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Review

A Comprehensive Review on the Fejér-Type Inequality Pertaining to Fractional Integral Operators

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Abstract: A review of the results on the fractional Fejér-type inequalities, associated with different families of convexities and different kinds of fractional integrals, is presented. In the numerous families of convexities, it includes classical convex functions, s-convex functions, quasi-convex functions, strongly convex functions, harmonically convex functions, harmonically quasi-convex functions, quasi-geometrically convex functions, p-convex functions, convexity with respect to strictly monotone function, co-ordinated-convex functions, $(\theta, h-m)-p$-convex functions, and h-preinvex functions. Included in the fractional integral operators are Riemann–Liouville fractional integral, $(k-p)$-Riemann–Liouville, k-Riemann–Liouville fractional integral, Riemann–Liouville fractional integrals with respect to another function, the weighted fractional integrals of a function with respect to another function, fractional integral operators with the exponential kernel, Hadamard fractional integral, Raina fractional integral operator, conformable integrals, non-conformable fractional integral, and Katugampola fractional integral. Finally, Fejér-type fractional integral inequalities for invex functions and (p, q)-calculus are also included.

Keywords: Fejér inequality; Hadamard fractional integral; conformable and non-conformable fractional integral; Katugampola fractional integral; Riemann–Liouville fractional integral; (p, q)-calculus

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1. Introduction

The literature about convexity had been investigated and discussed by many intellectual mathematicians before the 1960s, first of all by Fenchel and Minkowski. The efforts of Moreau and Rockafellar, who started a systematic examination of this new subject, considerably expanded and initiated the literature on convex theory at the beginning of the 1960s. Convexity and its assumptions have grown into an intriguing discipline of applied and pure mathematics over the past century. A lot of researchers have offered and contributed their expertise and insights into this area by offering updated versions of certain inequalities involving convex functions. The use of the concept of convexity in applications, of which convex optimization [1] is the primary one, is widespread. This concept has a lot of applications in applied sciences, such as finance [2], signal processing [3], control systems [4], computer science [5], mathematical optimization for modeling [6,7], engineering [8], and statistics [9]. In the subject of economics [10], this concept performs a fundamental influence on duality theory and equilibrium.

The study of integral inequalities along with convex analysis offers a fascinating and stimulating area of study in the realm of mathematical perception. Due to its importance, the literature of these concepts has recently become an amazing topic of research in both historical and contemporary times. The H-H (Hermite–Hadamard)-type and Fejér-type inequalities are the most frequently employed among all inequalities. These convex function-based inequalities are crucial and basic in practical mathematics. Thus, convexity and inequalities have been recommended as an engrossing area for researchers due to their vital role and fruitful importance. Integral inequalities have remarkable uses in integral operator theory, stochastic processes, probability, numerical integration, statistics, optimization theory and information technology. For the applications, see the references [11–15].

Many scholars are currently intrigued by the topic of convex functions, notably one famous inequality involving convexity known as the H-H inequality, which is stated as:

$$Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right) \le \frac{1}{\mathtt{w}_2-\mathtt{w}_1}\int_{\mathtt{w}_1}^{\mathtt{w}_2} Q(x)dx \le \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}. \tag{1}$$

The above inequality (1) was first time developed by C. Hermite [16] and explored by J. Hadamard [17] in 1893.

Fejér [18] was the first to introduce the following Fejér inequality (weighted version of H-H inequality), which is given by:

Theorem 1 ([18])**.** *Assume that $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is a convex function. Then, the inequality*

$$Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\int_{\mathtt{w}_1}^{\mathtt{w}_2} g(x)dx \le \int_{\mathtt{w}_1}^{\mathtt{w}_2} Q(x)g(x)dx \le \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\int_{\mathtt{w}_1}^{\mathtt{w}_2} g(x)dx \tag{2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is non-negative, integrable and symmetric to $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$.

Fractional calculus has captivated and motivated several researchers and mathematicians across a wide spectrum of practical and scientific disciplines. Fractional integrals and derivatives, which can interpolate between operators of integer order, have a long track record and are often employed in real-world applications, as can be seen in the references [19–22]. This calculus has enlarged to be a prominent field of investigation due to its utilization in the nonlinear systems (nonlocal) and modeling. Convex functions in the frame of the fractional integral operator have many real-world applications in modeling, circuit design, optimization, controller design, etc. This idea has attracted so much attention that it evolved into a fruitful subject for investigation and inspiration.

The intention and aim of this review manuscript are to offer an extensive and accurate overview of Fejér-type inequalities via multiple sorts of convexities pertaining to fractional calculus. In every part, we first set up the fundamental descriptions of fractional integral operators and different sorts of convexities, and then we present the results for Fejér-type fractional integral inequalities. We contend that compiling almost all current fractional Fejér-type inequalities in a single document will enable fresh scholars in the discipline to learn about previous work on the problem before creating new conclusions. We give outcomes without evidence but provide a comprehensive explanation for each outcome explored in this review for the reader's advantage.

Very recently, the authors in [23] provide an amazing review of H-H type inequalities involving convexities in the frame of fractional integral operators. The paper [23] was complimented with [24] by an up-to-date review of H-H-type inequalities pertaining to quantum calculus.

The construction of this review paper is as follows. In Section 2, we introduce the reader to the basic concepts of Riemann–Liouville fractional integrals. In Sections 2.2–2.11, we summarize Fejér-type fractional integral inequalities for various classes of convexities, including classical convex functions, s-convex functions, harmonically s-convex functions, quasi-convex functions, strongly convex functions, harmonically convex functions, harmonically quasi-convex functions, quasi-geometrically convex functions, p-convex functions, convexity with respect to strictly monotone function, co-ordinated-convex functions, $(\theta, h-m)-p$-convex functions, and h-preinvex function. In Section 3, we present Fejér-type fractional integral inequalities using the $(k-p)$-Riemann–Liouville fractional integrals; in Section 4, we present Fejér-type fractional integral inequalities via k-Riemann–Liouville fractional integral; in Section 5, we present Fejér-type fractional integral inequalities of fractional integrals with respect to another function; and in Section 6, we present Fejér-type fractional integral inequalities for exponential kernel. Fejér-type fractional integral inequalities via the Hadamard fractional integral are presented in Section 7; Fejér-type fractional integral inequalities via Raina integrals are presented in Section 8; Fejér-type fractional integral inequalities via conformable integrals are presented in Section 9; Fejér-type fractional integral inequalities are explored via non-conformable integrals in Section 10; Fejér-type fractional integral inequalities for Katugampola integral operators are included in Section 11; while Fejér-type fractional integral inequalities for invex functions are included in Section 12. Finally, Fejér-type fractional integral inequalities for (p,q)-calculus are included in the last Section 13.

It is essential to recognize that the intention here is to present a deeper and more extensive assessment, and we opted to cover as many consequences as possible to reflect progress on the topic. Any proofs are omitted for this matter, and the reader is referred to the relative article accordingly.

2. Fejér-Type Fractional Integral Inequalities via Riemann–Liouville Fractional Integral

First, we add the definitions of Riemann–Liouville fractional integrals in the left and right aspects.

Definition 1 ([25])**.** *Assume that $Q \in L[\mathtt{w}_1,\mathtt{w}_2]$. Then, Riemann–Liouville integrals in the left and right sense $J^{\alpha}_{\mathtt{w}_1+}Q$ and $J^{\alpha}_{\mathtt{w}_2-}Q$, $\alpha > 0$, $\mathtt{w}_1 \geq 0$ are stated by*

$$J^{\alpha}_{\mathtt{w}_1+}Q(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathtt{w}_1}^{x}(x-t)^{\alpha-1}Q(t)dt, \;\; x > \mathtt{w}_1,$$

and

$$J^{\alpha}_{\mathtt{w}_2-}Q(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\mathtt{w}_2}(t-x)^{\alpha-1}Q(t)dt, \;\; x < \mathtt{w}_2,$$

respectively. Here, $\Gamma(\alpha)$ represents the Euler Gamma function and $J^{0}_{\mathtt{w}_1+}Q(x) = J^{0}_{\mathtt{w}_2-}Q(x) = Q(x)$.

2.1. Fejér-Type Fractional Integral Inequalities for Convex Functions

In this subsection, we summarize Fejér-type fractional integral inequalities concerning convex functions.

Theorem 2 ([26])**.** *Let $Q : I \to \mathbb{R}$ be a differentiable function on the interior I° of I, such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$ for $\mathtt{w}_1,\mathtt{w}_2 \in I^{\circ}$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ be a continuous function. If $|Q'|$ is convex on $[\mathtt{w}_1,\mathtt{w}_2]$, then, for any $\alpha > 0$ and $x \in [\mathtt{w}_1,\mathtt{w}_2]$, the fractional inequality is given as:*

$$\begin{aligned} &\Gamma(\alpha)\Big[Q(x)\Big(J^{\alpha}_{x+}g(\mathtt{w}_2) + J^{\alpha}_{x-}g(\mathtt{w}_1)\Big) - \Big(J^{\alpha}_{x+}(Qg)(\mathtt{w}_2) + J^{\alpha}_{x-}(Qg)(\mathtt{w}_1)\Big)\Big] \\ \leq\; &\frac{(x-\mathtt{w}_1)^{\alpha+1}\|g\|_{[\mathtt{w}_1,x]}}{\Gamma(\alpha+3)}\Big[(\alpha+1)|Q'(x)| + Q'(\mathtt{w}_1)|\Big] \\ &+\frac{(\mathtt{w}_2-x)^{\alpha+1}\|g\|_{[x,\mathtt{w}_2]}}{\Gamma(\alpha+3)}\Big[(\alpha+1)|Q'(x)| + Q'(\mathtt{w}_2)|\Big]. \end{aligned}$$

Theorem 3 ([26])**.** *Let $Q : I \to \mathbb{R}$ be a differentiable function on the interior I° such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$ for $\mathtt{w}_1,\mathtt{w}_2 \in I^{\circ}$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ be a continuous function. If $|Q'|^q$ is convex on $[\mathtt{w}_1,\mathtt{w}_2]$ for some fixed $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then, for any $\alpha > 0$ and $x \in [\mathtt{w}_1,\mathtt{w}_2]$ the fractional inequality is given as:*

$$\Gamma(\alpha)\Big[Q(x)\Big(J^{\alpha}_{x+}g(\mathtt{w}_2) + J^{\alpha}_{x-}g(\mathtt{w}_1)\Big) - \Big(J^{\alpha}_{x+}(Qg)(\mathtt{w}_2) + J^{\alpha}_{x-}(Qg)(\mathtt{w}_1)\Big)\Big]$$

$$\leq \frac{\|g\|_{[\mathtt{w}_1,x]}}{\Gamma(\alpha+1)}\left(\frac{(x-\mathtt{w}_1)^{\alpha+1}}{(1+\alpha p)^{\frac{1}{p}}}\right)\left(\frac{|Q'(x)|^q+Q'(\mathtt{w}_1)|^q}{2}\right)^{\frac{1}{q}} + \frac{\|g\|_{[x,\mathtt{w}_2]}}{\Gamma(\alpha+1)}\left(\frac{(\mathtt{w}_2-x)^{\alpha+1}}{(1+\alpha p)^{\frac{1}{p}}}\right)\left(\frac{|Q'(x)|^q+Q'(\mathtt{w}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 4 ([26]). *Let $Q : I \to \mathbb{R}$ be a differentiable function on the interior I° such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$ for $\mathtt{w}_1,\mathtt{w}_2 \in I^\circ$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ be a continuous function. If $|Q'|^q$ is convex on $[\mathtt{w}_1,\mathtt{w}_2]$ for some fixed $q \geq 1$, then, for any $\alpha > 0$ and $x \in [\mathtt{w}_1,\mathtt{w}_2]$, the fractional inequality is given as:*

$$\Gamma(\alpha)\left[Q(x)\left(J^\alpha_{x+}g(\mathtt{w}_2)+J^\alpha_{x-}g(\mathtt{w}_1)\right)-\left(J^\alpha_{x+}(Qg)(\mathtt{w}_2)+J^\alpha_{x-}(Qg)(\mathtt{w}_1)\right)\right]$$
$$\leq \frac{\|g\|_{[\mathtt{w}_1,x]}}{\Gamma(\alpha+1)}(x-\mathtt{w}_1)^{\alpha+1}\left(\frac{(\alpha+1)|Q'(x)|^q+Q'(\mathtt{w}_1)|^q}{2}\right)^{\frac{1}{q}} + \frac{\|g\|_{[x,\mathtt{w}_2]}}{\Gamma(\alpha+1)}(\mathtt{w}_2-x)^{\alpha+1}\left(\frac{(\alpha+1)|Q'(x)|^q+Q'(\mathtt{w}_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 5 ([27]). *Let $Q : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ be a differentiable mapping on $(\mathtt{w}_1,\mathtt{w}_2)$ and $Q' \in [\mathtt{w}_1,\mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is convex on $[\mathtt{w}_1,\mathtt{w}_2]$ and $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is continuous and symmetric to $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$, the fractional inequality is given as:*

$$\left|Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_1-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha)}(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\left(\frac{1}{2^\alpha(\alpha+1)}+\frac{1}{\alpha+1}-\frac{1}{2\alpha}\right)[|Q'(\mathtt{w}_1)|+|Q'(\mathtt{w}_2)|],\quad \alpha>0,$$

where $\|g\|_\infty = \sup\{|g(t)|; t \in [\mathtt{w}_1,\mathtt{w}_2]\}$.

Theorem 6 ([27]). *Suppose that all the conditions of Theorem 5 hold. Then, the fractional inequality is given as:*

$$\left|Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}g(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_1-}g(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{\|g\|_\infty}{\Gamma(\alpha)}(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\left[\frac{|Q'(\mathtt{w}_1)|+|Q'(\mathtt{w}_2)|}{8}\right],\quad \alpha>0.$$

Theorem 7 ([27]). *Suppose that all the conditions of Theorem 5 hold. Additionally, we assume that $|Q'|^q, q > 1$ is convex on $[\mathtt{w}_1,\mathtt{w}_2]$. Then, the fractional inequality is given as:*

$$\left|Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_1-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \|g\|_\infty\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{3|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|Q'(\mathtt{w}_1)|^q+3|Q'(\mathtt{w}_2)|^q}{4}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 8 ([28]). *Let $Q : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\mathtt{w}_1,\mathtt{w}_2 \in I^\circ$ with $\mathtt{w}_1 < \mathtt{w}_2$, and let $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ be continuous on $[\mathtt{w}_1,\mathtt{w}_2]$. If $|Q'|$ is convex on $[\mathtt{w}_1,\mathtt{w}_2]$, then the fractional inequality is given as:*

$$\left| \left(\int_{\mathtt{w}_1}^{\mathtt{w}_2} g(u)du \right)^{\alpha} [Q(\mathtt{w}_1) + Q(\mathtt{w}_2)] - \alpha \int_{\mathtt{w}_1}^{\mathtt{w}_2} \left(\int_{\mathtt{w}_1}^{t} g(u)du \right)^{\alpha-1} g(t)Q(t)dt \right.$$
$$\left. -\alpha \int_{\mathtt{w}_1}^{\mathtt{w}_2} \left(\int_{t}^{\mathtt{w}_2} g(u)du \right)^{\alpha-1} g(t)Q(t)dt \right|$$
$$\leq \frac{\|g\|_{\infty}(\mathtt{w}_2 - \mathtt{w}_1)^{\alpha+1}}{\alpha+1} [|Q'(\mathtt{w}_1)| + |Q'(\mathtt{w}_2)|].$$

Theorem 9 ([28]). *Let $Q : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\mathtt{w}_1, \mathtt{w}_2 \in I^{\circ}$ with $\mathtt{w}_1 < \mathtt{w}_2$ and let $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be continuous on $[\mathtt{w}_1, \mathtt{w}_2]$. If $|Q'|^q$ is convex on $[\mathtt{w}_1, \mathtt{w}_2]$, $q > 1$, then the fractional inequality is given as:*

$$\left| \left(\int_{\mathtt{w}_1}^{\mathtt{w}_2} g(u)du \right)^{\alpha} [Q(\mathtt{w}_1) + Q(\mathtt{w}_2)] - \alpha \int_{\mathtt{w}_1}^{\mathtt{w}_2} \left(\int_{\mathtt{w}_1}^{t} g(u)du \right)^{\alpha-1} g(t)Q(t)dt \right.$$
$$\left. -\alpha \int_{\mathtt{w}_1}^{\mathtt{w}_2} \left(\int_{t}^{\mathtt{w}_2} g(u)du \right)^{\alpha-1} g(t)Q(t)dt \right|$$
$$\leq 2\|g\|_{\infty}^{\alpha} \frac{(\mathtt{w}_2 - \mathtt{w}_1)^{\alpha+1}}{(\alpha p + 1)^{\frac{1}{p}}} \left(\frac{|Q'(\mathtt{w}_1)|^q + |Q'(\mathtt{w}_2)|^q}{2} \right)^{\frac{1}{q}},$$

where $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 10 ([29]). *Let $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be a convex function with $\mathtt{w}_1 < \mathtt{w}_2$ and $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$. If $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is non-negative, integrable, and symmetric to $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$, then inequality in the frame of fractional operator is given as:*

$$Q\left(\frac{\mathtt{w}_1 + \mathtt{w}_2}{2} \right) \left[J_{\mathtt{w}_1+}^{\alpha} g(\mathtt{w}_2) + J_{\mathtt{w}_2-}^{\alpha} g(\mathtt{w}_1) \right] \leq \left[J_{\mathtt{w}_1+}^{\alpha} (Qg)(\mathtt{w}_2) + J_{\mathtt{w}_2-}^{\alpha} (Qg)(\mathtt{w}_1) \right]$$
$$\leq \frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2} \left[J_{\mathtt{w}_1+}^{\alpha} g(\mathtt{w}_2) + J_{\mathtt{w}_2-}^{\alpha} g(\mathtt{w}_1) \right], \quad \alpha > 0.$$

Theorem 11 ([29]). *Let $Q : \mathbb{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^{\circ}$ and $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is a convex function on $[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and symmetric to $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$, then inequality in the frame of the fractional operator is given as:*

$$\left| \left(\frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2} \right) \left[J_{\mathtt{w}_1+}^{\alpha} g(\mathtt{w}_2) + J_{\mathtt{w}_2-}^{\alpha} g(\mathtt{w}_1) \right] - \left[J_{\mathtt{w}_1+}^{\alpha} (Qg)(\mathtt{w}_2) + J_{\mathtt{w}_2-}^{\alpha} (Qg)(\mathtt{w}_1) \right] \right|$$
$$\leq \frac{(\mathtt{w}_2 - \mathtt{w}_1)^{\alpha+1} \|g\|_{\infty}}{(\alpha+1)\Gamma(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}} \right) \left[|Q'(\mathtt{w}_1)| + |Q'(\mathtt{w}_2)| \right], \quad \alpha > 0.$$

Theorem 12 ([30]). *Assume that $\mathtt{w}_1, \mathtt{w}_2 \in \mathbb{R}$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $Q : \mathbb{I} \to \mathbb{R}$ is a differentiable function on $\mathbb{I}^{\circ}$ such that $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous. If $|Q'|$ is a convex function on $\mathbb{I}$, then an inequality in the frame of the fractional operator is given as:*

$$\left| Q\left(\frac{\mathtt{w}_1 + \mathtt{w}_2}{2} \right) \left[J_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-}^{\alpha} g(\mathtt{w}_1) + J_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+}^{\alpha} g(\mathtt{w}_2) \right] \right.$$
$$\left. -\left[J_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-}^{\alpha} (Qg)(\mathtt{w}_1) + J_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+}^{\alpha} (Qg)(\mathtt{w}_2) \right] \right|$$
$$\leq \frac{(\mathtt{w}_2 - \mathtt{w}_1)^{\alpha+1} \|g\|_{\infty}}{2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)} [|Q'(\mathtt{w}_1)| + |Q'(\mathtt{w}_2)|],$$

where $\|g\|_\infty = \sup\{|g(t)|, t \in [\mathtt{w}_1, \mathtt{w}_2]\}$.

Theorem 13 ([30]). *Assume that Q is as in Theorem 12 and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous. If $|Q'|^q$, $q > 1$ is a convex function on $\mathbb{I}$, then an inequality in the frame of the fractional operator is given as:*

$$\begin{aligned} &\left| Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-} g(\mathtt{w}_1) + J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+} g(\mathtt{w}_2)\right]\right. \\ &\left. - \left[J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-}(Qg)(\mathtt{w}_1) + J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+}(Qg)(\mathtt{w}_2)\right]\right| \\ \leq\ & \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1+\frac{2}{q}}(\alpha p+1)^{1/p}\Gamma(\alpha+1)} \\ &\times\left[\left(3|Q'(\mathtt{w}_1)|^q + |Q'(\mathtt{w}_2)|^q\right)^{\frac{1}{q}} + \left(|Q'(\mathtt{w}_1)|^q + 3|Q'(\mathtt{w}_2)|^q\right)^{\frac{1}{q}}\right], \end{aligned}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

2.2. Fejér-Type Fractional Integral Inequalities for s-Convex Functions

Definition 2 ([31]). *A function $Q : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$Q(\lambda x + (1-\lambda)y) \leq \lambda^s Q(x) + (1-\lambda)^s Q(y)$$

for all $x, y \in [0, \infty)$, $\lambda \in [0,1]$ and for some fixed $s \in (0,1]$.

