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Stability of Stochastic Partial Differential Equations

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Abstract: In this paper, we study the stability of the stochastic parabolic differential equation with dependent coefficients. We consider the stability of an abstract Cauchy problem for the solution of certain stochastic parabolic differential equations in a Hilbert space. For the solution of the initial-boundary value problems (IBVPs), we obtain the stability estimates for stochastic parabolic equations with dependent coefficients in specific applications.

Keywords: stochastic parabolic equations; initial-boundary problems; Hilbert space; stability

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1. Introduction

Stochastic parabolic equations (SPEs) are relevant in applied sciences and engineering applications. It is widely acknowledged that the majority of issues involving uncertainties, such as heat flow, fusion processes, and models of financial instruments, including bonds, options, and interest rates, correspond to stochastic differential equations (SDEs). One of the applications of stochastic partial equations is the random evolution of systems with spatial extension like random interface growth or random evolution of surfaces and fluids, which are subject to random forcing. In particular, in mathematical finance, they have been used to model the term structure of finance, the term structure of interest rates, and volatility surfaces. Solutions of stochastic partial differential equations (SPDEs) and methods used to find these solutions have been studied extensively (see [1–17] and the references therein). A number of stochastic processes are described by SDEs, i.e., they represent solutions of the corresponding integral equations. For example, a function that depends on these stochastic processes is the portfolio of shares, which can also be represented using a stochastic integral equation. The stochastic integrals, which play a vital role in the stochastics of financial markets, are of paramount importance. SDEs contain a variable that involves random white noise, calculated as the derivative of the Wiener process or Brownian motion. In [1], the initial value problem

$$\begin{cases} d(\partial_t v(t)) + Av(t)dt = f(t)dw_t, \quad \text{with} \quad t \in (0,T), \\ v(0) = \varphi, \quad \partial_t v(0) = \psi \end{cases} \tag{1}$$

for stochastic hyperbolic differential equations in a Hilbert space H was examined. Further, $A : H \to H$ is a self-adjoint and positive definite (PD) operator with $A \geq \delta \mathbb{I}$, where $\delta > 0$ is a real number, and $\mathbb{I}$ is the identity operator on H. Here:

(i) A standard Wiener process w_t is specified on the probability space (Ω, F, P).
(ii) For any $z \in [0,T]$ and subset $H_1 \subseteq H$, we have $f(z) \in M_w^2([0,T], H_1)$.

Here, $M_w^2([0,T],H)$ represents the space of measurable, stochastic processes with values in H so that any $\rho \in M_w^2([0,T],H)$ satisfies the following statements:

(a) The function $\rho(t)$ has an F_t-measurable value.

(b) For the expectation value E, we have $E\int_0^T \|\rho(t)\|_H^2 dt < \infty$.

(iii) $\varphi, \psi \in M_w^2([0,T],H_2)$ are measurable processes with H_2 values for a subspace $H_2 \subseteq H$.

In [4], the multipoint nonlocal boundary value problem

$$\begin{cases} d\nu(t) = -A\nu(t)dt + f(t)dw_t \quad \text{with} \quad t \in (0,T), \\ \nu(0) = \sum_{j=1}^{J} \alpha_j \nu(\lambda_j) + \varphi(w_{\lambda_1,\ldots,w_{\lambda_J}}), \\ \sum_{j=1}^{J} |\alpha_j| \le 1 \quad \text{with} \quad 0 < \lambda_1 < \cdots < \lambda_j \le T \end{cases} \tag{2}$$

for a stochastic parabolic differential equation (SPDE) in H and self-adjoint PD operator A was examined with prerequisites (i) and (ii).

For the solution of Problem (2) *with* the standard Wiener process, the 1/2-th order of approximation of the RDS (Rothe difference scheme) is

$$\begin{cases} \mu_k - \mu_{k-1} + \tau A u_k = \varphi_k \quad \text{with} \quad \varphi_k = \int\limits_{t_{k-1}}^{t_k} f(s)dw_s,\ t_k = k\tau,\ k \in [1,N], \\ \mu_0 = \sum\limits_{j=1}^{J} \alpha_j \mu_{\left[\frac{\lambda_j}{\tau}\right]} + \varphi(w_{\lambda_1,\cdots,}w_{\lambda_J}). \end{cases}$$

For the same solution of Problem (2) *without* the standard Wiener process, the 1/2-th order of approximation of the RDS is

$$\begin{cases} \mu_k - \mu_{k-1} + \tau A \mu_k = f(t_{k-1})(w_{t_k} - w_{t_{k-1}}), \quad \text{with} \quad 1 \le k \le N, \\ \mu_0 = \sum\limits_{j=1}^{J} \alpha_j \mu_{\left[\frac{\lambda_j}{\tau}\right]} + \varphi(w_{\lambda_1,\cdots,}w_{\lambda_J}). \end{cases}$$

An estimate of the convergence of these difference schemes (DS) was established for the solution.

