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A Non-Local Non-Homogeneous Fractional Timoshenko System with Frictional and Viscoelastic Damping Terms

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Abstract: We are devoted to the study of a non-local non-homogeneous time fractional Timoshenko system with frictional and viscoelastic damping terms. We are concerned with the well-posedness of the given problem. The approach relies on some functional analysis tools, operator theory, a priori estimates and density arguments. This work can be considered as a contribution to the development of energy inequality methods, the so-called a priori estimate method inspired from functional analyses and used to prove the well-posedness of mixed problems with integral boundary conditions.

Keywords: fractional Timoshenko system; memory term damping; frictional damping; non-local constraint; a priori estimate; well-posedness

MSC: 35B45; 35R11; 35L55

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1. Introduction

Vibrations of beams are not always safe and welcomed because of their great and irreparable damaging effects. In this situation, researchers try to introduce some damping mechanisms (viscous, thermoelastic, modal, frictional, and Kelvin–Voigt damping) in such a way that these damaging and destructive vibrations are perfectly reduced. In other words, an intensive investigation was carried out to impose the minimal conditions to provide the stability of Timoshenko systems using several types of dissipative mechanisms. Several authors have studied and investigated problems involving the previously mentioned types of damping terms, where different kinds of stabilities were shown. In this regard, we refer the reader to [1,2] and the references therein.

As a classical and simple model [3], Timoshenko studied the following coupled hyperbolic system

$$\begin{cases} \rho_1\theta_{tt} - \kappa(\theta_x - \phi)_x = 0, & (x,t) \in (0,L) \times (0,\infty) \\ \rho_2\phi_{tt} = \kappa^*\phi_{xx} + \kappa(\theta_x - \phi) & (x,t) \in (0,L) \times (0,\infty), \\ (\theta_x - \phi)\mid_{x=0}^{x=L} = 0, \ \ \phi_x \mid_{x=0}^{x=L} = 0, \end{cases} \tag{1}$$

describing the transverse vibration of a beam of length L in its equilibrium position, with a transverse displacement θ, and a rotation angle ϕ. The parameters ρ_1, ρ_2, κ and κ^* denote the density, the polar moment of inertia of a cross-section, the shear modulus and the Young's modulus of elasticity, respectively. The Timoshenko system Equation (1) has been generalized and studied by many authors. As mentioned at the beginning of the introduction, different types of damping terms have been added to the Timoshenko system for the purpose of its stabilization. In the following problems the functions θ and ϕ always refer to the transverse displacement and rotation angle, respectively. For example, in [4], researchers investigated the exponential stability of a system with weak damping terms

$$\begin{cases} \rho_1\theta_{tt} = \kappa(\theta_x - \phi)_x - \theta_t, \quad \text{in } (0,L)\times(0,\infty), \\ \rho_2\phi_{tt} = \kappa^*\phi_{xx} - \kappa(\theta_x - \phi)_x - \phi_t, \quad \text{in } (0,L)\times(0,\infty), \\ \theta(0,t) = \theta(L,t) = \phi(0,t) = \phi(L,t) = 0, \;\; t > 0. \end{cases} \tag{2}$$

In [5], the authors obtained some exponential decay results for a system with a memory damping term

$$\begin{cases} \rho_1\theta_{tt} - \kappa_1(\theta_x + \phi)_x = 0, \quad \text{in } (0,L)\times(0,\infty) \\ \rho_2\phi_{tt} - \kappa_2\phi_{xx} + \kappa_1(\theta_x + \phi) + h * \phi_{xx}(x,t) = 0, \text{ in } (0,L)\times(0,\infty) \\ \theta(0,t) = \theta(L,t) = \phi(0,t) = \phi(L,t) = 0, \\ \theta(x,0) = \theta_0,\; \theta_t(x,0) = \theta_1,\; \phi(x,0) = \phi_0,\; \phi_t(x,0) = \phi_1. \end{cases} \tag{3}$$

Authors considered and studied in [6] a Timoshenko system with frictional and viscoelastic damping terms, showing some polynomial decay and exponential results

$$\begin{cases} \theta_{tt} - (\theta_x + \phi)_x = 0, \\ \phi_{tt} - \phi_{xx} + \theta_x + \phi + \int\limits_0^t g(t-s)(a(x)\phi_x(x,s))_x ds + b(x)h(\phi_t) = 0, \\ \theta(0,t) = \theta(1,t) = \phi(0,t) = \phi(1,t) = 0, \;\; t > 0. \end{cases} \tag{4}$$

We also mention that in [7], the authors investigated the exponential stabilization of a Timoshenko system by a thermal damping effect.

$$\begin{cases} \rho_1\theta_{tt} - \kappa_1(\theta_x + \phi)_x = 0, \quad \text{in } (0,L)\times(0,\infty) \\ \rho_2\phi_{tt} - \kappa_2\phi_{xx} + \kappa_1(\theta_x + \phi) + \gamma\omega_x, \text{ in } (0,L)\times(0,\infty) \\ \rho_2\omega_{tt} - \kappa_3\omega_{xx} + \beta\int\limits_0^t g(t-s)\omega_{xx}(x,s)ds + \gamma\phi_{ttx}, \text{ in } (0,L)\times(0,\infty). \end{cases} \tag{5}$$

In [8], the author considered a Timoshenko thermoelastic system with frictional damping and a distributed delay. They proved the existence and uniqueness of the solution of the system, showing that it is exponentially stable without taking into account the speeds of wave propagation. There are many other papers in the literature dealing with the stabilization of different version of Timoshenko systems. For other results dealing with the stabilization and controllability of Timoshenko systems, the reader can consult the papers [9–19].

Recently, a generalization of the Timoshenko system (1) into a fractional setting is studied in [20] by using a fractional version of resolvents. The author established the well-posedness of a fractional Timoshenko system, proving that lower-order fractional terms can stabilize the system in a Mittag–Leffler fashion. More precisely, the author considered the following fractional-order system

$$\begin{cases} \rho_1\partial_t^\alpha(\partial_t^\alpha\theta) - \kappa_1(\theta_x + \phi)_x = 0, \quad \text{in } (0,1)\times(0,\infty) \\ \rho_2\partial_t^\alpha(\partial_t^\alpha\phi + a\phi) - \kappa_2\phi_{xx} + \kappa_1(\theta_x + \phi), \text{ in } (0,1)\times(0,\infty) \\ \theta(0,t) = \theta(1,t) = 0,\; \phi(0,t) = \phi(1,t) = 0, \;\; t > 0 \\ \theta(x,0) = \theta_0(x)\;,\; \phi(x,0) = \psi(x). \end{cases}$$

On the other hand, it is known that the geological environment and surrounding (geological medium) with properties of a fractal nature can be described in terms of fractional calculus [21] by fractional-order equations with parameters, depending on the fractal dimension of the geomedium. For some other fractional- and integer-order Timoshenko systems, the reader can refer to [22–27]. For some other fractional and integer order Timoshenko systems, the reader can refer to [28–35]. Motivated by the above results on Timoshenko systems, we consider a non-local problem for a non-homogeneous fractional Timoshenko system with frictional damping in the first equation and a viscoelastic memory damping term in the second equation. The system is complemented with initial conditions and non-local purely boundary integral conditions. At the beginning of 1963, Cannon [36] was the first to investigate a non-local problem with a non-local constraint (energy specification) of the form $\int_0^l \chi(x)U(x,t)dt = \tau(t)$, where $\chi(x)$ and $\tau(t)$ are given functions. This kind of non-local conditions may represent a mean, a total flux, and total energy. It has many applications in different fields of science and engineering, such as underground water flow, vibration problems, heat conduction, medical science, nuclear reactor dynamics, thermoelasticity, and plasma physics and control theory. We can cite, for example, [19,37–46]. The presence of non-local integral conditions could create great difficulties in different computations, especially in the case of fractional-order derivatives as in our case. For the investigation of our posed problem, we apply the energy method. This is shown through the introduction of some multiplier operators, some classical and fractional inequalities, and the establishment of some properties involving fractional derivatives. We also mention that in the paper [47], the author established two-sided a priori estimates for eigenvalues of real components of second-order operators with fractional derivatives in their lower terms. Furthermore, in the paper [48], the author obtained some a priori estimates for some classes of fractional partial differential equations; precisely, the fractional diffusion equation and the fractional wave equation.

To the best of our knowledge, the fractional system in Problems (6)–(9) has never been studied nor explored in the literature. This paper can be considered a contribution to the development of traditional functional analysis methods used to prove the well-posedness of mixed problems with integral boundary conditions. For some classical cases, the reader can refer to [37,49–52], and for some fractional cases, to [48,53–62]. We should also mention here that there are some important papers dealing with the numerical aspects for Timoshenko systems, with many applications, for which the reader can refer to [63–66]. There are also some papers dealing with Timoshenko system with fractional operators in the memory [67,68].

