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Interval-Valued Topology on Soft Sets

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Abstract: In this paper, we study the concept of interval-valued fuzzy set on the family $\mathrm{SS}(X, E)$ of all soft sets over X with the set of parameters E and examine its basic properties. Later, we define the concept of interval-valued fuzzy topology (cotopology) τ on $\mathrm{SS}(X, E)$. We obtain that each interval-valued fuzzy topology is a descending family of soft topologies. In addition, we study some topological structures such as interval-valued fuzzy neighborhood system of a soft point, base and subbase of τ and investigate some relationships among them. Finally, we give some concepts such as direct sum, open mapping and continuous mapping and consider connections between them. A few examples support the presented results.

Keywords: interval-valued fuzzy topology (cotopology); interval-valued fuzzy neighborhood; base; subbase; continuous mapping; direct sum

MSC: 54A40; 54A05; 03E72; 06D72

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1. Introduction

The concept of interval-valued fuzzy set was given by Zadeh [1]. This set is an extension of fuzzy sets in the sense that the values of the membership degrees are intervals of numbers instead of the numbers. Chang [2] introduced the concept of fuzzy topology in 1968. But, since the concept of openness of a fuzzy set was not given, Samanta et al. [3,4] introduced the concept of gradation of openness (closedness) of a fuzzy set in 1992. Furthermore, the concept of intuitionistic gradation of openness of fuzzy sets in Sostak's sense [5] was defined by some researchers [6–8]. In [9], D. L. Shi et al. introduced the concept of ordinary interval-valued fuzzifying topology and investigated some of its important properties. It is known that to describe and deal with uncertainties, a lot of mathematical approaches put forward a proposal such as probability theory, fuzzy set theory, rough set theory, interval set theory etc. But all these theories have inherent difficulties. In [10], Molodtsov presented soft set theory in order to overcome difficulties affecting the existing methods. Later, many papers were written on soft set theory. Since soft set theory has many application areas, it has progressed very quickly until today. Maji et al. [11] defined some operations on soft sets. In recent years, topological structures of soft sets have been studied by some authors. M. Shabir and M. Naz [12] have initiated the concept of soft topological space. A large number of papers was devoted to the study of soft topological spaces from various aspects [13–22]. Moreover, C.G. Aras et al. [23] gave the definition of gradation of openness τ which is a mapping from $\mathrm{SS}(X, E)$ to $[0, 1]$ which satisfies some conditions and showed that a fuzzy topological space gives a parameterized family of soft topologies on X. Also, S. Bayramov et al. [24] gave the concepts of continuous mapping, open mapping and closed mapping by using soft points in intuitionistic fuzzy topological spaces.

The importance and applications of interval-valued analysis is given in the book [25]. Our aim in this paper is to demonstrate applications of interval-valued mathematics in the context of fuzzy and soft topologies. We study the concept of interval-valued fuzzy set on the family $\mathrm{SS}(X, E)$ of all soft sets over X and examine its basic properties. We also define the concept of interval-valued fuzzy topology τ (called also cotopology) on $\mathrm{SS}(X, E)$. We prove that each interval-valued fuzzy topology is actually a descending family of soft topologies. Further, we study some topological structures such as interval-valued fuzzy neighborhood system of a soft point, base and subbase of τ and investigate some relationships among them. Finally, we give some concepts such as direct sum, open mappings and continuous mappings and consider connections between them.

2. Preliminaries

In this section we give basic notions about soft sets and soft topology which will be used in the sequel.

Definition 1 ([10]). *Let X be a set, called an initial universal set, and E a nonempty set, called the set of parameters. A pair (F, E) is called a soft set over X, where $F : E \to P(X)$ is a mapping from E into a power set of X.*

The family of all soft sets over X with the set of parameters E is denoted by $\mathrm{SS}(X, E)$.

Definition 2 ([11]). *If for all $e \in E$, $F(e) = \emptyset$, (F, E) is said to be the null soft set denoted by Φ. If for all $e \in E$, $F(e) = X$, then (F, E) is said to be the* absolute soft set *denoted by $\widetilde{X}$.*

Definition 3 ([14,16]). *Let (F, E) be a soft set over X. The soft set (F, E) is called a soft point if for some element $e \in E$, $F(e) = \{x\}$ and $F(e') = \emptyset$ for all $e' \in E \setminus \{e\}$(briefly denoted by x_e).*

Note that since each soft set can be expressed as a union of soft points, to give the family of all soft sets on X it is sufficient to give only soft points on X.

Notice that in the literature there are other definitions of soft points, but we think that our approach gives an easier applications of these points.

Definition 4 ([14]). *The soft point x_e is said to belong to the soft set (F, E), denoted by $x_e \widetilde{\in} (F, E)$, if $x_e(e) \in F(e)$, i.e., $\{x\} \subseteq F(e)$.*

Definition 5 ([12]). *A soft topology on a non-empty set X is a collection τ of soft sets over X with a set of parameters E satisfying the following axioms:*
(ST1) $\widetilde{X}$ and Φ belong to τ;
(ST2) The soft intersection of finitely many members in τ belongs to τ;
(ST3) The soft union of any family of members in τ belongs to τ.

The triple (X, τ, E) is called a soft topological space. Members of τ are called soft open sets.

Notice that if (X, τ, E) is a soft topological space, then $\tau_e = \{F(e) : (F, E) \in \tau\}$ defines a topology on X, for each $e \in E$. This topology is called *e-parametric topology* [12].

Throughout this paper, I denotes the closed unit interval $[0, 1]$, and $[I]$ represents the set of all closed subintervals of I. The members of $[I]$ are called *interval numbers* and are denoted by $\widetilde{a}, \widetilde{b}, \widetilde{c}, \ldots$ Here $\widetilde{a} \in [I]$, $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. Especially, if $a^- = a^+$, then we take $\widetilde{a} = a$. Also, it is defined an order relation $\leq$, on $[I]$ as follows:

(1) $\left(\forall \widetilde{a}, \widetilde{b} \in [I]\right)$ $\widetilde{a} \leq \widetilde{b} \Longleftrightarrow a^- \leq b^-, a^+ \leq b^+$,

(2) $\left(\forall \widetilde{a}, \widetilde{b} \in [I]\right)$ $\widetilde{a} = \widetilde{b} \Longleftrightarrow \widetilde{a} \leq \widetilde{b}$ and $\widetilde{b} \leq \widetilde{a}$, (i.e., $a^- = b^-, a^+ = b^+$),

(3) For any $\widetilde{a}, \widetilde{b} \in [I]$, maximum and minimum of $\widetilde{a}, \widetilde{b}$, respectively

$$\begin{aligned} \widetilde{a} \vee \widetilde{b} &= \left[a^- \vee b^-, a^+ \vee b^+\right], \\ \widetilde{a} \wedge \widetilde{b} &= \left[a^- \wedge b^-, a^+ \wedge b^+\right]. \end{aligned}$$

Let $\{\widetilde{a}_i\}_{i\in J} \subset [I]$. Then inf and sup of $\{\widetilde{a}_i\}_{i\in J}$ are defined as follows:

$$\bigwedge_{i\in J}\widetilde{a}_i = \left[\bigwedge_{i\in J}a_j^-, \bigwedge_{i\in J}a_j^+\right],$$
$$\bigvee_{i\in J}\widetilde{a}_i = \left[\bigvee_{i\in J}a_j^-, \bigvee_{i\in J}a_j^+\right].$$

Also, for each $\widetilde{a} \in [I]$, the complement of $\widetilde{a}$, denoted by $\widetilde{a}^c$, is defined as:

$$\widetilde{a}^c = \left[1 - a^+, 1 - a^-\right].$$

3. Introduction to Interval-Valued Topology on Soft Sets

We introduce now the main notion in this paper, the notion of interval-valued fuzzy topology on the set $\mathsf{SS}(X,E)$.

Definition 6. *A mapping $A : \mathsf{SS}(X,E) \to [I]$ is called an interval-valued fuzzy set in $\mathsf{SS}(X,E)$ and is denoted briefly as $IVFS$.*

Let $[I]^{\mathsf{SS}(X,E)}$ represent the set of all $IVFSs$ in $\mathsf{SS}(X,E)$. For each $A \in [I]^{\mathsf{SS}(X,E)}$ and $(F,E) \in \mathsf{SS}(X,E)$. $A(F,E) = [A^-(F,E), A^+(F,E)]$ is a closed interval. Thus $A^-, A^+ : \mathsf{SS}(X,E) \to I$ are two fuzzy sets. For each $A \in [I]^{\mathsf{SS}(X,E)}$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}, \widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy whole set in $\mathsf{SS}(X,E)$, respectively.