A famous application of Ornstein–Uhlenbeck operators was described in [7] with time-periodic coefficients. In this paper, G. Da Prato and A. Lunardi studied applications of the differential operator $\mu \to \mu_t - \mathbb{L}(t)u$ in the space of time-periodic and continuous functions. The Ornstein–Uhlenbeck operator family $\{\mathbb{L}(t)\}_{t\in\mathbb{R}}$ is given by the time-dependent operators

$$\mathbb{L}(t)\phi(x) = \frac{1}{2}Tr\Big[\mathcal{B}(t)\mathcal{B}*(t)D^2\phi(x)\Big] + \langle \mathcal{A}(t)x + g(t), D\phi(x)\rangle,$$

where the maps $\mathcal{A} : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n)$, $\mathcal{B} : \mathbb{R} \to \mathcal{L}(\mathbb{R}^n)$ and $g : \mathbb{R} \to \mathbb{R}^n$ are continuous. In this paper, they consider the SDE in $\mathbb{R}^n$

$$\begin{cases} dX(t) = (\mathcal{A}(t)X(t) + g(t))dt + \mathcal{B}(t)dW(t), \\ X(s) = x, \end{cases} \tag{3}$$

where $\mathcal{W}(t)$ is an n-dimensional, standard Brownian motion and $x \in \mathbb{R}^n$. Problem (3) has a unique mild solution

$$X(t,s,x) = U(t,s)x + \int_s^t U(t,r)g(r)dr + \int_s^t (U(t,r)\mathcal{B}(r)d\mathcal{W}(r),$$

where $U(t,s)$ is the evolution operator defined in $\mathbb{R}^n$ that satisfies the operator-valued differential equation

$$\begin{cases} \frac{\partial U(t,s)}{\partial t} = \mathcal{A}(t)U(t,s) & \text{with} \quad t,s \in \mathbb{R} \\ U(s,s) = \mathbb{I}. \end{cases}$$

This paper aims to develop novel operator methods for estimating the stability of SPEs with coefficients in temporal and spatial variables. We also establish the stability of an abstract Cauchy problem for the solution of an SPDE in a Hilbert space with a time-dependent, positive operator and derive theorems concerning stability estimates for the solution of the SPE. We also provide numerical results for the $\frac{1}{2}$-th and the accuracy DS with $\frac{3}{2}$-th order of the approximate solution of mixed problems for SPEs with Dirichlet-boundary conditions (DBC).

2. Stochastic Parabolic Equations

It is well known that many initial-boundary value problems (IBVPs) for SPEs of the form

$$\begin{cases} d\nu(t) + A(t)\nu(t)dt = f(t,w(t))dt + g(t,w(t))dw_t & \text{with} \quad t \in (0,T), \\ \nu(0) = \varphi, \end{cases} \tag{4}$$

can be simplified to the Cauchy problem in the Hilbert space H. Additionally, the operator $A(t)$ is unbounded, $w_t = \sqrt{t}\xi$ is a standard Wiener process defined in the probability space (Π, F, P), and $\xi \in \mathcal{N}(0,1)$ is the standard normal distribution. SDEs contain a variable that involves random white noise, calculated as the derivative of the Wiener process or Brownian motion. According to the 1933 Paley–Wiener–Zygmund theorem, the Wiener process's path is almost nowhere differentiable. Therefore, it needs its own set of calculus rules, known as the Stratonovich stochastic calculus and the Ito stochastic calculus, respectively.

The probability space is related to the sample space (Π) and a sigma-algebra (F) of measurable events. Furthermore, w is a Wiener process in H. For more information, see A. Klenke [18]. Moreover, $\nu(t)$ is unknown, and $f(t,w(t))$ and $g(t,w(t))$ are given H-valued functions defined in $(0,T) \times \Pi$. Furthermore, suppose that $f, g \in M_w^2([0,T] \times \Pi, H_1)$ are H_1-valued, stochastic processes for which the conditions

$$E\int_0^T \|f(t,w(t))\|_{H_1}^2 dt < \infty \quad \text{and} \quad E\int_0^T \|g(t,w(t))\|_{H_1}^2 dt < \infty \tag{5}$$

are satisfied. Here, the integrals are Bochner integrals, $H_1 \subset H$ is a subspace, and E represents the expectation values. For more information, see H. Brezis [19]. The operator approach can be used to investigate Problem (4) for a parabolic-type SPDE with time-dependent coefficients. Assume that the operator $-A(t)$ generates an analytic semigroup $\exp\{-sA(t)\}$, with $s \geq 0$ and an exponentially decreasing norm for each $t \in [0,T]$. In the limit $s \to +\infty$, we have the following:

