\alpha\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{18}$$

Proof. Based on the expression as follows

$$\frac{1}{(m^{\alpha}+n^{\beta})^{\lambda+2}}=\frac{1}{\Gamma(\lambda+2)}\int_0^{\infty}t^{(\lambda+2)-1}e^{-(m^{\alpha}+n^{\beta})t}dt,$$

by (6) and (7), we have

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda+2)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^{\alpha-1}A_m)(n^{\beta-1}B_n)\int_0^{\infty}t^{\lambda+1}e^{-(m^{\alpha}+n^{\beta})t}dt \\ &= \frac{1}{\Gamma(\lambda+2)}\int_0^{\infty}t^{\lambda+1}\left(\sum_{m=1}^{\infty}e^{-m^{\alpha}t}m^{\alpha-1}A_m\right)\left(\sum_{n=1}^{\infty}e^{-n^{\beta}t}n^{\beta-1}B_n\right)dt \\ &\geq \frac{1}{\Gamma(\lambda+2)}\int_0^{\infty}t^{\lambda+1}\left(\frac{1}{t\alpha}\sum_{m=1}^{\infty}e^{-m^{\alpha}t}a_m\right)\left(\frac{1}{t\beta}\sum_{n=1}^{\infty}e^{-n^{\beta}t}b_n\right)dt \\ &= \frac{1}{\Gamma(\lambda+2)\alpha\beta}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_m b_n\int_0^{\infty}t^{\lambda-1}e^{-(m^{\alpha}+n^{\beta})t}dt = \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)\alpha\beta}I_{\lambda}. \end{aligned}$$

Then, by (15), inequality (17) follows.
The theorem is proved. □

In the following two theorems, we provide a few equivalent conditions on (17).

Theorem 2. *Assume that $\lambda_1\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda)$, $\lambda_2\in(0,\frac{2}{\beta})\cap(0,\lambda)$. $\lambda\in(0,4]$. If $\lambda_1+\lambda_2=\lambda$, then $\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)\alpha\beta}(\frac{1}{\beta}k_{\lambda}(\lambda_2))^{\frac{1}{p}}(\frac{1}{\alpha}k_{\lambda}(\lambda_1))^{\frac{1}{q}}$ in (17) is the best possible. constant factor.*

Proof. We now prove that the constant factor $\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}\left(\frac{1}{\alpha}\right)^{1+\frac{1}{q}}\left(\frac{1}{\beta}\right)^{1+\frac{1}{p}}B(\lambda_1,\lambda_2)$ in (18) is the best possible. For any $0<\varepsilon<\min\{p\lambda_1,|q|(\frac{2}{\beta}-\lambda_2)\}$, we set

$$\widetilde{a}_m := m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1},\ \widetilde{b}_n := n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}(m,n\in\mathrm{N}).$$

Since $0<\lambda_1-\frac{\varepsilon}{p}\leq\frac{2}{\alpha}-1,\ 0<\alpha(\lambda_1-\frac{\varepsilon}{p})\leq 2-\alpha<2$, by (2.2.24) (cf. [5]), we have

$$\begin{aligned} &\widetilde{A}_m := \sum_{i=1}^{m}\widetilde{a}_i=\sum_{i=1}^{m}i^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}=\int_1^m t^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}dt \\ &\quad+\frac{1}{2}[m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}+1]+\frac{\varepsilon_0}{12}[\alpha(\lambda_1-\frac{\varepsilon}{p})-1][m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-2}-1] \\ &\quad=\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}(m^{\alpha(\lambda_1-\frac{\varepsilon}{p})}+c_1+O_1(m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1})(\varepsilon_0\in(0,1);m\in\mathrm{N},\mathrm{m}\to\infty). \end{aligned}$$

In the same way, for $0 < \beta(\lambda_2 - \frac{\varepsilon}{q}) < 2$, we have

$$\widetilde{B}_n := \sum_{k=1}^{n} \widetilde{b}_k = \frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})}(n^{\beta(\lambda_2 - \frac{\varepsilon}{q})} + c_2 + O_2(n^{\beta(\lambda_2 - \frac{\varepsilon}{q})-1}))(n \in \mathrm{N}; n \to \infty).$$

We observe that $\widetilde{A}_m = o(e^{tm^\alpha}), \widetilde{B}_n = o(e^{tn^\beta})\,(t > 0; m, n \to \infty)$.