Now we give the relations $\subset$ and $=$ on $[I]^{\mathsf{SS}(X,E)}$ as follows:

$$\left(\forall A, B \in [I]^{\mathsf{SS}(X,E)}\right) [A \subset B \Longleftrightarrow (\forall (F,E) \in \mathsf{SS}(X,E))\ A(F,E) \leq B(F,E)],$$
$$\left(\forall A, B \in [I]^{\mathsf{SS}(X,E)}\right) [A = B \Longleftrightarrow (\forall (F,E) \in \mathsf{SS}(X,E))\ A(F,E) = B(F,E)].$$

Definition 7. *Let $A \in [I]^{\mathsf{SS}(X,E)}$ and $\{A_i\}_{i\in J}$ be arbitrary subfamily of $[I]^{\mathsf{SS}(X,E)}$. The complement, union and intersection of A are denoted by A^c, $\bigcup_{i\in J} A_i$ and $\bigcap_{i\in J} A_i$, respectively, are defined for each $(F,E) \in \mathsf{SS}(X,E)$ as follows respectively,*

$$A^c(F,E) = \left[1 - A^+(F,E), 1 - A^-(F,E)\right],$$
$$\left(\bigcup_{i\in J}A_i\right)(F,E) = \bigvee_{i\in J}A_i(F,E),$$
$$\left(\bigcap_{i\in J}A_i\right)(F,E) = \bigwedge_{i\in J}A_i(F,E).$$

Proposition 1. *Let $A, B, C \in [I]^{\mathsf{SS}(X,E)}$ and $\{A_i\}_{i\in J} \subset [I]^{\mathsf{SS}(X,E)}$. Then the following statements hold:*

(1) $\widetilde{0} \subset A \subset \widetilde{1}$,
(2) $A \cup B = B \cup A$, $A \cap B = B \cap A$,
(3) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$,
(4) $A, B \subset A \cup B$, $A \cap B \subset A, B$,
(5) $A \cap \left(\bigcup_{i\in J} A_i\right) = \bigcup_{i\in J}(A \cap A_i)$, $A \cup \left(\bigcap_{i\in J} A_i\right) = \bigcap_{i\in J}(A \cup A_i)$,
(6) $\left(\widetilde{0}\right)^c = \widetilde{1}$, $\left(\widetilde{1}\right)^c = \widetilde{0}$,
(7) $(A^c)^c = A$,

(8) $\left(\bigcup_{i\in J} A_i\right)^c = \bigcap_{i\in J} A_i^c$, $\left(\bigcap_{i\in J} A_i\right)^c = \bigcup_{i\in J} A_i^c$.

Proof. It is immediately obtained. □

Definition 8. *A mapping $\tau = [\tau^-, \tau^+] : \mathsf{SS}(X,E) \to [I]$ is called an interval-valued fuzzy topology over $\mathsf{SS}(X,E)$ if it satisfies the following conditions:*
(1) $\tau(\Phi) = \tau\left(\widetilde{X}\right) = 1$,
(2) $\tau((F,E) \cap (G,E)) \geq \tau(F,E) \wedge \tau(G,E), \forall (F,E),(G,E) \in \mathsf{SS}(X,E)$,
(3) $\tau\left(\bigcup_{i\in J}(F_i,E)\right) \geq \bigwedge_{i\in J} \tau(F_i,E), \forall \{(F_i,E)\}_{i\in J} \subset \mathsf{SS}(X,E)$.

The interval-valued fuzzy topology is denoted briefly $IVFT$ and the triple (X,τ,E) is called an *interval-valued fuzzy topological space over* $\mathsf{SS}(X,E)$ (in short $IVFTS$)

It is clear that $\tau \in IVFT$ consists of two fuzzy topologies over $\mathsf{SS}(X,E)$, τ^- and τ^+. Also, for each $(F,E) \in \mathsf{SS}(X,E)$, $\tau^-(F,E) \leq \tau^+(F,E)$.

Example 1. *Let $X = \{x,y,z\}$ and $E = \{e\}$. The set of all soft points on X is $\{x_e, y_e, z_e\}$. Then the soft sets are:*

$$F_1(e) = \{x\}, F_2(e) = \{y\}, F_3(e) = \{z\}, F_4(e) = \{x,y\},$$
$$F_5(e) = \{x,z\}, F_6(e)\{y,z\}, F_7(e) = \emptyset, F_8(e) = X.$$

Define the mapping $\tau : \mathsf{SS}(X,E) \to [I]$ as follows:

$$\begin{aligned}
\tau(\Phi) &= \tau\left(\widetilde{X}\right) = 1, \\
\tau(F_1) &= [0.2, 0.5], \\
\tau(F_2) &= [0.3, 0.4], \\
\tau(F_3) &= [0.4, 0.5], \\
\tau(F_4) &= [0.2, 0.4], \\
\tau(F_5) &= [0.3, 0.5], \\
\tau(F_6) &= [0.3, 0.5].
\end{aligned}$$

Then it is clear that τ is an $IVFT$.

Example 2. *Let $X = \{x\}$ and $E = \{a,b,c\}$. The set of all soft points on X is $\{x_a, x_b, x_c\}$. Then the soft sets are*

$$\begin{aligned}
F_1(a) &= \{x\}, F_1(b) = \emptyset, F_1(c) = \emptyset, \\
F_2(a) &= \emptyset, F_2(b) = \{x\}, F_2(c) = \emptyset, \\
F_3(a) &= \emptyset, F_3(b) = \emptyset, F_3(c) = \{x\}, \\
F_4(a) &= \{x\}, F_4(b) = \{x\}, F_4(c) = \emptyset, \\
F_5(a) &= \{x\}, F_5(b) = \emptyset, F_5(c) = \{x\}, \\
F_6(a) &= \emptyset, F_6(b) = \{x\}, F_6(c) = \{x\}, \\
F_7 &= \Phi,\ F_8 = \widetilde{X}.
\end{aligned}$$

We define the mapping $\tau : \mathsf{SS}(X,E) \to [I]$ as follows:

$$\begin{aligned}
\tau(\Phi) &= \tau\left(\widetilde{X}\right) = 1, \\
\tau(F_1) &= [0.3, 0.5], \\
\tau(F_2) &= [0.2, 0.4], \\
\tau(F_3) &= [0.3, 0.6], \\
\tau(F_4) &= [0.2, 0.5], \\
\tau(F_5) &= [0.4, 0.5], \\
\tau(F_6) &= [0.2, 0.4].
\end{aligned}$$

Then it is clear that τ is an IVFT.

Definition 9. *A mapping $C = (\mu_C, v_C) : \mathsf{SS}(X, E) \to [I]$ is called an interval-valued fuzzy cotopology (in short IVFCT) over $\mathsf{SS}(X, E)$ if it satisfies the following conditions:*

(1) $C(\Phi) = C\left(\widetilde{X}\right) = 1$,

(2) $C((F, E) \cup (G, E)) \geq C(F, E) \wedge C(G, E), \forall (F, E), (G, E) \in \mathsf{SS}(X, E)$,

(3) $C\left(\bigcap_{i \in J} (F_i, E)\right) \geq \bigwedge_{i \in J} C(F_i, E), \forall \{(F_i, E)\}_{i \in J} \subset \mathsf{SS}(X, E)$.

The triple (X, C, E) is called an *interval-valued fuzzy cotopological space over* $\mathsf{SS}(X, E)$ and denoted by IVFCTS.

Proposition 2. (1) *If $\tau : \mathsf{SS}(X, E) \to [I]$ is an IVFT, then $C(F, E) = \tau\big((F, E)^c\big)$ is a IVFCT, $\forall (F, E) \in \mathsf{SS}(X, E)$.*

(2) *If $C : \mathsf{SS}(X, E) \to [I]$ is an IVFCT, then $\tau(F, E) = C\big((F, E)^c\big)$ is an IVFT, $\forall (F, E) \in \mathsf{SS}(X, E)$.*

Proof. (1) It is clear that

$$C(\Phi) = \tau(\Phi^c) = \tau\left(\widetilde{X}\right) = 1,$$
$$C\left(\widetilde{X}\right) = \tau\left(\widetilde{X}^c\right) = \tau(\Phi) = 1.$$

$$\begin{aligned}
C((F, E) \cup (G, E)) &= \tau\big(((F, E) \cup (G, E))^c\big) = \tau\big((F, E)^c \cap (G, E)^c\big) \\
&\geq \tau\big((F, E)^c\big) \wedge \tau\big((G, E)^c\big) = C(F, E) \wedge C(G, E).
\end{aligned}$$

$$\begin{aligned}
C\left(\bigcap_{i \in J} (F_i, E)\right) &= \tau\left(\left(\bigcap_{i \in J} (F_i, E)\right)^c\right) = \tau\left(\bigcup_{i \in J} (F_i, E)^c\right) \\
&\geq \bigwedge_{i \in J} \tau(F_i, E)^c = \bigwedge_{i \in J} C(F_i, E).
\end{aligned}$$

(2) The proof is done similarly to (1). □

Definition 10. *Let (X, τ, E) be an IVFTS and $\widetilde{a} \in [I]$. We define two families $\tau_{\widetilde{a}}$ and $\tau^*_{\widetilde{a}}$ as follows, respectively:*

(1) $\tau_{\widetilde{a}} = \{(F, E) \in \mathsf{SS}(X, E) : \tau(F, E) \geq \widetilde{a}\}$,

(2) $\tau^*_{\widetilde{a}} = \{(F, E) \in \mathsf{SS}(X, E) : \tau(F, E) > \widetilde{a}\}$.