The paper is structured as follows: In Section 2, we set the problem and introduce different required function spaces. In Section 3, we provide some definitions and lemmas used throughout the following. Section 4 is devoted to the proof of the uniqueness and continuous dependence of the solutions to Problems (6)–(9) on the input data. Finally, in Section 5, we show and prove the existence of the solution and end with a conclusion.

2. Problem Setting and the Needed Function Spaces

Given the interval $I = (0, L)$, we consider the non-homogeneous fractional viscoelastic beam model (Timoshenko fractional system) with Timoshenko-type frictional damping

$$\begin{cases} \mathcal{L}_1(\theta,\phi) = \rho_1 \partial_t^{\alpha+1}\theta - \kappa_1(\theta_x+\phi)_x + \theta_t = F(x,t) \\ \mathcal{L}_2(\theta,\phi) = \rho_2 \partial_t^{\alpha+1}\phi - \kappa_2\phi_{xx} + \kappa_1(\theta_x+\phi) + \int_0^t m(t-s)\phi_{xx}(x,s)ds = G(x,t), \end{cases} \tag{6}$$

in the unknowns $(\theta,\phi) : (x,t) \in I \times [0,T] \to \mathbb{R}$, the strictly positive constants ρ_1, ρ_2, κ_1 and κ_2 satisfy the relation

$$\frac{\rho_1}{\kappa_1} = \frac{\rho_2}{\kappa_2},$$

and f, g, φ, ψ, F and G are given functions, and $m : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^2 class function such that

$$\kappa_2 - \int_0^T m(t)dt = l > 0, \quad m'(t) < 0, \ \forall t \geq 0. \tag{7}$$

System (6) is complemented with the initial conditions

$$\begin{cases} \Gamma_1\theta = \theta(x,0) = \varphi(x)\,,\ \Gamma_2\theta = \theta_t(x,0) = \psi(x)\,, \\ \Gamma_1\phi = \phi(x,0) = f(x)\,,\ \Gamma_2\phi = \phi_t(x,0) = g(x), \end{cases} \tag{8}$$

and the non-local boundary integral conditions

$$\int_0^L \theta dx = 0\,, \int_0^L x\theta dx = 0, \int_0^L \phi dx = 0\,, \int_0^L x\phi dx = 0. \tag{9}$$

Our main objective is to study the well-posedness of Problems (6), (8) and (9) by using the a priori estimate method (energy inequality method). We now introduce some function spaces needed throughout the following. Let $L^2(Q^T)$ be the Hilbert space of square integrable functions on $Q^T = (0,1) \times (0,T)$, $T < \infty$, with the respective scalar product and norm,

$$(Z,S)_{L^2(Q^T)} = \int_{Q^T} ZSdxdt \text{ and } \|Z\|^2_{L^2(Q^T)} = \int_{Q^T} Z^2 dxdt. \tag{10}$$

Let $B_2^1(0,L)$ be the Hilbert with the inner product

$$(\gamma,\gamma^*)_{B_2^1(0,L)} = \int_0^L \mathcal{I}_x\gamma.\Im_x\gamma^* dx,$$

where $\mathcal{I}_x\gamma = \int_0^x \gamma(\zeta)d\zeta$ for every fixed $x \in (0,L)$. The associated norm is $\|\gamma\|^2_{B_2^1(0,L)} = \sqrt{(\gamma,\gamma)_{B_2^1(0,L)}} = \int_0^L (\mathcal{I}_x\gamma)^2 dx$. We denote by $C(\bar{J}; L^2(0,L))$ with $J = (0,T)$ the set of all continuous functions $\gamma(.,t) : J \to L^2(0,L)$ with norm

$$\|\gamma\|^2_{C(J;L^2(0,L))} = \sup_{0 \le t \le T} \|\gamma(.,t)\|^2_{L^2(0,L)} < \infty, \tag{11}$$

and $C(\bar{J}; B_2^1(0,L))$ the set of functions $\gamma(.,t) : \bar{J} \to B_2^1(0,L)$ with norm

$$\|\gamma\|^2_{C(\bar{J};B_2^1(0,L))} = \sup_{0 \le t \le T} \|\mathcal{I}_x\gamma(.,t)\|^2_{L^2(0,L)} = \sup_{0 \le t \le T} \|\gamma(.,t)\|^2_{B_2^1(0,L)} < \infty. \tag{12}$$

Problems (6), (8) and (9) can be written in the operator form: $\mathcal{G}\mathcal{Z} = H$ with $\mathcal{Z} = (\theta,\phi)$, $\mathcal{G}\mathcal{Z} = (\mathcal{S}_1(\theta,\phi), \mathcal{S}_2(\theta,\phi))$ and $H = (H_1, H_2)$ where

$$\begin{cases} \mathcal{S}_1(\theta,\phi) = \{\mathcal{L}_1(\theta,\phi), \Gamma_1\theta, \Gamma_2\theta\} \\ \mathcal{S}_2(\theta,\phi) = \{\mathcal{L}_2(\theta,\phi), \Gamma_1\phi, \Gamma_2\phi\} \\ H_1 = \{F,\varphi,\psi\},\ H_2 = \{G,f,g\}. \end{cases}, \tag{13}$$

The operator $\mathcal{G}$ is unbounded with domain definition $D(\mathcal{G})$ consisting of elements $(\theta,\phi) \in \left(L^2(\bar{J}; L^2(0,L))\right)^2$ such that $\theta_x, \phi_x, \theta_t, \phi_t, \theta_{tt}, \phi_{tt}, \theta_{xx}, \phi_{xx}$ belong to $L^2(\bar{J}; L^2(0,L))$, verifying initial and boundary conditions in (8) and (9). The operator $\mathcal{G} : \mathcal{B} \to \mathcal{E}$, where $\mathcal{B}$ is the Banach space obtained by completing $D(\mathcal{G})$ with respect to the norm

$$\|\mathcal{Z}\|^2_{\mathcal{B}} = \|\theta(.,t)\|^2_{C(\bar{J};L^2(0,L))} + \|\phi(.,t)\|^2_{C(\bar{J};L^2(0,L))}. \tag{14}$$

Furthermore, $\mathcal{E} = \left[L^2(Q^T) \times (L^2(0,L))^2\right] \times \left[L^2(Q^T) \times (L^2(0,L))^2\right]$ is the Hilbert space consisting of vector-valued functions $H = (\{F,\varphi,\psi\}, \{G,f,g\})$ for which the norm

$$\begin{aligned} \|H\|^2_{\mathcal{E}} &= \|F\|^2_{L^2(Q^T)} + \|\varphi\|^2_{L^2(0,L)} + \|\psi\|^2_{L^2(0,L)} + \|G\|^2_{L^2(Q^T)} \\ &\quad + \|f\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)}. \end{aligned} \tag{15}$$

is finite.

3. Preliminaries (Definitions and Lemmas)

In this section we provide some definitions and lemmas needed to establish different proofs in the following.

Definition 1 ([69]). *The time fractional derivative of order β, with $\beta \in (1,2)$ for a function V is defined by*

$$^{C}\partial_t^{\beta} V(x,t) = \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{V_{\tau\tau}(x,\tau)}{(t-\tau)^{\beta-1}} d\tau, \tag{16}$$

and for $\beta \in (0,1)$ it is defined by

$$^{C}\partial_t^{\beta} V(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{V_{\tau}(x,\tau)}{(t-\tau)^{\beta}} d\tau$$

where $\Gamma(1-\beta)$ is the Gamma function.

Definition 2 ([70]). *The fractional Riemann–Liouville integral of order $0 < \beta < 1$ [1] is given by*

$$D_t^{-\beta} v(x,t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{v(x,\tau)}{(t-\tau)^{1-\beta}} d\tau. \tag{17}$$

Lemma 1 ([71]). *Let $E(s)$ be non-negative and absolutely continuous on $[0,T]$, and suppose that for almost all $s \in [0,T]$, R satisfies the inequality*

$$\frac{dE}{ds} \leq A(s)E(s) + B(s), \tag{18}$$

where the functions $A(s)$ and $B(s)$ are summable and non-negative on $[0,T]$. Then

$$E(s) \leq \exp\left\{ \int_0^s A(t)dt \right\} \left(E(0) + \int_0^s B(t)dt \right). \tag{19}$$

Lemma 2 ([48]). *For any absolutely continuous function $v(t)$ on $[0,T]$, the following inequality holds*

$$v(t)\, \partial_t^{\alpha} v(t) \geq \frac{1}{2} \partial_{0t}^{\alpha} v^2(t), \quad 0 < \alpha < 1. \tag{20}$$

Lemma 3 ([48]). *Let a non-negative absolutely continuous function $\mathcal{Q}(t)$ satisfy the inequality*

$$^{C}\partial_t^{\beta} \mathcal{Q}(t) \leq b_1 \mathcal{Q}(t) + b_2(t), \quad 0 < \beta < 1, \tag{21}$$

for almost all $t \in [0,T]$, where b_1 is a positive constant and $b_2(t)$ is an integrable non-negative function on $[0,T]$. Then

$$\mathcal{Q}(t) \leq \mathcal{Q}(0) E_{\beta}(b_1 t^{\beta}) + \Gamma(\beta) E_{\beta,\beta}(b_1 t^{\beta}) D_t^{-\beta} b_2(t), \tag{22}$$

where

$$E_{\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + 1)} \text{ and } E_{\beta,\mu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + \mu)},$$

are the Mittag–Leffler functions.