If there exists a constant $M \geq \left(\frac{1}{\alpha}\right)^{1+\frac{1}{q}} \left(\frac{1}{\beta}\right)^{1+\frac{1}{p}} \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)} B(\lambda_1, \lambda_2)$, such that (18) is valid when we replace $\left(\frac{1}{\alpha}\right)^{1+\frac{1}{q}} \left(\frac{1}{\beta}\right)^{1+\frac{1}{p}} \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)} B(\lambda_1, \lambda_2)$ by M, then in particular, we have

$$\begin{aligned} \widetilde{I} &:= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^{\alpha-1} n^{\beta-1}}{(m^\alpha + n^\beta)^{\lambda+2}} \widetilde{A}_m \widetilde{B}_n \\ &> M \left[\sum_{m=1}^{\infty} \left(1 - O(\tfrac{1}{m^{\alpha\lambda_2}})\right) m^{p(1-\alpha\lambda_1)-1} \widetilde{a}_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\beta\lambda_2)-1} \widetilde{b}_n^q \right]^{\frac{1}{q}}. \end{aligned} \quad (19)$$

By (19) and using the decreasingness property of series, it follows that

$$\begin{aligned} \widetilde{I} &> M \left[\sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1} - \sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1} O(\tfrac{1}{m^{\varepsilon\lambda_2}}) \right]^{\frac{1}{p}} \left(1 + \sum_{n=2}^{\infty} n^{-\beta\varepsilon-1}\right)^{\frac{1}{q}} \\ &> M\left(\int_1^\infty x^{-\alpha\varepsilon-1} dx - O(1)\right)^{\frac{1}{p}} \left(1 + \int_1^\infty y^{-\beta\varepsilon-1} dy\right)^{\frac{1}{q}} \\ &= \frac{M}{\varepsilon}\left(\frac{1}{\alpha} - \varepsilon O(1)\right)^{\frac{1}{p}} \left(\varepsilon + \frac{1}{\beta}\right)^{\frac{1}{q}}. \end{aligned}$$

We still find that

$$\begin{aligned} \widetilde{I} &< \frac{1}{\alpha(\lambda_1 - \frac{\varepsilon}{p})} \frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m^\alpha + n^\beta)^{\lambda+2}} \left[m^{\alpha-1} \left(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})} + |c_1| + |O_1(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1})| \right) \right] \\ &\times \left[n^{\beta-1} \left(n^{\beta(\lambda_2 - \frac{\varepsilon}{q})} + |c_2| + |O_2(n^{\beta(\lambda_2 - \frac{\varepsilon}{q})-1})| \right) \right] = \frac{1}{\alpha(\lambda_1 - \frac{\varepsilon}{p})} \frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})} \\ &\times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m^\alpha + n^\beta)^{\lambda+2}} \left[m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}+1)-1} + |c_1| m^{\alpha-1} + |O_1(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}+1)-2})| \right] \\ &\times \left[n^{\beta(\lambda_2 - \frac{\varepsilon}{q}+1)-1} + |c_2| n^{\beta-1} + |O_2(n^{\beta(\lambda_2 - \frac{\varepsilon}{q}+1)-2})| \right] \\ &= \frac{1}{\alpha(\lambda_1 - \frac{\varepsilon}{p})} \frac{1}{\beta(\lambda_2 - \frac{\varepsilon}{q})} (I_0 + I_1), \end{aligned}$$

where we indicate that $I_0 := \sum_{n=1}^{\infty} \left[n^{\beta(\lambda_2 - \frac{\varepsilon}{q}+1)} - \sum_{m=1}^{\infty} \frac{m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}+1)-1}}{(m^\alpha + n^\beta)^{\lambda+2}} \right]$, and