Proposition 3. *Let (X, τ, E) be an IVFTS and $\widetilde{a}, \widetilde{b} \in [I]$. Then:*

(1) *$\tau_{\widetilde{a}}$ is a soft topology.*

(2) *If $\widetilde{a} \leq \widetilde{b}$, then $\tau_{\widetilde{b}} \subset \tau_{\widetilde{a}}$.*

(3) *$\tau_{\widetilde{a}} = \bigcap_{\widetilde{b} < \widetilde{a}} \tau_{\widetilde{b}}$, where $\widetilde{a} \neq 0$.*

(4) *$\tau^*_{\widetilde{a}}$ is a soft topology.*

(5) *If* $\widetilde{a} \leq \widetilde{b}$, *then* $\tau^*_{\widetilde{b}} \subset \tau^*_{\widetilde{a}}$.
(6) $\tau^*_{\widetilde{a}}. = \bigcup_{\widetilde{a}<\widetilde{b}} \tau^*_{\widetilde{b}}$, *where* $\widetilde{a} \neq 1$.
Thus each interval-valued fuzzy topology is a descending family of soft topologies.

Proof. The proofs of (1), (2), (4) and (5) are clear.

(3) From (2), $\{\tau_{\widetilde{a}}\}_{\widetilde{a}\neq 0}$ is a descending family of soft topologies. Then for each $\widetilde{a} \neq 0$, $\mathsf{SS}(X, E)$

$$\tau_{\widetilde{a}} \subset \bigcap_{\widetilde{b}<\widetilde{a}} \tau_{\widetilde{b}} . \qquad \text{(i)}$$

Suppose that $(F, E) \notin \tau_{\widetilde{a}}$. Then $\tau(F, E) < \widetilde{a}$. Hence there exists $\widetilde{b} \neq 0$ such that $\tau(F, E) < \widetilde{b} < \widetilde{a}$. So $(F, E) \notin \tau_{\widetilde{b}}$ for $\widetilde{b} < \widetilde{a}$. Thus $(F, E) \notin \bigcap_{\widetilde{b}<\widetilde{a}} \tau_{\widetilde{b}}$ is obtained, i.e.,

$$\bigcap_{\widetilde{b}<\widetilde{a}} \tau_{\widetilde{b}} \subset \tau_{\widetilde{a}}. \qquad \text{(ii)}$$

Hence from (i) and (ii), $\tau_{\widetilde{a}} = \bigcap_{\widetilde{b}<\widetilde{a}} \tau_{\widetilde{b}}$, where $\widetilde{a} \neq 0$.

(6) The proof is obtained similarly to the proof of (3). □

Remark 1. *It is clear that for each* $\tau \in IVFT$, $\{\tau_{\widetilde{a}}\}_{\widetilde{a}\in[I]}$ *is a descending family of soft topologies.*

Proposition 4. *Let* $\{\tau_{\widetilde{a}}\}_{\widetilde{a}\in[I]}$ *be a descending family of soft topologies on* X. *We define the mapping* $\tau : \mathsf{SS}(X, E) \to [I]$ *as follows: for each* $(F, E) \in S\mathsf{SS}(X, E)$,

$$\tau(F, E) = \bigvee_{(F,E)\in\tau_{\widetilde{a}}} \widetilde{a}.$$

Then $\tau \in IVFT$.

Proof. Obviously $\tau(\Phi) = \tau\left(\widetilde{X}\right) = 1$ is met.

Suppose $(F, E), (G, E) \in \mathsf{SS}(X, E)$ such that $\tau(F, E) = \widetilde{a}$ and $\tau(G, E) = \widetilde{b}$. If $\widetilde{a} = 0$ or $\widetilde{b} = 0$, then

$$\begin{aligned}\tau^-((F, E) \cap (G, E)) &\geq \tau^-(F, E) \wedge \tau^-(G, E),\\ \tau^+((F, E) \cap (G, E)) &\geq \tau^+(F, E) \wedge \tau^+(G, E).\end{aligned}$$

Thus $\tau((F, E) \cap (G, E)) \geq \tau(F, E) \wedge \tau(G, E)$. Since

$$\tau(F, E) = \bigvee_{(F,E)\in\tau_{\widetilde{a}}} \widetilde{a} = \bigvee_{(F,E)\in\tau_{\widetilde{a}}} [a^-, a^+],$$

we can find $\varepsilon > 0$, $\widetilde{c_1}$ and $\widetilde{c_2}$ such that

$$\begin{aligned}a^- - \varepsilon &< c_1^- \leq a^-,\ a^+ - \varepsilon < c_1^+ \leq a^+,\\ b^- - \varepsilon &< c_2^- \leq b^-,\ b^+ - \varepsilon < c_2^+ \leq b^+\end{aligned}$$

and $(F, E) \in \tau_{\tau_{c_1^-}}, (G, E) \in \tau_{\tau_{c_2^-}}$. Let

$$\begin{aligned}c^- &= c_1^- \wedge c_2^-,\ c^+ = c_1^+ \wedge c_2^+,\\ d^- &= a^- \wedge b^-,\ d^+ = a^+ \wedge b^+.\end{aligned}$$

Then $\widetilde{c} \leq \widetilde{a}$, $\widetilde{c} \leq \widetilde{b}$. Since $\{\tau_{\widetilde{a}}\}_{\widetilde{a}\in[I]}$ is a descending family, then $\tau_{\widetilde{a}}, \tau_{\widetilde{b}} \subset \tau_{\widetilde{c}}$. Since $(F,E) \in \tau_{\widetilde{a}}$, $(G,E) \in \tau_{\widetilde{b}}$, then $(F,E) \cap (G,E) \in \tau_{\widetilde{c}}$. So we have

$$\begin{aligned} \tau^-((F,E)\cap(G,E)) &\geq c^- > d^- - \varepsilon, \\ \tau^+((F,E)\cap(G,E)) &\geq d^+ > d^+ - \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary,

$$\begin{aligned} \tau^-((F,E)\cap(G,E)) &\geq c^- \geq d^- = a^- \wedge b^-, \\ \tau^+((F,E)\cap(G,E)) &\geq d^+ = a^+ \wedge b^+. \end{aligned}$$

Hence $\tau((F,E)\cap(G,E)) \geq \tau(F,E) \wedge \tau(G,E)$ is obtained.

Finally, let $\{(F_i,E)\}_{i\in J} \subset \mathsf{SS}(X,E)$ and $\tau(F_i,E) = \widetilde{a}_i$, $\widetilde{a} = \bigwedge_{i\in J} \widetilde{a}_i$. If $\widetilde{a} = 0$, then obviously

$$\tau\left(\bigcup_{i\in J}(F_i,E)\right) \geq 0 = \bigwedge_{i\in J}\tau(F_i,E).$$

If $\widetilde{a} > 0$, choose $\varepsilon > 0$ such that $\widetilde{a} > \varepsilon$. Then for $i \in J$, $0 < a^- - \varepsilon < a_i^-$ and $0 < a^+ - \varepsilon < a_i^+$. Thus $(F_i,E) \in \tau_{[a^- - \varepsilon, a^+ - \varepsilon]}$. So $\bigcup_{i\in J}(F_i,E) \in \tau_{[a^- - \varepsilon, a^+ - \varepsilon]}$ and $\tau^-\left(\bigcup_{i\in J}(F_i,E)\right) \geq a^- - \varepsilon$, $\tau^+\left(\bigcup_{i\in J}(F_i,E)\right) \geq a^+ - \varepsilon$. Since $\varepsilon > 0$ was arbitrary,

$$\begin{aligned} \tau^-\left(\bigcup_{i\in J}(F_i,E)\right) &\geq a^- = \bigwedge_{i\in J}\tau^-(F_i,E), \\ \tau^+\left(\bigcup_{i\in J}(F_i,E)\right) &\geq a^+ = \bigwedge_{i\in J}\tau^+(F_i,E). \end{aligned}$$

Hence $\tau\left(\bigcup_{i\in J}(F_i,E)\right) \geq \bigwedge_{i\in J}\tau(F_i,E)$ is met. □

Theorem 1. *Let (X,τ,E) be an IVFTS and let $Y \subset X$. We define the mapping $\tau_Y : \mathsf{SS}(Y,E) \to [I]$ as follows: for each $(F,E) \in \mathsf{SS}(Y,E)$,*

$$\tau_Y(F,E) = \bigwedge_{\substack{(G,E)\in \mathsf{SS}(X,E) \\ (F,E)=(G,E)\cap\widetilde{Y}}} \tau(G,E).$$

Then $(X,\tau,E) \in IVFT$ and $\tau_Y(F,E) \geq \tau(F,E)$ for each $(F,E) \in \mathsf{SS}(Y,E)$. Then (Y,τ_Y,E) is said to be an interval-valued fuzzy subspace of (X,τ,E), and τ_Y is said to be induced interval-valued fuzzy topology on Y by τ.