$$\|\exp(-sA(t))\|_{H\to H} \leq Me^{-\delta s}, \tag{6}$$

$$\|s \cdot A(t) \cdot \exp(-sA(t))\|_{H\to H} \leq Me^{-\delta s}. \tag{7}$$

Here, let $M \in [1,+\infty)$ and $\delta \in (0,+\infty)$. Consequently, the operator $A^{-1}(t)$ is bounded and exists. Thus, the operator $A(t)$ is closed in the subspace $H_1 \subset H$ such that $A(t)(D(A(t))) \subseteq H_1$, $D(A(t)) = D(A(0))$, and $t \in [0,T]$, where $D(A(t))$ is the domain of $A(t)$. For more information, see ([20], Propositions 2.2 and 2.3).

If for some t in the uniform topology, the operator $A(t)A^{-1}(s)$ is Holder continuous, then

$$\left\| [A(t) - A(\tau)]A^{-1}(s) \right\|_{H\to H} \leq M|t-\tau|^{\varepsilon}, \quad \text{with} \quad \varepsilon \in (0,1], \tag{8}$$

where s is fixed and $M\varepsilon \in (0,+\infty)$ are independent of t, s, and τ for $0 \leq t,s,\tau \leq T$.

We call the operator-valued function $\nu(t,s)$ a *fundamental solution* of (4) if:

(1) For $0 \leq s < t \leq T$, the function $\nu(t,s)$ is strongly continuous in t and s.

(2) The time-evolution identities

$$\nu(t,s) = \nu(t,\tau)\nu(\tau,s) \quad \text{and} \quad \nu(t,t) = \mathbb{I} \tag{9}$$

for $0 \leq s \leq \tau \leq t \leq T$, where I is the identity operator.

(3) $\nu(t,s)$ maps the domain $D(A(t))$ of $A(t)$ into itself. In t and s for $0 \leq s < t \leq T$, the operator $u(t,s) = A(t)\nu(t,s)A^{-1}(s)$ is strongly continuous and bounded.

(4) In the domain $D(A(t))$, the operator-valued function $\nu(t,s)$ is differentiable with respect to both t and s, whereas

$$\frac{\partial \nu(t,s)}{\partial t} + A(t)\nu(t,s) = 0, \tag{10}$$

and

$$\frac{\partial \nu(t,s)}{\partial s} - \nu(t,s)A(s) = 0. \tag{11}$$

For more information, see [21]. By applying (9) and (10), we obtain

$$\nu(t) = \nu(t,0)\nu(0) + \int_0^t \nu(t,y)\{f(y,w(y))dy + g(y,w(y))dw_y\} \tag{12}$$

as a mild solution of SPDE (4), satisfying the assumption of (5) (see also [6,8]). Moreover, the solution of this problem requires the non-smoothness functions $f(t,w(t))$ and $g(t,w(t))$ for $t > 0$ since the functions $f(t,w(t))$ and $g(t,w(t))$ depend on the Wiener process $w(t)$.

Lemma 1. *For any $0 \leq s \leq t \leq T$ and $\beta \leq \alpha \in (0,1]$, the following inequalities hold*

$$\|\nu(t,s)\|_{H\to H} \leq M, \tag{13}$$

$$\|A^{\alpha}(t)\nu(t,s)A^{-\beta}(p)\|_{H\to H} \leq \frac{M}{(t-s)^{\alpha-\beta}}, \tag{14}$$

$$\|A(t)A^{-1}(s)\|_{H\to H} \leq M \tag{15}$$

This is discussed in [21,22]. The following Tubaro theorem was established in [9,15].

Theorem 1. *Assume that e^{-tA} is a C_0 semigroup on H with $\|e^{-tA}\|_{H\to H} \leq e^{-\delta t}$ for some $\delta \in [0,\infty)$ and for every $t \in [0,\infty)$. Then, there exists a constant $M_p(\delta) \in (0,\infty)$ for every $p \in (0,\infty]$ so that*

$$E \sup_{0\leq t\leq T} \left\| \int_0^t e^{-A(t-s)} f(y,w(y))dw_y \right\|^p \leq M_p(\delta) E \left(\int_0^T \|f(y,w(y))dw_y\|_H^2 dy \right)^{\frac{p}{2}} \tag{16}$$

for all $f \in M_w^2([0,T] \times \Pi, H_1)$.

The following theorem gives the stability estimate.