4. The A Priori Estimate

In this section an a priori estimate will be established in order to obtain the uniqueness and continuous dependence of the solution to Problems (6), (8) and (9) on the given data.

Theorem 1. *For any function $\mathcal{Z} = (\theta, \phi) \in D(\mathcal{G})$, the following a priori estimates hold*

$$\begin{aligned} & \|\theta(.,t)\|^2_{C(\bar{J};L^2(0,L))} + \|\phi(.,t)\|^2_{C(\bar{J};L^2(0,L))} \\ \leq \; & \mathcal{F}^*\Big(\|\varphi\|^2_{L^2(0,L)} + \|\psi\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)} + \|f\|^2_{L^2(0,L)} \\ & +\|F\|^2_{L^2(0,T;L^2(0,L))} + \|G\|^2_{L^2(0,T;L^2(0,L))}\Big), \end{aligned} \tag{23}$$

and

$$\begin{aligned} & D^{\alpha-1}\Big(\|\theta_t\|_{B_2^1(0,L)} + \|\phi_t\|_{B_2^1(0,L)}\Big) \\ \leq \; & \mathcal{F}^*\Big(\|\varphi\|^2_{L^2(0,L)} + \|\psi\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)} + \|f\|^2_{L^2(0,L)} \\ & +\|F\|^2_{L^2(0,T;L^2(0,L))} + \|G\|^2_{L^2(0,T;L^2(0,L))}\Big). \end{aligned} \tag{24}$$

where $\mathcal{F}^*$ is a positive constant independent of $\mathcal{Z} = (\theta, \phi)$ given by

$$\mathcal{F}^* = \mathcal{M}\omega \max\left\{1, \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right\},$$

with

$$\begin{aligned} \mathcal{M} &= \Gamma(\alpha)E_{\alpha,\alpha}(\omega t^\alpha)\left(\max\left\{1, \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right\}\right) \\ \omega &= \mathcal{W}^*(\mathcal{W}^* e^{\mathcal{W}^* T} + 1), \end{aligned}$$

and $\mathcal{W}^*$ is given by (46). The free terms F and G belong to $L^2(0,T;L^2(0,L))$, and the functions φ, ψ, f and gbelong to $L^2(0,L)$.

Proof. Define the integro-differential operators $\mathcal{M}_1\theta = -\mathcal{I}_x^2\theta_t$ and $\mathcal{M}_2\phi = -\mathcal{I}_x^2\phi_t$, where

$$\mathcal{I}_x^2\theta(x,t) = \int_0^x\int_0^\xi \theta(\eta,t)d\eta d\xi,\; \mathcal{I}_x^2\phi(x,t) = \int_0^x\int_0^\xi \phi(\eta,t)d\eta d\xi,$$

and consider the identity

$$\begin{aligned} & \left(\rho_1\partial_t^{\alpha+1}\theta, \mathcal{M}_1\theta\right)_{L^2(0,L)} - \kappa_1((\theta_x+\phi)_x, \mathcal{M}_1\theta)_{L^2(0,L)} + (\theta_t, \mathcal{M}_1\theta)_{L^2(0,L)} \\ & + (\rho_2\partial_t^{\alpha+1}\phi, \mathcal{M}_2\phi)_{L^2(0,L)} - \kappa_2((\phi_{xx}, \mathcal{M}_2\phi)_{L^2(0,L)} + \kappa_1((\theta_x+\phi), \mathcal{M}_2\phi)_{L^2(0,L)} \\ + & \left(\int_0^t m(t-s)\phi_{xx}(x,s)ds, \mathcal{M}_2\phi\right)_{L^2(0,L)} \\ = & (F(x,t), \mathcal{M}_1\theta)_{L^2(0,L)} + (G(x,t), \mathcal{M}_2\phi)_{L^2(0,L)}. \end{aligned} \tag{25}$$

The integration by parts of each term in (25) and the conditions in (8) and (9) give

$$\begin{aligned} \left(\rho_1\partial_t^{\alpha+1}\theta, \mathcal{M}_1\theta\right)_{L^2(0,L)} &= \frac{\rho_1}{2}(\partial_t^\alpha(\mathcal{I}_x\theta_t), \mathcal{I}_x\theta_t)_{L^2(0,L)} \\ &\geq \frac{\rho_1}{2}\partial_t^\alpha\|\mathcal{I}_x\theta_t\|_{L^2(0,L)}, \end{aligned} \tag{26}$$

$$\begin{aligned}\left(\rho_2\partial_t^{\alpha+1}\phi,\mathcal{M}_2\phi\right)_{L^2(0,L)} &= \frac{\rho_2}{2}(\partial_t^{\alpha}(\mathcal{I}_x\phi_t),\mathcal{I}_x\phi_t)_{L^2(0,L)}\\ &\geq \frac{\rho_2}{2}\partial_t^{\alpha}\|\mathcal{I}_x\phi_t\|_{L^2(0,L)},\end{aligned} \tag{27}$$

$$-(\theta_t,\mathcal{M}_1\theta)_{L^2(0,L)} = \|\mathcal{I}_x\theta_t\|^2_{L^2(0,L)}, \tag{28}$$

$$\begin{aligned}\kappa_1(\theta_{xx},\mathcal{I}_x^2\theta_t)_{L^2(0,L)} &= \kappa_1\mathcal{I}_x^2\theta_t.\theta_x]_0^L-\kappa_1\int_0^L\mathcal{I}_x\theta_t.\theta_x dx=\kappa_1\int_0^L\theta_t\theta dx\\ &=\frac{\kappa_1}{2}\frac{\partial}{\partial t}\|\theta\|^2_{L^2(0,L)},\end{aligned} \tag{29}$$

and in the same manner, we have

$$\kappa_2(\phi_{xx},\mathcal{I}_x^2\phi_t)_{L^2(0,L)} = \frac{\kappa_2}{2}\frac{\partial}{\partial t}\|\phi\|^2_{L^2(0,L)}, \tag{30}$$

$$\begin{aligned}\kappa_1(\phi_x,\mathcal{I}_x^2\theta_t)_{L^2((0,L))} &= \kappa_1\mathcal{I}_x^2\theta_t.\phi]_0^L dt-\kappa_1\int_0^L\mathcal{I}_x\theta_t.\phi dx\\ &=-\kappa_1\int_0^L\mathcal{I}_x\theta_t.\phi dx,\end{aligned} \tag{31}$$

$$-\kappa_1(\theta_x,\mathcal{I}_x^2\phi_t)_{L^2(Q^\tau)} = \kappa_1\int_0^L\mathcal{I}_x\phi_t.\theta dx, \tag{32}$$

$$\begin{aligned}&-\kappa_1(\phi,\mathcal{I}_x^2\phi_t)_{L^2(0,L)}\\ &=-\kappa_1\mathcal{I}_x^2\phi_t.\Im_x\phi]_0^L+\kappa_1\int_0^\tau\int_0^L\mathcal{I}_x\phi_t.\mathcal{I}_x\phi dx dt\\ &=\frac{1}{2}\frac{\partial}{\partial t}\|\mathcal{I}_x\phi\|^2_{L^2(0,L)},\end{aligned} \tag{33}$$

$$\begin{aligned}&-\left(\int_0^t m(t-s).\phi_{xx}(x,s)ds,\mathcal{I}_x^2\phi_t\right)_{L^2(0,L)}\\ &=-\int_0^L\left(\int_0^t m(t-s).\phi_{xx}(x,s)ds\right)\mathcal{I}_x^2\phi_t dx\\ &=-\int_0^t m(t-s).\phi_x(x,s)ds).\mathcal{I}_x^2\phi_t]_0^L dx+\int_0^L\left(\int_0^t m(t-s).\phi_x(x,s)ds\right)\mathcal{I}_x\phi_t dx\\ &=\left(\int_0^t m(t-s).\phi(x,s)ds)\right)\mathcal{I}_x\phi_t]_0^L dx-\int_0^L\left(\int_0^t m(t-s).\phi(x,s)ds\right)\phi_t dx\\ &=-\int_0^L\left(\int_0^t m(t-s).\phi(x,s)ds\right)\phi_t dx.\end{aligned} \tag{34}$$