Proof. It is obvious that $\tau_Y(\Phi) = \tau_Y\left(\widetilde{Y}\right) = 1$. Let $(F,E), (G,E) \in \mathsf{SS}(Y,E)$. Then

$$
\begin{aligned}
\tau_Y(F,E) \quad \wedge \quad \tau_Y(G,E) = & \left(\bigwedge_{\substack{(C_1,E)\in \mathsf{SS}(X,E)\\ (F,E)=(C_1,E)\cap \widetilde{Y}}} \tau(C_1,E) \right) \\
\wedge & \left(\bigwedge_{\substack{(C_2,E)\in \mathsf{SS}(X,E)\\ (G,E)=(C_2,E)\cap \widetilde{Y}}} \tau(C_2,E) \right) \\
= & \bigwedge_{\substack{(C_1,E),(C_2,E)\in \mathsf{SS}(X,E)\\ (F,E)\cap(G,E)=((C_1,E)\cap(C_2,E))\cap \widetilde{Y}}} [\tau(C_1,E)\wedge\tau(C_2,E)] \\
\leq & \bigwedge_{\substack{(C_1,E),(C_2,E)\in \mathsf{SS}(X,E)\\ (F,E)\cap(G,E)=((C_1,E)\cap(C_2,E))\cap \widetilde{Y}}} [\tau((C_1,E)\cap(C_2,E))] \\
= & \tau_Y((F,E)\cap(G,E)).
\end{aligned}
$$

Now, let $\{(F_i,E)\}_{i\in J}\subset \mathsf{SS}(Y,E)$. Then

$$
\begin{aligned}
\tau_Y\left(\bigcup_{i\in J}(F_i,E)\right) &= \bigwedge_{\substack{(B_i,E)\in \mathsf{SS}(X,E)\\ \bigcup_{i\in J}(F_i,E)=\bigcup_{i\in J}(B_i,E)\cap\widetilde{Y}}} \tau\left(\bigcup_{i\in J}(B_i,E)\right) \\
&\geq \bigwedge_{\substack{(B_i,E)\in \mathsf{SS}(X,E)\\ \bigcup_{i\in J}(F_i,E)=\bigcup_{i\in J}(B_i,E)\cap\widetilde{Y}}} \left(\bigwedge_{i\in J}\tau(B_i,E)\right) \\
&= \bigwedge_{i\in J}\left[\bigwedge_{\substack{(B_i,E)\in \mathsf{SS}(X,E)\\ \bigcup_{i\in J}(F_i,E)=\bigcup_{i\in J}(B_i,E)\cap\widetilde{Y}}} \tau(B_i,E)\right] \\
&= \bigwedge_{i\in J}\tau_Y(F_i,E).
\end{aligned}
$$

Also, for each $(F,E)\in \mathsf{SS}(Y,E)$, $\tau_Y(F,E)\geq\tau(F,E)$ is satisfied. □

4. Interval-Valued Neighborhood Structures

In this section we define and study the concept of interval-valued fuzzy neighborhood system of a soft point.

Definition 11. *Let (X,τ,E) be an $IVFTS$ and let x_e be a soft point. Then a mapping N_{x_e} : $\mathsf{SS}(X,E)\to[I]$ is called the interval-valued fuzzy neighborhood system of x_e if for each $(F,E)\in$ $\mathsf{SS}(X,E)$,*

$$N_{x_e} = \bigvee_{x_e\in(G,E)\subset(F,E)} \tau(G,E).$$

Proposition 5. *Let (X, τ, E) be an IVFTS and let $(F, E) \in \mathsf{SS}(X, E)$. Then*

$$\bigwedge_{x_e \in (F,E)} \bigvee_{x_e \in (G,E) \subset (F,E)} \tau(G, E) = \tau(F, E).$$

Proof. Since $(F, E) = \bigcup_{x_e \in (F,E)} \{x_e\}$, it is clear that

$$\bigwedge_{x_e \in (F,E)} \left(\bigvee_{x_e \in (G,E) \subset (F,E)} \tau(G, E) \right) \geq \tau(F, E).$$

Let $G_{x_e} = \{(G, E) \in \mathsf{SS}(X, E) : x_e \in (G, E) \subset (F, E)\}$. If $f \in \prod_{x_e \in (F,E)} G_{x_e}$, then obviously $\bigcup_{x_e \in (F,E)} f(x_e) = (F, E)$. Then

$$\bigwedge_{x_e \in (F,E)} \tau(f(x_e)) \leq \tau\left(\bigcup_{x_e \in (F,E)} f(x_e) \right) = \tau(F, E).$$

So

$$\bigwedge_{x_e \in (F,E)} \left(\bigvee_{x_e \in (G,E) \subset (F,E)} \tau(G, E) \right) = \bigvee_{f \in \prod_{x_e \in (F,E)} G_{x_e}} \left(\bigwedge_{x_e \in (F,E)} \tau(f(x_e)) \right) \leq \tau(F, E).$$

Hence

$$\bigwedge_{x_e \in (F,E)} \left(\bigvee_{x_e \in (G,E) \subset (F,E)} \tau(G, E) \right) = \tau(F, E).$$

□

Definition 12. *Let (X, τ, E) be an IVFTS.*

(1) $\beta : \mathsf{SS}(X, E) \to [I]$ is called a base of τ if β satisfies the following condition: $(F, E) \in \mathsf{SS}(X, E)$

$$\tau(F, E) = \bigvee_{\bigcup_{i \in J} (G_i, E) = (F,E)} \bigwedge_{i \in J} \beta(G_i, E).$$

(2) $\varphi : \mathsf{SS}(X, E) \to [I]$ is called a subbase of τ if $\widetilde{\varphi} : \mathsf{SS}(X, E) \to [I]$ is a base of τ, where

$$\widetilde{\varphi}(F, E) = \bigvee_{\bigcap_{i \in J} (G_i, E) = (F,E)} \bigwedge_{i \in J} \varphi(G_i, E)$$

and J is a finite set.

We now give an example of a base for a topology.

Example 3. *Let $X = \{x\}$, $E = \{e_1, e_2, e_3\}$. The set of soft points in X is $\{x_{e_1}, x_{e_2}, x_{e_3}\}$. Let $\widetilde{a} \in [I]$ be fixed. We define the mapping $\tau : \mathsf{SS}(X, E) \to [I]$ as follows: for $(F, E) \in \mathsf{SS}(X, E)$ we set*

$$\tau(F, E) = \begin{cases} 1, & \text{if } (F, E) \in \{\Phi, \widetilde{X}, \{x_{e_2}\}, \{x_{e_1}, x_{e_2}\}, \{x_{e_2}, x_{e_3}\}\}, \\ \widetilde{a}, & \text{otherwise.} \end{cases}$$

The mapping $\beta : \mathsf{SS}(X, E) \to [I]$ defined by

$$\beta(F, E) = 1, \text{ if } (F, E) \in \{\{x_{e_2}\}, \{x_{e_1}, x_{e_2}\}, \{x_{e_2}, x_{e_3}\}\}$$

is a base for τ.

Theorem 2. *Let* (X, τ, E) *be an* $IVFTS$ *and let* $\beta : \mathsf{SS}(X, E) \to [I]$ *be a mapping such that* $\beta \subset \tau$. *Then* β *is an interval-valued fuzzy base for* τ *if and only if for each soft point* x_e *and each* $(F, E) \in \mathsf{SS}(X, E)$,

$$N_{x_e}(F, E) \leq \bigvee_{x_e \in (G,E) \subset (F,E)} \beta(G, E).$$

Proof. Let β be an interval-valued fuzzy base for τ, x_e be a soft point and $(F, E) \in \mathsf{SS}(X, E), x_e \in (F, E)$. Thus from the definition of interval-valued fuzzy neighborhood system of x_e,

$$\begin{aligned} N_{x_e}(F, E) &= \bigvee_{x_e \in (G,E) \subset (F,E)} \tau(G, E) \\ &= \bigvee_{x_e \in (G,E) \subset (F,E)} \; \bigvee_{\bigcup_{i \in J}(G_i,E) = (G,E)} \; \bigwedge_{i \in J} \beta(G_i, E). \end{aligned}$$

If $x_e \in (G, E) = \bigcup_{i \in J}(G_i, E)$, then there is $i_0 \in J$ such that $x_e \in \big(G_{i_0}, E\big)$. Hence

$$\bigwedge_{i \in J} \beta(G_i, E) \leq \beta\big(G_{i_0}, E\big) \leq \bigvee_{x_e \in (G,E) \subset (F,E)} \beta(G, E).$$

So

$$N_{x_e}(F, E) \leq \bigvee_{x_e \in (G,E) \subset (F,E)} \beta(G, E)$$

is obtained.