Theorem 2. *Suppose that*

$$E\|\nu(0)\|_H < \infty,$$

$$E\int_0^T \|f(y,w(y))\|_H dy < \infty,$$

$$E\int_0^T \|g(y,w(y))\|_H^2 dy < \infty.$$

Then, for the solution of (4), we have the inequality

$$\max_{t\in[0,T]} E\|\nu(t)\|_H \leq M\left[E\|\nu(0)\|_H + E\int_0^T \|f(y,w(y))\|_H dy + \left(E\int_0^T \|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}\right]. \tag{17}$$

Proof. The triangle inequality and Formula (12) imply that

$$\max_{t\in[0,T]} E\|\nu(t)\|_H^2 \leq \max_{t\in[0,T]} E\|\nu(t,0)\nu(0)\|_H$$

$$+\max_{t\in[0,T]} E\left\|\int_0^t \nu(t,y)f(y,w(y))dy\right\|_H + \max_{t\in[0,T]} E\left\|\int_0^t \nu(t,y)g(y,w(y))dw_y\right\|_H$$

$$= I_1 + I_2 + I_3.$$

Here,

$$I_1 = \max_{t\in[0,T]} E\|\nu(t,0)\nu(0)\|_H,$$

$$I_2 = \max_{t\in[0,T]} E\left\|\int_0^t \nu(t,y)f(y,w(y))dy\right\|_H,$$

$$I_3 = \max_{t\in[0,T]} E\left\|\int_0^t \nu(t,y)g(y,w(y))dw_y\right\|_H.$$

For all $r \in \{1,2,3\}$, we estimate I_r separately. We start with I_1. By applying Estimate (13), we can write

$$I_1 = \max_{t\in[0,T]} E\|\nu(t,0)\|_{H\to H}\|\nu(0)\|_H \leq M_3\|\nu(0)\|_H.$$

For I_2, using estimate (13), we obtain

$$I_2 \leq \max_{t\in[0,T]} E\int_0^t \|\nu(t,y)\|_{H\to H}\|f(y,w(y))\|_H dy \leq M_4 E\int_0^t \|f(y,w(y))\|_H dy.$$

Finally, let us estimate I_3. Using the estimate from Theorem 1, we obtain

$$I_3 \leq \max_{t\in[0,T]} E\left(\int_0^t \|\nu(t,y)g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}$$

$$\leq \max_{t\in[0,T]} E\left(\int_0^t \|\nu(t,y)\|_{H\to H}^2\|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}} \leq M_5 E\left(\int_0^T \|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}.$$

By combining the estimates for I_r for all $r = 1,2,$ and 3, we obtain (17). Theorem 2 is established. □

Theorem 3. *Suppose that*

$$E\|\nu(0)\|_H^2 < \infty,$$
$$E\int_0^T \|f(y,w(y))\|_H^2 dy < \infty,$$
$$E\int_0^T \|g(y,w(y))\|_H^2 dy < \infty.$$

Hence, in order to solve Problem (4), we obtain the estimate as follows

$$\left(E\int_0^T \|\nu(t)\|_H^2 dt\right)^{\frac{1}{2}} \le M(\delta)\left[\left(E\|\nu(0)\|_H^2\right)^{\frac{1}{2}} \right. \tag{18}$$
$$\left. + \left(E\int_0^T \|f(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}} + \left(E\int_0^T \|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}\right].$$

Proof. By applying (12) and the triangle inequality, we obtain

$$\left(E\int_0^T \|\nu(t)\|_H^2 dt\right)^{\frac{1}{2}} \le \left(E\int_0^T \|\nu(t,0)\nu(0)\|_H^2 dt\right)^{\frac{1}{2}}$$
$$+\left(E\int_0^T \left\|\int_0^t \nu(t,y)f(y,w(y))dy\right\|_H^2 dt\right)^{\frac{1}{2}} + \left(E\int_0^T \left\|\int_0^t \nu(t,y)g(y,w(y))dw_y\right\|_H^2 dt\right)^{\frac{1}{2}}$$
$$= J_1 + J_2 + J_3.$$

Here,

$$J_1 = \left(E\int_0^T \|\nu(t,0)\nu(0)\|_H^2 dt\right)^{\frac{1}{2}},$$
$$J_2 = \left(E\int_0^T \left\|\int_0^t \nu(t,y)f(y,w(y))dy\right\|_H^2 dt\right)^{\frac{1}{2}},$$
$$J_3 = \left(E\int_0^T \left\|\int_0^t \nu(t,y)g(y,w(y))dw_y\right\|_H^2 dt\right)^{\frac{1}{2}}.$$

For all $r \in \{1,2,3\}$, we estimate J_r separately. First, we estimate J_1 by applying Estimate (13) and obtain

$$J_1 \le \left(E\int_0^T \|\nu(t,0)\|_{H\to H}^2 \|\nu(0)\|_H^2 dt\right)^{\frac{1}{2}} \le M_1 TE\left(\|\nu(0)\|_H^2\right)^{\frac{1}{2}}.$$