$$\begin{aligned} I_1 &:= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m^\alpha + n^\beta)^{\lambda+2}} \Big[(|c_1| m^{\alpha-1} + |O_1(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}+1)-2})) n^{\beta(\lambda_2 - \frac{\varepsilon}{q}+1)-1}| \\ &+ (|c_1| m^{\alpha-1} + |O_1(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}+1)-2}))(|c_2| n^{\beta-1} + |O_2(n^{\beta(\lambda_2 - \frac{\varepsilon}{q}+1)-2})|) \\ &+ m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}+1)-1} (|c_2| n^{\beta-1} + |O_2(n^{\beta(\lambda_2 - \frac{\varepsilon}{q}+1)-2})|) \Big] \\ &\leq \sum_{n=1}^{\infty} \frac{n^{\beta(\lambda_2 - \frac{\varepsilon}{q}+1)-1}}{(n^\beta)^{\lambda_2+1}} \sum_{m=1}^{\infty} \frac{1}{(m^\alpha)^{\lambda_1+1}} (|c_1| m^{\alpha-1} + |O_1(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}+1)-2})|) \\ &+ \sum_{n=1}^{\infty} \frac{(|c_2| n^{\beta-1} + |O_2(n^{\beta(\lambda_2 - \frac{\varepsilon}{q}+1)-2})|)}{(n^\beta)^{\lambda_2+1}} \sum_{m=1}^{\infty} \frac{1}{(m^\alpha)^{\lambda_1+1}} (|c_1| m^{\alpha-1} + |O_1(m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}+1)-2}|) \\ &+ \sum_{n=1}^{\infty} \frac{(|c_2| n^{\beta-1} + |O_2(n^{\beta(\lambda_2 - \frac{\varepsilon}{q}+1)-2})|)}{(n^\beta)^{\lambda_2+1}} \sum_{m=1}^{\infty} \frac{1}{(m^\alpha)^{\lambda_1+1}} m^{\alpha(\lambda_1 - \frac{\varepsilon}{p}+1)-1} \leq M_1 < \infty. \end{aligned}$$

By (14), for $s = \lambda + 2 \in (0, 6], s_1 = \lambda_1 + 1 - \frac{\varepsilon}{p}$ $(\in (0, \frac{2}{\alpha}] \cap (0, \lambda + 2))$, we have

$$\begin{aligned}
I_0 &= \frac{1}{\alpha} \sum_{n=1}^{\infty} \left[n^{\beta(\lambda_2 + 1 + \frac{\varepsilon}{p})} \sum_{m=1}^{\infty} \frac{\alpha m^{\alpha(\lambda_1 + 1 - \frac{\varepsilon}{p}) - 1}}{(m+n)^{\lambda+2}} \right] n^{-\beta\varepsilon - 1} \\
&= \frac{1}{\alpha} \sum_{n=1}^{\infty} \omega_{\lambda+2}(\lambda_1 + 1 - \frac{\varepsilon}{p}, n) n^{-\beta\varepsilon - 1} \\
&< \frac{1}{\alpha} k_{\lambda+2}(\lambda_1 + 1 - \frac{\varepsilon}{p}) \left(1 + \sum_{n=2}^{\infty} n^{-\beta\varepsilon - 1} \right) \\
&< \frac{1}{\alpha} k_{\lambda+2}(\lambda_1 + 1 - \frac{\varepsilon}{p}) (1 + \int_1^{\infty} y^{-\beta\varepsilon - 1} dy) \\
&= \frac{1}{\varepsilon} \frac{1}{\alpha\beta} B(\lambda_1 + 1 - \frac{\varepsilon}{p}, \lambda_2 + 1 + \frac{\varepsilon}{p})(1 + \beta\varepsilon).
\end{aligned}$$

Based on the above results, we have

$$\frac{1}{\alpha^2(\lambda_1 - \frac{\varepsilon}{p})} \frac{1}{\beta^2(\lambda_2 - \frac{\varepsilon}{q})} [B(\lambda_1 + 1 - \frac{\varepsilon}{p}, \lambda_2 + 1 + \frac{\varepsilon}{p})(1 + \beta\varepsilon) + \varepsilon M_1] > \varepsilon \widetilde{I} > M \left(\frac{1}{\alpha} - \varepsilon O(1) \right)^{\frac{1}{p}} \left(\varepsilon + \frac{1}{\beta} \right)^{\frac{1}{q}}.$$