Conversely, suppose the condition of necessary holds and for $(F, E) \in \mathsf{SS}(X, E)$,

$$(F, E) = \bigcup_{i \in J}(G_i, E).$$

Then

$$\tau(F, E) \geq \bigwedge_{i \in J} \tau(G_i, E) \geq \bigwedge_{i \in J} \beta(G_i, E).$$

Hence

$$\tau(F, E) \geq \bigvee_{\bigcup_{i \in J}(G_i,E) = (F,E)} \; \bigwedge_{i \in J} \beta(G_i, E) \quad \text{(i)}$$

On the other hand, from Proposition 5,

$$\begin{aligned} \tau(F, E) &= \bigwedge_{x_e \in (F,E)} \; \bigvee_{x_e \in (G,E) \subset (F,E)} \tau(G, E) \\ &= \bigwedge_{x_e \in (F,E)} N_{x_e}(F, E) \\ &\leq \bigwedge_{x_e \in (F,E)} \; \bigvee_{x_e \in (G,E) \subset (F,E)} \beta(G, E) \\ &= \bigvee_{f \in \prod_{x_e \in (F,E)} G_{x_e}} \left(\bigwedge_{x_e \in (F,E)} \beta(f(x_e)) \right). \end{aligned}$$

So, for each $f \in \prod_{x_e \in (F,E)} G_{x_e}$, since $(F,E) = \bigcup_{x_e \in (F,E)} f(x_e)$,

$$\bigvee_{f \in \prod_{x_e \in (F,E)} G_{x_e}} \left(\bigwedge_{x_e \in (F,E)} \beta(f(x_e)) \right) = \bigvee_{\bigcup_{i \in J} (G_i,E)=(F,E)} \bigwedge_{i \in J} \beta(G_i, E).$$

Therefore

$$\tau(F,E) \leq \bigvee_{\bigcup_{i \in J} (G_i,E)=(F,E)} \bigwedge_{i \in J} \beta(G_i, E). \quad \text{(ii)}$$

Hence from (i) and (ii),

$$\tau(F,E) = \bigvee_{\bigcup_{i \in J} (G_i,E)=(F,E)} \bigwedge_{i \in J} \beta(G_i, E),$$

i.e., β is an interval-valued fuzzy base for τ. □

Theorem 3. *If $\beta : \mathsf{SS}(X,E) \to [I]$ satisfies the following conditions:*
(1) $\beta(\Phi) = \beta\left(\widetilde{X}\right) = 1$,
(2) $\beta((F,E) \cap (G,E)) \geq \beta(F,E) \wedge \beta(G,E), \forall (F,E), (G,E) \in \mathsf{SS}(X,E)$, then

$$\tau_\beta(F,E) = \bigvee_{\bigcup_{i \in J} (G_i,E)=(F,E)} \bigwedge_{i \in J} \beta(G_i, E)$$

is an interval-valued fuzzy topology and β is a base of τ_β.

Proof. From the condition (1), $\tau_\beta(\Phi) = \tau_\beta\left(\widetilde{X}\right) = 1$ hold. For $\forall (F,E), (G,E) \in \mathsf{SS}(X,E)$,

$$\begin{aligned}
\tau_\beta(F,E) \quad \wedge \quad \tau_\beta(G,E) &= \left(\bigvee_{\bigcup_{i \in J_1} (F_i,E)=(F,E)} \bigwedge_{i \in J} \beta(F_i, E) \right) \\
\wedge \quad & \left(\bigvee_{\bigcup_{j \in J_2} (G_j,E)=(G,E)} \bigwedge_{j \in J} \beta(G_j, E) \right) \\
&= \bigvee_{\bigcup_{i \in J_1} (F_i,E)=(F,E)} \bigvee_{\bigcup_{j \in J_2} (G_j,E)=(G,E)} \left(\left(\bigwedge_{i \in J_1} \beta(F_i, E) \right) \wedge \left(\bigwedge_{j \in J_2} \beta(G_j, E) \right) \right) \\
&\leq \bigvee_{\bigcup_{i \in J_1, j \in J_2} ((F_i,E) \cap (G_j,E))=(F,E)\cap(G,E)} \left(\bigwedge_{i \in J_1, j \in J_2} \beta((F_i,E) \cap (G_j,E)) \right) \\
&\leq \bigvee_{\bigcup_{k \in J_3} (H_k,E)(H_k,E)=(F,E)\cap(G,E)} \left(\bigwedge_{k \in J_3} \beta(H_k, E) \right) \\
&= \tau_\beta((F,E) \cap (G,E)).
\end{aligned}$$

is obtained. Let let $\{(F_i, E)\}_{i\in J} \subset \mathsf{SS}(X, E)$. We consider a family

$$B_i = \left\{ \{(G_{l_i}, E)\} : l_i \in J_i : \bigcup_{l_i \in J_i} (G_{l_i}, E) = (F_i, E) \right\}.$$

Then

$$(F, E) = \bigcup_{i\in J} (F_i, E) = \bigcup_{i\in J} \bigcup_{l_i \in K_i} (G_{l_i}, E).$$

For an arbitrary $f \in \prod_{i\in J} B_i$, since $\bigcup_{i\in J} \bigcup_{(G_{l_i}, E)\in f(i)} (G_{l_i}, E) = \bigcup_{i\in J} (F_i, E)$,

$$\begin{aligned}
\tau_\beta(F, E) &= \bigvee_{\bigcup_{l\in J}(G_l, E)=(F,E)} \bigwedge_{l\in J} \beta(G_l, E) \\
&\geq \bigvee_{f\in \prod_{i\in J} B_i} \bigwedge_{i\in J} \bigwedge_{(G_{l_i}, E)\in f(i)} \beta(G_{l_i}, E) \\
&= \bigwedge_{i\in J} \bigwedge_{\{(G_{l_i}, E): l_i\in J_i\}} \bigwedge_{l_i\in J_i} \beta(G_{l_i}, E) \\
&= \bigwedge_{i\in J} \tau_\beta(F_i, E)
\end{aligned}$$

is obtained. Thus τ_β is an interval-valued fuzzy topology. It is clear that β is a base of τ_β. $\square$

5. Mappings

In this section we define and study continuous and open mappings between interval-valued fuzzy topological spaces.

Definition 13. *Let (X, τ, E) and (Y, ζ, E^*) be two IVFTSs and $(f, \varphi) : (X, \tau, E) \to (Y, \zeta, E^*)$ be a mapping. Then (f, φ) is called a continuous mapping at the soft point $x_e \in (X, E)$ if for each arbitrary soft set $(f, \varphi)(x_e) = (f(x))_{\varphi(e)} \in (G, E^*) \in \mathsf{SS}(Y, E^*)$, there exists $(F, E) \in \mathsf{SS}(X, E)$ such that*

$$\tau(F, E) \geq \zeta(G, E^*) \text{ and } (f, \varphi)(F, E) \subset (G, E^*).$$

(f, φ) is called a continuous mapping *if (f, φ) is a continuous mapping for each soft point.*

The following example illustrates the definition of continuity.

Example 4. *Let $X = \{x, y\}$, $E = \{e\}$. The set of all soft points in X is $\{x_e, y_e\}$, and the soft sets are*

$$F_1(e) = \{x\},\ F_2(e) = \{y\},\ F_3(e) = \Phi,\ F_4(e) = \tilde{X}.$$

Let $Y = \{u\}$, $E' = \{e'_1, e'_2\}$. The soft sets in Y are:

$$\begin{gathered}
G_1(e'_1) = u,\ G_1(e'_2) = \Phi, \\
G_2(e'_1) = \Phi,\ G_2(e'_2) = u, \\
G_3(e'_1) = G_3(e'_2) = \tilde{Y}, \\
G_4(e'_1) = G_4(e'_2) = \Phi.
\end{gathered}$$

Define $\tau : \mathsf{SS}(X, E) \to [I]$ and $\tau' : \mathsf{SS}(Y, E') \to [I]$ by

$$\begin{gathered}
\tau(F_1, E) = [0.1, 0.6],\ \tau(F_2, E) = [0.3, 0.5],\ \tau(F_3, E) = \tau(F_4, E) = 1; \\
\tau'(G_1, E') = [0.2, 0.7],\ \tau'(G_2, E') = [0.4, 0.6],\ \tau'(G_3, E') = \tau'(G_4, E') = 1.
\end{gathered}$$

Consider mappings $f : X \to Y$ and $\varphi : E \to E'$ defined by

$$f(x) = f(y) = u,$$
$$\varphi(e) = e'_1.$$

Then $(f,\varphi) : (X,\tau,E) \to (Y,\tau',E')$ *is a continuous mapping. Indeed, we have*

$$(f,\varphi)^{-1}(G_1,E')(e) = f^{-1}(G_1(\varphi(e))) = f^{-1}(u) = \{x,y\},$$
$$\tau\big((f,\varphi)^{-1}(G_1,E')\big) = 1 \geq \tau'(G_1,E');$$
$$(f,\varphi)^{-1}(G_2,E')(e) = f^{-1}(G_2(\varphi(e))) = f^{-1}(\Phi) = \Phi,$$
$$\tau\big((f,\varphi)^{-1}(G_2,E')\big) = 1 \geq \tau'(G_2,E').$$

Theorem 4. *Let* (X,τ,E) *and* (Y,ζ,E^*) *be two IVFTSs and* $(f,\varphi) : (X,\tau,E) \to (Y,\zeta,E^*)$ *be a mapping. Then* (f,φ) *is a continuous mapping if and only if*

$$\tau\Big((f,\varphi)^{-1}(G,E^*)\Big) \geq \zeta(G,E^*)$$

is satisfied for each $(G,E^*) \in \mathsf{SS}(Y,E^*)$.