Now, let us estimate J_2. By making the substitution $s = t - y$, we can write

$$\int_0^t \nu(t,y) f(y,w(y))dy = \int_0^t \nu(t,t-s) f(t-s,w(t-s))ds$$
$$= \int_0^T \nu(t,t-s) f_*(t-s,w(t-s))ds.$$

By using the Minkowski inequality and Estimate (13), we obtain

$$J_2 \leq E\int_0^T Me^{-\delta s}\left(\int_0^T \|f_*(t-s,w(t-s))\|_H^2 dt\right)^{\frac{1}{2}} ds$$
$$\leq E\int_0^T Me^{-\delta s}\left(\int_0^T \|f(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}} ds$$
$$= M_4(\delta)\left(E\int_0^T \|f(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}.$$

Finally, let us estimate J_3. By making the substitution $s = t - y$, we obtain

$$\int_0^t \nu(t,y) g(y,w(y))dw_y = \int_0^t \nu(t,t-s) g(t-s,w(t-s))dw_{t-s}$$
$$= \int_0^T \nu(t,t-s) g_*(t-s,w(t-s))dw_{t-s}.$$

Here,

$$g_*(t-s,w(t-s)) = \begin{cases} g_*(t-s,w(t-s)), & s \in [0,t], \\ 0, & t-s \notin [0,T]. \end{cases}$$

By using the Minkowski inequality, Estimate (13), and the estimate from Theorem 2, it follows that

$$J_3 \leq E\int_0^T Me^{-\delta s}\left(\int_0^T \|g_*(t-s,w(t-s))\|_H^2 (dw_{t-s})^2\right)^{\frac{1}{2}} ds$$
$$\leq E\int_0^T Me^{-\delta s}\left(\int_0^T \|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}} ds$$
$$= M_4(\delta)\left(E\int_0^T \|g(y,w(y))\|_H^2 dy\right)^{\frac{1}{2}}.$$

By combining the estimates for J_r for all $r = 1, 2$, and 3, we obtain (18), which completes the proof of Theorem 3. □

3. Applications

We discuss the applications of abstract theorems in this section.

First, let $\Omega = (0,1)^n$ be the unit open cube with boundary $S = \overline{\Omega} \setminus \Omega$. The mixed problem for the multidimensional SPE is considered in $[0,T] \times \overline{\Omega}$

$$\begin{cases} du(t,x,w(t)) + \left[-\sum\limits_{r=1}^{n} (a_r(t,x)u_{x_r})_{x_r} + \delta u(t,x,w(t)) \right] dt \\ = g(t,x,w(t))dt + f(t,x,w(t))dw_t, \\ 0 < t < T, x = (x_1, \dots, x_n) \in \Omega, \\ u(0,x,w(0)) = \varphi(x),\ x \in \overline{\Omega}, \\ u(t,x,w(t)) = 0,\ x \in S, t \in [0,T]. \end{cases} \tag{19}$$

with DBC. Let $\delta \in (0,+\infty)$ and $(t,x) \in (0,T) \times \Omega$ so that $a_1(t,x), ..., a_n(t,x) \geq \delta$, and let the functions $g(t,x,w(t))$ and $f(t,x,w(t))$ be smooth in x. Assuming compatibility conditions, Problem (19) has a unique weak solution $u(t,x,w(t))$ and Problem (19) can be regarded as a Cauchy problem in (4) in a Hilbert space $H = L_2(\overline{\Omega})$. The differential expression

$$A(t)v(x) = -\sum_{r=1}^{n} (a_r(t,x)v_{x_r})_{x_r} + \delta v(x) \tag{20}$$

defines a PD and self-adjoint operator $A(t)$ with the domain

$$D(A(t)) = \{v(x) \mid (a_r(x)v_{x_r})_{x_r} \in L_2(\Omega), 1 \leq r \leq n, u(x) = 0, x \in S\}.$$

Theorem 4. *Assume that*

$$E\|\varphi\|^2_{L_2(\Omega)}, E\int_0^T \|f(t,\cdot,w_t)\|^2_{L_2(\Omega)}dt < \infty,$$

$$E\int_0^T \|g(t,.,w_t)\|^2_{L_2(\Omega)}dt < \infty.$$

Therefore, the solution of Problem (19) is given as

$$\max_{t\in[0,T]} E\|u(t)\|_{L_2(\Omega)} \leq M\Big[E\|\varphi\|_{L_2(\Omega)}$$

$$+E\int_0^T \|f(y,w(y))\|_{L_2(\Omega)}dy + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}}\Bigg],$$

$$\left(E\int_0^T \|u(t)\|^2_{L_2(\Omega)}dt\right)^{\frac{1}{2}} \leq M(\delta)\left[\left(E\|\varphi\|^2_{L_2(\Omega)}\right)^{\frac{1}{2}}\right.$$

$$\left.+\left(E\int_0^T \|f(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}} + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}}\right].$$

Proof. The proof is based on Theorems 2 and 3. Consider the operator $A(t)$ defined by (20) self-adjointness and positivity. Additionally, it is defined on the coercivity equality for the solution of the elliptic problem in $L_2(\overline{\Omega})$ (see also [23]).