In the next theorems, Fejér-type fractional integral inequalities for s-convex functions are presented.

Theorem 14 ([32]). *Let $Q : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\mathtt{w}_1, \mathtt{w}_2 \in I^\circ$ with $\mathtt{w}_1 < \mathtt{w}_2$ and let $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be continuous on $[\mathtt{w}_1, \mathtt{w}_2]$. If $|Q'|$ is s-convex in the second sense on $[\mathtt{w}_1, \mathtt{w}_2]$, then the fractional inequality is given as:*

$$\begin{aligned} &\left|\left(\int_{\mathtt{w}_1}^{\mathtt{w}_2} g(u)du\right)^{\alpha}[Q(\mathtt{w}_1)+Q(\mathtt{w}_2)] - \alpha\int_{\mathtt{w}_1}^{\mathtt{w}_2}\left(\int_{\mathtt{w}_1}^{t} g(u)du\right)^{\alpha-1} g(t)Q(t)dt\right. \\ &\left. -\alpha\int_{\mathtt{w}_1}^{\mathtt{w}_2}\left(\int_{t}^{\mathtt{w}_2} g(u)du\right)^{\alpha-1} g(t)Q(t)dt\right| \\ \leq\ & \|g\|_\infty(\mathtt{w}_2-\mathtt{w}_1)^{\alpha-1}A(\alpha,s)[|Q'(\mathtt{w}_1)|+|Q'(\mathtt{w}_2)|] \end{aligned}$$

where

$$A(\alpha,s) = \frac{1}{\alpha+s+1}\left(1-\frac{1}{2^{\alpha+s}}\right) + B_{1/2}(s+1,\alpha+1) - B_{1/2}(\alpha+1,s+1),$$

and B_x is the incomplete beta function defined as follows

$$B_x(m,n) = \int_0^x t^{m-1}(1-t)^{n-1}dt, \quad m,n > 0,\ 0 < x < 1.$$

Theorem 15 ([32]). *Let $Q : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\mathtt{w}_1, \mathtt{w}_2 \in I^\circ$ with $\mathtt{w}_1 < \mathtt{w}_2$ and let $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be continuous on $[\mathtt{w}_1, \mathtt{w}_2]$. If $|Q'|^q$ is s-convex in the second sense on $[\mathtt{w}_1, \mathtt{w}_2]$, $q > 1$, then the fractional inequality be given as:*

$$\left|\left(\int_{\mathtt{w}_1}^{\mathtt{w}_2} g(u)du\right)^{\alpha}[Q(\mathtt{w}_1)+Q(\mathtt{w}_2)] - \alpha\int_{\mathtt{w}_1}^{\mathtt{w}_2}\left(\int_{\mathtt{w}_1}^{t} g(u)du\right)^{\alpha-1} g(t)Q(t)dt\right.$$

$$-\alpha\int_{\mathtt{w}_1}^{\mathtt{w}_2}\left(\int_t^{\mathtt{w}_2}g(u)du\right)^{\alpha-1}g(t)Q(t)dt\Bigg|$$
$$\leq \|g\|_\infty^\alpha\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\frac{|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q}{s+1}\right)^{\frac{1}{q}},$$

where $\alpha > 0$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 16 ([33]). *Let* $Q : I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *and* $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$ *where* $\mathtt{w}_1 < \mathtt{w}_2$ *and* $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ *is continuous and symmetric to* $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$. *If* $|Q'|$ *is s-convex in the second sense on* $[\mathtt{w}_1,\mathtt{w}_2]$ *for some fixed* $s \in (0,1]$, *then the fractional inequality is given as:*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{(\alpha+s+1)\Gamma(\alpha+1)}\left[(1-(\alpha+s+1)\left(B_{1/2}(s+1,\alpha+1)-B_{1/2}(\alpha+1,s+1)\right)\right]$$
$$\times\left(|Q'(\mathtt{w}_1)|+|Q'(\mathtt{w}_2)|\right).$$

Theorem 17 ([33]). *Let* $Q : I \subseteq [0,\infty) \to \mathbb{R}$ *be differentiable function on* I° *and* $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$ *where* $\mathtt{w}_1 < \mathtt{w}_2$ *and* $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ *be continuous and symmetric to* $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$. *If* $|Q'|^q$, $q > 1$ *is s-convex in the second sense on* $[\mathtt{w}_1,\mathtt{w}_2]$ *for some fixed* $s \in (0,1]$, *then the fractional inequality is given as:*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{2(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{(\alpha+1)^{1-\frac{1}{q}}(\alpha+s+1)^{\frac{1}{q}}\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)^{1-\frac{1}{q}}\left(\frac{|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q}{2}\right)^{\frac{1}{q}}$$
$$\times\left[1-(\alpha+s+1)\left(B_{1/2}(s+1,\alpha+1)-B_{1/2}(\alpha+1,s+1)\right)\right]^{\frac{1}{q}}.$$

Theorem 18 ([33]). *Assume that the conditions of Theorem 17 hold true. Then:*

(i) For $\alpha > 0$, *we have:*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{2^{\frac{1}{p}}(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)^{\frac{1}{p}}\left(\frac{|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q}{s+1}\right)^{\frac{1}{q}}.$$

(ii) For $0 < \alpha < 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *we have:*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\frac{|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q}{s+1}\right)^{\frac{1}{q}}.$$

Theorem 19 ([34]). *Let $Q : I \subseteq [0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ are continuous. If $|Q'|$ is s-convex in the second sense on $[\mathtt{w}_1, \mathtt{w}_2]$ for some fixed $s \in (0,1]$, then the fractional inequality is given as:*

$$\begin{aligned}&\bigg|Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\Big)\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+}Q(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-}Q(\mathtt{w}_1)\Big]\\&-\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+}(Qg)(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-}(Qg)(\mathtt{w}_1)\Big]\bigg|\\ \leq\ &\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{\Gamma(\alpha+1)}\Big[B_{1/2}(\alpha+1,s+1)+\frac{1}{2^{\alpha+s+1}(\alpha+s+1)}\Big]\Big(|Q'(\mathtt{w}_1)|+|Q'(\mathtt{w}_2)|\Big).\end{aligned}$$

Theorem 20 ([34]). *Let $Q : I \subseteq [0,\infty) \to \mathbb{R}$ be differentiable function on I° and $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ where $\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be continuous. If $|Q'|^q, q \geq 1$ is s-convex in the second sense on $[\mathtt{w}_1, \mathtt{w}_2]$ for some fixed $s \in (0,1]$, then the fractional inequality is given as:*

$$\begin{aligned}&\bigg|Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\Big)\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+}Q(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-}Q(\mathtt{w}_1)\Big]\\&-\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+}(Qg)(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-}(Qg)(\mathtt{w}_1)\Big]\bigg|\\ \leq\ &\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1+\frac{1}{q}}(\alpha+1)(\alpha+s+1)^{\frac{1}{q}}\Gamma(\alpha+1)}\\&\times\Big[(2^{\alpha+2}(\alpha+1)(\alpha+s+1)B_{1/2}(\alpha+1,s+1)|Q'(\mathtt{w}_1)|^q+2^{1-s}(\alpha+1)|Q'(\mathtt{w}_2)|^q)^{\frac{1}{q}}\\&+(2^{1-s}(\alpha+1)|Q'(\mathtt{w}_1)|^q+2^{\alpha+2}(\alpha+1)(\alpha+s+1)B_{1/2}(\alpha+1,s+1)|Q'(\mathtt{w}_2)|^q)^{\frac{1}{q}}\Big].\end{aligned}$$

Theorem 21 ([34]). *Let $Q : I \subseteq [0,\infty) \to \mathbb{R}$ be differentiable function on I° and $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ where $\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be continuous. If $|Q'|^q, q > 1$ is s-convex in the second sense on $[\mathtt{w}_1, \mathtt{w}_2]$ for some fixed $s \in (0,1]$, then the fractional inequality is given as:*

$$\begin{aligned}&\bigg|Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\Big)\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+}Q(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-}Q(\mathtt{w}_1)\Big]\\&-\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+}(Qg)(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-}(Qg)(\mathtt{w}_1)\Big]\bigg|\\ \leq\ &\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1+\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}\Gamma(\alpha+1)}\\&\times\Big[((2^{s+1}-1)|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q)^{\frac{1}{q}}+(|Q'(\mathtt{w}_1)|^q+(2^{s+1}-1)|Q'(\mathtt{w}_2)|^q)^{\frac{1}{q}}\Big].\end{aligned}$$

Theorem 22 ([35]). *Let $Q : \mathbb{I} \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on $\mathbb{I}^\circ$ such that $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1, \mathtt{w}_2 \in \mathbb{I}$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is s-convex in the second sense on $[\mathtt{w}_1, \mathtt{w}_2]$, then inequality in the frame of the fractional operator is given as:*

$$\begin{aligned}&\bigg|Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\Big)\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-}g(\mathtt{w}_1)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+}g(\mathtt{w}_2)\Big]\\&-\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-}(Qg)(\mathtt{w}_1)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+}(Qg)(\mathtt{w}_2)\Big]\bigg|\\ \leq\ &\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{\Gamma(\alpha+1)}\Big\{B_{1/2}(\alpha+1,s+1)+\frac{1}{2^{\alpha+s+1}(\alpha+s+1)}\Big\}[|Q'(\mathtt{w}_1)|+|Q'(\mathtt{w}_2)|].\end{aligned}$$

Theorem 23 ([35]). *Let $Q : \mathbb{I} \subset [0,\infty) \to \mathbb{R}$ be a differentiable function on $\mathbb{I}^\circ$ such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$, where $\mathtt{w}_1,\mathtt{w}_2 \in \mathbb{I}$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is continuous. If $|Q'|^q$ is s-convex in the second sense on $[\mathtt{w}_1,\mathtt{w}_2]$, then an inequality in the frame of fractional operator is given as:*

$$\begin{aligned} & \Bigg| Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-}g(\mathtt{w}_1) + J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+}g(\mathtt{w}_2)\right] \\ & -\left[J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-}(Qg)(\mathtt{w}_1) + J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+}(Qg)(\mathtt{w}_2)\right]\Bigg| \\ \leq\ & \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1+\frac{1}{q}}(\alpha+1)(\alpha+2)^{\frac{1}{q}}(\alpha+s+q)^{\frac{1}{q}}\Gamma(\alpha+1)} \\ & \times\left\{\left((\alpha+s+1)(\alpha+3)|Q'(\mathtt{w}_1)|^q + 2^{1-s}(\alpha+1)(\alpha+2)|Q'(\mathtt{w}_2)|^q\right)^{\frac{1}{q}}\right. \\ & \left.+\left(2^{1-s}(\alpha+1)(\alpha+2)|Q'(\mathtt{w}_1)|^q + (\alpha+s+1)(\alpha+3)|Q'(\mathtt{w}_2)|^q\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

2.3. Fejér-Type Fractional Integral Inequalities for Harmonically s-Convex Functions

In this subsection, some Fejér-type integral inequalities for harmonically s-convex functions are presented.

Definition 3 ([36]). *Assume that $\mathbb{I} \subset \mathbb{R}\setminus\{0\}$ is a real interval. A function $Q : \mathbb{I} \to \mathbb{R}$ is harmonically s-convex if*

$$Q\left(\frac{xy}{tx+(1-t)y}\right) \leq t^s Q(y) + (1-t)^s Q(x)$$

for all $x,y \in \mathbb{I}$, $t \in [0,1]$ and $s \in (0,1]$.

Theorem 24 ([37]). *Let $Q : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$, where $\mathtt{w}_1,\mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is harmonically s-convex on $[\mathtt{w}_1,\mathtt{w}_2]$, $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}$, then the fractional inequality is given as:*

$$\begin{aligned} & \Bigg| \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\alpha}_{1/\mathtt{w}_2+}(g\circ h)(1/\mathtt{w}_1) + J^{\alpha}_{1/\mathtt{w}_1-}(g\circ h)(1/\mathtt{w}_2)\right] \\ & -\left[J^{\alpha}_{1/\mathtt{w}_2+}(Qg\circ h)(1/\mathtt{w}_1) + J^{\alpha}_{1/\mathtt{w}_1-}(Qg\circ h)(1/\mathtt{w}_2)\right]\Bigg| \\ \leq\ & \frac{\|g\|_\infty \mathtt{w}_1\mathtt{w}_2(\mathtt{w}_2-\mathtt{w}_1)}{\Gamma(\alpha+1)}\left(\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1\mathtt{w}_2}\right)^{\alpha}[C_1(\alpha)|Q'(\mathtt{w}_1)| + C_2(\alpha)|Q'(\mathtt{w}_2)|, \end{aligned}$$

where

$$\begin{aligned} C_1(\alpha) &= \frac{\mathtt{w}_2^{-2}}{\alpha+s+1}\,{}_2F_1\left(2,1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right) \\ &\quad -\frac{\beta(\alpha+1,s+1)}{\mathtt{w}_2^2}\,{}_2F_1\left(2,\alpha+1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right) \\ &\quad +\frac{\beta(\alpha+1,s+1)}{2^{s-2}(\mathtt{w}_1+\mathtt{w}_2)^2}\,{}_2F_1\left(2,\alpha+1;\alpha+s+2;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_2+\mathtt{w}_1}\right) \\ C_2(\alpha) &= \frac{\mathtt{w}_2^{-2}}{\alpha+s+1}\,{}_2F_1\left(2,\alpha+s+1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{\beta(s+1,\alpha+1)}{\mathtt{w}_2^2}\,{}_2F_1(2,s+1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}) \\
&+\frac{1}{2^s}\frac{\beta(s+1,\alpha+1)}{\mathtt{w}_2^2}\,{}_2F_1(2,s+1;-\alpha+s+2;\frac{1}{2}\Big(1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\Big)\Big),
\end{aligned}
$$

with $0<\alpha\le 1$ *and* $h(x)=\frac{1}{x}, x\in\Big[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\Big]$.

Theorem 25 ([37])**.** *Let $Q : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$, where $\mathtt{w}_1,\mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q, q \ge 1$ is harmonically s-convex on $[\mathtt{w}_1,\mathtt{w}_2]$, $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}$, then the fractional inequality is given as:*

$$
\begin{aligned}
&\Bigg|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^\alpha_{1/\mathtt{w}_2+}(g\circ h)(1/\mathtt{w}_1)+J^\alpha_{1/\mathtt{w}_1-}(g\circ h)(1/\mathtt{w}_2)\Big] \\
&-\Big[J^\alpha_{1/\mathtt{w}_2+}(Qg\circ h)(1/\mathtt{w}_1)+J^\alpha_{1/\mathtt{w}_1-}(Qg\circ h)(1/\mathtt{w}_2)\Big]\Bigg| \\
\le\;& \frac{\|g\|_\infty \mathtt{w}_1\mathtt{w}_2(\mathtt{w}_2-\mathtt{w}_1)}{\Gamma(\alpha+1)}\Big(\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1\mathtt{w}_2}\Big)^\alpha\Big\{C_3^{1-\frac{1}{q}}(\alpha)\Big[C_4(\alpha)|Q'(\mathtt{w}_1)|^q+C_5(\alpha)|Q'(\mathtt{w}_2)|^q\Big]^{\frac{1}{q}} \\
&+C_6^{1-\frac{1}{q}}(\alpha)\Big[C_7(\alpha)|Q'(\mathtt{w}_1)|^q+C_8(\alpha)|Q'(\mathtt{w}_2)|^q\Big]^{\frac{1}{q}}\Big\}
\end{aligned}
$$

where

$$
\begin{aligned}
C_3(\alpha) &= \frac{2(\mathtt{w}_1+\mathtt{w}_2)^{-2}}{\alpha+1}\,{}_2F_1\Big(2,\alpha+1;\alpha+3;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1+\mathtt{w}_2}\Big), \\
C_4(\alpha) &= \frac{(\mathtt{w}_1+\mathtt{w}_2)^{-2}}{2^{s-1}(\alpha+1)(\alpha+2)}\,{}_2F_1\Big(2,\alpha+1;\alpha+3;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1+\mathtt{w}_2}\Big), \\
C_5(\alpha) &= \frac{\mathtt{w}_2^{-2}}{\alpha+s+1}\,{}_2F_1\Big(2,1;\alpha+s+1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\Big) \\
&\quad-\frac{\beta(\alpha+1,s+1)}{\mathtt{w}_2^2}\,{}_2F_1(2,s+1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}) \\
&\quad+\frac{1}{2^{s+1}}\frac{\beta(\alpha+1,s+1)}{\mathtt{w}_2^2}\,{}_2F_1\Big(2,s+1;\alpha+s+2;\frac{1}{2}\Big(1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\Big)\Big), \\
C_6(\alpha) &= \frac{\mathtt{w}_2^{-2}}{\alpha+1}\,{}_2F_1\Big(2,1;\alpha+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\Big) \\
&\quad-\frac{\mathtt{w}_2^{-2}}{\alpha+1}\,{}_2F_1\Big(2,\alpha+1;\alpha+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\Big)+C_3(\alpha), \\
C_7(\alpha) &= \frac{\mathtt{w}_2^{-2}}{\alpha+s+1}\,{}_2F_1\Big(2,1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\Big) \\
&\quad-\frac{\beta(\alpha+1,s+1)}{\mathtt{w}_2^2}\,{}_2F_1\Big(2,\alpha+1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\Big)+C_4(\alpha), \\
C_8(\alpha) &= \frac{\beta(\alpha+1,s+1)}{\mathtt{w}_2^2}\,{}_2F_1\Big(2,s+1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\Big) \\
&\quad-\frac{\mathtt{w}_2^{-2}}{\alpha+s+1}\,{}_2F_1\Big(2,\alpha+s+1;\alpha+s+2;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\Big)+C_5(\alpha),
\end{aligned}
$$

with $0<\alpha\le 1$ *and* $h(x)=\frac{1}{x}, x\in\Big[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\Big]$.

Theorem 26 ([37]). *Let $Q : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$, where $\mathtt{w}_1,\mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q, q > 1$ is harmonically s-convex on $[\mathtt{w}_1,\mathtt{w}_2]$, $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}$, then the fractional inequality is given as:*

$$\begin{aligned}
&\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^\alpha_{1/\mathtt{w}_2+}(g\circ h)(1/\mathtt{w}_1)+J^\alpha_{1/\mathtt{w}_1-}(g\circ h)(1/\mathtt{w}_2)\right]\right.\\
&\left.-\left[J^\alpha_{1/\mathtt{w}_2+}(Qg\circ h)(1/\mathtt{w}_1)+J^\alpha_{1/\mathtt{w}_1-}(Qg\circ h)(1/\mathtt{w}_2)\right]\right|\\
\leq\ &\frac{\|g\|_\infty\mathtt{w}_1\mathtt{w}_2(\mathtt{w}_2-\mathtt{w}_1)}{\Gamma(\alpha+1)}\left(\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1\mathtt{w}_2}\right)^\alpha\left\{C_9^{\frac{1}{p}}(\alpha)\left[\frac{|Q'(\mathtt{w}_1)|^q+(2^{s+1}-1)|Q'(\mathtt{w}_2)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\right.\\
&\left.+C_{10}^{\frac{1}{p}}(\alpha)\left[\frac{(2^{s+1}-1)|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q}{2^{s+1}(s+1)}\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

where

$$\begin{aligned}
C_9(\alpha) &= \left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)^{-2p}\frac{1}{2(\alpha p+1)}{}_2F_1\left(2p,\alpha p+1;\alpha p+2;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1+\mathtt{w}_2}\right),\\
C_{10}(\alpha) &= \mathtt{w}_2^{-2p}\frac{1}{2(\alpha p+1)}{}_2F_1\left(2p,\alpha p+1;\alpha p+2;\frac{1}{2}\left(1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right)\right)
\end{aligned}$$

with $0 < \alpha \leq 1$ and $h(x) = \frac{1}{x}, x \in \left[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\right]$.

2.4. Fejér-Type Fractional Integral Inequalities for Quasi-Convex Functions

Definition 4 ([38]). *A function $Q : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is said to be quasi-convex on $[\mathtt{w}_1,\mathtt{w}_2]$ if*

$$Q(\lambda x+(1-\lambda)y) \leq \sup\{Q(x),Q(y)\},$$

for all $x,y \in [\mathtt{w}_1,\mathtt{w}_2]$ and $\lambda \in [0,1]$.

In the following, we present theorems including Fejér-type fractional integral inequalities for quasi-convex functions.

Theorem 27 ([39]). *Let $Q : I \to \mathbb{R}$ be a differentiable mapping on I° and $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ be continuous. If $|Q'|^q$ is quasi convex on $[\mathtt{w}_1,\mathtt{w}_2]$, $q > 1$, then the fractional inequality is given as:*

$$\begin{aligned}
&\left|Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^\alpha_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-}g(\mathtt{w}_1)+J^\alpha_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+}g(\mathtt{w}_2)\right]\right.\\
&\left.-\left[J^\alpha_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-}(Qg)(\mathtt{w}_1)+J^\alpha_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+}(Qg)(\mathtt{w}_2)\right]\right|\\
\leq\ &\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^\alpha(\alpha+1)\Gamma(\alpha+1)}\left(\sup\{|Q'(\mathtt{w}_1)|^q,|Q'(\mathtt{w}_2)|^q\}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\alpha > 0$.