Proof. Let (f,φ) be a continuous mapping and $\forall(G,E^*) \in \mathsf{SS}(Y,E^*)$. Suppose $x_e \in (f,\varphi)^{-1}(G,E^*)$ be an arbitrary soft point. Since (f,φ) is a continuous mapping, there exists $x_e \in (F,E) \in \mathsf{SS}(X,E)$ such that

$$\tau(F,E) \geq \zeta(G,E^*) \text{ and } (f,\varphi)(F,E) \subset (G,E^*).$$

Then

$$(f,\varphi)^{-1}(G,E^*) = \bigcup_{x_e \in (f,\varphi)^{-1}(G,E^*)} x_e \subset \bigcup_{x_e \in (f,\varphi)^{-1}(G,E^*)} (F,E) \subset (f,\varphi)^{-1}(G,E^*).$$

We have

$$\tau\Big((f,\varphi)^{-1}(G,E^*)\Big) = \tau\left(\bigcup_{x_e \in (f,\varphi)^{-1}(G,E^*)} (F,E)\right) \geq \wedge\tau(F,E) \geq \zeta(G,E^*).$$

Conversely, let $x_e \in \mathsf{SS}(X,E)$ be an arbitrary soft point and $(f,\varphi)(x_e) \in (G,E^*)$. From the condition of the theorem, $x_e \in (f,\varphi)^{-1}(G,E^*)$,

$$\tau\Big((f,\varphi)^{-1}(G,E^*)\Big) \geq \zeta(G,E^*),$$

and $(f,\varphi)\Big((f,\varphi)^{-1}(G,E^*)\Big) \subset (G,E^*)$ hold. So (f,φ) is a continuous mapping. □

Theorem 5. *Let* (X,τ,E) *and* (Y,ζ,E^*) *be two IVFTSs and* $(f,\varphi) : (X,\tau,E) \to (Y,\zeta,E^*)$ *be a mapping. Then* (f,φ) *is a continuous mapping if and only if for* $\forall\widetilde{a} = [a^-,a^+] \in [I]$,

$$\begin{aligned}(f_{a^-},\varphi_{a^-}) &: (X,\tau_{a^-},E) \to (Y,\zeta_{a^-},E^*),\\ (f_{a^+},\varphi_{a^+}) &: (X,\tau_{a^+},E) \to (Y,\zeta_{a^+},E^*)\end{aligned}$$

are soft continuous mappings.

Proof. Let (f,φ) be a continuous mapping and $(G,E^*) \in \zeta_{\widetilde{a}}$. Then $\zeta(G,E^*) \geq \widetilde{a}$. For each $(G,E^*) \in \mathsf{SS}(Y,E^*)$, $\zeta^-(G,E^*) \geq a^-$, $\zeta^+(G,E^*) \geq a^+$. Since

$$\tau\Big((f,\varphi)^{-1}(G,E^*)\Big) \geq \zeta(G,E^*) \geq \widetilde{a},$$

then

$$\begin{aligned}\tau^-\Big((f,\varphi)^{-1}(G,E^*)\Big) &\geq a^-, \tau^+\Big((f,\varphi)^{-1}(G,E^*)\Big) \geq a^+,\\ (f,\varphi)^{-1}(G,E^*) &\in \tau_{a^-}, (f,\varphi)^{-1}(G,E^*) \in \tau_{a^+}.\end{aligned}$$

Conversely, suppose that for $\forall\widetilde{a} = [a^-, a^+] \in [I]$,

$$\begin{array}{rcl} (f_{a^-}, \varphi_{a^-}) & : & (X, \tau_{a^-}, E) \to (Y, \zeta_{a^-}, E^*), \\ (f_{a^+}, \varphi_{a^+}) & : & (X, \tau_{a^+}, E) \to (Y, \zeta_{a^+}, E^*) \end{array}$$

are soft continuous mappings. If for each $(G, E^*) \in \mathsf{SS}(Y, E^*)$, $\zeta(G, E^*) = \widetilde{a}$, then $(G, E^*) \in \zeta_{\widetilde{a}}$, so $(G, E^*) \in \zeta_{a^-}$ and $(G, E^*) \in \zeta_{a^+}$. Since (f_{a^-}, φ_{a^-}), (f_{a^+}, φ_{a^+}) are continuous mappings, $(f_{a^-}, \varphi_{a^-})^{-1}(G, E^*) \in \tau_{a^-}$, $(f_{a^+}, \varphi_{a^+})^{-1}(G, E^*) \in \tau_{a^+}$. Then

$$\tau\Big((f, \varphi)^{-1}(G, E^*)\Big) \geq \widetilde{a} = \zeta(G, E^*),$$

i.e., (f, φ) is a continuous mapping. □

Theorem 6. *Let (X, τ, E) and (Y, ζ, E^*) be two $IVFTSs$ and β^* be an interval-valued fuzzy base for ζ. Then $(f, \varphi) : (X, \tau, E) \to (Y, \zeta, E^*)$ is a continuous mapping if and only if $\beta^*(G, E^*) \leq \tau\Big((f, \varphi)^{-1}(G, E^*)\Big)$ for each $(G, E^*) \in \mathsf{SS}(Y, E^*)$.*

Proof. Let $(f, \varphi) : (X, \tau, E) \to (Y, \zeta, E^*)$ be a continuous mapping and $(G, E^*) \in \mathsf{SS}(Y, E^*)$. Then $\zeta(G, E^*) \geq \beta^*(G, E^*)$. So,

$$\tau\Big((f, \varphi)^{-1}(G, E^*)\Big) \geq \zeta(G, E^*) \geq \beta^*(G, E^*)$$

is obtained.

Conversely, let $\beta^*(G, E^*) \leq \tau\Big((f, \varphi)^{-1}(G, E^*)\Big)$ for each $(G, E^*) \in \mathsf{SS}(Y, E^*)$. Let $(G, E^*) = \bigcup\limits_{i\in J}(G_j, E^*)$. Hence

$$\begin{array}{rcl} \tau\Big((f, \varphi)^{-1}(G, E^*)\Big) & = & \tau\left((f, \varphi)^{-1}\left(\bigcup\limits_{i\in J}(G_j, E^*)\right)\right) \\ & = & \tau\left(\bigcup\limits_{i\in J}(f, \varphi)^{-1}(G_j, E^*)\right) \\ & \geq & \bigwedge\limits_{i\in J}\tau\Big((f, \varphi)^{-1}(G_j, E^*)\Big) \\ & \geq & \bigwedge\limits_{i\in J}\beta^*(G_j, E^*). \end{array}$$

So we have $\tau\Big((f, \varphi)^{-1}(G, E^*)\Big) \geq \bigvee\limits_{(G,E^*)=\bigcup\limits_{i\in J}(G_j,E^*)} \bigwedge\limits_{i\in J}\beta^*(G_j, E^*) = \zeta(G, E^*)$. □

Theorem 7. *Let (X, τ, E) and $(Y, \zeta\zeta, E^*)$ be two $IVFTSs$ and δ^* be a subbase for ζ. Then $(f, \varphi) : (X, \tau, E) \to (Y, \zeta, E^*)$ is a continuous mapping if $\delta^*(G, E^*) \leq \tau\Big((f, \varphi)^{-1}(G, E^*)\Big)$ is satisfied, for each $(G, E^*) \in \mathsf{SS}(Y, E^*)$.*

Proof. For each $(G, E^*) \in \mathsf{SS}(Y, E^*)$,

$$
\begin{aligned}
\delta^*(G,E^*) &= \bigvee_{\bigcup_{i\in J}(G_i,E^*)=(G,E^*)} \bigwedge_{i\in J} \bigvee_{\bigcap_{\mu\in A}(F_i,E^*)=(G_i,E^*)} \bigwedge_{\mu\in A} \zeta(F_\mu, E^*) \\
&\leq \bigvee_{\bigcup_{i\in J}(G_i,E^*)=(G,E^*)} \bigwedge_{i\in J} \bigvee_{\bigcap_{\mu\in A}(F_i,E^*)=(G_i,E^*)} \bigwedge_{\mu\in A} \tau\Big((f,\varphi)^{-1}(F_\mu, E^*)\Big) \\
&\leq \bigvee_{\bigcup_{i\in J}(G_i,E^*)=(G,E^*)} \bigwedge_{i\in J} \tau\Big((f,\varphi)^{-1}(G_i,E^*)\Big) \\
&\leq \bigvee_{\bigcup_{i\in J}(G_i,E^*)=(G,E^*)} \tau\left((f,\varphi)^{-1}\left(\bigcup_{i\in J}(G_i,E^*)\right)\right) \\
&= \tau\Big((f,\varphi)^{-1}(G,E^*)\Big)
\end{aligned}
$$

is obtained. □

Definition 14. *Let (X, τ, E) and (Y, ζ, E^*) be two IVFTSs and $(f, \varphi) : (X, \tau, E) \to (Y, \zeta, E^*)$ be a mapping. Then (f, φ) is called an open mapping if it the following condition*

$$\tau(F, E) \leq \zeta((f, \varphi)(F, E))$$

is satisfied for each $(F, E) \in \mathsf{SS}(X, E)$.

Now we give an example of an open mapping.