Substituting Equalities (26)–(34) into (25), we obtain

$$\begin{aligned}
&\frac{\rho_1}{2}\partial_t^\alpha \|\mathcal{I}_x\theta_t\|_{L^2(0,L)} + \frac{\rho_2}{2}\partial_t^\alpha \|\mathcal{I}_x\phi_t\|_{L^2(0,L)} + \frac{\kappa_1}{2}\frac{\partial}{\partial t}\|\theta\|^2_{L^2(0,L)} \\
&+ \frac{\kappa_2}{2}\frac{\partial}{\partial t}\|\phi\|^2_{L^2(0,L)} + \|\mathcal{I}_x\theta_t\|^2_{L^2(0,L)} + \frac{1}{2}\frac{\partial}{\partial t}\|\mathcal{I}_x\phi\|^2_{L^2(0,L)} \\
=&\,\kappa_1 \int_0^L \phi\mathcal{I}_x\theta_t dx - \kappa_1 \int_0^L \theta I_x\phi_t dx - \int_0^L \left(\int_0^t m(t-s).\phi(x,s)ds\right)\phi_t dx \\
&- \int_0^L F\mathcal{I}_x^2\theta_t dx - \int_0^L G\mathcal{I}_x^2\phi_t dx.
\end{aligned} \tag{35}$$

Upon integrating in (35) and using the given conditions, we obtain

$$\begin{aligned}
&\frac{\rho_1}{2}D^{\alpha-1}\|\mathcal{I}_x\theta_t\|_{L^2(0,L)} + \frac{\rho_2}{2}D^{\alpha-1}\|\mathcal{I}_x\phi_t\|_{L^2(0,L)} + \frac{\kappa_1}{2}\|\theta(.,t)\|^2_{L^2(0,L)} \\
&+ \frac{\kappa_2}{2}\|\phi(.,t)\|^2_{L^2(0,L)} + \|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))} + \frac{1}{2}\|\mathcal{I}_x\phi(.,t)\|^2_{L^2(0,L)} \\
=&\,\kappa_1(\phi,\mathcal{I}_x\theta_\tau)_{L^2(0,t;L^2(0,L))} - \kappa_1(\theta,\mathcal{I}_x\phi_\tau)_{L^2(0,t;L^2(0,L))} \\
&- \left(F,\mathcal{I}_x^2\theta_\tau\right)_{L^2(0,t;L^2(0,L))} - \left(G,\mathcal{I}_x^2\phi_\tau\right)_{L^2(0,t;L^2(0,L))} \\
&+ \frac{\rho_1 t^{1-\alpha}}{2(\Gamma(1-\alpha)(1-\alpha))}\|\mathcal{I}_x\psi\|^2_{L^2(0,L)} + \frac{\kappa_1}{2}\|\varphi\|^2_{L^2(0,L)} \\
&+ \frac{\rho_2 t^{1-\alpha}}{2(\Gamma(1-\alpha)(1-\alpha))}\|\mathcal{I}_x g\|^2_{L^2(0,L)} + \frac{\kappa_2}{2}\|f\|^2_{L^2(0,L)} \\
&+ \frac{1}{2}\|\mathcal{I}_x f\|^2_{L^2(0,L)} - \int_0^t\int_0^L\left(\int_0^\tau m(\tau-s).\phi(x,s)ds\right)\phi_\tau dx d\tau.
\end{aligned} \tag{36}$$

The last term on the right-hand side needs to be evaluated as follows

$$\begin{aligned}
&-\int_0^t\int_0^L\left(\int_0^\tau m(\tau-s).\phi(x,s)ds\right)\phi_\tau dx d\tau \\
=&-\int_0^L\left(\int_0^\tau m(\tau-s).\phi(x,s)ds\right)\phi dx]_0^t + \int_0^\tau\int_0^L m(0)\phi^2 dx d\tau \\
&+\int_0^t\int_0^L\left(\int_0^\tau m'(\tau-s).\phi(x,s)ds\right)\phi(x,\tau)dx d\tau \\
=&-\int_0^L\left(\int_0^t m(t-s).\phi(x,s)ds\right)\phi(x,t)dx + m(0)\|\phi\|^2_{L^2(0,t;L^2(0,L))} \\
&+\int_0^t\int_0^L\left(\int_0^\tau m'(\tau-s).\phi(x,s)ds\right)\phi dx d\tau.
\end{aligned} \tag{37}$$

We then replace (37) with (36), and estimate the different terms on the right-hand side of (36) (using the Cauchy ϵ inequality, a Poincaré-type inequality) as follows

$$\begin{aligned}
&\kappa_1(\phi,\mathcal{I}_x\theta_t)_{L^2(0,t;L^2(0,L))} \\
\le&\frac{\kappa_1\epsilon_1}{2}\|\phi\|^2_{L^2(0,t;L^2(0,L))} + \frac{\kappa_1}{2\epsilon_1}\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))},
\end{aligned} \tag{38}$$

$$
\begin{aligned}
&-\kappa_1(\theta, \mathcal{I}_x\phi_\tau)_{L^2(0,t;L^2(0,L))} \\
&\leq \frac{\kappa_1\epsilon_2}{2}\|\theta\|^2_{L^2(0,t;L^2(0,L))} + \frac{\kappa_1}{2\epsilon_2}\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))},
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&-\int_0^L \left(\int_0^t m(t-s).\phi(x,s)ds \right) \phi(x,t)dx \\
&\leq \frac{\epsilon_3}{2}\int_0^L \phi^2(x,t)dx + \frac{1}{2\epsilon_3}\int_0^L \left(\int_0^t m(t-s)\phi(x,s)ds \right)^2 dx \\
&\leq \frac{\epsilon_3}{2}\int_0^L \phi^2(x,t)dx + \frac{1}{2\epsilon_3}\int_0^L \left(\int_0^t m^2(t-s)ds \right)\left(\int_0^t \phi^2(x,s)ds \right) dx \\
&\leq \frac{\epsilon_3}{2}\int_0^L \phi^2(x,t)dx + \frac{T}{2\epsilon_3}\sup_{0\leq t\leq T} m^2(t)\int_0^L\int_0^t \phi^2 dxd\tau,
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&\int_0^t\int_0^L \left(\int_0^\tau m'(\tau-s).\phi(x,s)ds \right) \phi dxd\tau \\
&\leq \frac{\epsilon_4}{2}\int_0^t\int_0^L \phi^2 dxd\tau + \frac{1}{2\epsilon_4}\int_0^t\int_0^L \left(\int_0^\tau m'^2(\tau-s)ds \right)\left(\int_0^\tau \phi^2(x,s)ds \right) dxd\tau \\
&\leq \frac{\epsilon_4}{2}\int_0^t\int_0^L \phi^2 dxd\tau + \frac{T}{2\epsilon_4}\sup_{0\leq t\leq T} m'^2(t)\int_0^t\int_0^L \left(\int_0^\tau \phi^2(x,s)ds \right) dxd\tau \\
&= \frac{\epsilon_4}{2}\int_0^t\int_0^L \phi^2 dxd\tau + \frac{T}{2\epsilon_4}\sup_{0\leq t\leq T} m'^2(t)\int_0^L \left[\left(\tau\int_0^\tau \phi^2(x,s)ds \right)_0^t - \int_0^t \tau\phi^2 d\tau \right] dx \\
&= \frac{\epsilon_4}{2}\int_0^t\int_0^L \phi^2 dxd\tau + \frac{T}{2\epsilon_4}\sup_{0\leq t\leq T} m'^2(t)\int_0^L \left[\int_0^t (t-\tau)\phi^2(x,\tau)d\tau \right] dx \\
&\leq \frac{\epsilon_4}{2}\int_0^t\int_0^L \phi^2 dxd\tau + \frac{T^2}{2\epsilon_4}\sup_{0\leq t\leq T} m'^2(t)\int_0^L\int_0^t \phi^2 d\tau dx \\
&= \left(\frac{\epsilon_4}{2} + \frac{T^2}{2\epsilon_4}\sup_{0\leq t\leq T} m'^2(t) \right)\int_0^t\int_0^L \phi^2 dxd\tau,
\end{aligned} \tag{41}
$$

$$
-\left(F, \mathcal{I}_x^2\theta_\tau\right)_{L^2(0,t;L^2(0,L))} \leq \frac{\epsilon_5}{2}\|F\|^2_{L^2(0,t;L^2(0,L))} + \frac{L^2}{2\epsilon_5}\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}, \tag{42}
$$

$$
-\left(G, \mathcal{I}_x^2\phi_\tau\right)_{L^2(0,t;L^2(0,L))} \leq \frac{\epsilon_6}{2}\|G\|^2_{L^2(0,t;L^2(0,L))} + \frac{L^2}{4\epsilon_6}\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))}, \tag{43}
$$