Theorem 28 ([39]). *Suppose that all the conditions of Theorem 27 hold. Then, the fractional inequality is given as:*

$$\left|Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^\alpha_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-}g(\mathtt{w}_1)+J^\alpha_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+}g(\mathtt{w}_2)\right]\right.$$

$$-\left[J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-}(Qg)(\mathtt{w}_1)+J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+}(Qg)(\mathtt{w}_2)\right]\Bigg|$$
$$\leq \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^\alpha(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\sup\{|Q'(\mathtt{w}_1)|^q,|Q'(\mathtt{w}_2)|^q\}\right)^{\frac{1}{q}},$$

where $\alpha>0$, *and* $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 29 ([39])**.** *Let* $Q:I\to\mathbb{R}$ *be a differentiable mapping on* I° *and* $Q'\in L[\mathtt{w}_1,\mathtt{w}_2]$ *with* $\mathtt{w}_1<\mathtt{w}_2$ *and* $g:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ *is continuous and symmetric to* $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$. *If* $|Q'|$ *is quasi convex on* $[\mathtt{w}_1,\mathtt{w}_2]$, *then the fractional inequality is given as:*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\alpha}_{\mathtt{w}_1-}g(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{2(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\sup\{|Q'(\mathtt{w}_1)|,|Q'(\mathtt{w}_2)|\},$$

where $\alpha>0$.

Theorem 30 ([39])**.** *Let* $Q:I\to\mathbb{R}$ *be a differentiable mapping on* I° *and* $Q'\in L[\mathtt{w}_1,\mathtt{w}_2]$ *with* $\mathtt{w}_1<\mathtt{w}_2$ *and* $g:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ *be continuous and symmetric to* $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$. *If* $|Q'|^q$ *is quasi convex on* $[\mathtt{w}_1,\mathtt{w}_2]$, *then the fractional inequality is given as:*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\alpha}_{\mathtt{w}_1-}g(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_2)\right]-\left[J^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{2(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left(\sup\{|Q'(\mathtt{w}_1)|^q,|Q'(\mathtt{w}_2)|^q\}\right)^{\frac{1}{q}},$$

where $\alpha>0$.

Theorem 31 ([39])**.** *Suppose that all the conditions of Theorem 30 hold. Then, the fractional inequality is given as:*

(i)

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\alpha}_{\mathtt{w}_1-}g(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{2^{\frac{1}{p}}(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\sup\{|Q'(\mathtt{w}_1)|^q,|Q'(\mathtt{w}_2)|^q\}\right)^{\frac{1}{q}},$$

where $\alpha>0$.

(ii)

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\alpha}_{\mathtt{w}_1-}g(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\sup\{|Q'(\mathtt{w}_1)|^q,|Q'(\mathtt{w}_2)|^q\}\right)^{\frac{1}{q}},$$

where $\alpha>0$, *and* $\frac{1}{p}+\frac{1}{q}=1$.

2.5. Fejér-Type Fractional Integral Inequalities for Harmonically Convex Functions

Definition 5 ([40])**.** *Assume that $\mathbb{I} \subset \mathbb{R} \setminus \{0\}$ is a real interval. A function $Q : \mathbb{I} \to \mathbb{R}$ is harmonically convex, if*

$$Q\left(\frac{xy}{tx + (1-t)y}\right) \leq tQ(y) + (1-t)Q(x)$$

for all $x, y \in \mathbb{I}$ and $t \in [0, 1]$.

Definition 6 ([40])**.** *A function $g : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\}$ is said to be harmonically symmetric with respect to $\dfrac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1 + \mathtt{w}_2}$ if*

$$g(x) = g\left(\frac{1}{\frac{1}{\mathtt{w}_1} + \frac{1}{\mathtt{w}_2} - \frac{1}{x}}\right)$$

holds for all $x \in [\mathtt{w}_1, \mathtt{w}_2]$.

Fejér-type fractional integral inequalities for harmonically convex functions are presented now.

Theorem 32 ([41])**.** *Let $Q : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1, \mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is harmonically convex on $[\mathtt{w}_1, \mathtt{w}_2]$, $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $\dfrac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1 + \mathtt{w}_2}$, then the fractional inequality is given as:*

$$\begin{aligned}
&\Bigg| Q\left(\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1 + \mathtt{w}_2}\right)\left[J^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}+}(Q \circ h)(1/\mathtt{w}_1) + J^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}-}(Q \circ h)(1/\mathtt{w}_2)\right] \\
&\quad - \left[J^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}+}(Qg \circ h)(1/\mathtt{w}_1) + J^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}-}(Qg \circ h)(1/\mathtt{w}_2)\right]\Bigg| \\
\leq\ & \frac{\|g\|_\infty \mathtt{w}_1\mathtt{w}_2(\mathtt{w}_2 - \mathtt{w}_1)}{\Gamma(\alpha+1)}\left(\frac{\mathtt{w}_2 - \mathtt{w}_1}{\mathtt{w}_1\mathtt{w}_2}\right)^{\alpha}[C_1(\alpha)|Q'(\mathtt{w}_1)| + C_2(\alpha)|Q'(\mathtt{w}_2)|,
\end{aligned}$$

where

$$\begin{aligned}
C_1(\alpha) &= \frac{\mathtt{w}_2^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2, \alpha+1; \alpha+3; 1 - \frac{\mathtt{w}_1}{\mathtt{w}_2}\right) \\
&\quad - \frac{(\mathtt{w}_1 + \mathtt{w}_2)^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2, \alpha+1; \alpha+3; 1 - \frac{\mathtt{w}_2 - \mathtt{w}_1}{\mathtt{w}_2 + \mathtt{w}_1}\right), \\
C_2(\alpha) &= \frac{\mathtt{w}_2^{-2}}{\alpha+2}\,{}_2F_1\left(2, \alpha+2; \alpha+3; 1 - \frac{\mathtt{w}_1}{\mathtt{w}_2}\right) \\
&\quad - \frac{2(\mathtt{w}_1 + \mathtt{w}_2)^{-2}}{\alpha+1}\,{}_2F_1\left(2, \alpha+1; \alpha+2; 1 - \frac{\mathtt{w}_2 - \mathtt{w}_1}{\mathtt{w}_2 + \mathtt{w}_1}\right) \\
&\quad + \frac{(\mathtt{w}_1 + \mathtt{w}_2)^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2, \alpha+1; \alpha+3; 1 - \frac{\mathtt{w}_2 - \mathtt{w}_1}{\mathtt{w}_2 + \mathtt{w}_1}\right),
\end{aligned}$$

with $0 < \alpha \leq 1$ and $h(x) = \dfrac{1}{x}, x \in \left[\dfrac{1}{\mathtt{w}_2}, \dfrac{1}{\mathtt{w}_1}\right]$.

Theorem 33 ([41])**.** *Assume that the function g is as in Theorem 32. Let $Q : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1, \mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q, q \geq 1$ is harmonically convex on $[\mathtt{w}_1, \mathtt{w}_2]$, then the fractional inequality is given as:*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(g\circ h)(1/\mathtt{w}_1)+J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(Q\circ h)(1/\mathtt{w}_2)\right]\right.$$
$$\left.-\left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(Qg\circ h)(1/\mathtt{w}_1)+J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(Qg\circ h)(1/\mathtt{w}_2)\right]\right|$$
$$\leq \frac{\|g\|_\infty \mathtt{w}_1\mathtt{w}_2(\mathtt{w}_2-\mathtt{w}_1)}{\Gamma(\alpha+1)}\left(\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1\mathtt{w}_2}\right)^{\alpha}\left[C_3^{1-\frac{1}{q}}(\alpha)\left[C_4(\alpha)|Q'(\mathtt{w}_1)|^q+C_5(\alpha)|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+C_6^{1-\frac{1}{q}}(\alpha)\left[C_7(\alpha)|Q'(\mathtt{w}_1)|^q+C_8(\alpha)|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}}\right],$$

where

$$\begin{aligned}
C_3(\alpha) &= \frac{(\mathtt{w}_1+\mathtt{w}_2)^{-2}}{2^{\alpha-1}(\alpha+1)}\,{}_2F_1\left(2,1,\alpha+2;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_2+\mathtt{w}_1}\right)\\
C_4(\alpha) &= \frac{(\mathtt{w}_1+\mathtt{w}_2)^{-2}}{2^{\alpha}(\alpha+2)}\,{}_2F_1\left(2,1,\alpha+3;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_2+\mathtt{w}_1}\right)\\
C_5(\alpha) &= C_3(\alpha)-C_4(\alpha)\\
C_6(\alpha) &= \frac{\mathtt{w}_2^{-2}}{2^{\alpha+1}(\alpha+1)}\,{}_2F_1\left(2,\alpha+1;\alpha+1;\frac{1}{2}\left(1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right)\right),\\
C_7(\alpha) &= \frac{\mathtt{w}_2^{-2}}{2^{\alpha+1}(\alpha+1)}\,{}_2F_1\left(2,\alpha+1;\alpha+2;\frac{1}{2}\left(1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right)\right)\\
&\quad -\frac{\mathtt{w}_2^{-2}}{2^{\alpha+2}(\alpha+2)}\,{}_2F_1\left(2,\alpha+2;\alpha+3;\frac{1}{2}\left(1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right)\right)\Big],\\
C_8(\alpha) &= C_6(\alpha)-C_7(\alpha),
\end{aligned}$$

with $0<\alpha\leq 1$ *and* $h(x)=\frac{1}{x}, x\in\left[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\right]$.

Theorem 34 ([41]). *Assume that the function g is as in Theorem 32. Let $Q: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q'\in L[\mathtt{w}_1,\mathtt{w}_2]$, where $\mathtt{w}_1,\mathtt{w}_2\in I$ and $\mathtt{w}_1<\mathtt{w}_2$. If $|Q'|^q, q>1$ is harmonically convex on $[\mathtt{w}_1,\mathtt{w}_2]$, then the fractional inequality is given as:*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(g\circ h)(1/\mathtt{w}_1)+J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(Q\circ h)(1/\mathtt{w}_2)\right]\right.$$
$$\left.-\left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(Qg\circ h)(1/\mathtt{w}_1)+J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(Qg\circ h)(1/\mathtt{w}_2)\right]\right|$$
$$\leq \frac{\|g\|_\infty \mathtt{w}_1\mathtt{w}_2(\mathtt{w}_2-\mathtt{w}_1)}{\Gamma(\alpha+1)}\left(\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1\mathtt{w}_2}\right)^{\alpha}\left[C_9^{\frac{1}{p}}(\alpha)\left[\frac{|Q'(\mathtt{w}_1)|^q+3|Q'(\mathtt{w}_2)|^q}{8}\right]^{\frac{1}{q}}\right.$$
$$\left.+C_{10}^{\frac{1}{p}}(\alpha)\left[\frac{3|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q}{8}\right]^{\frac{1}{q}}\right],$$

where

$$\begin{aligned}
C_9(\alpha) &= \frac{(\mathtt{w}_1+\mathtt{w}_2)^{-2p}}{2^{\alpha p-2p+1}(\alpha p+1)}\,{}_2F_1\left(2p,1;\alpha p+2;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_2+\mathtt{w}_1}\right)\\
C_{10}(\alpha) &= \frac{\mathtt{w}_2^{-2p}}{2^{\alpha p+1}(\alpha p+1)}\,{}_2F_1\left(2,\alpha p+1;\alpha p+2;\frac{1}{2}\left(1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right)\right),
\end{aligned}$$

with $\alpha>1$ *and* $h(x)=\frac{1}{x}, x\in\left[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\right]$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 35 ([42]). *Let $Q : I \subseteq (0,\infty) \to \mathbb{R}$ be differentiable mapping I°, where $\mathtt{w}_1, \mathtt{w}_2 \in I$ with $\mathtt{w}_1 < \mathtt{w}_2$. Assume also that $g : [\mathtt{w}_1, \mathtt{w}_2] \to [0,\infty)$ is a continuous positive mapping symmetric to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}$. If the mapping $|Q'|$ is harmonically-convex on $[\mathtt{w}_1, \mathtt{w}_2]$, then the fractional inequality is given as:*

$$\begin{aligned}&\Bigg|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^{\alpha}_{1/\mathtt{w}_2+}g\circ h(1/\mathtt{w}_1)+J^{\alpha}_{1/\mathtt{w}_1-}g\circ h(1/\mathtt{w}_2)\Big]\\&-\Big[J^{\alpha}_{1/\mathtt{w}_2+}(Qg\circ h)(1/\mathtt{w}_1)+J^{\alpha}_{1/\mathtt{w}_1-}(Qg\circ h)(1/\mathtt{w}_2)\Big]\Bigg|\\ \leq\;&\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1}(\mathtt{w}_1\mathtt{w}_2)^{\alpha+1}\Gamma(\alpha+1)}\Big[C_1(\alpha)|Q'(\mathtt{w}_1)|+C_2(\alpha)|Q'(H)|+C_3(\alpha)|Q'(\mathtt{w}_2)|\Big],\end{aligned}$$

with $h(x)=\frac{1}{x}, x\in\left[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\right]$,

$$\begin{aligned}C_1(\alpha)&=\int_0^1(1-t)\big[(1+t)^\alpha-(1-t)^\alpha\big](L(t))^2dt,\\C_2(\alpha)&=\int_0^1 t\big[(1+t)^\alpha-(1-t)^\alpha\big][(L(t))^2+(U(t))^2]dt,\\C_3(\alpha)&=\int_0^1(1-t)\big[(1+t)^\alpha-(1-t)^\alpha\big](U(t))^2dt.\end{aligned}$$

and

$$L(t)=\frac{\mathtt{w}_1H}{tH+(1-t)\mathtt{w}_1},\quad U(t)=\frac{\mathtt{w}_2H}{tH+(1-t)\mathtt{w}_2},\quad H=H(\mathtt{w}_1,\mathtt{w}_2)=\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}.$$

Theorem 36 ([42]). *Let g be as in Theorem 35. Let $Q : I \subseteq (0,\infty) \to \mathbb{R}$ be differentiable mapping I°, where $\mathtt{w}_1, \mathtt{w}_2 \in I$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $|Q'|^q$ is harmonically convex on $[\mathtt{w}_1, \mathtt{w}_2]$, then the fractional inequality is given as:*

$$\begin{aligned}&\Bigg|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^{\alpha}_{1/\mathtt{w}_2+}g\circ h(1/\mathtt{w}_1)+J^{\alpha}_{1/\mathtt{w}_1-}g\circ h(1/\mathtt{w}_2)\Big]\\&-\Big[J^{\alpha}_{1/\mathtt{w}_2+}(Qg\circ h)(1/\mathtt{w}_1)+J^{\alpha}_{1/\mathtt{w}_1-}(Qg\circ h)(1/\mathtt{w}_2)\Big]\Bigg|\\ \leq\;&\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha+1}(\mathtt{w}_1\mathtt{w}_2)^{\alpha+1}\Gamma(\alpha+1)}\left(\frac{2^2(2^\alpha-1)}{\alpha+1}\right)^{1-\frac{1}{q}}\\&\times\Big[C_1(\alpha,q)|Q'(\mathtt{w}_1)|^q+C_2(\alpha,q)|Q'(H)|^q+C_3(\alpha,q)|Q'(\mathtt{w}_2)|^q\Big]^{\frac{1}{q}},\end{aligned}$$

with $h(x)=\frac{1}{x}, x\in\left[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\right]$,

$$\begin{aligned}C_1(\alpha,q)&=\int_0^1(1-t)\big[(1+t)^\alpha-(1-t)^\alpha\big]t(L(t))^{2q}dt,\\C_2(\alpha,q)&=\int_0^1\big[(1+t)^\alpha-(1-t)^\alpha\big](1-t)[(L(t))^{2q}+(U(t))^{2q}]dt,\\C_3(\alpha,q)&=\int_0^1(1-t)\big[(1+t)^\alpha-(1-t)^\alpha\big]t(U(t))^{2q}dt,\end{aligned}$$

and L, U, H are given in Theorem 35.

Theorem 37 ([43]). *Assume that $\mathtt{w}_1, \mathtt{w}_2 \in \mathbb{I}$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is a harmonically convex function and $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$. If $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is harmonically symmetric with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}$, integrable and non-negative then the fractional inequalities are given as:*

$$\begin{aligned} & Q\left(\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}\right)\left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(g\circ h)\left(\frac{1}{\mathtt{w}_1}\right)+J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(g\circ h)\left(\frac{1}{\mathtt{w}_2}\right)\right] \\ \leq\ & \left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(g\circ h)\left(\frac{1}{\mathtt{w}_1}\right)+J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(g\circ h)\left(\frac{1}{\mathtt{w}_2}\right)\right] \\ \leq\ & \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(g\circ h)\left(\frac{1}{\mathtt{w}_1}\right)+J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(g\circ h)\left(\frac{1}{\mathtt{w}_2}\right)\right], \end{aligned}$$

with $\alpha > 0$ and $h(x) = \frac{1}{x}, x \in \left[\frac{1}{\mathtt{w}_2}, \frac{1}{\mathtt{w}_1}\right]$.

Theorem 38 ([43]). *Assume that Q is as in Theorem 37. If $|Q'|$ is harmonically convex on $[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is harmonically symmetric with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}$ and continuous, then an inequality in the frame of the fractional operator is given as:*

$$\begin{aligned} & \Bigg|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}-\left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(g\circ h)\left(\frac{1}{\mathtt{w}_1}\right)+J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(g\circ h)\left(\frac{1}{\mathtt{w}_2}\right)\right] \\ & -\left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(Qg\circ h)\left(\frac{1}{\mathtt{w}_1}\right)+J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(Qg\circ h)\left(\frac{1}{\mathtt{w}_2}\right)\right]\Bigg| \\ \leq\ & \frac{\|g\|_\infty \mathtt{w}_1\mathtt{w}_2(\mathtt{w}_2-\mathtt{w}_1)}{\Gamma(\alpha+1)}\left(\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1\mathtt{w}_2}\right)^{\alpha}\Big[E_1(a)|Q'(\mathtt{w}_1)|+E_2(a)|Q'(\mathtt{w}_2)|\Big], \end{aligned}$$

where

$$\begin{aligned} E_1(a) &= \frac{x_2^{-2}}{\alpha+2}\,{}_2F_1\left(2,1;\alpha+3;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right)-\frac{x_2^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right) \\ &\quad +\frac{2(\mathtt{w}_1+\mathtt{w}_2)^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1+\mathtt{w}_2}\right), \\ E_2(a) &= \frac{x_2^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,2;\alpha+3;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right)-\frac{x_2^{-2}}{\alpha+2}\,{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{\mathtt{w}_1}{\mathtt{w}_2}\right) \\ &\quad +\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)^{-2}\frac{1}{\alpha+1}\,{}_2F_1\left(2,\alpha+1;\alpha+2;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1+\mathtt{w}_2}\right) \\ &\quad -\frac{2(\mathtt{w}_1+\mathtt{w}_2)^{-2}}{(\alpha+1)(\alpha+2)}\,{}_2F_1\left(2,\alpha+1;\alpha+3;\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1+\mathtt{w}_2}\right), \end{aligned}$$

with $0 < \alpha \leq 1$ and $h(x) = \frac{1}{x}, x \in \left[\frac{1}{\mathtt{w}_2}, \frac{1}{\mathtt{w}_1}\right]$.

2.6. Fejér-Type Fractional Integral Inequalities for Harmonically Quasi-Convex Functions

Definition 7 ([44]). *A function $Q : \mathbb{I} \subseteq (0,\infty) \to [0,\infty)$ is said to be harmonically quasi-convex, if*

$$Q\left(\frac{xy}{tx+(1-t)y}\right) \leq \max\{Q(x), Q(y)\},$$

for all $x, y \in \mathbb{I}$ and $t \in [0,1]$.