Example 5. *Let $X = \{x\}$, $E = \{e_1, e_2\}$; $Y = \{u, v\}$, $E' = \{e'\}$.*

The soft sets in X are

$$
\begin{gathered}
F_1(e_1) = \{x\},\ F_1(e_2) = \Phi, \\
F_2(e_1) = \Phi,\ F_2(e_2) = \{x\}, \\
F_3(e_1) = F_3(e_2) = \{x\}, \\
F_4(e_1) = F_4(e_2) = \Phi,
\end{gathered}
$$

and the soft sets in Y are

$$G_1(e') = \{u\},\ G_2(e') = \{v\},\ G_3(e') = \widetilde{Y},\ G_4(e') = \Phi.$$

Define topologies τ on X and τ' on Y by

$$
\begin{gathered}
\tau(F_1, E) = [0.2, 0.5],\ \tau(F_2, E) = [0.1, 0.7],\ \tau(F_3, E) = \tau(F_4, E) = 1; \\
\tau'(G_1, E') = 1,\ \tau'(G_2, E') = [0.4, 0.8], \tau'(G_3, E') = \tau'(G_4, E') = 1.
\end{gathered}
$$

Consider mappings $f : X \to Y$ and $\varphi : E \to E'$ given by

$$
\begin{gathered}
f(x) = u, \\
\varphi(e_1) = \varphi(e_2) = e'.
\end{gathered}
$$

Then

$$
\begin{gathered}
(f, \varphi)(F_1, E)(e') = f(F_1(e_1)) \cup f(F_1(e_2)) = f(x) = u = G_1(e'), \\
\tau(F_1, E) = [0.2, 0.5] \leq \tau'(G_1, e') = [1, 1]; \\
(f, \varphi)(F_2, E)(e') = f(F_2(e_1)) \cup f(F_2(e_2)) = f(x) = u = G_1(e'), \\
\tau(F_2, E) = [0.1, 0.7] \leq \tau'(G_1, e') = [1, 1].
\end{gathered}
$$

It follows that (f, φ) is an open mapping.

Theorem 8. *Let (X,τ,E) and (Y,ζ,E^*) be two IVFTSs and $(f,\varphi):(X,\tau,E)\to(Y,\zeta,E^*)$ be a mapping and β be a base of τ. If*

$$\beta(F,E)\leq\zeta((f,\varphi)(F,E))$$

is satisfied for each $(F,E)\in\mathsf{SS}(X,E)$, then (f,φ) is an open mapping.

Proof. For each $(F,E)\in\mathsf{SS}(X,E)$,

$$\begin{aligned}\tau(F,E) &= \tau(F,E)=\bigvee_{\bigcup_{i\in J}(F_i,E)=(F,E)}\bigwedge_{i\in J}\beta(F_i,E)\\ &\leq \bigvee_{\bigcup_{i\in J}(F_i,E)=(F,E)}\bigwedge_{i\in J}\zeta((f,\varphi)(F_i,E))\\ &\leq \bigvee_{\bigcup_{i\in J}(F_i,E)=(F,E)}\zeta\left((f,\varphi)\left(\bigcup_{i\in J}(F_i,E)\right)\right)\\ &= \zeta((f,\varphi)(F,E))\end{aligned}$$

is satisfied. □

Theorem 9. *Let (Y,ζ,E^*) be an IVFTS and $(f,\varphi):\mathsf{SS}(X,E)\to(Y,\zeta,E^*)$ be a mapping of soft sets. Then define $\tau:\mathsf{SS}(X,E)\to[I]$ as follows:*

$$\tau(F,E)=\bigvee_{(f,\varphi)^{-1}(G,E^*)=(F,E)}\zeta(G,E^*).$$

Then τ is an interval-valued fuzzy topology over $\mathsf{SS}(X,E)$ and (f,φ) is a continuous mapping.

Proof. It is obvious that $\tau(\Phi)=\tau\left(\widetilde{X}\right)=1$.

$$\begin{aligned}&\tau((F_1,E)\cap(F_2,E))=\bigvee\left\{\zeta(G,E^*):(f,\varphi)^{-1}(G,E^*)=(F_1,E)\cap(F_2,E)\right\}\\ &\geq\bigvee\left\{\zeta((G_1,E^*)\cap(G_2,E^*)):(f,\varphi)^{-1}((G_1,E^*)\cap(G_2,E^*))=(F_1,E)\cap(F_2,E)\right\}\\ &\geq\left(\bigvee\left\{\zeta(G_1,E^*):(f,\varphi)^{-1}(G_1,E^*)=(F_1,E)\right\}\right)\\ &\wedge\bigvee\left\{\zeta(G_2,E^*):(f,\varphi)^{-1}(G_2,E^*)=(F_2,E)\right\}\\ &=\tau(F_1,E)\wedge\tau(F_2,E)\end{aligned}$$

is obtained. Also,

$$\begin{aligned}&\tau\left(\bigcup_{i\in J}(F_i,E)\right)=\bigvee\left\{\zeta(G,E^*):(f,\varphi)^{-1}(G,E^*)=\bigcup_{i\in J}(F_i,E)\right\}\\ &\geq\bigvee\left\{\zeta\left(\bigcup_{i\in J}(G_i,E^*)\right):(f,\varphi)^{-1}\left(\bigcup_{i\in J}(G_i,E^*)\right)=\bigcup_{i\in J}(F_i,E)\right\}\\ &\geq\bigvee\left\{\left(\bigwedge_{i\in J}\zeta(G_i,E^*)\right):(f,\varphi)^{-1}(G_i,E^*)=(F_i,E)\right\}\\ &=\bigwedge_{i\in J}\left(\bigvee\left\{\zeta(G_i,E^*):(f,\varphi)^{-1}(G_i,E^*)=(F_i,E)\right\}\right)=\bigwedge_{i\in J}\tau(F_i,E).\end{aligned}$$

So τ is an interval-valued fuzzy topology over $\mathsf{SS}(X,E)$ and (f,φ) is a continuous mapping. □

Theorem 10. *Let (X, τ, E) be an $IVFTS$ and $(f, \varphi) : (X, \tau, E) \to \mathsf{SS}(Y, E^*)$ be a mapping of soft sets. Then define $\zeta : \mathsf{SS}(Y, E^*) \to [I]$ as follows:*

$$\zeta(G, E^*) = \tau\Big((f, \varphi)^{-1}(G_i, E^*)\Big).$$

Then ζ is an interval-valued fuzzy topology over $\mathsf{SS}(Y, E^*)$ and (f, φ) is a continuous mapping.

Proof. It is clear that $\zeta(\Phi) = \zeta\Big(\widetilde{Y}\Big) = 1$.

$$\zeta((G_1, E^*) \cap (G_2, E^*)) = \tau\Big((f, \varphi)^{-1}((G_1, E^*) \cap (G_2, E^*))\Big)$$
$$= \tau\Big(\Big((f, \varphi)^{-1}(G_1, E^*)\Big) \cap \Big((f, \varphi)^{-1}(G_2, E^*)\Big)\Big)$$
$$\geq \tau\Big((f, \varphi)^{-1}(G_1, E^*)\Big) \wedge \tau\Big((f, \varphi)^{-1}(G_2, E^*)\Big) = \zeta(G_1, E^*) \wedge \zeta(G_2, E^*)$$

is obtained. Furthermore,

$$\zeta\left(\bigcup_{i\in J}(G_i, E^*)\right) = \tau\left((f, \varphi)^{-1}\left(\bigcup_{i\in J}(G_i, E^*)\right)\right) = \tau\left(\bigcup_{i\in J}(f, \varphi)^{-1}(G_i, E^*)\right)$$
$$\geq \bigwedge_{i\in J}\tau\Big((f, \varphi)^{-1}(G_i, E^*)\Big) = \bigwedge_{i\in J}\zeta(G_i, E^*).$$

So ζ is an interval-valued fuzzy topology over $\mathsf{SS}(Y, E^*)$ and (f, φ) is a continuous mapping. □

6. Direct Sum

Now let $\{(X_\lambda, \tau_\lambda, E_\lambda)\}_{\lambda\in\Lambda}$ be a family of fuzzy topological spaces, $X_\lambda \cap X_\nu = \varnothing$ and $E_\lambda \cap E_\nu = \varnothing$ for $\lambda \neq \nu$. Let $\widetilde{X}$ be union of all soft points which belong to this space and $E = \bigcup_{\lambda\in\Lambda} E_\lambda$. Then $\Big(\widetilde{X}, E\Big)$ is the family of soft sets on $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ with parameters E. For soft point $x_e \in \Big(\widetilde{X}, E\Big)$ if $x \in X_\lambda$, then $e \in E_\lambda$. If $e \in E_\lambda$, then $x \in X_\lambda$ is satisfied. For an arbitrary $(F, E) \in \Big(\widetilde{X}, E\Big)$, $(F, E)_\lambda = \{F(e) \cap X_\lambda\}_{\lambda\in\Lambda}$ [23].

Theorem 11. *Let $\{(X_\lambda, \tau_\lambda, E_\lambda)\}_{\lambda\in\Lambda}$ be a family of interval-valued fuzzy topological spaces, $X'_\lambda s$ be pairwise disjoint. Then τ defined by*

$$\tau(F, E) = \bigwedge_{\lambda\in\Lambda} \tau_\lambda((F, E)_\lambda),$$

for each $(F, E) \in \Big(\widetilde{X}, E\Big)$ is an interval-valued fuzzy topology on $\Big(\widetilde{X}, E\Big)$.

Proof. Let $(F_1, E), (F_2, E) \in \Big(\widetilde{X}, E\Big)$. Then

$$\tau((F_1, E) \cap (F_2, E)) = \bigwedge_{\lambda\in\Lambda} \tau_\lambda(((F_1, E) \cap (F_2, E))_\lambda)$$
$$= \bigwedge_{\lambda\in\Lambda} \tau_\lambda((F_1, E)_\lambda \cap (F_2, E)_\lambda)$$
$$\geq \bigwedge_{\lambda\in\Lambda} (\tau_\lambda((F_1, E)_\lambda) \wedge \tau_\lambda((F_2, E)_\lambda))$$
$$= \left(\bigwedge_{\lambda\in\Lambda} \tau_\lambda((F_1, E)_\lambda)\right) \wedge \left(\bigwedge_{\lambda\in\Lambda} \tau_\lambda((F_2, E)_\lambda)\right) = \tau(F_1, E) \wedge \tau(F_2, E)$$

holds.