A combination of (38)–(43) and (36), leads to

$$
\begin{aligned}
&\frac{\rho_1}{2}D^{\alpha-1}\|\mathcal{I}_x\theta_t\|_{L^2(0,L)}+\frac{\rho_2}{2}D^{\alpha-1}\|\mathcal{I}_x\phi_t\|_{L^2(0,L)}+\frac{\kappa_1}{2}\|\theta(.,t)\|^2_{L^2(0,L)}\\
&+\frac{\kappa_2}{2}\|\phi(.,t)\|^2_{L^2(0,L)}+\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\frac{1}{2}\|\mathcal{I}_x\phi(.,t)\|^2_{L^2(0,L)}\\
\leq&\frac{\rho_1T^{1-\alpha}L^2}{4(\Gamma(1-\alpha)(1-\alpha))}\|\psi\|^2_{L^2(0,L)}+\frac{\kappa_1}{2}\|\varphi\|^2_{L^2(0,L)}\\
&+\frac{\rho_2T^{1-\alpha}L^2}{4(\Gamma(1-\alpha)(1-\alpha))}\|g\|^2_{L^2(0,L)}+\left(\frac{\kappa_2}{2}+\frac{L^2}{4}\right)\|f\|^2_{L^2(0,L)}\\
&+\left(\frac{\kappa_1\epsilon_1}{2}+\frac{T}{2\epsilon_3}\sup_{0\leq t\leq T}m^2(t)+\frac{\epsilon_4}{2}+\frac{T^2}{2\epsilon_4}\sup_{0\leq t\leq T}m'^2(t)+m(0)\right)\|\phi\|^2_{L^2(0,t;L^2(0,L))}\\
&+\left(\frac{\kappa_1}{2\epsilon_1}+\frac{L^2}{2\epsilon_5}\right)\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\left(\frac{\kappa_1}{2\epsilon_2}+\frac{L^2}{4\epsilon_6}\right)\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))}\\
&+\frac{\kappa_1\epsilon_2}{2}\|\theta\|^2_{L^2(0,t;L^2(0,L))}+\frac{\epsilon_3}{2}\|\phi(.,t)\|^2_{L^2(0,L)}+\frac{\epsilon_5}{2}\|F\|^2_{L^2(0,t;L^2(0,L))}\\
&+\frac{\epsilon_6}{2}\|G\|^2_{L^2(0,t;L^2(0,L))}.
\end{aligned}
\tag{44}
$$

The choice $\varepsilon_1=\kappa_1$, $\varepsilon_5=L^2/2$, $\varepsilon_3=\kappa_2/2$, $\varepsilon_2=\varepsilon_4=\varepsilon_6=1$, and cancellation of the last term on the left-hand side of (44) reduces it to the following estimate

$$
\begin{aligned}
&D^{\alpha-1}\|\mathcal{I}_x\theta_t\|_{L^2(0,L)}+D^{\alpha-1}\|\mathcal{I}_x\phi_t\|_{L^2(0,L)}+\|\theta(.,t)\|^2_{L^2(0,L)}+\|\phi(.,t)\|^2_{L^2(0,L)}\\
\leq&\mathcal{W}^*\Big(\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))}+\|\theta\|^2_{L^2(0,t;L^2(0,L))}+\|\phi\|^2_{L^2(0,t;L^2(0,L))}\\
&+\|\varphi\|^2_{L^2(0,L)}+\|\psi\|_{L^2(0,L)}+\|f\|^2_{L^2(0,L)}+\|g\|^2_{L^2(0,L)}+\|F\|^2_{L^2(0,t;L^2(0,L))}\\
&+\|G\|^2_{L^2(0,t;L^2(0,L))}\Big),
\end{aligned}
\tag{45}
$$

where

$$
\begin{aligned}
\mathcal{W}^*=&\max\left(\frac{\kappa_1^2}{2}+\frac{T}{\kappa_2}\sup_{0\leq t\leq T}m^2(t)+\frac{1}{2}+\frac{T^2}{2}\sup_{0\leq t\leq T}m'^2(t)+m(0),\ \frac{3}{2},\ \frac{\kappa_1}{2}+\frac{L^2}{4},\frac{\kappa_2}{4},\right.\\
&\left.\frac{\rho_1T^{1-\alpha}L^2}{4(\Gamma(1-\alpha)(1-\alpha))},\frac{\rho_2T^{1-\alpha}L^2}{4(\Gamma(1-\alpha)(1-\alpha))}\right)/\min\left(\frac{\rho_1}{2},\frac{\rho_2}{2},\frac{\kappa_1}{2},\frac{\kappa_2}{2},\frac{1}{2}\right).
\end{aligned}
\tag{46}
$$

By omitting the first and second terms on the left-hand side of (45), and applying the Gronwall–Bellman lemma ([72]), where

$$
\begin{cases}
E(t)=\|\theta\|^2_{L^2(0,t;L^2(0,L))}+\|\phi\|^2_{L^2(0,t;L^2(0,L))}\\
\frac{dE}{dt}=\|\theta(.,t)\|^2_{L^2(0,L)}+\|\phi(.,t)\|^2_{L^2(0,L)},\\
\mathcal{Q}(0)=0.
\end{cases}
\tag{47}
$$

we obtain

$$
\begin{aligned}
E(t)\quad\leq\quad&\mathcal{W}^*e^{\mathcal{W}^*t}\Big(\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))}\\
&+\|\varphi\|^2_{L^2(0,L)}+\|\psi\|_{L^2(0,L)}+\|f\|^2_{L^2(0,L)}+\|g\|^2_{L^2(0,L)}\\
&+\|F\|^2_{L^2(0,t;L^2(0,L))}+\|G\|^2_{L^2(0,t;L^2(0,L))}\Big).
\end{aligned}
\tag{48}
$$

Then by discarding the last two terms on the left-hand side of (45), and using (48), we have

$$
\begin{aligned}
& D^{\alpha-1}\Big(\|\mathcal{I}_x\theta_t\|_{L^2(0,L)} + \|\mathcal{I}_x\phi_t\|_{L^2(0,L)}\Big) \\
\leq\ & \mathcal{W}^*(\mathcal{W}^* e^{\mathcal{W}^* T}+1)\Big(\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))} \\
& + \|\varphi\|^2_{L^2(0,L)} + \|\psi\|_{L^2(0,L)} + \|f\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)} \\
& + \|F\|^2_{L^2(0,t;L^2(0,L))} + \|G\|^2_{L^2(0,t;L^2(0,L))}\Big).
\end{aligned} \tag{49}
$$

Now, Lemma 3, can be applied to remove the first two terms on the right-hand side of (49), by taking

$$
\begin{cases}
\mathcal{Q}(t) = \|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))} \\
{}^C\partial_t^\beta \mathcal{Q}(t) = D^{\alpha-1}\Big(\|\mathcal{I}_x\theta_t\|_{L^2(0,L)} + \|\mathcal{I}_x\phi_t\|_{L^2(0,L)}\Big) \\
\mathcal{Q}(0) = 0.
\end{cases} \tag{50}
$$

From (49), it follows that

$$
\begin{aligned}
\mathcal{Q}(t) \leq & \mathcal{M}\Big\{D^{-1-\alpha}\Big(\|F\|^2_{L^2(0,L)} + \|G\|^2_{L^2(0,L)}\Big) \\
& + \|\varphi\|^2_{L^2(0,L)} + \|\psi\|_{L^2(0,L)} + \|f\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)}\Big\},
\end{aligned} \tag{51}
$$

where

$$
\mathcal{M} = \Gamma(\alpha)E_{\alpha,\alpha}(\omega t^\alpha)\Big(\max\Big\{1, \frac{T^\alpha}{\alpha\Gamma(\alpha)}\Big\}\Big),
$$

with

$$
\omega = \mathcal{W}^*(\mathcal{W}^* e^{\mathcal{W}^* T}+1).
$$

Now, since

$$
D^{-1-\alpha}\Big(\|F\|^2_{L^2(0,L)} + \|G\|^2_{L^2(0,L)}\Big) \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\int_0^t \Big(\|F\|^2_{L^2(0,L)} + \|G\|^2_{L^2(0,L)}\Big)d\tau, \tag{52}
$$

then we infer from (51), (52) and (45) the following inequality

$$
\begin{aligned}
& D^{\alpha-1}\|\mathcal{I}_x\theta_t\|_{L^2(0,L)} + D^{\alpha-1}\|\mathcal{I}_x\phi_t\|_{L^2(0,L)} + \|\theta(.,t)\|^2_{L^2(0,L)} + \|\phi(.,t)\|^2_{L^2(0,L)} \\
\leq & \mathcal{F}^*\Big(\|\psi\|^2_{L^2(0,L)} + \|\varphi\|^2_{L^2(0,L)} + \|f\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)} \\
& + \|F\|^2_{L^2(0,T;L^2(0,L))} + \|G\|^2_{L^2(0,T;L^2(0,L))}\Big).
\end{aligned} \tag{53}
$$

□

The first estimate in (24) follows, if we disregard the first and second terms on the left-hand side of (53), and passes to the supremum over $(0,T)$. The second estimate in (25) follows from (53) if we drop the last two terms on the left-hand side of Inequality (53).