Theorem 39 ([45]). *Assume that $Q : \mathbb{I} \subset (0,\infty) \to \mathbb{R}$ is a differentiable function on $\mathbb{I}^\circ$ such that $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1, \mathtt{w}_2 \in \mathbb{I}$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is harmonically quasi-convex function on*

$[\mathtt{w}_1, \mathtt{w}_2]$, $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1 + \mathtt{w}_2}$, then the fractional inequality is given as:

$$\begin{aligned}
&\left| \frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2} \left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(g \circ h)\left(\frac{1}{\mathtt{w}_1}\right) + J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(g \circ h)\left(\frac{1}{\mathtt{w}_2}\right) \right] \right. \\
&\left. - \left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(Qg \circ h)\left(\frac{1}{\mathtt{w}_1}\right) + J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(Qg \circ h)\left(\frac{1}{\mathtt{w}_2}\right) \right] \right| \\
\leq\ & \frac{\|g\|_\infty \mathtt{w}_1\mathtt{w}_2(\mathtt{w}_2 - \mathtt{w}_1)}{\Gamma(\alpha+1)} \left(\frac{\mathtt{w}_2 - \mathtt{w}_1}{a}\right)^{\alpha} \sup\{|Q'(\mathtt{w}_1)|, |Q'(\mathtt{w}_2)|\} \left[\frac{x_2^{-2}}{\alpha+1}\, {}_2F_1\left(2, 1, \alpha+2, 1 - \frac{\mathtt{w}_1}{\mathtt{w}_2}\right) \right. \\
&\left. - \frac{x_2^{-2}}{\alpha+1}\, {}_2F_1\left(2, \alpha+1, \alpha+2, 1 - \frac{\mathtt{w}_1}{\mathtt{w}_2}\right) + \frac{4(\mathtt{w}_1+\mathtt{w}_2)^{-2}}{\alpha+1}\, {}_2F_1\left(2, \alpha+1, \alpha+2, 1 - \frac{\mathtt{w}_2 - \mathtt{w}_1}{\mathtt{w}_1 + \mathtt{w}_2}\right) \right],
\end{aligned}$$

with $0 < \alpha \leq 1$ and $h(x) = \frac{1}{x}, x \in \left[\frac{1}{\mathtt{w}_2}, \frac{1}{\mathtt{w}_1}\right]$.

Theorem 40 ([45]). *Assume that Q is as in Theorem 39. If $|Q'|^q, q > 1$ is harmonically quasi-convex function on $[\mathtt{w}_1, \mathtt{w}_2]$, $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and harmonically symmetric with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1 + \mathtt{w}_2}$, then the fractional inequality is given as:*

$$\begin{aligned}
&\left| \frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2} \left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(g \circ h)\left(\frac{1}{\mathtt{w}_1}\right) + J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(g \circ h)\left(\frac{1}{\mathtt{w}_2}\right) \right] \right. \\
&\left. - \left[J^{\alpha}_{\frac{1}{\mathtt{w}_2}+}(Qg \circ h)\left(\frac{1}{\mathtt{w}_1}\right) + J^{\alpha}_{\frac{1}{\mathtt{w}_1}-}(Qg \circ h)\left(\frac{1}{\mathtt{w}_2}\right) \right] \right| \\
\leq\ & \frac{\|g\|_\infty \mathtt{w}_1\mathtt{w}_2(\mathtt{w}_2 - \mathtt{w}_1)}{2^{\frac{1}{q}}\Gamma(\alpha+1)} \left(\frac{\mathtt{w}_2 - \mathtt{w}_1}{a}\right)^{\alpha} \sup\{|Q'(\mathtt{w}_1)|^q, |Q'(\mathtt{w}_2)|^q\} \\
&\times \left[\left(\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}\right)^{-2p} \frac{1}{2(\alpha p+1)}\, {}_2F_1\left(2p, \alpha p+1, \alpha p+2, 1 - \frac{\mathtt{w}_2 - \mathtt{w}_1}{\mathtt{w}_1 + \mathtt{w}_2}\right) \right. \\
&\left. + \mathtt{w}_2^{-2p} \frac{1}{2(\alpha p+1)}\, {}_2F_1\left(2p, 1, \alpha p+2, \frac{1}{2}\left(1 - \frac{\mathtt{w}_1}{\mathtt{w}_2}\right)\right) \right],
\end{aligned}$$

with $0 < \alpha \leq 1$ and $h(x) = \frac{1}{x}, x \in \left[\frac{1}{\mathtt{w}_2}, \frac{1}{\mathtt{w}_1}\right]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

2.7. Fejér-Type Fractional Integral Inequalities for p-Convex Functions

Definition 8 ([46]). *The function Q is strongly p-convex with modulus μ if*

$$Q(tx^p + (1-t)y^p)^{1/p} \leq Q(y) + t\Phi(Q(x), Q(y)) - \mu t(1-t)(y^p - x^p)^2$$

holds for $t \in [0, 1]$.

We include in the next theorem a Fejér-type fractional integral inequality for strongly convex functions in a generalized sense.

Theorem 41 ([47]). *Let the strongly generalized p-convex function $Q : I \to \mathbb{R}$ with magnitude $\mu > 0$ and $\Phi(.)$ be bounded above in $Q(I) \times Q(I)$ and $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$. Also, let $w : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be an integrable, non-negative and symmetric with respect to $\left[\frac{\mathtt{w}_1^p + \mathtt{w}_2^p}{2}\right]^{\frac{1}{p}}$, then*

$$\frac{\Gamma(\alpha)}{2} Q\left[\frac{\mathtt{w}_1^p + \mathtt{w}_2^p}{2}\right]^{\frac{1}{p}} \left[J^{\alpha}_{\mathtt{w}_1^p+} w \circ g(\mathtt{w}_2^p) + J^{\alpha}_{\mathtt{w}_2^p-} w \circ g(\mathtt{w}_1^p) \right]$$

$$-\frac{M_\Phi\Gamma(\alpha)}{2}\Big[J^\alpha_{\mathtt{w}_1^p+}w\circ g(\mathtt{w}_2^p)+J^\alpha_{\mathtt{w}_2^p-}w\circ g(\mathtt{w}_1^p)\Big]$$
$$+\frac{\mu}{2}\int_{\mathtt{w}_1^p}^{\mathtt{w}_2^p}(2x-\mathtt{w}_2^p-\mathtt{w}_1^p)^2(\mathtt{w}_2^p-x)^{\alpha-1}w\circ g(x)dx$$
$$\leq \frac{\Gamma(\alpha)}{2}\Big[J^\alpha_{\mathtt{w}_1^p+}Qw\circ g(\mathtt{w}_2^p)+J^\alpha_{\mathtt{w}_2^p-}Qw\circ g(\mathtt{w}_1^p)\Big]\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\frac{\Gamma(\alpha)}{2}$$
$$\times\Big[J^\alpha_{\mathtt{w}_1^p+}w\circ g(\mathtt{w}_2^p)+J^\alpha_{\mathtt{w}_2^p-}w\circ g(\mathtt{w}_1^p)\Big]+\frac{M_\Phi}{\mathtt{w}_2^p-\mathtt{w}_1^p}\int_{\mathtt{w}_1^p}^{\mathtt{w}_2^p}(\mathtt{w}_2^p-x)^\alpha w\circ g(x)dx$$
$$-\mu\int_{\mathtt{w}_1^p}^{\mathtt{w}_2^p}(\mathtt{w}_2^p-x)^2 w\circ g(x)dx.$$

Definition 9 ([48]). *Let $p\in\mathbb{R}\setminus\{0\}$. A function $Q:[\mathtt{w}_1,\mathtt{w}_2]\subset(0,\infty)\to\mathbb{R}$ is said to be p-symmetric with respect to $\Big[\frac{\mathtt{w}_1^p+\mathtt{w}_2^p}{2}\Big]^{\frac{1}{p}}$ if*

$$Q(x)=Q\Big([\mathtt{w}_1^p+\mathtt{w}_2^p-x^p]^{\frac{1}{p}}\Big),$$

holds for all $x\in[\mathtt{w}_1,\mathtt{w}_2]$.

Theorem 42 ([48]). *Assume that $Q:\mathbb{I}\subset(0,\infty)\to\mathbb{R}$ is a p-convex function, $p\in\mathbb{R}\setminus\{0\}$, $\alpha>0$ and $\mathtt{w}_1,\mathtt{w}_2\in\mathbb{I}$ with $\mathtt{w}_1<\mathtt{w}_2$. If $Q\in L[\mathtt{w}_1,\mathtt{w}_2]$ and $w:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ is non-negative, integrable and p-symmetric with respect to $\Big[\frac{\mathtt{w}_1^p+\mathtt{w}_2^p}{2}\Big]^{\frac{1}{p}}$, then the fractional inequalities are given as:*

$$Q\Big(\Big[\frac{\mathtt{w}_1^p+\mathtt{w}_2^p}{2}\Big]^{\frac{1}{p}}\Big)\Big[J^\alpha_{\mathtt{w}_1^p+}(w\circ g)(\mathtt{w}_2^p)+J^\alpha_{\mathtt{w}_2^p-}(w\circ g)(\mathtt{w}_1^p)\Big]$$
$$\leq \Big[J^\alpha_{\mathtt{w}_1^p+}(Qw\circ g)(\mathtt{w}_2^p)+J^\alpha_{\mathtt{w}_2^p-}(Qw\circ g)(\mathtt{w}_1^p)\Big]$$
$$\leq \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^\alpha_{\mathtt{w}_1^p+}(w\circ g)(\mathtt{w}_2^p)+J^\alpha_{\mathtt{w}_2^p-}(w\circ g)(\mathtt{w}_1^p)\Big],\ \ p>0,$$

with $g(x)=x^{\frac{1}{p}},\ x\in[\mathtt{w}_1^p,\mathtt{w}_2^p]$, and

$$Q\Big(\Big[\frac{\mathtt{w}_1^p+\mathtt{w}_2^p}{2}\Big]^{\frac{1}{p}}\Big)\Big[J^\alpha_{\mathtt{w}_2^p+}(w\circ g)(\mathtt{w}_1^p)+J^\alpha_{\mathtt{w}_1^p-}(w\circ g)(\mathtt{w}_2^p)\Big]$$
$$\leq \Big[J^\alpha_{\mathtt{w}_2^p+}(Qw\circ g)(\mathtt{w}_1^p)+J^\alpha_{\mathtt{w}_1^p-}(Qw\circ g)(\mathtt{w}_2^p)\Big]$$
$$\leq \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^\alpha_{\mathtt{w}_2^p+}(w\circ g)(\mathtt{w}_1^p)+J^\alpha_{\mathtt{w}_1^p-}(w\circ g)(\mathtt{w}_2^p)\Big],\ \ p<0,$$

with $g(x)=x^{\frac{1}{p}},\ x\in[\mathtt{w}_2^p,\mathtt{w}_1^p]$.

2.8. Fejér-Type Fractional Integral Inequalities via Convexity with Respect to Strictly Monotone Function

Definition 10 ([38]). *Let I,J be intervals in $\mathbb{R}$ and $Q:I\to\mathbb{R}$ be the convex function, also let $\sigma:J\subset I\to\mathbb{R}$ be strictly monotone function. Then, Q is called convex with respect to σ if*

$$Q(\sigma^{-1}(tx+(1-t)y))\leq tQ(\sigma^{-1}(x))+(1-t)Q(\sigma^{-1}(y)),$$

for all $t\in[0,1]$, $x,y\in Range(\sigma)$, provided $Range(\sigma)$ is a convex set.

We give a Fejér-type fractional integral inequality for convex function Q with respect to a strictly monotone function σ.

Theorem 43 ([49]). *Let I, J be intervals in $\mathbb{R}$ and $Q, \Phi : [\mathtt{w}_1, \mathtt{w}_2] \subset I \to \mathbb{R}$ be real valued functions. Let Q be convex and g be positive and symmetric about $\frac{\sigma(\mathtt{w}_1)+\sigma(\mathtt{w}_2)}{2}$. Let $\sigma : J \supset [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be a strictly monotone function. If Q is convex with respect to σ, then the fractional inequality is given as:*

$$\begin{aligned} & Q\left(\sigma^{-1}\left(\frac{\sigma(\mathtt{w}_1)+\sigma(\mathtt{w}_2)}{2}\right)\right)\left(J^{\alpha}_{\sigma(\mathtt{w}_1)+}\Phi(\mathtt{w}_2) + J^{\alpha}_{\sigma(\mathtt{w}_2)-}\Phi(\mathtt{w}_1)\right) \\ \leq\ & J^{\alpha}_{\sigma(\mathtt{w}_1)+}(Q\Phi)(\mathtt{w}_2) + J^{\alpha}_{\sigma(\mathtt{w}_2)-}(Q\Phi)(\mathtt{w}_1) \\ \leq\ & \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left(J^{\alpha}_{\sigma(\mathtt{w}_1)+}\Phi(\mathtt{w}_2) + J^{\alpha}_{\sigma(\mathtt{w}_2)-}\Phi(\mathtt{w}_1)\right). \end{aligned}$$

In the next theorem, we establish another version of the Fejér-type fractional integral inequality for a convex function with respect to a strictly monotone function.

Theorem 44 ([49]). *Under the conditions of Theorem 43, the fractional inequality is given as:*

$$\begin{aligned} & Q\left(\sigma^{-1}\left(\frac{\sigma(\mathtt{w}_1)+\sigma(\mathtt{w}_2)}{2}\right)\right)\left(J^{\alpha}_{\left(\frac{\sigma(\mathtt{w}_1)+\sigma(\mathtt{w}_2)}{2}\right)+}\Phi(\mathtt{w}_2) + J^{\alpha}_{\left(\frac{\sigma(\mathtt{w}_1)+\sigma(\mathtt{w}_2)}{2}\right)-}\Phi(\mathtt{w}_1)\right) \\ \leq\ & J^{\alpha}_{\left(\frac{\sigma(\mathtt{w}_1)+\sigma(\mathtt{w}_2)}{2}\right)+}(Q\Phi)(\mathtt{w}_2) + J^{\alpha}_{\left(\frac{\sigma(\mathtt{w}_1)+\sigma(\mathtt{w}_2)}{2}\right)-}(Q\Phi)(\mathtt{w}_1) \\ \leq\ & \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left(J^{\alpha}_{\left(\frac{\sigma(\mathtt{w}_1)+\sigma(\mathtt{w}_2)}{2}\right)+}\Phi(\mathtt{w}_2) + J^{\alpha}_{\left(\frac{\sigma(\mathtt{w}_1)+\sigma(\mathtt{w}_2)}{2}\right)-}\Phi(\mathtt{w}_1)\right). \end{aligned}$$

2.9. Fejér-Type Fractional Integral Inequalities for Co-Ordinated Convex Functions

This subsection includes the Fejér-type fractional integral inequalities for co-ordinated convex functions via fractional integral.

Definition 11 ([50]). *A function $Q : \Delta = [\mathtt{w}_1, \mathtt{w}_2] \times [\mathtt{z}_1, \mathtt{z}_2] \to \mathbb{R}$ will be called co-ordinated convex on Δ, for all $t, s \in [0,1]$ and $(x, y), (u, w) \in \Delta$, if the following inequality holds:*

$$\begin{aligned} & Q(tx + (1-t)y, su + (1-s)w) \\ \leq\ & tsQ(x,u) + s(1-t)Q(y,u) + t(1-s)Q(x,w) + (1-t)(1-s)Q(y,w). \end{aligned}$$

Definition 12 ([51]). *Assume that $Q \in L_1([\mathtt{w}_1, \mathtt{w}_2] \times [\mathtt{z}_1, \mathtt{z}_2])$. The Riemann–Liouville integrals $J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_1+}Q$, $J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_2-}Q$, $J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_1+}Q$, and $J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_2-}Q$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by*

$$\begin{aligned} J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_1+}Q(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\mathtt{w}_1}^{x}\int_{\mathtt{z}_1}^{y}(x-t)^{\alpha-1}(y-s)^{\beta-1}Q(s,t)dsdt, \quad x > \mathtt{w}_1, y > \mathtt{z}_1, \\ J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_2-}Q(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{\mathtt{w}_1}^{x}\int_{y}^{\mathtt{z}_2}(x-t)^{\alpha-1}(s-y)^{\beta-1}Q(s,t)dsdt, \quad x > \mathtt{w}_1, y < \mathtt{z}_2, \\ J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_1+}Q(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\mathtt{w}_2}\int_{\mathtt{z}_1}^{y}(t-x)^{\alpha-1}(y-s)^{\beta-1}Q(s,t)dsdt, \quad x < \mathtt{w}_2, y > \mathtt{z}_1, \\ J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_2-}Q(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{x}^{\mathtt{w}_2}\int_{y}^{\mathtt{z}_2}(t-x)^{\alpha-1}(s-y)^{\beta-1}Q(s,t)dsdt, \quad x < \mathtt{w}_2, y < \mathtt{z}_2, \end{aligned}$$

respectively.

Theorem 45 ([51]). *Let $Q : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a co-ordinated convex on $\Delta = [\mathtt{w}_1, \mathtt{w}_2] \times [\mathtt{z}_1, \mathtt{z}_2]$ in $\mathbb{R}^2$ with $0 \leq \mathtt{w}_1 < \mathtt{w}_2$, $0 \leq \mathtt{z}_1 < \mathtt{z}_2$ and $Q \in L_1(\Delta)$. If $g : \Delta \to \mathbb{R}$ is non-negative, integrable and symmetric to $\dfrac{\mathtt{w}_1+\mathtt{w}_2}{2}$ and $\dfrac{\mathtt{z}_1+\mathtt{z}_2}{2}$, then for all $\alpha, \beta > 0$ and $\lambda, \gamma \in [0,1]$, we have the inequalities:*

$$\begin{aligned} & Q\left(\frac{\lambda\mathtt{w}_2 + (2-\lambda)\mathtt{w}_1}{2}, \frac{\gamma\mathtt{z}_2 + (2-\gamma)\mathtt{z}_1}{2}\right)J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_2-}(g)(\lambda\mathtt{w}_1 + (1-\lambda)\mathtt{w}_2, \gamma\mathtt{z}_1 + (1-\gamma)\mathtt{z}_2) \\ \leq\ & J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_1+}(Qg)(\mathtt{w}_1 + \lambda(\mathtt{w}_2 - \mathtt{w}_1), \mathtt{z}_1 + \gamma(\mathtt{z}_2 - \mathtt{z}_1)) \end{aligned}$$

$$
\begin{aligned}
&+J^{\alpha,\beta}_{w_1+,z_2-}(Qg)(w_1+\lambda(w_2-w_1),\gamma z_1+(1-\gamma)z_2)\\
&+J^{\alpha,\beta}_{w_2-,z_1+}(Qg)(\lambda w_1+(1-\lambda)w_2,z_1+\gamma(z_2-z_1))\\
&+J^{\alpha,\beta}_{w_2-,z_2-}(Qg)(\lambda w_1+(1-\lambda)w_2,\gamma z_1+(1-\gamma)z_2)\\
\leq\ & J^{\alpha,\beta}_{w_2-,z_2-}(g)(\lambda w_1+(1-\lambda)w_2,\gamma z_1+(1-\gamma)z_2)\frac{Q(w_1,z_1)+Q(w_2,z_1)+Q(w_1,z_2)+Q(w_2,z_2)}{4}.
\end{aligned}
$$

Theorem 46 ([51]). *Assume that the conditions of Theorem 45 are satisfied. Then, we have the inequalities:*

$$
\begin{aligned}
&J^{\alpha,\beta}_{w_1+,[z_2-\gamma(z_2-z_1)]+}Q\Big(w_2,\frac{\gamma z_2+(2-\gamma)z_1}{2}\Big)g(w_2,z_2)\\
&+J^{\alpha,\beta}_{w_1+,z_2-}Q\Big(w_2,\frac{\gamma z_2+(2-\gamma)z_1}{2}\Big)g(w_2,z_2-\gamma(z_2-z_1))\\
&+J^{\alpha,\beta}_{w_2-,[z_2-\gamma(z_2-z_1)]+}Q\Big(w_1,\frac{\gamma z_2+(2-\gamma)z_1}{2}\Big)g(w_1,z_2-\gamma(z_2-z_1))\\
&+J^{\alpha,\beta}_{w_2-,z_2-}Q\Big(w_1,\frac{\gamma z_2+(2-\gamma)z_1}{2}\Big)g(w_1,z_2-\gamma(z_2-z_1))\\
&+J^{\beta,\alpha}_{z_1+,[w_2-\lambda(w_2-w_1)]+}Q\Big(\frac{\lambda w_2+(2-\lambda)w_1}{2},z_2\Big)g(w_2,z_2)\\
&+J^{\beta,\alpha}_{z_1+,w_2-}Q\Big(\frac{\lambda w_2+(2-\lambda)w_1}{2},z_2\Big)g(w_2-\lambda(w_2-w_1),z_2)\\
&+J^{\beta,\alpha}_{z_2-,[w_2-\lambda(w_2-w_1)]+}Q\Big(\frac{\lambda w_2+(2-\lambda)w_1}{2},z_1\Big)g(w_2,z_1)\\
&+J^{\beta,\alpha}_{z_2-,w_2-}Q\Big(\frac{\lambda w_2+(2-\lambda)w_1}{2},z_1\Big)g(w_2,z_1)\\
\leq\ & J^{\alpha,\beta}_{w_1+,z_1+}(Qg)(w_2,z_1+\gamma(z_2-z_1))+J^{\alpha,\beta}_{w_1+,z_2-}(Qg)(w_2,z_2-\gamma(z_2-z_1))\\
&+J^{\alpha,\beta}_{w_2-,z_1+}(Qg)(w_1,z_1+\gamma(z_2-z_1))+J^{\alpha,\beta}_{w_2-,z_2+}(Qg)(w_1,z_2-\gamma(z_2-z_1))\\
&+J^{\beta,\alpha}_{z_1+,w_1+}(Qg)(w_1+\lambda(w_2-w_1),z_2)+J^{\beta,\alpha}_{z_1+,w_2-}(Qg)(z_2,w_2-\lambda(w_2-w_1))\\
&+J^{\beta,\alpha}_{z_2-,w_1+}(Qg)(w_1+\lambda(w_2-w_1),z_1)+J^{\beta,\alpha}_{z_2-,w_2-}(Qg)(w_2-\lambda(w_2-w_1),z_1)\\
\leq\ & \frac{1}{2}\{J^{\alpha,\beta}_{w_1+,z_1+}[Q(w_2,z_1)+Q(w_2,z_2)]g(w_2,z_1+\gamma(z_2-z_1))\}\\
&+\frac{1}{2}\{J^{\alpha,\beta}_{w_1+,z_2-}[Q(w_2,z_1)+Q(w_2,z_2)]g(w_2,z_2-\gamma(z_2-z_1))\}\\
&+\frac{1}{2}\{J^{\alpha,\beta}_{w_2-,z_1+}[Q(w_1,z_1)+Q(w_1,z_2)]g(w_1,z_1+\gamma(z_2-z_1))\}\\
&+\frac{1}{2}\{J^{\alpha,\beta}_{w_2-,z_2-}[Q(w_1,z_1)+Q(w_1,z_2)]g(w_1,z_2-\gamma(z_2-z_1))\}\\
&+\frac{1}{2}\{J^{\beta,\alpha}_{z_1+,w_1+}[Q(w_1,z_2)+Q(w_2,z_2)](Qg)(w_1+\lambda(w_2-w_1),z_2)\}\\
&+\frac{1}{2}\{J^{\beta,\alpha}_{z_1+,w_2-}[Q(w_1,z_2)+Q(w_2,z_2)](Qg)(w_2+\lambda(w_2-w_1),z_2)\}\\
&+\frac{1}{2}\{J^{\beta,\alpha}_{z_2-,w_1+}[Q(w_1,z_1)+Q(w_2,z_1)](Qg)(w_1+\lambda(w_2-w_1),z_1)\}\\
&+\frac{1}{2}\{J^{\beta,\alpha}_{z_2-,w_2-}[Q(w_1,z_1)+Q(w_2,z_1)](Qg)(w_2-\lambda(w_2-w_1),z_1)\}.
\end{aligned}
$$