Second, in $[0,T]\times\overline{\Omega}$, we consider the mixed problem involving a multidimensional SPDE

$$\begin{cases} du(t,x,w(t)) + \left(-\sum\limits_{r=1}^{n}(a_r(t,x)u_{x_r})_{x_r} + \delta u(t,x,w(t))\right)dt \\ = g(t,x,w(t))dt + f(t,x,w(t))dw_t, \\ 0<t<T, x=(x_1,\ldots,\ x_n)\in\Omega, \\ u(0,x,w(0)) = \varphi(x),\ x\in\overline{\Omega}, \\ \frac{\partial}{\partial\mu}u(t,x,w(t)) = 0,\ x\in S, t\in[0,T]. \end{cases} \tag{21}$$

Here, the Neumann boundary conditions are given. Let μ be the normal vector to Ω. We have the same parameter settings as in the previous problem (19), and under compatibility conditions, Problem (21) has a weak unique solution $u(t,x,w(t))$. Equation (21) can be formulated as a Cauchy problem in (4) in a Hilbert space $H=L_2(\overline{\Omega})$, and we obtain the PD, self-adjoint operator $A(t)$ in the same way. □

Theorem 5. *To ensure that all assumptions in Theorem 4 are satisfied, for the solution of Problem (21), the following estimates hold*

$$\max_{t\in[0,T]} E\|u(t)\|_{L_2(\Omega)} \le M\left[E\|\varphi\|_{L_2(\Omega)} + E\int_0^T \|f(y,w(y))\|_{L_2(\Omega)}dy + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}}\right],$$

$$\left(E\int_0^T \|u(t)\|^2_{L_2(\Omega)}dt\right)^{\frac{1}{2}} \le M(\delta)\left[\left(E\|\varphi\|^2_{L_2(\Omega)}\right)^{\frac{1}{2}}\right.$$

$$\left. + \left(E\int_0^T \|f(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}} + \left(E\int_0^T \|g(y,w(y))\|^2_{L_2(\Omega)}dy\right)^{\frac{1}{2}}\right].$$

Proof. The proof of Theorem 5 makes use of Theorems 2 and 3, as well as the positivity and self-adjointness of the operator $A(t)$, which is defined by (20). Additionally, the proof requires the theorem of coercivity inequality for the solution of the elliptic problem in $L_2(\overline{\Omega})$ (also see [23]). □

4. Numerical Results

In applied mathematics, numerical methods are important when analytical techniques are ineffective for obtaining approximate solutions for SPDEs. In this paragraph, we present the single-step DS in time for the solution of one-dimensional stochastic SPDEs and provide numerical results for the mixed problem with DBC. The problem is solved by applying modified Gauss elimination. The outcomes of our numerical experiments support the theoretical statements regarding the solution of the DS (see [24,25]).

In the following, we regard the IBVP

$$\begin{cases} dv(t,x,w_t) - 2(1+t)v_{xx}(t,x,w_t)dt = e^{-(t+1)^2}\sin(x)dw_t, \\ t\in(0,t) \text{ and } x\in(0,\pi), \\ v(0,x,0) = 0,\ x\in[0,\pi], \\ v(t,0,w_t) = v(t,\pi,w_t) = 0, \quad \xi\in\mathcal{N}(0,1) \text{ and } t\in[0,1] \end{cases} \tag{22}$$

with DBC for the one-dimensional SPDE, where $w_t=\sqrt{t}\xi$. We have

$$u(t,x,w_t) = e^{-(t+1)^2}\sin(x)w_t.$$

as the exact solution of Problem (22). In the remainder of this document, for $\tau \in [0,1]$, $h \in [0,\pi]$, we discretize the parameter space $[0,1] \times [0,\pi]$ into $N+1$ time points and $M+1$ spatial points and obtain the uniform grid space

$$[0,1]_\tau \times [0,\pi]_h = \{(t_k, x_n) : t_n = n\tau \text{ for } n \in \{0,1,...,N\},\ N\tau = 1;\ x_m = mh,\ m \in \{0,1,...,M\}\},$$

where $N\tau = 1$ and $Mh = \pi$. To estimate the solution of the IBVP (22), we first examine the DS of the 1/2-th order of accuracy in t and the second order of accuracy in x. The equation that we have is as follows