Since the range of the operator $\mathcal{G}$ is a subset of $\mathcal{E}$, that is $R(\mathcal{G}) \subset \mathcal{E}$, we extend $\mathcal{G}$ so that Inequality (53) holds for the extension, and $R(\overline{\mathcal{G}}) = \mathcal{E}$. We can easily show the following

Proposition 1. *The operator $\mathcal{G} : \mathcal{B} \to \mathcal{E}$ has a closure $\overline{\mathcal{G}}$ with domain $D(\overline{\mathcal{G}})$.*

Definition 3. *We call the solution to the equation $\overline{\mathcal{G}}\,\mathcal{Z} = H = (\{F,\varphi,\psi\},\{G,f,g\})$ a strong solution to Problems (6), (8), and (9).*

The a priori estimate in (24) can be extended to

$$\|\mathcal{Z}\|^2_{\mathcal{B}} \leq \mathcal{F}^* \|\overline{\mathcal{G}}\,\mathcal{Z}\|^2_{\mathcal{E}}, \quad \forall \mathcal{Z} \in D(\overline{\mathcal{G}}). \tag{54}$$

The estimate in (54) shows that the operator $\overline{\mathcal{G}}$ is one-to-one and that $\overline{\mathcal{G}}^{-1}$ is continuous from $R(\overline{\mathcal{G}})$ onto $\mathcal{B}$. Consequently, if a strong solution to Problems (6), (8) and (9) exists, it is unique and depends continuously on the input data φ, ψ, f, g and the external forces F and G. Furthermore, as a consequence of (54), the set $R(\overline{\mathcal{G}}) \subset \mathcal{E}$ is closed and $R(\overline{\mathcal{G}}) = \overline{R(\mathcal{G})}$.

5. Existence of Solution

Theorem 2. *Problems (6), (8) and (9) allow a unique strong solution* $\mathcal{Z} = \left(\overline{\mathcal{G}}\right)^{-1}(\{F, \varphi, \psi\}, \{G, f, g\}) = \overline{\mathcal{G}^{-1}}(\{F, \varphi, \psi\}, \{G, f, g\})$*, which continuously depends on the given data, for all* $F, G \in L^2(0, T; L^2(0, L))$*, and* $\varphi, \psi, f, g \in L^2(0, L)$*.*

Proof. From the above, we see that in order to show the existence of the generalized solution to Problems (6), (8), and (9), it suffices to prove that $\overline{R(\mathcal{G})} = \mathcal{E}$. Let us first prove the following. □

Theorem 3. *(Density in a special case). If for some function* $W(x,t) = (\Lambda_1(x,t), \Lambda_2(x,t)) \in (L^2(0, T; L^2(0, L)))^2$ *and for elements* $\mathcal{Z} \in D_0(\mathcal{G}) = \{\mathcal{Z} : \mathcal{Z} \in D(\mathcal{G}) \text{ and } \Gamma_i\theta = \Gamma_i\phi = 0\,,\ i = 1,2\,\}$ *we have*

$$(\mathcal{S}_1(\theta, \phi), \Lambda_1)_{L^2(0,T;L^2(0,L))} + (\mathcal{S}_2(\theta, \phi), \Lambda_2)_{L^2(0,T;L^2(0,L))} = 0, \tag{55}$$

then $W(x,t) = (\Lambda_1(x,t), \Lambda_2(x,t)) = (0,0)$ *almost everywhere in* Q^T*.*

Proof. Identity (55) is equivalent to

$$\begin{aligned}
&\int_0^T (\rho_1 \partial_t^{\alpha+1}\theta, \Lambda_1)_{L^2(0,L)}dt - \kappa_1 \int_0^T (\theta_{xx}, \Lambda_1)_{L^2(0,L)}dt - \kappa_1 \int_0^T (\phi_x, \Lambda_1)_{L^2(0,L)}dt + \int_0^T (\theta_t, \Lambda_1)_{L^2(0,L)}dt \\
&+ \int_0^T (\rho_2 \partial_t^{\alpha+1}\phi, \Lambda_2)_{L^2(0,L)}dt - \kappa_2 \int_0^T (\phi_{xx}, \Lambda_2)_{L^2(0,L)}dt + \kappa_1 \int_0^T ((\theta_x, \Lambda_2)_{L^2(0,L)}dt \\
&+ \kappa_1 \int_0^T (\phi, \Lambda_2)_{L^2(0,L)}dt + \int_0^T (\int_0^t m(t-s)\phi_{xx}(x,s)ds, \Lambda_2)_{L^2(0,L)}dsdt \\
=&\, 0.
\end{aligned} \tag{56}$$

Assume that the functions $\xi(x,t)$ and $\eta(x,t)$ satisfy the boundary and initial conditions in (8) and (9), such that ξ, η, ξ_x, η_x, $\mathcal{I}_t\xi$, $\mathcal{I}_t\eta$, $\mathcal{I}_t\mathcal{I}_x^2\xi$, $\mathcal{I}_t\mathcal{I}_x^2\eta$, $\mathcal{I}_t^2\xi$, $\mathcal{I}_t^2\eta$ and $\partial_t^{\beta+1}\xi$, $\partial_t^{\beta+1}\eta \in L^2(0, T; L^2(0, L))$. We then set

$$\theta(x,t) = \mathcal{I}_t^2\xi = \int_0^t \int_0^s \xi(x,z)dzds,\ \phi(x,t) = \mathcal{I}_t^2\eta = \int_0^t \int_0^s \eta(x,z)dzds, \tag{57}$$

and introduce the functions

$$\Lambda_1(x,t) = \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi,\ \Lambda_2(x,t) = \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta. \tag{58}$$

Equation (56) then reduces to

$$\begin{aligned}
&\int_0^T (\rho_1 \partial_t^{\alpha+1} \mathcal{I}_t^2 \xi, \mathcal{I}_t \xi + \mathcal{I}_x^2 \mathcal{I}_t \xi)_{L^2(0,L)} dt - \kappa_1 \int_0^T (\mathcal{I}_t^2 \xi_{xx}, \mathcal{I}_t \xi + \mathcal{I}_x^2 \mathcal{I}_t \xi)_{L^2(0,L)} dt \\
&- \kappa_1 \int_0^T (\mathcal{I}_t^2 \eta_x, \mathcal{I}_t \xi + \mathcal{I}_x^2 \mathcal{I}_t \xi)_{L^2(0,L)} dt + \int_0^T (\mathcal{I}_t \xi, \mathcal{I}_t \xi + \mathcal{I}_x^2 \mathcal{I}_t \xi)_{L^2(0,L)} dt \\
&+ \int_0^T (\rho_2 \partial_t^{\alpha+1} \mathcal{I}_t^2 \eta, \mathcal{I}_t \eta + \mathcal{I}_x^2 \mathcal{I}_t \eta)_{L^2(0,L)} dt - \kappa_2 \int_0^T (\mathcal{I}_t^2 \eta_{xx}, \mathcal{I}_t \eta + \mathcal{I}_x^2 \mathcal{I}_t \eta)_{L^2(0,L)} dt \\
&+ \kappa_1 \int_0^T ((\mathcal{I}_t^2 \xi_x, \mathcal{I}_t \eta + \mathcal{I}_x^2 \mathcal{I}_t \eta)_{L^2(0,L)} dt + \kappa_1 \int_0^T (\mathcal{I}_t^2 \eta, \mathcal{I}_t \eta + \mathcal{I}_x^2 \mathcal{I}_t \eta)_{L^2(0,L)} dt \\
&+ \int_0^T (\int_0^t m(t-s) \mathcal{I}_s^2 \eta_{xx}(x,s) ds, \mathcal{I}_t \eta + \mathcal{I}_x^2 \mathcal{I}_t \eta)_{L^2(0,L)} dt \\
&= 0.
\end{aligned} \tag{59}$$

Invoking boundary conditions and calculating different terms by integrations by parts, we have

$$\begin{aligned}
&(\rho_1 \partial_t^{\alpha+1} \mathcal{I}_t^2 \xi, \mathcal{I}_t \xi + \mathcal{I}_x^2 \mathcal{I}_t \xi)_{L^2(0,L)} \\
&= (\rho_1 \partial_t^{\alpha} \mathcal{I}_t \xi, \mathcal{I}_t \xi) + (\rho_1 \partial_t^{\alpha} \mathcal{I}_x \mathcal{I}_t \xi, \mathcal{I}_x \mathcal{I}_t \xi)_{L^2(0,L)},
\end{aligned} \tag{60}$$

$$\begin{aligned}
&- \kappa_1 (\mathcal{I}_t^2 \xi_{xx}, \mathcal{I}_t \xi + \mathcal{I}_x^2 \mathcal{I}_t \xi)_{L^2(0,L)} \\
&= \frac{\kappa_1 \partial}{2\partial t} \|\mathcal{I}_t^2 \xi_x\|_{L^2(0,L)}^2 + \frac{\kappa_1 \partial}{2\partial t} \|\mathcal{I}_t^2 \xi\|_{L^2(0,L)}^2,
\end{aligned} \tag{61}$$

$$\begin{aligned}
&- \kappa_1 (\mathcal{I}_t^2 \eta_x, \mathcal{I}_t \xi + \mathcal{I}_x^2 \mathcal{I}_t \xi)_{L^2(0,L)} \\
&= - \kappa_1 (\mathcal{I}_t^2 \eta_x, \mathcal{I}_t \xi)_{L^2(0,L)} + \kappa_1 (\mathcal{I}_t^2 \eta, \mathcal{I}_x \mathcal{I}_t \xi)_{L^2(0,L)},
\end{aligned} \tag{62}$$