Setting $\varepsilon \to 0^+$ in the above inequality, in virtue of the continuity of the beta function, we find

$$(\frac{1}{\alpha})^{1+\frac{1}{q}} (\frac{1}{\beta})^{1+\frac{1}{p}} \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)} B(\lambda_1, \lambda_2) = (\frac{1}{\alpha})^{1+\frac{1}{q}} (\frac{1}{\beta})^{1+\frac{1}{p}} \frac{1}{\lambda_1 \lambda_2} B(\lambda_1 + 1, \lambda_2 + 1) \geq M.$$

Hence, $M = \left(\frac{1}{\alpha}\right)^{1+\frac{1}{q}} \left(\frac{1}{\beta}\right)^{1+\frac{1}{p}} \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)} B(\lambda_1, \lambda_2)$ is the best possible constant factor in (18).

The theorem is proved. □

Theorem 3. *Suppose that* $\lambda_1 \in \left(0, \frac{2}{\alpha}\right] \cap (0, \lambda)$, $\lambda_2 \in \left(0, \frac{2}{\beta}\right] \cap (0, \lambda)$. $\lambda \in (0, 6]$. *If the constant factor* $\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)\alpha\beta} (\frac{1}{\beta} k_\lambda(\lambda_2))^{\frac{1}{p}} (\frac{1}{\alpha} k_\lambda(\lambda_1))^{\frac{1}{q}}$ *in (17) is the best possible, then for*

$$\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1)) \cap [q(\frac{2}{\beta} - \lambda_2), p(\frac{2}{\alpha} - \lambda_1)],$$

we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. For $\hat{\lambda}_1 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} = \frac{\lambda - \lambda_1 - \lambda_2}{p} + \lambda_1$, $\hat{\lambda}_2 = \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda - \lambda_1 - \lambda_2}{q} + \lambda_2$, we find $\hat{\lambda}_1 + \hat{\lambda}_2 = \lambda$. For $\lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1))$, we have $\hat{\lambda}_1 \in (0, \lambda)$, and then $\hat{\lambda}_2 = \lambda - \hat{\lambda}_1 \in (0, \lambda)$; for $\lambda - \lambda_1 - \lambda_2 \in [q(\frac{2}{\beta} - \lambda_2), p(\frac{2}{\alpha} - \lambda_1)]$, we still have $\hat{\lambda}_1 \leq \frac{2}{\alpha}$, $\hat{\lambda}_2 \leq \frac{2}{\beta}$. Then, for $\lambda_1 + \lambda_2 = \lambda$ in (17), substitution of $\lambda_i = \hat{\lambda}_i$ $(i = 1, 2)$, we still have

$$\begin{aligned}
&\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{A_m B_n}{(m^\alpha + n^\beta)^{\lambda+2}} > (\frac{1}{\alpha})^{1+\frac{1}{q}} (\frac{1}{\beta})^{1+\frac{1}{p}} \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)} B(\hat{\lambda}_1, \hat{\lambda}_2) \\
&\times \left[\sum_{m=1}^{\infty} (1 - O(\frac{1}{m^{\alpha\hat{\hat{\lambda}}_2}})) m^{p(1-\alpha\hat{\lambda}_1)-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\beta\hat{\lambda}_2)-1} b_n^q \right]^{\frac{1}{q}}.
\end{aligned} \tag{20}$$

By using the reverse Hölder's inequality (cf. [31]), we still obtain

$$
\begin{aligned}
& B(\hat{\lambda}_1,\hat{\lambda}_2)=k_\lambda(\tfrac{\lambda-\lambda_2}{p}+\tfrac{\lambda_1}{q}) \\
& =\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du=\int_0^\infty \frac{1}{(1+u)^\lambda}\left(u^{\frac{\lambda-\lambda_2-1}{p}}\right)\left(u^{\frac{\lambda_1-1}{q}}\right)du \\
& \geq \left[\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\lambda-\lambda_2-1}du\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\lambda_1-1}du\right]^{\frac{1}{q}} \\
& =\left[\int_0^\infty \frac{1}{(1+v)^\lambda}v^{\lambda_2-1}dv\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{1}{(1+u)^\lambda}u^{\lambda_1-1}du\right]^{\frac{1}{q}} \\
& =(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}.
\end{aligned} \tag{21}
$$