$$\begin{cases} u_m^n - u_m^{n-1} - \frac{\tau}{h^2} 2(1+n\tau)(u_{m+1}^n - 2u_m^n + u_{m-1}^n) \\ = f(t_n, x_m)(\sqrt{n\tau} - \sqrt{(n-1)\tau})\xi, \\ f(t_n, x_m) = e^{-(t_n+1)^2} \sin(x_m) \text{ with } n \in \{1,2,...,N-1\} \text{ and } m \in \{1,2,...,M-1\}, \\ u_m^0 = 0,\ m \in \{0,1,...,M\}, \\ u_0^n = u_M^n = 0, \quad n \in \{0,1,...,N\}. \end{cases} \tag{23}$$

This is a system of linear equations with an $(N+1) \times (M+1)$ dimension. In matrix form, we represent it as the following equation

$$\begin{cases} \mathcal{A}u_{m+1} + \mathcal{B}u_m + \mathcal{C}u_{m-1} = \mathcal{D}\varphi_m, \quad m \in \{1,...,M-1\}, \\ u_0 = \vec{0},\ u_M = \vec{0}. \end{cases} \tag{24}$$

Here,

$$\varphi_m = \begin{pmatrix} \varphi_m^0 \\ \varphi_m^1 \\ \varphi_m^2 \\ \vdots \\ \varphi_m^N \end{pmatrix}_{(N+1)\times 1},$$

$\varphi_m^0 = 0$, $\varphi_m^n = f(t_n, x_m)\left(\sqrt{n\tau} - \sqrt{(n-1)\tau}\right)\xi$, where $n \in \{1,2,...,N\}$ and $m \in \{1,2,...,M\}$ and

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & a_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{N-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & a_N \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & b_1 & 0 & \dots & 0 & 0 \\ 0 & -1 & b_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-1} & 0 \\ 0 & 0 & 0 & \dots & -1 & b_N \end{bmatrix}_{(N+1)\times(N+1)},$$

$a_n = -\frac{\tau}{h^2}(2(1+t_n))$, $b_n = 1 + \frac{2\tau}{h^2}(2(1+t_n))$, $\mathcal{C} = \mathcal{A}$, and $\mathcal{D} = \mathbb{I}_{(N+1)\times(N+1)}$ is the identity matrix,

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

with $s \in \{m-1,\, m\, m+1\}$. The final matrix equation is solved using the modified Gauss elimination technique to obtain the matrix equation as follows

$$u_m = \alpha_{m+1} u_{m+1} + \beta_{m+1} \quad \text{for} \quad m = M-1, \ldots, 1, \tag{25}$$

where $u_M = \vec{0}$, α_j are $(N+1) \times (N+1)$ square matrices, and β_j are $(N+1) \times 1$ column matrices for $j \in \{1, \ldots, M-1\}$, which are determined by

$$\begin{aligned} \alpha_{m+1} &= -(\mathcal{B} + \mathcal{C}\alpha_m)^{-1}\mathcal{A}, \\ \beta_{m+1} &= (\mathcal{B} + \mathcal{C}\alpha_m)^{-1}(\mathcal{D}\varphi_m - \mathcal{C}\beta_m) \text{ for } m \in \{1, 2, \ldots, M-1\}. \end{aligned} \tag{26}$$

Here, $\alpha_1 = [0]_{(N+1)\times(N+1)}$, $\beta_1 = [0]_{(N+1)\times 1}$. Finally, we generate 1000 random numbers with a mean of 0 and a variance of 1. We set $\xi = [y_1, y_2, \ldots, y_{1000}]^T$ and define $\xi_j = y_j$, $j : 1$ to 1000 as the result of error analysis. The errors are given by

$$E_M^N = \left(\sum_{n=0}^{N} \frac{1}{1000} \sum_{m_\xi=1}^{1000} \sum_{m=1}^{M-1} \left| u(t_n, x_m, m_\xi) - u_m^n(m_\xi) \right|^2 h \right)^{\frac{1}{2}} \tag{27}$$

The functions $u(t_n, x_m) = u(t_n, x_m, m_\xi)$ are shown in the exact solutions at ξ, and $u_m^n = u_m^n(m_\xi)$ represents the numerical solution at (t_n, x_m, ξ). The results for different grid resolutions are specified in the table below.

DS $/N, M$	$10, 10$	$20, 20$	$40, 40$
DS (23)	0.0018	0.0010	0.0004782

(28)

As N and M are doubled, the value of the error decreases by a factor of approximately $1/\sqrt{2}$ for the $\frac{1}{2}$-th order of accuracy DS.