Theorem 47 ([52]). *Let $Q:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be co-ordinated convex on $\Delta=[w_1,w_2]\times[z_1,z_2]$ in $\mathbb{R}^2$ with $0\leq w_1<w_2$, $0\leq z_1<z_2$ and $Q\in L_1(\Delta)$. If $g:\Delta\to\mathbb{R}$ is non-negative, integrable, and symmetric to $\frac{w_1+w_2}{2}$ and $\frac{z_1+z_2}{2}$, then for all $\alpha,\beta>0$, we have the inequalities:*

$$
Q\Big(\frac{w_1+w_2}{2},\frac{z_1+z_2}{2}\Big)\Big[J^{\alpha,\beta}_{w_1+,z_1+}g(w_2,z_2)+J^{\alpha,\beta}_{w_1+,z_2-}g(w_2,z_1)
$$

$$
\begin{aligned}
&+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_1+}g(\mathtt{w}_1,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_2-}g(\mathtt{w}_1,\mathtt{z}_1)\Big]\\
\leq\;&\frac{1}{4}\Big[J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_1+}(Qg)(\mathtt{w}_2,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_2-}(Qg)(\mathtt{w}_2,\mathtt{z}_1)\\
&+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_1+}(Qg)(\mathtt{w}_1,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_2-}(Qg)(\mathtt{w}_1,\mathtt{z}_1)\Big]\\
\leq\;&\frac{Q(\mathtt{w}_1,\mathtt{z}_1)+Q(\mathtt{w}_1,\mathtt{z}_2)+Q(\mathtt{w}_2,\mathtt{z}_1)+Q(\mathtt{w}_2,\mathtt{z}_2)}{4}\\
&\times\Big[J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_1+}g(\mathtt{w}_2,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_2-}g(\mathtt{w}_2,\mathtt{z}_1)+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_1+}g(\mathtt{w}_1,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_2-}g(\mathtt{w}_1,\mathtt{z}_1)\Big].
\end{aligned}
$$

Theorem 48 ([52]). *Under the conditions of Theorem 47, we have the fractional inequalities:*

$$
\begin{aligned}
&Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2},\frac{\mathtt{z}_1+\mathtt{z}_2}{2}\Big)\Big[J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_1+}g(\mathtt{w}_2,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_2-}g(\mathtt{w}_2,\mathtt{z}_1)\\
&+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_1+}g(\mathtt{w}_1,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_2-}g(\mathtt{w}_1,\mathtt{z}_1)\Big]\\
\leq\;&J^{\alpha}_{\mathtt{w}_1+}\Big[Q\Big(\mathtt{w}_2,\frac{\mathtt{z}_1+\mathtt{z}_2}{2}\Big)J^{\beta}_{\mathtt{z}_1+}g(\mathtt{w}_2,\mathtt{z}_2)\Big]+J^{\alpha}_{\mathtt{w}_1+}\Big[Q\Big(\mathtt{w}_2,\frac{\mathtt{z}_1+\mathtt{z}_2}{2}\Big)J^{\beta}_{\mathtt{z}_2-}g(\mathtt{w}_2,\mathtt{z}_1)\Big]\\
&+J^{\alpha}_{\mathtt{w}_2-}\Big[Q\Big(\mathtt{w}_1,\frac{\mathtt{z}_1+\mathtt{z}_2}{2}\Big)J^{\beta}_{\mathtt{z}_1+}g(\mathtt{w}_1,\mathtt{z}_2)\Big]+J^{\alpha}_{\mathtt{w}_2-}\Big[Q\Big(\mathtt{w}_1,\frac{\mathtt{z}_1+\mathtt{z}_2}{2}\Big)J^{\beta}_{\mathtt{z}_2-}g(\mathtt{w}_1,\mathtt{z}_1)\Big]\\
&+J^{\beta}_{\mathtt{z}_1+}\Big[Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2},\mathtt{z}_2\Big)J^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2,\mathtt{z}_2)\Big]+J^{\beta}_{\mathtt{z}_1+}\Big[Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2},\mathtt{z}_2\Big)J^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1,\mathtt{z}_2)\Big]\\
&+J^{\beta}_{\mathtt{z}_2-}\Big[Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2},\mathtt{z}_1\Big)J^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2,\mathtt{z}_1)\Big]+J^{\beta}_{\mathtt{z}_2-}\Big[Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2},\mathtt{z}_1\Big)J^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1,\mathtt{z}_1)\Big]\\
&2\Big[J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_1+}(Qg)(\mathtt{w}_2,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_2-}(Qg)(\mathtt{w}_2,\mathtt{z}_1)\\
&+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_1+}(Qg)(\mathtt{w}_1,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_2-}(Qg)(\mathtt{w}_1,\mathtt{z}_1)\Big]\\
\leq\;&J^{\alpha}_{\mathtt{w}_1+}\Big[Q(\mathtt{w}_2,\mathtt{z}_1)J^{\beta}_{\mathtt{z}_1+}g(\mathtt{w}_2,\mathtt{z}_2)\Big]+J^{\alpha}_{\mathtt{w}_1+}\Big[Q(\mathtt{w}_2,\mathtt{z}_2)J^{\beta}_{\mathtt{z}_2-}g(\mathtt{w}_2,\mathtt{z}_1)\Big]\\
&+J^{\alpha}_{\mathtt{w}_2-}\Big[Q(\mathtt{w}_1,\mathtt{z}_1)J^{\beta}_{\mathtt{z}_1+}g(\mathtt{w}_1,\mathtt{z}_2)\Big]+J^{\alpha}_{\mathtt{w}_2-}\Big[Q(\mathtt{w}_1,\mathtt{z}_2)J^{\beta}_{\mathtt{z}_2-}g(\mathtt{w}_1,\mathtt{z}_1)\Big]\\
&+J^{\beta}_{\mathtt{z}_1+}\Big[Q(\mathtt{w}_1,\mathtt{z}_2)J^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2,\mathtt{z}_2)\Big]+J^{\beta}_{\mathtt{z}_1+}\Big[Q(\mathtt{w}_2,\mathtt{z}_2)J^{\alpha}_{\mathtt{z}_2-}g(\mathtt{w}_1,\mathtt{z}_1)\Big]\\
&+J^{\beta}_{\mathtt{z}_2-}\Big[Q(\mathtt{w}_1,\mathtt{z}_1)J^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2,\mathtt{z}_1)\Big]+J^{\beta}_{\mathtt{z}_2-}\Big[Q(\mathtt{w}_2,\mathtt{z}_1)J^{\alpha}_{\mathtt{z}_2-}g(\mathtt{w}_1,\mathtt{z}_1)\Big]\\
\leq\;&\frac{Q(\mathtt{w}_1,\mathtt{z}_1)+Q(\mathtt{w}_1,\mathtt{z}_2)+Q(\mathtt{w}_2,\mathtt{z}_1)+Q(\mathtt{w}_2,\mathtt{z}_2)}{4}\\
&\times\Big[J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_1+}g(\mathtt{w}_2,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_1+,\mathtt{z}_2-}g(\mathtt{w}_2,\mathtt{z}_1)+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_1+}g(\mathtt{w}_1,\mathtt{z}_2)+J^{\alpha,\beta}_{\mathtt{w}_2-,\mathtt{z}_2-}g(\mathtt{w}_1,\mathtt{z}_1)\Big].
\end{aligned}
$$

2.10. Fejér-Type Fractional Integral Inequalities for $(\theta,h-m)-p$-Convex Functions

Definition 13 ([53]). *Let $J\subseteq\mathbb{R}$ be an interval containing $(0,1)$ and let $h:J\to\mathbb{R}$ be a non-negative function. Let $\mathbb{I}\subset(0,\infty)$ be an interval and $p\in\mathbb{R}\setminus\{0\}$. A function $Q:\mathbb{I}\to\mathbb{R}$ is said to be $(\theta,h-m)-p$-convex, if*

$$Q\Big(\big(t\mathtt{w}_1^p+m(1-t)\mathtt{w}_2^p\big)^{\frac{1}{p}}\Big)\leq h(t^\theta)Q(\mathtt{w}_1)+mh(1-t^\theta)Q(\mathtt{w}_2),$$

holds, provided that $\big(t\mathtt{w}_1^p+m(1-t)\mathtt{w}_2^p\big)^{\frac{1}{p}}\in\mathbb{I}$ for $t\in[0,1]$ and $(\theta,m)\in[0,1]^2$.

Theorem 49 ([53]). *Let $Q:\mathbb{I}\to\mathbb{R}$ be a positive $(\theta,h-m)-p$-convex function with $\big(t\mathtt{w}_2^p+m(1-t)(\mathtt{w}_1^p/m)\big)^{\frac{1}{p}}\in\mathbb{I}$, $m\neq0$, $\mathtt{w}_1^p<m\mathtt{w}_2^p$. If $\mathcal{F}:I\longrightarrow\mathbb{R}$ is a positive function, then*

$$Q\left(\left(\frac{\mathtt{w}_1^p+m\mathtt{w}_1^p}{2}\right)^{1/p}\right)\left(I^{\tau}_{\mathtt{w}_1^{p}+}\mathcal{F}\circ\xi\right)\left(m\mathtt{w}_1^p\right)$$

$$\leq h\left(\frac{1}{2^{\alpha}}\right)\left(I^{\tau}_{\mathtt{w}_1^{p}+}Q\mathcal{F}\circ\xi\right)\left(m\mathtt{w}_2^{p}\right)+m^{\mu+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left(I^{\tau}_{\mathtt{w}_2^{p}}Q\mathcal{F}\circ\xi\right)\left(\frac{\mathtt{w}_1^{p}}{m}\right)$$
$$\leq \frac{\left(m\mathtt{w}_2^{p}-\mathtt{w}_1^{p}\right)^{\tau}}{\Gamma(\tau)}\left\{\left(h\left(\frac{1}{2^{\alpha}}\right)Q(\mathtt{w}_1)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)Q(\mathtt{w}_2)\right)\right.$$
$$\times\int_0^1 t^{\tau-1}\mathcal{F}\left(\left(t\mathtt{w}_1^{p}+m(1-t)\mathtt{w}_2^{p}\right)\frac{1}{p}\right)h(t^{\alpha})\mathrm{d}t$$
$$+m\left(h\left(\frac{1}{2^{\alpha}}\right)Q(\mathtt{w}_2)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)Q\left(\frac{\mathtt{w}_1}{m^2}\right)\right)$$
$$\left.\times\int_0^1 t^{\tau-1}\mathcal{F}\left(\left(t\mathtt{w}_1^{p}+m(1-t)\mathtt{w}_2^{p}\right)^{1/p}\right)h(1-t^{\alpha})\mathrm{d}t\right\},$$
$$\xi(t)=t^{1/p},Q\mathcal{F}\circ\xi=(Q\circ\xi)(\mathcal{F}\circ\xi).$$

Theorem 50 ([53]). *Assume that Q is as in Theorem 49. Then, the following inequalities hold:*

$$Q\left(\left(\frac{\mathtt{w}_1^{p}+m\mathtt{w}_2^{p}}{2}\right)^{1/p}\right)\left(I^{\tau}_{((\mathtt{w}_1^{p}+m\mathtt{w}_2^{p})/2)^{+}}\mathcal{F}\circ\xi\right)\left(m\mathtt{w}_2^{p}\right)$$
$$\leq h\left(\frac{1}{2^{\alpha}}\right)\left(I^{\tau}_{((\mathtt{w}_1^{p}+m\mathtt{w}_2^{p})/2)^{+}}Q\mathcal{F}\circ\xi\right)\left(m\mathtt{w}_2^{p}\right)$$
$$+m^{\mu+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left(I^{\tau}_{((\mathtt{w}_1^{p}+m\mathtt{w}_2\mathtt{w}_2^{p})/2)^{-}}Q\mathcal{F}\circ\xi\right)\left(\frac{\mathtt{w}_1^{p}}{m}\right)$$
$$\leq\frac{1}{\Gamma(\tau)}\left(\frac{m\mathtt{w}_2^{p}-\mathtt{w}_1^{p}}{2}\right)^{\tau}\left\{\left(h\left(\frac{1}{2^{\alpha}}\right)Q(\mathtt{w}_1)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)Q(\mathtt{w}_2)\right)\right.$$
$$\times\int_0^1 t^{\tau-1}\mathcal{F}\left(\left(\frac{t}{2}\mathtt{w}_1^{p}+m\left(\frac{2-t}{2}\right)\mathtt{w}_2^{p}\right)^{1/p}\right)h\left(\left(\frac{t}{2}\right)^{\alpha}\right)\mathrm{d}t$$
$$+m\left(h\left(\frac{1}{2^{\alpha}}\right)Q(\mathtt{w}_2)+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)Q\left(\frac{\mathtt{w}_1}{m^2}\right)\right)$$
$$\left.\times\int_0^1 t^{\tau-1}\mathcal{F}\left(\left(\frac{t}{2}\mathtt{w}_1^{p}+m\left(\frac{2-t}{2}\right)\mathtt{w}_2^{p}\right)^{1/p}\right)h\left(\frac{2^{\alpha}-t^{\alpha}}{2^{\alpha}}\right)\mathrm{d}t\right\},$$
$$\xi(t)=t^{1/p},Q\mathcal{F}\circ\xi=(Q\circ\xi)(\mathcal{F}\circ\xi).$$

2.11. Fejér-Type Fractional Integral Inequalities for h-Preinvex Functions

Definition 14 ([54]). *Let $Q: X\to\mathbb{R}$ and $\Phi: X\times X\to\mathbb{R}^n$, where X is a non-empty closed set in $\mathbb{R}^n$, be continuous function. Assume that $h:[0,1]\to\mathbb{R}$. Then, Q is said to be h-preinvex with respect to Φ, if*

$$Q(\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1))\leq h(1-\wp)Q(\mathtt{w}_1)+h(\wp)Q(\mathtt{w}_2),$$

for all $\mathtt{w}_1,\mathtt{w}_2\in X$ and $\wp\in[0,1]$, where $Q(\cdot)>0$.

Theorem 51 ([55]). *Let $Q:[\mathtt{w}_1,\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)]\to\mathbb{R}$ be a h-preinvex function, condition C for Φ holds, and $\Phi(\mathtt{w}_2,\mathtt{w}_1)>0$, $h(\frac{1}{2})>0$ and $\mathcal{F}:A\to[0,\infty)$ be differentiable and symmetric to $\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)$, then we have*

$$\frac{\Gamma(\alpha)}{2\cdot h\left(\frac{1}{2}\right)\cdot\Phi(\mathtt{w}_2,\mathtt{w}_1)^{\alpha}}Q\left(\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)\right)\left[\mathrm{I}^{\alpha}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^{-}}\mathcal{F}(\mathtt{w}_1)+\mathrm{I}^{\alpha}_{\mathtt{w}_1^{+}}\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\right]$$
$$\leq\frac{\Gamma(\alpha)}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{\alpha}}\left[\mathrm{I}^{\alpha}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^{\alpha}}\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)+\mathrm{I}^{\alpha}_{\mathtt{w}_1^{+}}\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\right]$$

$$\leq [Q(\mathtt{w}_1)+Q(\mathtt{w}_2)]\cdot\int_0^1 \wp^{\alpha-1}[h(\wp)+h(1-\wp)]\mathcal{F}(\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1))d\wp.$$

Theorem 52 ([55]). *Let $A\subset\mathbb{R}$ be an open invex subset with respect to $\Phi: A\times A\to\mathbb{R}$ and $\mathtt{w}_1,\mathtt{w}_2\in A$ with $\Phi(\mathtt{w}_2,\mathtt{w}_1)>0$. Suppose that $Q:A\to\mathbb{R}$ is a differentiable mapping on A and $\mathcal{F}:A\to[0,\infty)$ is differentiable and symmetric to $\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)$. If $|Q'|$ is h-preinvex on A, we have*

$$\begin{aligned}&\Bigg|\frac{\Gamma(\alpha+1)}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{\alpha+1}}\Big[\mathrm{I}^{\alpha}_{\mathtt{w}_1^+}\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+\mathrm{I}^{\alpha}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)\\&-\mathrm{I}^{\alpha+1}_{\mathtt{w}_1^+}\mathcal{F}'(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+\mathrm{I}^{\alpha+1}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}'(\mathtt{w}_1)Q(\mathtt{w}_1)\Big]\\&-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}[Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+Q(\mathtt{w}_1)\mathcal{F}(\mathtt{w}_1)]\Bigg|\\&\leq\big[|Q'(\mathtt{w}_1)|+|Q'(\mathtt{w}_2)|\big]\cdot\int_0^1\wp^{\alpha}\mathcal{F}(\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1))[h(\wp)+h(1-\wp)]d\wp.\end{aligned}$$

Theorem 53 ([55]). *Let $A\subset\mathbb{R}$ be an open invex subset with respect to $\Phi: A\times A\to\mathbb{R}$ and $\mathtt{w}_1,\mathtt{w}_2\in A$ with $\Phi(\mathtt{w}_2,\mathtt{w}_1)>0$. Suppose that $Q:A\to\mathbb{R}$ is a differentiable mapping on A and $\mathcal{F}:A\to[0,\infty)$ is differentiable and symmetric to $\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)$. If $|Q'|$ is h-preinvex on A and $q\geq1$, we have*

$$\begin{aligned}&\Bigg|\frac{\Gamma(\alpha+1)}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{\alpha+1}}\Big[\mathrm{I}^{\alpha}_{\mathtt{w}_1^+}\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+\mathrm{I}^{\alpha}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)\\&-\mathrm{I}^{\alpha+1}_{\mathtt{w}_1^+}\mathcal{F}'(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+\mathrm{I}^{\alpha+1}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}'(\mathtt{w}_1)Q(\mathtt{w}_1)\Big]\\&-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}[Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+Q(\mathtt{w}_1)\mathcal{F}(\mathtt{w}_1)]\Bigg|\\&\leq\left(\frac{2}{\alpha+1}\right)^{1-\frac{1}{q}}\Big(\big[|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q\big]\\&\times\int_0^1\wp^{\alpha}[\mathcal{F}(\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1))]^q[h(\wp)+h(1-\wp)]d\wp\Big)^{\frac{1}{q}}.\end{aligned}$$

Theorem 54 ([55]). *Let $A\subset\mathbb{R}$ be an open invex subset with respect to $\Phi: A\times A\to\mathbb{R}$ and $\mathtt{w}_1,\mathtt{w}_2\in A$ with $\Phi(\mathtt{w}_2,\mathtt{w}_1)>0$. Suppose that $Q:A\to\mathbb{R}$ is a differentiable mapping on A and $\mathcal{F}:A\to[0,\infty)$ is differentiable and symmetric to $\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)$. If $|Q'|$ is h-preinvex on A and $\dfrac{1}{p}+\dfrac{1}{q}=1$, we have*

$$\begin{aligned}&\Bigg|\frac{\Gamma(\alpha+1)}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{\alpha+1}}\Big[\mathrm{I}^{\alpha}_{\mathtt{w}_1^+}\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+\mathrm{I}^{\alpha}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)\\&-\mathrm{I}^{\alpha+1}_{\mathtt{w}_1^+}\mathcal{F}'(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+\mathrm{I}^{\alpha}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}'(\mathtt{w}_1)Q(\mathtt{w}_1)\Big]\\&-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}[Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+Q(\mathtt{w}_1)\mathcal{F}(\mathtt{w}_1)]\Bigg|\\&\leq\frac{2}{(\alpha\rho+1)^{\frac{1}{p}}}\left(\big[|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q\big]\cdot\int_0^1[\mathcal{F}(\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1))]^q h(\wp)d\wp\right)^{\frac{1}{q}}.\end{aligned}$$

3. Fejér-Type Fractional Integral Inequalities Using $(k-p)$-Riemann–Liouville Fractional Integrals

In this section, we present Fejér-type fractional integral inequalities using $(k-p)$-Riemann–Liouville fractional integrals.