$$\begin{aligned}
&(\mathcal{I}_t \xi, \mathcal{I}_t \xi + \mathcal{I}_x^2 \mathcal{I}_t \xi)_{L^2(0,L)} \\
&= \|\Im_t \xi\|_{L^2(0,L)}^2 - \|\mathcal{I}_x \Im_t \xi\|_{L^2(0,L)}^2,
\end{aligned} \tag{63}$$

$$\begin{aligned}
&(\rho_2 \partial_t^{\alpha+1} \mathcal{I}_t^2 \eta, \mathcal{I}_t \eta + \mathcal{I}_x^2 \mathcal{I}_t \eta)_{L^2(0,L)} \\
&= (\rho_2 \partial_t^{\alpha} \mathcal{I}_t \eta, \mathcal{I}_t \eta) + (\rho_2 \partial_t^{\alpha} \mathcal{I}_x \mathcal{I}_t \eta, \mathcal{I}_x \mathcal{I}_t \eta)_{L^2(0,L)},
\end{aligned} \tag{64}$$

$$\begin{aligned}
&- \kappa_2 (\mathcal{I}_t^2 \eta_{xx}, \mathcal{I}_t \eta + \mathcal{I}_x^2 \mathcal{I}_t \eta)_{L^2(0,L)} \\
&= \frac{\kappa_2 \partial}{2\partial t} \|\mathcal{I}_t^2 \eta_x\|_{L^2(0,L)}^2 + \frac{\kappa_2 \partial}{2\partial t} \|\mathcal{I}_t^2 \eta\|_{L^2(0,L)}^2
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\kappa_1 (\mathcal{I}_t^2 \xi_x, \mathcal{I}_t \eta + \mathcal{I}_x^2 \mathcal{I}_t \eta)_{L^2(0,L)} \\
&= \kappa_1 (\mathcal{I}_t^2 \xi_x, \mathcal{I}_t \eta)_{L^2(0,L)} - \kappa_1 (\mathcal{I}_t^2 \xi, \mathcal{I}_x \mathcal{I}_t \eta)_{L^2(0,L)},
\end{aligned} \tag{66}$$

$$
\begin{aligned}
&(\int_0^t m(t-s)\mathcal{I}_s^2\eta_{xx}(x,s)ds, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)} \\
&= -(\int_0^t m(t-s)\mathcal{I}_s^2\eta_x(x,s)ds, \mathcal{I}_t\eta_x)_{L^2(0,L)} + (\int_0^t m(t-s)\mathcal{I}_s^2\eta(x,s)ds, \mathcal{I}_t\eta)_{L^2(0,L)} \\
&= -\frac{d}{dt}\int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m(t-s)(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx \\
&\quad + \int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m'(t-s)(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx + \int_0^L m(0)\left(\mathcal{I}_t^2\eta_x\right)^2 dx \\
&\quad + (\int_0^t m(t-s)\mathcal{I}_s^2\eta(x,s)ds, \mathcal{I}_t\eta)_{L^2(0,L)}.
\end{aligned} \tag{67}
$$

Insertion of Equations (60)–(67) into (59), and using Lemma 2, yields

$$
\begin{aligned}
&\frac{\rho_1}{2}{}^C\partial_t^\alpha\|\mathcal{I}_t\xi)\|_{L^2(0,L)}^2 + \frac{\rho_1}{2}{}^C\partial_t^\alpha\|\mathcal{I}_x\mathcal{I}_t\xi)\|_{L^2(0,L)}^2 + \frac{\kappa_1\partial}{2\partial t}\|\mathcal{I}_t^2\xi_x\|_{L^2(0,L)}^2 \\
&+ \frac{\kappa_1\partial}{2\partial t}\|\mathcal{I}_t^2\xi\|_{L^2(0,L)}^2 + \|\mathcal{I}_t\xi\|_{L^2(0,L)}^2 + \frac{\rho_2}{2}{}^C\partial_t^\alpha\|\mathcal{I}_t\eta)\|_{L^2(0,L)}^2 \\
&+ \frac{\rho_2}{2}{}^C\partial_t^\alpha\|\mathcal{I}_x\mathcal{I}_t\eta)\|_{L^2(0,L)}^2 + \frac{\kappa_2\partial}{2\partial t}\|\mathcal{I}_t^2\eta_x\|_{L^2(0,L)}^2 + \frac{\kappa_2\partial}{2\partial t}\|\mathcal{I}_t^2\eta\|_{L^2(0,L)}^2 \\
&+ \int_0^L h(0)\left(\mathcal{I}_t^2\eta_x\right)^2 dx \\
&\le \kappa_1(\mathcal{I}_t^2\eta_x, \mathcal{I}_t\xi)_{L^2(0,L)} - \kappa_1(\mathcal{I}_t^2\eta, \mathcal{I}_x\mathcal{I}_t\xi)_{L^2(0,L)} + \|\mathcal{I}_x\mathcal{I}_t\xi\|_{L^2(0,L)}^2 \\
&- \kappa_1(\mathcal{I}_t^2\xi_x, \mathcal{I}_t\eta)_{L^2(0,L)} + \kappa_1(\mathcal{I}_t^2\xi, \mathcal{I}_x\mathcal{I}_t\eta)_{L^2(0,L)} \\
&+ \frac{d}{dt}\int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m(t-s)(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx \\
&- \int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m'(t-s)(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx \\
&- (\int_0^t m(t-s)\mathcal{I}_s^2\eta(x,s)ds, \mathcal{I}_t\eta)_{L^2(0,L)}.
\end{aligned} \tag{68}
$$

The Cauchy ϵ-inequality, gives

$$\kappa_1(\mathcal{I}_t^2\eta_x, \mathcal{I}_t\xi)_{L^2(0,L)} \le \frac{\kappa_1}{2}\|\mathcal{I}_t^2\eta_x\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2}\|\mathcal{I}_t\xi\|_{L^2(0,L)}^2, \tag{69}$$

$$-\kappa_1(\mathcal{I}_t^2\eta, \mathcal{I}_x\mathcal{I}_t\xi)_{L^2(0,L)} \le \frac{\kappa_1}{2}\|\mathcal{I}_t^2\eta\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2}\|\mathcal{I}_x\mathcal{I}_t\xi\|_{L^2(0,L)}^2, \tag{70}$$

$$-\kappa_1(\mathcal{I}_t^2\xi_x, \mathcal{I}_t\eta)_{L^2(0,L)} \le \frac{\kappa_1}{2}\|\mathcal{I}_t^2\xi_x\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2}\|\mathcal{I}_t\eta\|_{L^2(0,L)}^2, \tag{71}$$

$$\kappa_1(\mathcal{I}_t^2\xi, \mathcal{I}_x\mathcal{I}_t\eta)_{L^2(0,L)} \le \frac{\kappa_1}{2}\|\mathcal{I}_t^2\xi\|_{L^2(0,L)}^2 + \frac{\kappa_1}{2}\|\mathcal{I}_x\mathcal{I}_t\eta\|_{L^2(0,L)}^2, \tag{72}$$

$$
\begin{aligned}
&-\int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m'(t-s)(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx \\
&\leq \frac{1}{2}\sup_{0\leq t\leq T}|m'|\left(1+\frac{T^2}{2}\right)\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)},
\end{aligned} \tag{73}
$$

$$
\begin{aligned}
&-(\int_0^t m(t-s)\mathcal{I}_s^2\eta(x,s)ds, \mathcal{I}_t\eta)_{L^2(0,L)} \\
&\frac{1}{2}\sup_{0\leq t\leq T}|m|\|\mathcal{I}_t\eta\|^2_{L^2(0,L)}+\frac{T^2}{2}\sup_{0\leq t\leq T}|m|\|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)}.
\end{aligned} \tag{74}
$$

By combining Equality (68) and Inequalities (69)–(74), we obtain

$$
\begin{aligned}
&{}^C\partial_t^\alpha\|\mathcal{I}_t\xi)\|^2_{L^2(0,L)}+{}^C\partial_t^\alpha\|\mathcal{I}_t\eta)\|^2_{L^2(0,L)}+{}^C\partial_t^\alpha\|\mathcal{I}_x\mathcal{I}_t\xi)\|^2_{L^2(0,L)} \\
&+{}^C\partial_t^\alpha\|\mathcal{I}_x\mathcal{I}_t\eta)\|^2_{L^2(0,L)}+\frac{\partial}{\partial t}\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)}+\frac{\partial}{\partial t}\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)} \\
&+\frac{\partial}{\partial t}\|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)}+\frac{\partial}{\partial t}\|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)}+\int_0^L\left(\mathcal{I}_t^2\eta_x\right)^2dx+\|\mathcal{I}_t\xi\|^2_{L^2(0,L)} \\
&\leq\mathcal{W}\Big(\|\mathcal{I}_t\xi\|^2_{L^2(0,L)}+\|\mathcal{I}_t\eta\|^2_{L^2(0,L)}+\|\mathcal{I}_x\mathcal{I}_t\xi\|^2_{L^2(0,L)}+\|\mathcal{I}_x\mathcal{I}_t\eta\|^2_{L^2(0,L)} \\
&+\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)}+\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)}+\|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)}+\|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)}\Big),
\end{aligned} \tag{75}
$$

where

$$
\mathcal{W}=\frac{\max\left\{\frac{\kappa_1}{2}+(1+\frac{T^2}{2})\frac{1}{2}\sup|m'|,\frac{\kappa_1}{2}+\frac{T^2}{2}\frac{1}{2}\sup|m|\right\}}{\min\left\{1,\frac{\rho_1}{2},\frac{\rho_2}{2},\frac{\kappa_1}{2},\frac{\kappa_2}{2},h(0)\right\}}. \tag{76}
$$