Finally, we apply the Crank–Nicholson DS to the IBVP (22) in t with $3/2$-th order accuracy and in x with second-order accuracy. Then, we obtain the following equation

$$\begin{cases} u_m^n - u_m^{n-1} - \frac{\tau}{h^2}(1 + t_{n-\frac{\tau}{2}})(u_{m+1}^n - 2u_m^n + u_{m-1}^n) \\ -\frac{\tau}{h^2}(1 + t_{n-\frac{\tau}{2}})(u_{m+1}^{n-1} - 2u_m^{n-1} + u_{m-1}^{n-1}) \\ = f(t_{n-\frac{1}{2}}, x_m)\left(\sqrt{n\tau} - \sqrt{(n-1)\tau}\right)\xi, \\ f(t_{n-\frac{1}{2}}, x_m) = e^{-(t_{n-\frac{1}{2}}+1)^2} \sin(x_m), \\ t_{n-\frac{1}{2}} = \left(n - \frac{1}{2}\right)\tau,\ x_m = mh, m \in \{1, \ldots, M-1\} \text{ and } n \in \{1, \ldots, N-1\}, \\ u_m^0 = 0,\ n \in \{0, \ldots, N\}, \\ u_0^n = u_M^n = 0,\ n \in \{0, \ldots, N\}. \end{cases} \tag{29}$$

to determine an approximate solution.

Again, we have an $(N+1) \times (M+1)$ system of linear equations and obtain the matrix representation (24). We have similar expressions as follows:

$$\varphi_m = \begin{pmatrix} \varphi_m^0 \\ \varphi_m^1 \\ \varphi_m^2 \\ \vdots \\ \varphi_m^N \end{pmatrix}_{(N+1)\times 1},$$

where $\varphi_m^0 = 0$ and $\varphi_m^n = f(t_{n-\frac{\tau}{2}}, x_m)\left(\sqrt{n\tau} - \sqrt{(n-1)\tau}\right)\xi$ for $n \in \{1, \ldots, N\}$ and $m \in \{1, \ldots, M\}$, and

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ a_1 & a_1 & 0 & \dots & 0 & 0 \\ 0 & a_2 & a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{N-1} & 0 \\ 0 & 0 & 0 & \dots & a_N & a_N \end{bmatrix}_{(N+1)\times(N+1)},$$

$$\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ b_1 & c_1 & 0 & \dots & 0 & 0 \\ 0 & b_2 & c_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_{N-1} & 0 \\ 0 & 0 & 0 & \dots & b_N & c_N \end{bmatrix}_{(N+1)\times(N+1)},$$

with $a_k = -\frac{\tau}{h^2}(1 + t_k - \frac{\tau}{2})$, $b_k = -1 + \frac{\tau}{h^2}(2(1 + t_{k-\frac{1}{2}}))$, $c_k = 1 + \frac{2\tau}{h^2}(1 + t_{k-\frac{\tau}{2}})$, $\mathcal{C} = \mathcal{A}$, and $\mathcal{D} = \mathbb{I}_{(N+1)\times(N+1)}$. Additionally, we define

$$u_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1},$$

for $s \in \{m-1,\ m\, m+1\}$. We used the method of modified Gauss elimination to find the solution of the last matrix equation, following the form of (25) and (26), as before. We estimated the error in the same way as before and obtained the table below.

DS $/N, M$	$10, 10$	$20, 20$	$40, 40$
DS (29)	0.00077724	0.00034105	0.00015924

(30)

As N and M are doubled, the value of error decreases by a factor of approximately $\sqrt{2}/4$ for the $\frac{3}{2}$-th order of accuracy DS.

5. Conclusions

In this paper, SDEs with dependent coefficients are studied. We prove stability estimates for the solution of an abstract Cauchy problem of certain SDEs in a Hilbert space. In practice, we obtain the stability estimates for the solution of IBVPs for SPEs with dependent coefficients. Notably, the method in this paper can be applied to other problems. In [1,3,13,26], the absolutely stable DS for the numerical solution of stochastic parabolic and hyperbolic differential equations is studied. The future roadmap encompasses investigating higher-order accuracy absolute stable DS for the numerical solution of stochastic hyperbolic equations with dependence in time and space variables as

$$d\dot{v}(t) + A(t)v(t)dt = f(t, w(t))dt + g(t, w(t))dw_t,\ 0 < t < T,\ v(0) = \varphi,\ \dot{v}(0) = \psi$$

in a Hilbert space H with the unbounded operator $A(t)$.

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Abbreviations

The following abbreviations are used in this manuscript:

SPE	stochastic partial equation
SDE	stochastic differential equation
SPDE	stochastic parabolic differential equation
IBVP	initial-boundary value problem
DBC	Dirichlet-boundary conditions
DS	difference scheme
RDS	Rothe difference scheme
PD	positive definite

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