If $\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)\alpha\beta}(\frac{1}{\beta}k_\lambda(\lambda_2))^{\frac{1}{p}}(\frac{1}{\alpha}k_\lambda(\lambda_1))^{\frac{1}{q}}$ in (17) is the best possible constant factor, then compare it with the constant factors in (17) and (20), and we have the following inequality:

$$
\begin{aligned}
& \frac{\Gamma(\lambda)}{\Gamma(\lambda+2)\alpha\beta}(\tfrac{1}{\beta}k_\lambda(\lambda_2))^{\frac{1}{p}}(\tfrac{1}{\alpha}k_\lambda(\lambda_1))^{\frac{1}{q}} \\
& \geq (\tfrac{1}{\alpha})^{1+\frac{1}{q}}(\tfrac{1}{\beta})^{1+\frac{1}{p}}\frac{\Gamma(\lambda)}{\Gamma(\lambda+2)}B(\hat{\lambda}_1,\hat{\lambda}_2)(\in \mathrm{R}_+),
\end{aligned}
$$

namely, $B(\hat{\lambda}_1,\hat{\lambda}_2)\leq(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}}$. Then, by (21), we have

$$B(\hat{\lambda}_1,\hat{\lambda}_{21})=(k_\lambda(\lambda_2))^{\frac{1}{p}}(k_\lambda(\lambda_1))^{\frac{1}{q}},$$

which follows that (21) protains the form of equality.

We observe that (21) protains the form of equality if and only if there exist A and B, such that they are not both zero and (cf. [31]) $Au^{\lambda-\lambda_2-1}=Bu^{\lambda_1-1}a.e.$ in R_+. Assume that $A\neq 0$. It follows that $u^{\lambda-\lambda_2-\lambda_1}=\frac{B}{A}a.e.$ in R_+, and then $\lambda-\lambda_2-\lambda_1=0$. Hence, we have $\lambda_1+\lambda_2=\lambda$.

The theorem is proved. □

Remark 1. *(i) For $\alpha=\beta=1,\lambda\in(0,4],\lambda_1\in(0,1]\cap(0,\lambda),\lambda_2\in(0,2)\cap(0,\lambda)$ in (18), we have the following reverse inequality with $\frac{1}{\lambda(\lambda+1)}B(\lambda_1,\lambda_2)$ as the best possible constant factor:*

$$
\begin{aligned}
& \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{A_mB_n}{(m+n)^{\lambda+2}}>\frac{1}{\lambda(\lambda+1)}B(\lambda_1,\lambda_2) \\
& \times\left[\sum_{m=1}^\infty (1-O(\tfrac{1}{m^{\lambda_2}}))m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{22}
$$

(ii) For $\alpha=\beta=\frac{1}{2},\lambda\in(0,4],\lambda_1\in(0,3]\cap(0,\lambda),\lambda_2\in(0,\lambda)$ in (18), we have the following reverse inequality with $\frac{8}{\lambda(\lambda+1)}B(\lambda_1,\lambda_2)$ as the best possible constant factor:

$$
\begin{aligned}
& \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{A_mB_n}{(\sqrt{m}+\sqrt{n})^{\lambda+2}\sqrt{mn}}>\frac{8}{\lambda(\lambda+1)}B(\lambda_1,\lambda_2) \\
& \times\left[\sum_{m=1}^\infty (1-O(\tfrac{1}{m^{\lambda_2/2}}))m^{p(1-\frac{\lambda_1}{2})-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty n^{q(1-\frac{\lambda_2}{2})-1}b_n^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{23}
$$

4. Conclusions

In this article, by using the techniques of real analysis, the way of weight coefficients and the idea of introduced parameters, applying the mid-value theorem. We estimate some lemmas and obtain a new reverse extended inequality (2) with two partial sums and multi-parameters in Theorem 1. We consider a few equivalent statements of the best possible constant factor related to several parameters in Theorems 2 and 3. We also deduce some inequalities for setting particular parameters in Remark 1. The theorems and lemmas in this paper provide a useful extensive account of this type of inequality. Further studies should be using the idea of this article to build some other kinds of Hilbert-type inequalities with partial sums and parameters.

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