By dropping the last two terms from the left-hand side of (75), replacing t with τ in (76) and then integrating with respect to τ, we have

$$
\begin{aligned}
&D_t^{\alpha-1}\|\mathcal{I}_t\xi)\|^2_{L^2(0,L)}+D_t^{\alpha-1}\|\mathcal{I}_t\eta)\|^2_{L^2(0,L)}+D_t^{\alpha-1}\|\mathcal{I}_x\mathcal{I}_t\xi)\|^2_{L^2(0,L)} \\
&+D_t^{\alpha-1}\|\mathcal{I}_x\mathcal{I}_t\eta)\|^2_{L^2(0,L)}+\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)}+\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)} \\
&+\|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)}+\|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)} \\
&\leq\mathcal{W}\Big(\|\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_x\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_x\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))} \\
&+\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_t^2\xi\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_t^2\eta\|^2_{L^2(0,t;L^2(0,L))}\Big).
\end{aligned} \tag{77}
$$

If we omit the first four terms on the left-hand side of (77), and use the Gronwall–Bellman lemma, by taking

$$
\begin{cases}
\mathcal{R}(t)=\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,t;L^2(0,L))} \\
\quad+\|\mathcal{I}_t^2\xi\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_t^2\eta\|^2_{L^2(0,t;L^2(0,L))}, \\
\frac{\partial\mathcal{R}(t)}{\partial t}=\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)}+\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)} \\
\quad+\|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)}+\|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)}, \\
\mathcal{R}(t)=0,
\end{cases} \tag{78}
$$

we obtain

$$
\begin{aligned}
\mathcal{R}(t)\leq Te^{T\mathcal{W}}&\Big(\|\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_x\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))} \\
&+\|\mathcal{I}_x\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))}\Big).
\end{aligned} \tag{79}
$$

Next, if we disregard the last four terms on the left-hand side and take into account Inequality (79), we end with

$$\begin{aligned} &D_t^{\alpha-1}\|\mathcal{I}_t\xi)\|^2_{L^2(0,L)} + D_t^{\alpha-1}\|\mathcal{I}_t\eta)\|^2_{L^2(0,L)} + D_t^{\alpha-1}\|\mathcal{I}_x\mathcal{I}_t\xi)\|^2_{L^2(0,L)} \\ &+ D_t^{\alpha-1}\|\mathcal{I}_x\mathcal{I}_t\eta)\|^2_{L^2(0,L)} \\ \leq &\mathcal{W}\Big(Te^{T\mathcal{W}}+1\Big)\Big(\|\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))} \\ &+ \|\mathcal{I}_t^2\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t^2\eta\|^2_{L^2(0,t;L^2(0,L))}\Big). \end{aligned} \tag{80}$$

Now we are able to apply Lemma 2, by letting

$$\begin{cases} Q(t) = \|\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))} \\ \quad \|\mathcal{I}_t^2\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t^2\eta\|^2_{L^2(0,t;L^2(0,L))}, \\ {}^C\partial_t^\alpha Q(t) = D_t^{\alpha-1}\|\mathcal{I}_t\xi)\|^2_{L^2(0,L)} + D_t^{\alpha-1}\|\mathcal{I}_t\eta)\|^2_{L^2(0,L)} \\ \quad +D_t^{\alpha-1}\|\mathcal{I}_x\mathcal{I}_t\xi)\|^2_{L^2(0,L)} + D_t^{\alpha-1}\|\mathcal{I}_x\mathcal{I}_t\eta)\|^2_{L^2(0,L)}, \\ Q(0) = 0, \end{cases} \tag{81}$$

and we infer from (80) that

$$Q(t) \leq \Gamma(\alpha)E_{\alpha,\alpha}(\mathcal{W}\Big(Te^{T\mathcal{W}}+1\Big)t^\alpha)D_t^{-\alpha}(0) = 0. \tag{82}$$

We conclude from (82) and (81) that $\xi = 0$, and $\eta = 0$. Consequently, $W(x,t) = (\Lambda_1(x,t), \text{and } \Lambda_2(x,t)) = (0,0)$ almost everywhere in Q^T.

We now consider the general case for density.

Since $\mathcal{E}$ is a Hilbert space, then $\overline{R(\mathcal{G})} = \mathcal{E} \Leftrightarrow R(\mathcal{G})^\perp = \{0\} \Leftrightarrow (\mathcal{G}\mathcal{Z},\mathcal{K})_\mathcal{E} = 0$, for all $\mathcal{Z} \in \mathcal{B}$, and $\mathcal{K} \in \mathcal{E}$, then $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2) = \{(J_1, J_3, J_4), (J_2, J_5, J_6)\} = (0,0)$, that is $J_1 = J_2 = J_3 = J_4 = J_5 = J_6 = 0$. Therefore, suppose that for some element $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2) = \{(J_1, J_3, J_4), (J_2, J_5, J_6)\} \in R(\mathcal{G})^\perp$

$$\begin{aligned} &(\mathcal{G}\mathcal{Z},\mathcal{K})_\mathcal{E} \\ =&(\{\mathcal{S}_1(\theta,\phi), \mathcal{S}_2(\theta,\phi\}, \{\mathcal{K}_1,\mathcal{K}_2\})_\mathcal{E} \\ =&(\{\mathcal{S}_1(\theta,\phi), \Gamma_1\theta, \Gamma_2\theta\}, \{\mathcal{S}_2(\theta,\phi), \Gamma_1\phi, \Gamma_2\phi\}\}, \{\{J_1,J_2,J_3\},\{J_4,J_5,J_6\}\})_\mathcal{E} \\ =&(\mathcal{S}_1(\theta,\phi), J_1)_{L^2(Q^T)} + (\Gamma_1\theta, J_2)_{L^2(0,L)} + (\Gamma_2\theta, J_3)_{L^2(0,L)} \\ &+ (\mathcal{S}_2(\theta,\phi), J_4)_{L^2(Q^T)} + (\Gamma_1\phi, J_5)_{L^2(0,L)} + (\Gamma_2\phi, J_6)_{L^2(0,L)} \\ =&0, \end{aligned} \tag{83}$$

where $\mathcal{Z}$ runs over the space $\mathcal{B}$, we have to prove that $\mathcal{K} = 0$.

Let $\mathcal{Z} \in D_0(\mathcal{G})$, then Equation (83) becomes

$$(\mathcal{S}_1(\theta,\phi), J_1)_{L^2(Q^T)} + (\mathcal{S}_2(\theta,\phi), J_4)_{L^2(Q^T)} = 0. \tag{84}$$

From Theorem 3, it follows from (84) that $J_1 = J_4 = 0$. Then, Equation (83) takes the form

$$(\Gamma_1\theta, J_2)_{L^2(0,L)} + (\Gamma_2\theta, J_3)_{L^2(0,L)} + (\Gamma_1\phi, J_5)_{L^2(0,L)} + (\Gamma_2\phi, J_6)_{L^2(0,L)} = 0. \tag{85}$$

Since the four terms in (85) vanish independently and the ranges $R(\Gamma_1), R(\Gamma_2)$ of the trace operators Γ_1 and Γ_2 are dense everywhere in $L^2(0,L)$, then it follows from (85) that $J_2 = J_3 = J_5 = J_6 = 0$. Consequently, $\mathcal{K} = 0$, that is $R(\mathcal{G})^\perp = \{0\}$. Thus, $\overline{R(\mathcal{G})} = \mathcal{E}$. This completes the proof of Theorem 2 . □

6. Conclusions

In this paper, we investigated a non-local non-homogeneous fractional Timoshenko system with frictional and viscoelastic damping terms. This fractional-order system was supplemented with some initial conditions and classical and non-local boundary integral-type conditions. The well-posedness of the given non-local initial boundary value problem is established. The used approach relies on some functional analysis tools, operator theory, a priori estimates and density arguments. To the best of our knowledge, the treated fractional Timoshenko system problem has never been studied nor explored in the literature. This work can be considered as a contribution to the development of traditional functional analysis methods, the so-called a priori estimate method or the energy inequalities method used to prove the existence, uniqueness and stability of initial boundary value problems with non-local boundary integral-type conditions.

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