Definition 15 ([56])**.** *The left- and right-sided $(k-p)$-Riemann–Liouville fractional integral operator for a real-valued function Q is defined by*

$$({}^{p}_{k}J^{\alpha}_{\mathtt{w}_1+}Q)(x) = \frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_{\mathtt{w}_1}^{x} [x^{p+1} - \tau^{p+1}]^{\frac{\alpha}{k}-1} \tau^p Q(\tau) d\tau,$$

and

$$({}^{p}_{k}J^{\alpha}_{\mathtt{w}_2-}Q)(x) = \frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_{x}^{\mathtt{w}_2} [\tau^{p+1} - x^{p+1}]^{\frac{\alpha}{k}-1} \tau^p Q(\tau) d\tau.$$

Theorem 55 ([57])**.** *Let $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be a convex function with $\mathtt{w}_1 < \mathtt{w}_2$ and $g \in L[\mathtt{w}_1, \mathtt{w}_2]$. Then, $G(x) = g(x) + g(\mathtt{w}_1 + \mathtt{w}_2 - x)$ is also convex and $G \in L[\mathtt{w}_1, \mathtt{w}_2]$. If $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is non-negative and integrable, then the following inequalities holds true*

$$\begin{aligned}
& G\left(\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}\right) \left[{}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_1+}Q(\mathtt{w}_2) + {}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_2-}Q(\mathtt{w}_1)\right] \\
\leq \; & {}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_1+}(QG)(\mathtt{w}_2) + {}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_2-}(QG)(\mathtt{w}_1) \\
\leq \; & \frac{G(\mathtt{w}_1) + G(\mathtt{w}_2)}{2} \left[{}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_1+}Q(\mathtt{w}_2) + {}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_2-}Q(\mathtt{w}_1)\right].
\end{aligned}$$

Theorem 56 ([57])**.** *Let $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be a differentiable mapping on $(\mathtt{w}_1, \mathtt{w}_2)$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $g \in L[\mathtt{w}_1, \mathtt{w}_2]$. Then, $G(x) = g(x) + g(\mathtt{w}_1 + \mathtt{w}_2 - x)$ is also differentiable and $G' \in L[\mathtt{w}_1, \mathtt{w}_2]$. If g' is convex on $[\mathtt{w}_1, \mathtt{w}_2]$ and $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous, then the fractional inequality is given as:*

$$\begin{aligned}
& \frac{G(\mathtt{w}_1) + G(\mathtt{w}_2)}{2} \left[{}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_1+}Q(\mathtt{w}_2) + {}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_2-}Q(\mathtt{w}_1)\right] \\
& - \left[{}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_1+}(QG)(\mathtt{w}_2) + {}^{p-1}_{k}J^{\alpha}_{\mathtt{w}_2-}(QG)(\mathtt{w}_1)\right] \\
\leq \; & \frac{\|Q\|_\infty (\mathtt{w}_2 - \mathtt{w}_1)}{r^{\frac{\alpha}{k}} \alpha \Gamma_k(\alpha)} \left[|g'(\mathtt{w}_1)| + |g'(\mathtt{w}_2)|\right] \int_0^1 |M(t)| dt,
\end{aligned}$$

where $\alpha > 0$ and

$$M(t) = ([1-\tau)\mathtt{w}_1 + \tau \mathtt{w}_2]^p - \mathtt{w}_1^p)^{\frac{\alpha}{k}} - (\mathtt{w}_2^p - [(1-\tau)\mathtt{w}_1 + \tau \mathtt{w}_2]^p)^{\frac{\alpha}{k}}.$$

4. Fejér-Type Fractional Integral Inequalities via k-Riemann–Liouville Fractional Integral

Definition 16 ([58])**.** *Let $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$, $a \geq 0$, and $k > 0$. The k-Riemann–Liouville fractional integrals $I^{\alpha}_{\mathtt{w}_1+,k}Q$ and $I^{\alpha}_{\mathtt{w}_2-,k}Q$ of order $\alpha > 0$ for a real-valued function Π are defined by*

$$I^{\alpha}_{\mathtt{w}_1+,k}Q(t) = \frac{1}{k\Gamma_k(\alpha)} \int_{\mathtt{w}_1}^{t} (t-s)^{\frac{\alpha}{k}-1} Q(s) ds, \;\; t > \mathtt{w}_1,$$

and

$$I^{\alpha}_{\mathtt{w}_2-,k}Q(t) = \frac{1}{k\Gamma_k(\alpha)} \int_{t}^{\mathtt{w}_2} (s-t)^{\frac{\alpha}{k}-1} Q(s) ds, \;\; t < \mathtt{w}_2,$$

respectively, where Γ_k is the k-Gamma function $Gamma_k(t) = \int_0^\infty s^{t-1} e^{-\frac{s^k}{k}} ds$.

In the following theorems, we give Fejér-type inequalities for quasi-convex functions via k-Riemann–Liouville fractional integrals.

Theorem 57 ([59]). *Assume that $Q : \mathbb{I} \to \mathbb{R}$ be a differentiable mapping on $\mathbb{I}^\circ$ and $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous. If $|Q'|^q$ is quasi-convex on $[\mathtt{w}_1, \mathtt{w}_2]$, $q > 1$, then the fractional inequality is given as:*

$$\begin{aligned} &\Bigg| Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-,k} g(\mathtt{w}_1) + J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+,k} g(\mathtt{w}_2)\right] \\ &\quad - \left[J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)-,k}(Qg)(\mathtt{w}_1) + J^{\alpha}_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)+,k}(Qg)(\mathtt{w}_2)\right]\Bigg| \\ \leq\ & \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\frac{\alpha}{k}+1}\|g\|_\infty}{2^{\frac{\alpha}{k}}\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left[\max\{|Q'(\mathtt{w}_1)|^q, |Q'(\mathtt{w}_2)|^q\}\right]^{\frac{1}{q}}, \end{aligned}$$

where $\frac{\alpha}{k} > 0$.

Theorem 58 ([59]). *Assume that Q is as in Theorem 57. Then, an inequality in the frame of the k-fractional operator is given as:*

$$\begin{aligned} &\left| \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\alpha}_{\mathtt{w}_2-,k} g(\mathtt{w}_1) + J^{\alpha}_{\mathtt{w}_1+,k} g(\mathtt{w}_2)\right] - \left[J^{\alpha}_{\mathtt{w}_2-,k}(Qg)(\mathtt{w}_1) + J^{\alpha}_{\mathtt{w}_1+,k}(Qg)(\mathtt{w}_2)\right]\right| \\ \leq\ & \frac{2(\mathtt{w}_2-\mathtt{w}_1)^{\frac{\alpha}{k}+1}\|g\|_\infty}{\left(\frac{\alpha}{k}+1\right)\Gamma_k(\alpha+k)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\max\{|Q'(\mathtt{w}_1)|^q, |Q'(\mathtt{w}_2)|^q\}, \end{aligned}$$

where $\frac{\alpha}{k} > 0$.

5. Fejér-Type Fractional Integral Inequalities of Fractional Integrals with Respect to Another Function

Now, we recall the definition of fractional integrals of real-valued function concerning to another function.

Definition 17 ([25]). *Let $\psi : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be an increasing and positive function on $(\mathtt{w}_1, \mathtt{w}_2]$, having a continuous derivative ψ' on $(\mathtt{w}_1, \mathtt{w}_2)$. The left- and right-sided Riemann–Liouville fractional integrals of Q with respect to the function ψ on $[\mathtt{w}_1, \mathtt{w}_2]$ of order $\alpha > 0$ are defined, respectively, by*

$$J^{\alpha}_{\mathtt{w}_1+;\psi} Q(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathtt{w}_1}^{x}[\psi(x)-\psi(v)]^{\alpha-1}\psi'(v)Q(v)dv, \quad x > \mathtt{w}_1,$$

and

$$J^{\alpha}_{\mathtt{w}_2-;\psi} Q(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\mathtt{w}_2}[\psi(v)-\psi(x)]^{\alpha-1}\psi'(v)Q(v)dv, \quad x < \mathtt{w}_2,$$

provided that the integrals exists.

In the following, we state Fejér-type fractional integral inequalities of fractional integrals with respect to another function.

Theorem 59 ([60]). *Let $\alpha > 0$. Let $\psi : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be an increasing and positive monotone function on $(\mathtt{w}_1, \mathtt{w}_2]$, having a continuous derivative $\psi'(x)$ on $(\mathtt{w}_1, \mathtt{w}_2)$ and let $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be non-negative, integrable. If Q is a convex function on $[\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}]$, then the fractional inequalities are given as:*

$$Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^{\alpha}_{\mathtt{w}_1+;\psi} g(\mathtt{w}_2) + J^{\alpha}_{\mathtt{w}_2-;\psi} g(\mathtt{w}_1)\right] \leq \frac{1}{2}\left[J^{\alpha}_{\mathtt{w}_1+;\psi}(gG)(\mathtt{w}_2) + J^{\alpha}_{\mathtt{w}_2-;\psi}(gG)(\mathtt{w}_1)\right]$$

$$\leq \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^{\alpha}_{\mathtt{w}_1+;\psi}g(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-;\psi}g(\mathtt{w}_1)\Big],$$

where $G(x)=g(x)+g(\mathtt{w}_1+\mathtt{w}_2-x)$.

Theorem 60 ([61])**.** *Let* $\alpha>0$ *and* $\psi:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ *be an increasing and positive function on* $(\mathtt{w}_1,\mathtt{w}_2]$*, having a continuous derivative* ψ' *on* $(\mathtt{w}_1,\mathtt{w}_2)$*. Let* $Q:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ *be non-negative, integrable, and* g *be a convex function on* $[\mathtt{w}_1,\mathtt{w}_2]$*, the fractional inequalities are given as:*

$$\begin{aligned}
&G\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\Big)\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+;\psi}Q(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-;\psi}Q(\mathtt{w}_1)\Big]\\
\leq&\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+;\psi}(QG)(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-;\psi}(QG)(\mathtt{w}_1)\Big]\\
\leq&\frac{G(\mathtt{w}_1)+G(\mathtt{w}_2)}{2}\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+;\psi}Q(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-;\psi}Q(\mathtt{w}_1)\Big],
\end{aligned}$$

where $G(x)=g(x)+g(\mathtt{w}_1+\mathtt{w}_2-x), x\in[\mathtt{w}_1,\mathtt{w}_2]$.

Theorem 61 ([61])**.** *Let* $\psi:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ *be as in Theorem 60. Let* $g:I\to\mathbb{R}$ *be a differentiable function on* I° *and* $g'\in L[\mathtt{w}_1,\mathtt{w}_2]$ *and* $Q:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ *is continuous function. If* $|g'|$ *is s-convex in the second sense on* $[\mathtt{w}_1,\mathtt{w}_2]$ *for some fixed* $s\in(0,1]$*, then the fractional inequality is given as:*

$$\begin{aligned}
&G\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\Big)\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+;\psi}Q(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-;\psi}Q(\mathtt{w}_1)\Big]\\
&-\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+;\psi}(QG)(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-;\psi}(QG)(\mathtt{w}_1)\Big]\\
\leq&\frac{\|Q\|_{\infty}(|g'(\mathtt{w}_1|+|g(\mathtt{w}_2|)}{(\mathtt{w}_2-\mathtt{w}_1)^s\Gamma(\alpha+1)}\Big[\Big(\int_{\mathtt{w}_1}^{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}}[\psi(v)-\psi(\mathtt{w}_1]^{\alpha}\\
&+\int_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}}^{\mathtt{w}_2}[\psi(\mathtt{w}_2)-\psi(v)]^{\alpha}\Big)\Big((\mathtt{w}_2-v)^s+(v-\mathtt{w}_1)^s\Big)dv\Big],
\end{aligned}$$

where $G(x)=g(x)+g(\mathtt{w}_1+\mathtt{w}_2-x), x\in[\mathtt{w}_1,\mathtt{w}_2]$.

Theorem 62 ([61])**.** *Let* ψ,Q,g *be as in Theorem 60. If* $|g'|^q$ *is s-convex in the second sense on* $[\mathtt{w}_1,\mathtt{w}_2]$ *for some fixed* $s\in(0,1]$ *and* $q\geq1$*, then the fractional inequality is given as:*

$$\begin{aligned}
&g\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\Big)\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+;\psi}Q(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-;\psi}Q(\mathtt{w}_1)\Big]\\
&-\Big[J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})+;\psi}(Qg)(\mathtt{w}_2)+J^{\alpha}_{(\frac{\mathtt{w}_1+\mathtt{w}_2}{2})-;\psi}(Qg)(\mathtt{w}_1)\Big]\\
\leq&\frac{\|Q\|_{\infty}}{(\mathtt{w}_2-\mathtt{w}_1)^{\frac{s}{q}}\Gamma(\alpha+1)}\Big[\Big(\int_{\mathtt{w}_1}^{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}}[\psi(v)-\psi(\mathtt{w}_1)]^{\alpha}dv\Big)^{1-\frac{1}{q}}\\
&\times\Big(\int_{\mathtt{w}_1}^{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}}[\psi(v)-\psi(\mathtt{w}_1)]^{\alpha}\Big((\mathtt{w}_2-v)^s|g'(\mathtt{w}_1)|^q+(v-\mathtt{w}_1)^s|g'(\mathtt{w}_2)|^q\Big)dv\Big)^{\frac{1}{q}}\\
&+\Big(\int_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}}^{\mathtt{w}_2}[\psi(\mathtt{w}_2)-\psi(v)]^{\alpha}dv\Big)^{1-\frac{1}{q}}\\
&\times\Big(\int_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}}^{\mathtt{w}_2}[\psi(\mathtt{w}_2)-\psi(v)]^{\alpha}\Big((\mathtt{w}_2-v)^s|g'(\mathtt{w}_1)|^q+(v-\mathtt{w}_1)^s|g'(\mathtt{w}_2)|^q\Big)dv\Big)^{\frac{1}{q}}\Big].
\end{aligned}$$

Theorem 63 ([61])**.** *Let* ψ,Q,g *be as in Theorem 60. If* $|g'|^q$ *is s-convex in the second sense on* $[\mathtt{w}_1,\mathtt{w}_2]$ *for some fixed* $s\in(0,1]$ *and* $q>1$*, then the fractional inequality is given as:*

$$
\begin{aligned}
& g\Big(\frac{w_1+w_2}{2}\Big)\Big[J^{\alpha}_{\left(\frac{w_1+w_2}{2}\right)+;\psi}Q(w_2)+J^{\alpha}_{\left(\frac{w_1+w_2}{2}\right)-;\psi}Q(w_1)\Big] \\
& -\Big[J^{\alpha}_{\left(\frac{w_1+w_2}{2}\right)+;\psi}(Qg)(w_2)+J^{\alpha}_{\left(\frac{w_1+w_2}{2}\right)-;\psi}(Qg)(w_1)\Big] \\
\leq\ & \frac{\|Q\|_\infty (w_2-w_1)^{\frac{1}{q}}}{2^{\frac{s}{q}+\frac{1}{q}}(s+1)^{\frac{1}{q}}\Gamma(\alpha+1)}\Big[\Big(\int_{w_1}^{\frac{w_1+w_2}{2}}[\psi(v)-\psi(w_1)]^{p\alpha}dv\Big)^{\frac{1}{p}}\Big((2^{s+1}-1)|g'(w_1)|^q+|g'(w_2)|^q\Big)^{\frac{1}{q}} \\
& +\Big(\int_{\frac{w_1+w_2}{2}}^{w_2}[\psi(w_2)-\psi(v)]^{p\alpha}dv\Big)^{1-\frac{1}{p}}\Big(|g'(w_1)|^q+(2^{s+1}-1)|g'(w_2)|^q\Big)^{\frac{1}{q}}\Big].
\end{aligned}
$$

Fejér-Type Fractional Integral Inequalities for Weighted Fractional Integrals of a Function with Respect to Another Function

Fejér-type fractional integral inequalities are presented in this subsection concerning the weighted fractional integrals of a function with respect to another function.

Definition 18 ([62]). *Let $(w_1, w_2) \subset \mathbb{R}$ and ψ be an increasing positive monotonic function on the interval $(w_1, w_2]$ with a continuous derivative $\psi'(x)$ on the interval (w_1, w_2) with $\psi(0) = 0$, $0 \in [w_1, w_2]$. Then, the left-side and right-side of the weighted fractional integrals of a function Q with respect to another function $\psi(x)$ on $[w_1, w_2]$ are defined by*

$$
\big({}_{w_1+}J_w^{\alpha,\psi}Q\big)(x)=\frac{w^{-1}(x)}{\Gamma(\alpha)}\int_{w_1}^{x}\psi'(s)(\psi(x)-\psi(s))^{\alpha-1}Q(s)w(s)ds,
$$

and

$$
\big({}_{w_2-}J_w^{\alpha,\psi}Q\big)(x)=\frac{w^{-1}(x)}{\Gamma(\alpha)}\int_{x}^{w_2}\psi'(s)(\psi(s)-\psi(x))^{\alpha-1}Q(s)w(s)ds,
$$

where $\alpha > 0$ and $w^{-1}(x) = \frac{1}{w(x)}, w(x) \neq 0$.

Theorem 64 ([63]). *Let $Q : I \subseteq (0,\infty) \to \mathbb{R}$ be an integrable convex function with $0 < w_1 < w_2, w_1, w_2 \in I$ and $g : [w_1, w_2] \to \mathbb{R}$ be an integrable, positive, and weighted symmetric function with respect to $\dfrac{2w_1w_2}{w_1+w_2}$. If ψ an increasing and positive function on $[w_1, w_2)$ and ψ' is continuous on (w_1, w_2), then, we have for $\alpha > 0$:*

$$
\begin{aligned}
& Q\Big(\frac{2w_1w_2}{w_1+w_2}\Big)\Big[J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{w_2}\right)+}(g\circ h\circ\psi)\Big)\Big(\psi^{-1}\Big(\frac{1}{w_1}\Big)\Big)+\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{w_1}\right)-}(g\circ h\circ\psi)\Big)\Big(\psi^{-1}\Big(\frac{1}{w_2}\Big)\Big) \\
\leq\ & g\Big(\frac{1}{w_1}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{w_2}\right)+}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{w_1}\Big)\Big)\Big)+g\Big(\frac{1}{w_2}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{w_1}\right)-}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{w_2}\Big)\Big)\Big) \\
\leq\ & \frac{Q(w_1)+Q(w_2)}{2}\Big[J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{w_2}\right)+}(g\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{w_1}\Big)\Big)+J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{w_1}\right)-}(g\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{w_2}\Big)\Big)\Big],
\end{aligned}
$$

where $h(x) = \dfrac{1}{x}, x \in \Big[\dfrac{1}{w_2}, \dfrac{1}{w_1}\Big]$.