Now, let $\{(F_i, E_i)\}_{i\in J}$ be a family of soft sets. Then

$$\tau\left(\bigcup_{i\in J}(F_i, E_i)\right) = \bigwedge_{\lambda\in\Lambda} \tau_\lambda\left(\left(\bigcup_{i\in J}(F_i, E_i)\right)_\lambda\right) = \bigwedge_{\lambda\in\Lambda} \tau_\lambda\left(\bigcup_{i\in J}(F_i, E_i)_\lambda\right)$$
$$\geq \bigwedge_{\lambda\in\Lambda} \bigwedge_{i\in J}\tau_\lambda((F_i, E_i)_\lambda) = \bigwedge_{i\in J}\left(\bigwedge_{\lambda\in\Lambda} \tau_\lambda((F_i, E_i))_\lambda\right) = \bigwedge_{i\in J}\tau(F_i, E_i)$$

is satisfied. Hence, (X, τ, E) is an $IVFTS$. □

Definition 15. *The interval-valued fuzzy topological space* (X, τ, E) *in the previous theorem is called the direct sum of* $\{(X_\lambda, \tau_\lambda, E_\lambda)\}_{\lambda\in\Lambda}$*, denoted by* $(X, \tau, E) = \underset{\lambda\in\Lambda}{\oplus}(X_\lambda, \tau_\lambda, E_\lambda)$*.*

It is obvious that $i_\lambda : X_\lambda \to X = \bigcup_{\lambda\in\Lambda} X_\lambda$ and $j_\lambda : E_\lambda \to E = \bigcup_{\lambda\in\Lambda} E_\lambda$ are embedding mappings for all $\lambda \in \Lambda$. Then

$$(i_\lambda, j_\lambda) : (X_\lambda, \tau_\lambda, E_\lambda) \to (X, \tau, E)$$

is a continuous mapping.

Theorem 12. *Let* $\{(X_\lambda, \tau_\lambda, E_\lambda)\}_{\lambda\in\Lambda}$ *be a family of interval-valued fuzzy topological spaces,* $X = \prod_{\lambda\in\Lambda} X_\lambda$ *be a set,* $E = \prod_{\lambda\in\Lambda} E_\lambda$ *be a parameter set and* $p_\lambda : X \to X_\lambda$*,* $q_\lambda : E \to E_\lambda$ *be two projections mappings for* $\forall\lambda \in \Lambda$*. Define* $\beta : \mathsf{SS}(Y, E) \to [I]$ *as follows:*

$$\beta(G, E^*) = \bigvee\left\{\bigwedge_{j=1}^{n}\tau_{\alpha_i}(F_{\alpha_i}, E_{\alpha_i}) : (F, E) = \bigcap_{j=1}^{n}(p_{\alpha_i}, q_{\alpha_i})^{-1}(F_{\alpha_i}, E_{\alpha_i})\right\}.$$

Then β *is a base of the topology* $\tau_\beta R$ *on* (X, E)*, and* $(p_\lambda, q_\lambda) : \big(X, \tau_\beta, E\big) \to (X_\lambda, \tau_\lambda, E_\lambda)$ *are continuous mapping for* $\forall\lambda \in \Lambda$*.*

Proof. We show that β is a base. Indeed,

$$\begin{aligned}\beta\Big(\widetilde{X}\Big) &= \bigvee\left\{\bigwedge_{j=1}^{n}\tau_{\alpha_i}(F_{\alpha_i}, E_{\alpha_i}) : \widetilde{X} = \bigcap_{j=1}^{n}(p_{\alpha_i}, q_{\alpha_i})^{-1}(F_{\alpha_i}, E_{\alpha_i})\right\} \\ &= \bigvee\left\{\bigwedge_{j=1}^{n}\tau_{\alpha_i}(X_{\alpha_i}, E_{\alpha_i})\right\} = 1\end{aligned}$$

is satisfied. Similarly, $\beta(\Phi)$ is obtained.

$$\beta(F, E) \wedge \beta(G, E) = \left(\bigvee_{(F,E)=\bigcap_{j=1}^{n}\left(p_{\alpha_i}, q_{\alpha_i}\right)^{-1}\left(F_{\alpha_i}, E_{\alpha_i}\right)} \bigwedge_{j=1}^{n}\tau_{\alpha_i}(F_{\alpha_i}, E_{\alpha_i})\right)$$

$$\wedge\left(\bigvee_{(G,E)=\bigcap_{j=1}^{k}\left(p_{\delta_i}, q_{\delta_i}\right)^{-1}\left(G_{\delta_i}, E_{\delta_i}\right)} \bigwedge_{j=1}^{k}\tau_{\delta_i}\big(G_{\delta_i}, E_{\delta_i}\big)\right)$$

$$= \bigvee_{(F,E)=\bigcap_{j=1}^{n}\left(p_{\alpha_i}, q_{\alpha_i}\right)^{-1}\left(F_{\alpha_i}, E_{\alpha_i}\right)} \bigvee_{(G,E)=\bigcap_{j=1}^{k}\left(p_{\delta_i}, q_{\delta_i}\right)^{-1}\left(G_{\delta_i}, E_{\delta_i}\right)} \left(\left(\bigwedge_{j=1}^{n}\tau_{\alpha_i}(F_{\alpha_i}, E_{\alpha_i})\right)\right.$$

$$\left.\wedge\left(\bigwedge_{j=1}^{k}\tau_{\delta_i}\big(G_{\delta_i}, E_{\delta_i}\big)\right)\right)$$

$$= \bigvee_{\left(\bigcap_{j=1}^{n}\left(p_{\alpha_i}, q_{\alpha_i}\right)^{-1}\left(F_{\alpha_i}, E_{\alpha_i}\right)\right)\cap\left(\bigcap_{j=1}^{k}\left(p_{\delta_i}, q_{\delta_i}\right)^{-1}\left(G_{\delta_i}, E_{\delta_i}\right)\right)=(F,E)\cap(G,E)} \left(\left(\bigwedge_{j=1}^{n}\tau_{\alpha_i}(F_{\alpha_i}, E_{\alpha_i})\right)\right.$$

$$\left.\wedge\left(\bigwedge_{j=1}^{k}\tau_{\delta_i}\big(G_{\delta_i}, E_{\delta_i}\big)\right)\right)$$

$$\leq \bigvee_{\bigcap_{\lambda\in\Lambda}\left(p_{\varpi_\lambda}, q_{\varpi_\lambda}\right)^{-1}\left(H_{\varpi_\lambda}, E_{\varpi_\lambda}\right)=(F,E)\cap(G,E)} \tau_{\varpi_\lambda}(H_{\varpi_\lambda}, E_{\varpi_\lambda}) = \beta((F, E) \cap (G, E)).$$

Hence β is a base.

Now we check that the projection mapping $(p_\lambda, q_\lambda) : (X, \tau_\beta, E) \to (X_\lambda, \tau_\lambda, E_\lambda)$ are continuous mapping for $\forall\lambda \in \Lambda$. Indeed, for each $(F_\lambda, E_\lambda) \in \mathrm{SS}(X_\lambda, E_\lambda)$,

$$\tau\Big((p_\lambda, q_\lambda)^{-1}(F_\lambda, E_\lambda)\Big) \geq \beta\Big((p_\lambda, q_\lambda)^{-1}(F_\lambda, E_\lambda)\Big)$$
$$= \left\{\bigwedge_{j=1}^{n}\tau_{\alpha_i}(F_{\alpha_i}, E_{\alpha_i}) : (p_{\alpha_i}, q_{\alpha_i})^{-1}(F_{\alpha_i}, E_{\alpha_i}) = (p_\lambda, q_\lambda)^{-1}(F_\lambda, E_\lambda)\right\}$$
$$\geq \tau_\lambda(F_\lambda, E_\lambda)$$

is satisfied. □

Remark 2. *In general, we cannot obtain an interval-valued fuzzy topology by utilizing τ^- and τ^+, with τ^- and τ^+ being fuzzy topologies. If τ^-, τ^+ are two fuzzy topologies and $(F, E) \in \mathrm{SS}(X, E)$, $\tau^-(F, E) \leq \tau^+(F, E)$, then $\tau = [\tau^-, \tau^+]$ is an interval-valued fuzzy topology.*

7. Conclusions

We introduce the interval-valued fuzzy set on the family of all soft sets over X. Later we give interval-valued fuzzy topology (cotopology) on $\mathrm{SS}(X, E)$. We obtain that each interval-valued fuzzy topology is a descending family of soft topologies. In addition to, we study some topological structures such as interval-valued fuzzy neighborhood system of a soft point, base and subbase of τ and investigate some relationship between them. Finally, we give some concepts such as direct sum, open mapping and continuous mapping,consider relationships between them and illustrate it by examples.

The relations between soft topologies and crisp topologies explained in the paper [19,21] may be used for the future research in this field. Also, the relations between fuzzy soft and soft topologies might suggest a new lines of investigation related to our article.

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