Theorem 65 ([63]). *Let $Q : I \subseteq (0,\infty) \to \mathbb{R}$ be an integrable convex function with $Q' \in L[w_1, w_2]$ for $0 < w_1 < w_2, w_1, w_2 \in I^\circ$ and $g : [w_1, w_2] \to \mathbb{R}$ be an integrable, positive, and weighted symmetric function with respect to $\dfrac{2w_1w_2}{w_1+w_2}$. If $|(Q\circ h)'$ is harmonically convex on $[w_1, w_2]$, ψ an increasing and positive function on $[w_1, w_2)$ and ψ is continuous on (w_1, w_2), then we have for $\alpha > 0$:*

$$\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)+}(g\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)\Big)+J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)-}(g\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)\Big)\Big]$$
$$-\Big[g\Big(\frac{1}{\mathtt{w}_1}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)+}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)\Big)\Big)+g\Big(\frac{1}{\mathtt{w}_2}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)-}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)\Big)\Big)\Big]$$
$$\leq \frac{\|g\circ h\circ\psi\|_\infty}{\Gamma(\alpha+1)}\Big[A_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2)\Big|\Big(Q'\circ h\Big)\Big(\frac{1}{\mathtt{w}_1}\Big)\Big|+B_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2)\Big|\Big(Q'\circ h\Big)\Big(\frac{1}{\mathtt{w}_2}\Big)\Big|\Big],$$

where $h(x)=\frac{1}{x}, x\in\Big[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\Big]$ *and*

$$A_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2) = \int_{\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)}^{\frac{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)+\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)}{2}}\frac{\mathtt{w}_1(\psi(t)-\mathtt{w}_2)}{\psi(t)(\mathtt{w}_2-\mathtt{w}_1)}$$
$$\times\Big[\Big(\frac{1}{\mathtt{w}_2}-\psi(t)\Big)^\alpha-\Big(\frac{1}{\mathtt{w}_1}-\psi\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)+\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)-\psi(t)\Big)\Big)^\alpha\Big]\psi'(t)dt$$
$$+\int^{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)}_{\frac{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)+\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)}{2}}\frac{\mathtt{w}_1(\psi(t)-\mathtt{w}_2)}{\psi(t)(\mathtt{w}_2-\mathtt{w}_1)}$$
$$\times\Big[\Big(\frac{1}{\mathtt{w}_1}-\psi\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)+\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)-t\Big)\Big)^\alpha-\Big(\frac{1}{\mathtt{w}_1}-\psi(t)\Big)^\alpha\Big]\psi'(t)dt$$
$$B_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2) = \int_{\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)}^{\frac{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)+\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)}{2}}\frac{\mathtt{w}_2(\mathtt{w}_1-\psi(t))}{\psi(t)(\mathtt{w}_2-\mathtt{w}_1)}$$
$$\times\Big[\Big(\frac{1}{\mathtt{w}_1}-\psi(t)\Big)^\alpha-\Big(\frac{1}{\mathtt{w}_1}-\psi\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)+\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)-\psi(t)\Big)\Big)^\alpha\Big]\psi'(t)dt$$
$$+\int^{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)}_{\frac{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)+\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)}{2}}\frac{\mathtt{w}_2(\mathtt{w}_2-\psi(t))}{\psi(t)(\mathtt{w}_2-\mathtt{w}_1)}$$
$$\times\Big[\Big(\frac{1}{\mathtt{w}_1}-\psi\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)+\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)-t\Big)\Big)^\alpha-\Big(\frac{1}{\mathtt{w}_1}-\psi(t)\Big)^\alpha\Big]\psi'(t)dt\Big].$$

Theorem 66 ([63])**.** *Let* $Q: I\subseteq(0,\infty)\to\mathbb{R}$ *be an integrable convex function with* $Q'\in L[\mathtt{w}_1,\mathtt{w}_2]$ *for* $0<\mathtt{w}_1<\mathtt{w}_2, \mathtt{w}_1,\mathtt{w}_2\in I^\circ$ *and* $g:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ *be an integrable, positive and weighted symmetric function with respect to* $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}$*. If* $|Q'|^q$ *is harmonically convex on* $[\mathtt{w}_1,\mathtt{w}_2]$ *for* $q\geq 1$*,* ψ *an increasing and positive function on* $[\mathtt{w}_1,\mathtt{w}_2)$ *and* ψ *is continuous on* $(\mathtt{w}_1,\mathtt{w}_2)$*, then, we have for* $\alpha>0$*:*

$$\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)+}(g\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)\Big)+J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)-}(g\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)\Big)\Big]$$
$$-\Big[g\Big(\frac{1}{\mathtt{w}_1}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)+}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)\Big)\Big)+g\Big(\frac{1}{\mathtt{w}_2}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)-}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)\Big)\Big)\Big]$$
$$\leq \frac{\|g\circ h\circ\psi\|_\infty}{\Gamma(\alpha+1)}\Big[C_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2)\Big]^{1-\frac{1}{q}}\Big[A_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2)\Big|\Big(Q'\circ h\Big)\Big(\frac{1}{\mathtt{w}_1}\Big)\Big|^q+B_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2)\Big|\Big(Q'\circ h\Big)\Big(\frac{1}{\mathtt{w}_2}\Big)\Big|^q\Big]^{\frac{1}{q}},$$

where $h(x)=\frac{1}{x}, x\in\Big[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\Big]$*,* $A_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2), B_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2)$ *are defined in Theorem 65 and*

$$C_\psi(\alpha,\mathtt{w}_1,\mathtt{w}_2) = \frac{2}{\alpha+1}\Big[\Big(\frac{1}{\mathtt{w}_1}-\frac{1}{\mathtt{w}_2}\Big)^{\alpha+1}-\Big(\frac{1}{\mathtt{w}_1}-\psi\Big(\frac{\psi^{-1}\left(\frac{1}{\mathtt{w}_1}\right)+\psi^{-1}\left(\frac{1}{\mathtt{w}_2}\right)}{2}\Big)\Big)^{\alpha+1}\Big]$$

$$
\begin{aligned}
&-\int_{\psi^{-1}\left(\frac{1}{w_2}\right)}^{\frac{\psi^{-1}\left(\frac{1}{w_1}\right)+\psi^{-1}\left(\frac{1}{w_2}\right)}{2}}\left(\frac{1}{w_1}-\psi\left(\psi^{-1}\left(\frac{1}{w_1}\right)+\psi^{-1}\left(\frac{1}{w_2}\right)-t\right)\right)^{\alpha}\psi'(t)dt\\
&+\int_{\frac{\psi^{-1}\left(\frac{1}{w_1}\right)+\psi^{-1}\left(\frac{1}{w_2}\right)}{2}}^{\psi^{-1}\left(\frac{1}{w_1}\right)}\left(\frac{1}{w_1}-\psi\left(\psi^{-1}\left(\frac{1}{w_1}\right)+\psi^{-1}\left(\frac{1}{w_2}\right)-t\right)\right)^{\alpha}\psi'(t)dt
\end{aligned}
$$

Theorem 67 ([64]). *Assume that $Q:[w_1,w_2]\subset[0,\infty)\to\mathbb{R}$ is a convex function with $0\le w_1<w_2$ and $w:[w_1,w_2]\to\mathbb{R}$ be an integrable, positive and weighted symmetric function with respect to $\frac{w_1+w_2}{2}$. If ψ is an increasing and positive function on $[w_1,w_2)$ and $\psi'(x)$ is continuous on (w_1,w_2), then, we have for $\alpha>0$:*

$$
\begin{aligned}
&Q\left(\frac{w_1+w_2}{2}\right)\left[\left({}_{\psi^{-1}(w_1)+}J^{\alpha,\psi}(w\circ\psi)\right)(\psi^{-1}(w_2))+\left({}_{\psi^{-1}(w_2)-}J^{\alpha,\psi}(w\circ\psi)\right)(\psi^{-1}(w_1))\right]\\
\le\;& w(w_2)\left({}_{\psi^{-1}(w_1)+}J^{\alpha,\psi}_{w\circ\psi}(Q\circ\psi)\right)(\psi^{-1}(w_2))+w(w_1)\left({}_{\psi^{-1}(w_2)-}J^{\alpha,\psi}_{w\circ\psi}(Q\circ\psi)\right)(\psi^{-1}(w_1))\\
\le\;& \frac{Q(w_1)+Q(w_2)}{2}\left[\left({}_{\psi^{-1}(w_1)+}J^{\alpha,\psi}(w\circ\psi)\right)(\psi^{-1}(w_2))+\left({}_{\psi^{-1}(w_2)-}J^{\alpha,\psi}(w\circ\psi)\right)(\psi^{-1}(w_1))\right].
\end{aligned}
$$

Theorem 68 ([65]). *Let $Q:[w_1,w_2]\to\mathbb{R}$ be an integrable harmonically convex function with $0<w_1<w_2, w_1,w_2\in I$ and $g:[w_1,w_2]\to\mathbb{R}$ is an integrable, positive and weighted symmetric function with respect to $\frac{2w_1w_2}{w_1+w_2}$. If ψ is an increasing and positive function on $[w_1,w_2)$ and ψ' is continuous on (w_1,w_2), then, we have for $\alpha>0$:*

$$
\begin{aligned}
&Q\left(\frac{2w_1w_2}{w_1+w_2}\right)\left[J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)+}(g\circ h\circ\psi)\right)\left(\psi^{-1}\left(\frac{1}{w_1}\right)\right)+\left(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)-}(g\circ h\circ\psi)\right)\left(\psi^{-1}\left(\frac{1}{w_2}\right)\right)\\
\le\;& g\left(\frac{1}{w_1}\right)\left(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)+}(Q\circ h\circ\psi)\left(\psi^{-1}\left(\frac{1}{w_1}\right)\right)\right)+g\left(\frac{1}{w_2}\right)\left(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)-}(Q\circ h\circ\psi)\left(\psi^{-1}\left(\frac{1}{w_2}\right)\right)\right)\\
\le\;& \frac{Q(w_1)+Q(w_2)}{2}\left[J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)+}(g\circ h\circ\psi)\left(\psi^{-1}\left(\frac{1}{w_1}\right)\right)+J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)-}(g\circ h\circ\psi)\left(\psi^{-1}\left(\frac{1}{w_2}\right)\right)\right],
\end{aligned}
$$

where $h(x)=\frac{1}{x}, x\in\left[\frac{1}{w_2},\frac{1}{w_1}\right]$.

Theorem 69 ([65]). *Let $Q:[w_1,w_2]\to\mathbb{R}$ with $0<w_1<w_2$ be continuous with a derivative $Q'\in L[w_1,w_2]$ such that $Q(\tau)=Q\left(\frac{1}{w_1}\right)+\int_{w_1}^{\tau}Q'(s)ds$ and $|Q'|$ is harmonically convex on $[w_1,w_2]$ and let $g:[w_1,w_2]\to\mathbb{R}$ be an integrable, positive non-negative and weighted symmetric function with respect to $\frac{2w_1w_2}{w_1+w_2}$. If ψ an increasing and positive function on $[w_1,w_2)$ and ψ' is continuous on (w_1,w_2), then, we have for $\alpha>0$:*

$$
\begin{aligned}
&\left|Q\left(\frac{2w_1w_2}{w_1+w_2}\right)\left[J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)+}(g\circ h\circ\psi)\right)\left(\psi^{-1}\left(\frac{1}{w_1}\right)\right)+\left(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)-}(g\circ h\circ\psi)\right)\left(\psi^{-1}\left(\frac{1}{w_2}\right)\right)\right.\\
&\left.-\left[g\left(\frac{1}{w_1}\right)\left(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)+}(Q\circ h\circ\psi)\left(\psi^{-1}\left(\frac{1}{w_1}\right)\right)\right)+g\left(\frac{1}{w_2}\right)\left(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{w_1+w_2}{2w_1w_2}\right)-}(Q\circ h\circ\psi)\left(\psi^{-1}\left(\frac{1}{w_2}\right)\right)\right)\right]\right|\\
\le\;& \frac{\|g\|_{\left[\frac{1}{w_2},\frac{w_1+w_2}{2w_1w_2}\right],\infty}}{(w_1-w_2)\alpha\Gamma(\alpha)}\left[\mathbb{A}_1(w_1,w_2)|(Q'\circ h)(w_1)|+\mathbb{A}_2(w_1,w_2)|(Q'\circ h)(w_2)|\right]\\
&+\frac{\|g\|_{\left[\frac{w_1+w_2}{2w_1w_2},\frac{1}{w_1}\right],\infty}}{(w_1-w_2)\alpha\Gamma(\alpha)}\left[\mathbb{A}_3(w_1,w_2)|(Q'\circ h)(w_1)|+\mathbb{A}_4(w_1,w_2)|(Q'\circ h)(w_2)|\right]\\
\le\;& \frac{\|g\|_{\left[\frac{1}{w_1},\frac{1}{w_2}\right],\infty}}{(w_1-w_2)\alpha\Gamma(\alpha)}\Big\{(\mathbb{A}_1(w_1,w_2)+\mathbb{A}_3(w_1,w_2))|(Q'\circ h)(w_1)|
\end{aligned}
$$

$$+(\mathbb{A}_2(\mathtt{w}_1,\mathtt{w}_2)+\mathbb{A}_4(\mathtt{w}_1,\mathtt{w}_2))|(Q'\circ h)(\mathtt{w}_2)|\Big]\Big\}$$

where

$$\begin{aligned}
\mathbb{A}_1(\mathtt{w}_1,\mathtt{w}_2) &= \frac{1}{\alpha}\Big[2^{-\alpha}(\mathtt{w}_1\mathtt{w}_2)^{1-\alpha}(\mathtt{w}_1+\mathtt{w}_2)^{\alpha}\,{}_2F_1\Big(-\alpha,-\alpha;1-\alpha;\frac{2\mathtt{w}_1}{\mathtt{w}_1+\mathtt{w}_2}\Big)\Big] \\
&\quad+\frac{2^{-\alpha-1}(\mathtt{w}_1-\mathtt{w}_2)\Big(\frac{1}{\mathtt{w}_1}-\frac{1}{\mathtt{w}_2}\Big)^{\alpha}}{\mathtt{w}_2(\alpha+1)}-Q\mathtt{w}_1\Big(\frac{1}{\mathtt{w}_2}\Big)^{\alpha-1}csc(Q\alpha), \\
\mathbb{A}_2(\mathtt{w}_1,\mathtt{w}_2) &= \frac{1}{\mathtt{w}_1(\alpha+1)}\Big[2^{-\alpha-1}(\mathtt{w}_1\mathtt{w}_2)^{-\alpha}\Big(\alpha\big(2^{\alpha+1}(\alpha+1)Q\mathtt{w}_2\mathtt{w}_1^{\alpha+2}csc(Q\alpha)+(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\big) \\
&\quad-2(\alpha+1)\mathtt{w}_1^2\mathtt{w}_2(\mathtt{w}_1+\mathtt{w}_2)^{\alpha}\,{}_2F_1\Big(-\alpha,-\alpha;1-\alpha;\frac{2\mathtt{w}_1}{\mathtt{w}_1+\mathtt{w}_2}\Big)\Big], \\
\mathbb{A}_3(\mathtt{w}_1,\mathtt{w}_2) &= \frac{1}{\mathtt{w}_2(\alpha+1)}\Big[2^{-\alpha-1}(\mathtt{w}_2-\mathtt{w}_1)\Big(\frac{1}{\mathtt{w}_1}-\frac{1}{\mathtt{w}_2}\Big)^{\alpha}\Big(\mathtt{w}_1\mathtt{w}_2\,{}_2F_1\Big(1,\alpha+1;\alpha+2;\frac{\mathtt{w}_2-\mathtt{w}_1}{2\mathtt{w}_2}\Big)-1\Big)\Big], \\
\mathbb{A}_4(\mathtt{w}_1,\mathtt{w}_2) &= \frac{1}{\mathtt{w}_1(\alpha+1)}\Big[2^{-\alpha-1}(\mathtt{w}_1-\mathtt{w}_2)\Big(\frac{1}{\mathtt{w}_1}-\frac{1}{\mathtt{w}_2}\Big)^{\alpha}\Big(\mathtt{w}_1^2\,{}_2F_1\Big(1,\alpha+1;\alpha+2;\frac{\mathtt{w}_2-\mathtt{w}_1}{2\mathtt{w}_2}\Big)-1\Big)\Big],
\end{aligned}$$

with $h(x)=\frac{1}{x}, x\in\Big[\frac{1}{\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\Big]$.

Theorem 70 ([65]). *Assume that all the conditions of Theorem 69 hold and* $|Q'|^q$ *is harmonically convex on* $[\mathtt{w}_1,\mathtt{w}_2]$ *with* $q\geq 1$*. Then, we have:*

$$\begin{aligned}
&\Bigg|Q\Big(\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}\Big)\Big[J^{\alpha,\psi}_{\psi^{-1}\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right)+}(g\circ h\circ\psi)\Big)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)\Big)+\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right)-}(g\circ h\circ\psi)\Big)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)\Big) \\
&-\Big[g\Big(\frac{1}{\mathtt{w}_1}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right)+}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)\Big)\Big)+g\Big(\frac{1}{\mathtt{w}_2}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right)-}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)\Big)\Big)\Big]\Bigg| \\
\leq\;&\Big(\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}}{(2\mathtt{w}_1\mathtt{w}_1)^{\alpha+1}\alpha(\alpha+1)}\Big)^{1-\frac{1}{q}}\Bigg[\frac{\|g\|_{\left[\frac{1}{\mathtt{w}_2},\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right],\infty}}{(\mathtt{w}_1-\mathtt{w}_2)\alpha\Gamma(\alpha)}\Big(\mathbb{A}_1(\mathtt{w}_1,\mathtt{w}_2)|(Q'\circ h)(\mathtt{w}_1)|^q+\mathbb{A}_2(\mathtt{w}_1,\mathtt{w}_2)|(Q'\circ h)(\mathtt{w}_2)|^q\Big)^{\frac{1}{q}} \\
&+\frac{\|g\|_{\left[\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\right],\infty}}{(\mathtt{w}_1-\mathtt{w}_2)\alpha\Gamma(\alpha)}\Big(\mathbb{A}_3(\mathtt{w}_1,\mathtt{w}_2)|(Q'\circ h)(\mathtt{w}_1)|^q+\mathbb{A}_4(\mathtt{w}_1,\mathtt{w}_2)|(Q'\circ h)(\mathtt{w}_2)|^q\Big)^{\frac{1}{q}}\Bigg] \\
\leq\;&\Big(\frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}}{(2\mathtt{w}_1\mathtt{w}_1)^{\alpha+1}\alpha(\alpha+1)}\Big)^{1-\frac{1}{q}}\Big(\frac{\|g\|_{\left[\frac{1}{\mathtt{w}_1},\frac{1}{\mathtt{w}_2}\right],\infty}}{(\mathtt{w}_1-\mathtt{w}_2)\alpha\Gamma(\alpha)}\Big)\Big\{\Big((\mathbb{A}_1(\mathtt{w}_1,\mathtt{w}_2)+\mathbb{A}_3(\mathtt{w}_1,\mathtt{w}_2))|(Q'\circ h)(\mathtt{w}_1)|^q\Big)^{\frac{1}{q}} \\
&+\Big((\mathbb{A}_2(\mathtt{w}_1,\mathtt{w}_2)+\mathbb{A}_4(\mathtt{w}_1,\mathtt{w}_2))|(Q'\circ h)(\mathtt{w}_2)|^q\Big)^{\frac{1}{q}}\Big\},
\end{aligned}$$

where $\mathbb{A}_1(\mathtt{w}_1,\mathtt{w}_2),\mathbb{A}_2(\mathtt{w}_1,\mathtt{w}_2),\mathbb{A}_3(\mathtt{w}_1,\mathtt{w}_2)$*, and* $\mathbb{A}_4(\mathtt{w}_1,\mathtt{w}_2)$ *are defined in Theorem 69*

Theorem 71 ([65]). *Assume that all the conditions of Theorem 69 hold and* $|Q'|^q$ *are harmonically convex on* $[\mathtt{w}_1,\mathtt{w}_2]$ *with* $q>1$*. Then, we have:*

$$\begin{aligned}
&\Bigg|Q\Big(\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}\Big)\Big[J^{\alpha,\psi}_{\psi^{-1}\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right)+}(g\circ h\circ\psi)\Big)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)\Big)+\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right)-}(g\circ h\circ\psi)\Big)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)\Big) \\
&-\Big[g\Big(\frac{1}{\mathtt{w}_1}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right)+}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_1}\Big)\Big)\Big)+g\Big(\frac{1}{\mathtt{w}_2}\Big)\Big(J^{\alpha,\psi}_{\psi^{-1}\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right)-}(Q\circ h\circ\psi)\Big(\psi^{-1}\Big(\frac{1}{\mathtt{w}_2}\Big)\Big)\Big)\Big]\Bigg| \\
\leq\;&\Big(\frac{(\mathtt{w}_2-\mathtt{w}_1)^{p\alpha+1}}{(2\mathtt{w}_1\mathtt{w}_1)^{p\alpha+1}p\alpha(p\alpha+1)}\Big)^{\frac{1}{p}}\Bigg[\frac{\|g\|_{\left[\frac{1}{\mathtt{w}_2},\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}\right],\infty}}{(\mathtt{w}_1-\mathtt{w}_2)\Gamma(\alpha)}\Big(\mathbb{B}_1(\mathtt{w}_1,\mathtt{w}_2;\psi(k))|(Q'\circ h)(\mathtt{w}_1)|^q
\end{aligned}$$

$$+\mathbb{B}_2(\mathtt{w}_1,\mathtt{w}_2;\psi(k))|(Q'\circ h)(\mathtt{w}_2)|^q\Big)^{\frac{1}{q}}$$
$$+\frac{\|g\|_{\left[\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2},\frac{1}{\mathtt{w}_1}\right],\infty}}{(\mathtt{w}_1-\mathtt{w}_2)\Gamma(\alpha)}\Big(\mathbb{B}_3(\mathtt{w}_1,\mathtt{w}_2;\psi(k))|(Q'\circ h)(\mathtt{w}_1)|^q+\mathbb{B}_4(\mathtt{w}_1,\mathtt{w}_2;\psi(k))|(Q'\circ h)(\mathtt{w}_2)|^q\Big)^{\frac{1}{q}}\Bigg]$$
$$\leq \Big(\frac{(\mathtt{w}_2-\mathtt{w}_1)^{p\alpha+1}}{(2\mathtt{w}_1\mathtt{w}_1)^{p\alpha+1}p\alpha(p\alpha+1)}\Big)^{\frac{1}{p}}\Big(\frac{\|g\|_{\left[\frac{1}{\mathtt{w}_1},\frac{1}{\mathtt{w}_2}\right],\infty}}{(\mathtt{w}_1-\mathtt{w}_2)\Gamma(\alpha)}\Big)\Big\{\Big((\mathbb{B}_1(\mathtt{w}_1,\mathtt{w}_2;\psi(k))+\mathbb{B}_3(\mathtt{w}_1,\mathtt{w}_2;\psi(k)))|(Q'\circ h)(\mathtt{w}_1)|^q\Big)^{\frac{1}{q}}$$
$$+\Big((\mathbb{B}_2(\mathtt{w}_1,\mathtt{w}_2;\psi(k))+\mathbb{B}_4(\mathtt{w}_1,\mathtt{w}_2;\psi(k)))|(Q'\circ h)(\mathtt{w}_2)|^q\Big)^{\frac{1}{q}}\Big\},$$

where

$$\begin{aligned}
\mathbb{B}_1(\mathtt{w}_1,\mathtt{w}_2;\psi(k)) &= \frac{1}{2}\Big(\frac{\mathtt{w}_1}{\mathtt{w}_2}-1\Big)+\mathtt{w}_1\mathtt{w}_2\Big(\ln\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{\mathtt{w}_1\mathtt{w}_1}\Big)-\ln\Big(\frac{2}{\mathtt{w}_2}\Big)\Big),\\
\mathbb{B}_2(\mathtt{w}_1,\mathtt{w}_2;\psi(k)) &= \frac{1}{2}\Big(\frac{\mathtt{w}_2}{\mathtt{w}_1}-2\mathtt{w}_1\mathtt{w}_2\Big(\ln\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{\mathtt{w}_1\mathtt{w}_1}\Big)-\ln\Big(\frac{2}{\mathtt{w}_2}\Big)\Big)-1\Big),\\
\mathbb{B}_3(\mathtt{w}_1,\mathtt{w}_2;\psi(k)) &= \frac{1}{2}\Big(\frac{\mathtt{w}_1}{\mathtt{w}_2}-1\Big)+\mathtt{w}_1\mathtt{w}_2\Big(-\ln\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{\mathtt{w}_1\mathtt{w}_1}\Big)+\ln\Big(\frac{1}{\mathtt{w}_1}\Big)+\ln 2\Big),\\
\mathbb{B}_4(\mathtt{w}_1,\mathtt{w}_2;\psi(k)) &= \frac{1}{2}\Big(\frac{\mathtt{w}_2}{\mathtt{w}_1}-2\mathtt{w}_1\mathtt{w}_2\Big(-\ln\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{\mathtt{w}_1\mathtt{w}_1}\Big)-\ln\mathtt{w}_1+\ln 2\Big)-1\Big).
\end{aligned}$$

6. Fejér-Type Fractional Integral Inequalities for Exponential Kernel

In the following, we give the definition of a fractional integral with an exponential kernel and present Fejér-type fractional integral inequalities for this new fractional integral.

Definition 19 ([66]). *Let $Q\in L(\mathtt{w}_1,\mathtt{w}_2)$. The left and right fractional integrals of order $\alpha\in(0,1)$ are, respectively, defined by*

$$I^{\alpha}_{\mathtt{w}_1}Q(x)=\frac{1}{\alpha}\int_{\mathtt{w}_1}^{x}\exp\Big(-\frac{1-\alpha}{\alpha}(x-s)\Big)Q(s)ds,\ \ x>\mathtt{w}_1,$$

and

$$I^{\alpha}_{\mathtt{w}_2}Q(x)=\frac{1}{\alpha}\int_{x}^{\mathtt{w}_2}\exp\Big(-\frac{1-\alpha}{\alpha}(s-x)\Big)Q(s)ds,\ \ x<\mathtt{w}_2.$$

Theorem 72 ([66]). *Let $Q:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ be a convex and integrable function with $\mathtt{w}_1<\mathtt{w}_2$. If $g:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$, then the fractional inequality is given as:*

$$\begin{aligned}
Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\Big)\Big[I^{\alpha}_{\mathtt{w}_1}g(\mathtt{w}_2)+I^{\alpha}_{\mathtt{w}_2}g(\mathtt{w}_1)\Big] &\leq I^{\alpha}_{\mathtt{w}_1}(gQ)(\mathtt{w}_2)+I^{\alpha}_{\mathtt{w}_2}(fQ)(\mathtt{w}_1)\\
&\leq \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[I^{\alpha}_{\mathtt{w}_1}g(\mathtt{w}_2)+I^{\alpha}_{\mathtt{w}_2}g(\mathtt{w}_1)\Big].
\end{aligned}$$

Theorem 73 ([67]). *Let $Q:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ be convex function with $\mathtt{w}_1<\mathtt{w}_2$ and $Q\in L[\mathtt{w}_1,\mathtt{w}_2]$. If $g:[\mathtt{w}_1,\mathtt{w}_2]$ is non-negative, integrable and symmetric to $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$, then A_g is convex and monotonically increasing on $[0,1]$, and the fractional inequality is given as:*

$$\begin{aligned}
Q\Big(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\Big)\Big[J^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big] &= A_g(0)\leq A_g(t)\leq A_g(1)\\
&= \Big[J^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\Big],
\end{aligned}$$

with $\alpha > 0$ *and* A_g *defined by*

$$A_g(t) = \frac{1}{\Gamma(\alpha)} \int_{\mathtt{w}_1}^{\mathtt{w}_2} Q\Big(tx + (1-t)\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}\Big)\Big[(\mathtt{w}_2 - x)^{\alpha-1} + (x - \mathtt{w}_1)^{\alpha-1}\Big] g(x)dx.$$

Theorem 74 ([67])**.** *Let* $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *be convex function with* $\mathtt{w}_1 < \mathtt{w}_2$ *and* $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$*. If* $g : [\mathtt{w}_1, \mathtt{w}_2]$ *is non-negative, integrable and symmetric to* $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$*, then* B_g *is convex and monotonically increasing on* $[0, 1]$*, and the fractional inequality is given as:*

$$\begin{aligned} \Big[J^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2) + J^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\Big] &= B_g(0) \le B_g(t) \le B_g(1) \\ &= \frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2}\Big[J^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2) + J^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big], \end{aligned}$$

with $\alpha > 0$ *and* B_g *defined by*

$$\begin{aligned} B_g(t) &= \frac{1}{2\Gamma(\alpha)} \int_{\mathtt{w}_1}^{\mathtt{w}_2} Q\Big(\frac{1+t}{2}\mathtt{w}_1 + \frac{1-t}{2}x\Big)\Big(\Big(\frac{2\mathtt{w}_2 - \mathtt{w}_1 - x}{2}\Big)^{\alpha-1} + \Big(\frac{x - \mathtt{w}_1}{2}\Big)^{\alpha-1}\Big) g\Big(\frac{\mathtt{w}_1 + x}{2}\Big)dx \\ &\quad + \frac{1}{2\Gamma(\alpha)} \int_{\mathtt{w}_1}^{\mathtt{w}_2} Q\Big(\frac{1+t}{2}\mathtt{w}_2 + \frac{1-t}{2}x\Big)\Big(\Big(\frac{\mathtt{w}_2 - x}{2}\Big)^{\alpha-1} + \Big(\frac{x + \mathtt{w}_2 - 2\mathtt{w}_1}{2}\Big)^{\alpha-1}\Big) g\Big(\frac{\mathtt{w}_2 + x}{2}\Big)dx. \end{aligned}$$

Theorem 75 ([67])**.** *Let* $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *be a positive convex function with* $\mathtt{w}_1 < \mathtt{w}_2$ *and* $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$*. If* $g : [\mathtt{w}_1, \mathtt{w}_2]$ *is non-negative, integrable and symmetric to* $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$*, then* C_g *is convex and monotonically increasing on* $[0, 1]$*, and the fractional inequality is given as:*

$$\begin{aligned} Q\Big(\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}\Big)\Big[I^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2) + I^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big] &= C_g(0) \le C_g(t) \le C_g(1) \\ &= \Big[I^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2) + I^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\Big], \end{aligned}$$

with $\alpha > 0$ *and*

$$C_g(t) = \frac{1}{\alpha} \int_{\mathtt{w}_1}^{\mathtt{w}_2} Q\Big(tx + (1-t)\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}\Big)\Big[\exp\Big(-\frac{1-\alpha}{\alpha}(\mathtt{w}_2 - x)\Big) + \exp\Big(-\frac{1-\alpha}{\alpha}(x - \mathtt{w}_1)\Big)\Big] g(x)dx.$$

Theorem 76 ([67])**.** *Let* $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *be a positive convex function with* $\mathtt{w}_1 < \mathtt{w}_2$ *and* $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$*. If* $g : [\mathtt{w}_1, \mathtt{w}_2]$ *is non-negative, integrable and symmetric to* $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$*, then* D_g *is convex and monotonically increasing on* $[0, 1]$*, and the fractional inequality is given as:*

$$\begin{aligned} \Big[I^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2) + I^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\Big] &= D_g(0) \le D_g(t) \le D_g(1) \\ &= \frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2}\Big[I^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2) + I^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big], \end{aligned}$$

with $\alpha > 0$ *and* D_g *defined by*

$$\begin{aligned} D_g(t) &= \frac{1}{2\alpha} \int_{\mathtt{w}_1}^{\mathtt{w}_2} Q\Big(\frac{1+t}{2}\mathtt{w}_1 + \frac{1-t}{2}x\Big) g\Big(\frac{\mathtt{w}_1 + x}{2}\Big) \\ &\quad \times \Big[\exp\Big(-\frac{1-\alpha}{\alpha}\Big(\frac{2\mathtt{w}_2 - \mathtt{w}_1 - x}{2}\Big)\Big) + \exp\Big(-\frac{1-\alpha}{\alpha}\Big(\frac{x - \mathtt{w}_1}{2}\Big)\Big)\Big]dx \\ &\quad + \frac{1}{2\alpha} \int_{\mathtt{w}_1}^{\mathtt{w}_2} Q\Big(\frac{1+t}{2}\mathtt{w}_2 + \frac{1-t}{2}x\Big) g\Big(\frac{\mathtt{w}_2 + x}{2}\Big) \\ &\quad \times \Big[\exp\Big(-\frac{1-\alpha}{\alpha}\Big(\frac{\mathtt{w}_2 - x}{2}\Big)\Big) + \exp\Big(-\frac{1-\alpha}{\alpha}\Big(\frac{x + \mathtt{w}_2 - 2\mathtt{w}_1}{2}\Big)\Big)\Big]dx. \end{aligned}$$

Now, we prove both the first and second kind Fejér-type inequalities in a different approach.

Theorem 77 ([68]). *Let $g : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $\mathtt{w}_1 < \mathtt{w}_2$ be a convex function. If $Q : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a convex symmetric and integrable function with respect to $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$, then for $\alpha > 0$, then the fractional inequality is given as:*

$$g\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-}Q(\mathtt{w}_1)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+}Q(\mathtt{w}_2)\right]\leq\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-}(gQ)(\mathtt{w}_1)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+}(gQ)(\mathtt{w}_2)\right].$$

Theorem 78 ([68]). *Let $g : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $\mathtt{w}_1 < \mathtt{w}_2$ be a convex function. If $Q : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a convex symmetric and integrable function with respect to $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$, then for $\alpha > 0$, then the fractional inequality is given as:*

$$\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-}(gQ)(\mathtt{w}_1)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+}(gQ)(\mathtt{w}_2)\right]\leq\frac{g(\mathtt{w}_1)+g(\mathtt{w}_2)}{2}\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-}Q(\mathtt{w}_1)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+}Q(\mathtt{w}_2)\right].$$

Theorem 79 ([68]). *Let $g : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $\mathtt{w}_1 < \mathtt{w}_2$ be a convex function. Then for $\alpha > 0$, the fractional inequality is given as:*

$$g\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\leq\frac{1-\alpha}{2\left(1-\exp\left(-\frac{1-\alpha}{\alpha}(\mathtt{w}_2-\mathtt{w}_1)\right)\right)}\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-}Q(\mathtt{w}_1)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+}Q(\mathtt{w}_2)\right]\leq\frac{g(\mathtt{w}_1)+g(\mathtt{w}_2)}{2}.$$

In the next we present Hadamard-Fejér type inequalities of both first and second kind for harmonically convex functions. Let us begin with the Hadamard-Fejér type inequality of the first kind.

Theorem 80 ([68]). *Let $g : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $\mathtt{w}_1 < \mathtt{w}_2$ be a harmonically convex function. If $Q : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a harmonically symmetric and integrable function with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1 + \mathtt{w}_2}$, then for $\alpha > 0$, the fractional inequality is given as:*

$$\begin{aligned}&g\left(\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}\right)\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}-}Q\circ h\left(\frac{1}{\mathtt{w}_2}\right)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}+}Q\circ h\left(\frac{1}{\mathtt{w}_1}\right)\right]\\ \leq\ &\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}-}gQ\circ h\left(\frac{1}{\mathtt{w}_2}\right)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}+}gQ\circ h\left(\frac{1}{\mathtt{w}_1}\right)\right],\end{aligned}$$

where $h(x) = \frac{1}{x}, x \in \left[\frac{1}{\mathtt{w}_2}, \frac{1}{\mathtt{w}_1}\right]$.

Theorem 81 ([68]). *Let $g : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $\mathtt{w}_1 < \mathtt{w}_2$ be a harmonically convex function. Then, for $\alpha > 0$, we have:*

$$\begin{aligned}&\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}-}gQ\circ h\left(\frac{1}{\mathtt{w}_2}\right)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}+}gQ\circ h\left(\frac{1}{\mathtt{w}_1}\right)\right]\\ \leq\ &\frac{g(\mathtt{w}_1)+g(\mathtt{w}_2)}{2}\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}-}Q\circ h\left(\frac{1}{\mathtt{w}_2}\right)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}+}Q\circ h\left(\frac{1}{\mathtt{w}_1}\right)\right].\end{aligned}$$

Theorem 82 ([68]). *Let $g : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $\mathtt{w}_1 < \mathtt{w}_2$ be a harmonically convex function. If $Q : [\mathtt{w}_1, \mathtt{w}_2] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is a harmonically symmetric and integrable function with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1 + \mathtt{w}_2}$, then for $\alpha > 0$, the fractional inequality is given as:*

$$g\left(\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}\right)\ \leq\ \frac{1-\alpha}{2\left(1-\exp\left(\frac{1-\alpha}{\alpha}\frac{\mathtt{w}_2-\mathtt{w}_1}{\mathtt{w}_1\mathtt{w}_2}\right)\right)}\left[I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}-}g\circ h\left(\frac{1}{\mathtt{w}_2}\right)+I^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2\mathtt{w}_1\mathtt{w}_2}+}g\circ h\left(\frac{1}{\mathtt{w}_1}\right)\right]$$

$$\leq \frac{g(\mathtt{w}_1)+g(\mathtt{w}_2)}{2}.$$

The Fejeér–Hadamard–Mercer-type inequality for harmonically convex function is presented in the next result.

Theorem 83 ([69]). *Let $Q : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ be a harmonically convex function for $\mathtt{w}_1, \mathtt{w}_2 \in \mathbb{I}$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$ and $\omega : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is non-negative, integrable, and harmonically symmetric with respect to $\frac{2\mathtt{w}_1\mathtt{w}_2}{\mathtt{w}_1+\mathtt{w}_2}$, then:*

$$\begin{aligned}
& Q\left(\frac{1}{(1/\mathtt{w}_1)+(1/\mathtt{w}_2)-\frac{x+y}{2xy}}\right)\left[I^{\alpha}_{(1/\mathtt{w}_1)+(1/\mathtt{w}_2)-(1/x)}(\omega\circ h)\left(\frac{1}{\mathtt{w}_1}+\frac{1}{\mathtt{w}_2}-\frac{1}{y}\right)\right.\\
& \left.+I^{\alpha}_{(1/\mathtt{w}_1)+(1/\mathtt{w}_2)-(1/y)}(\omega\circ h)\left(\frac{1}{\mathtt{w}_1}+\frac{1}{\mathtt{w}_2}-\frac{1}{x}\right)\right]\\
\leq & \left[I^{\alpha}_{(1/\mathtt{w}_1)+(1/\mathtt{w}_2)-(1/x)}(Q\omega\circ h)\left(\frac{1}{\mathtt{w}_1}+\frac{1}{\mathtt{w}_2}-\frac{1}{y}\right)\right.\\
& \left.+I^{\alpha}_{(1/\mathtt{w}_1)+(1/\mathtt{w}_2)-(1/y)}(Q\omega\circ h)\left(\frac{1}{\mathtt{w}_1}+\frac{1}{\mathtt{w}_2}-\frac{1}{x}\right)\right]\\
\leq & \frac{1}{2}\left[Q\left(\frac{1}{(1/\mathtt{w}_1)+(1/\mathtt{w}_2)-(1/x)}\right)+Q\left(\frac{1}{(1/\mathtt{w}_1)+(1/\mathtt{w}_2)-(1/y)}\right)\right]\\
& \times\left[I^{\alpha}_{(1/\mathtt{w}_1)+(1/\mathtt{w}_2)-(1/x)}(\omega\circ h)\left(\frac{1}{\mathtt{w}_1}+\frac{1}{\mathtt{w}_2}-\frac{1}{y}\right)\right.\\
& \left.+I^{\alpha}_{(1/\mathtt{w}_1)+(1/\mathtt{w}_2)-(1/y)}(\omega\circ h)\left(\frac{1}{\mathtt{w}_1}+\frac{1}{\mathtt{w}_2}-\frac{1}{x}\right)\right]
\end{aligned}$$

for all $x, y \in [\mathtt{w}_1, \mathtt{w}_2]$, $\alpha > 0$ and $h(x) = \frac{1}{x}$, $x \in \left[\frac{1}{\mathtt{w}_2}, \frac{1}{\mathtt{w}_1}\right]$.

7. Fejér-Type Fractional Integral Inequalities via Hadamard Fractional Integral

In this section, Fejér-type fractional integral inequalities concerning the Hadamard fractional integral are presented.

Definition 20 ([25]). *The left-sided and right-sided Hadamard fractional integrals of order $\alpha \in \mathbb{R}^+$ of function Q are defined by*

$$\left({}_{H}J^{\alpha}_{\mathtt{w}_1+}Q\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{\mathtt{w}_1}^{x}\left(\ln\frac{x}{t}\right)^{\alpha-1}Q(t)\frac{dt}{t},\ \ 0<\mathtt{w}_1<x\leq \mathtt{w}_2,$$

and

$$\left({}_{H}J^{\alpha}_{\mathtt{w}_2-}Q\right)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\mathtt{w}_2}\left(\ln\frac{t}{x}\right)^{\alpha-1}Q(t)\frac{dt}{t},\ \ 0<\mathtt{w}_1\leq x< \mathtt{w}_2.$$

Definition 21 ([70]). *A function $Q : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ is said to be geometric–arithmetically convex (GA-convex) on $\mathbb{I}$, if*

$$Q\left(x^t y^{1-t}\right) \leq tQ(x) + (1-t)Q(y),$$

for any $x, y \in \mathbb{I}$ and $t \in [0,1]$.

Theorem 84 ([71]). *Assume that $Q : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ is a GA-convex function such that $Q \in L_1[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1, \mathtt{w}_2 \in \mathbb{I}$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $\alpha > 0$. If $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is non-negative, integrable and geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$, then the fractional integral inequalities are given as:*

$$Q(\sqrt{\mathtt{w}_1\mathtt{w}_2})\left[{}_{H}J^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2)+{}_{H}J^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\right] \leq \left[{}_{H}J^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+{}_{H}J^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]$$

$$\leq \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[{}_HJ^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2)+{}_HJ^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big].$$

Theorem 85 ([71]). *Assume that Q and g are defined as in Theorem 84. If $|Q'|$ is GA-convex on $[\mathtt{w}_1,\mathtt{w}_2]$, then fractional integral inequality is given as:*

$$\left|\left(\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\right)\Big[{}_HJ^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2)+{}_HJ^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big]-\Big[{}_HJ^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+{}_HJ^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\Big]\right|$$
$$\leq \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{\mathtt{w}_2}{\mathtt{w}_1}\right)}{\Gamma(\alpha+1)}\Big[|Q'(\mathtt{w}_1)|\int_0^{1/2}[(1-u)^\alpha-u^\alpha][(1-u)a^{1-u}\mathtt{w}_2^u+ua^u\mathtt{w}_2^{1-u}]du$$
$$+|Q'(\mathtt{w}_2)|\int_0^{1/2}[(1-u)^\alpha-u^\alpha][ua^{1-u}\mathtt{w}_2^u+(1-u)a^u\mathtt{w}_2^{1-u}]du.$$

Theorem 86 ([71]). *Assume that Q and g are defined as in the Theorem 84. If $|Q'|^q, q>1$ is GA-convex on $[\mathtt{w}_1,\mathtt{w}_2]$, then fractional integral inequality is given as:*

$$\left|\left(\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\right)\Big[{}_HJ^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2)+{}_HJ^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big]-\Big[{}_HJ^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+{}_HJ^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\Big]\right|$$
$$\leq \frac{\|g\|_\infty \ln^{\alpha+1-\frac{2}{q}}\left(\frac{\mathtt{w}_2}{\mathtt{w}_1}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}$$
$$\times\left[(\mathtt{w}_2^q-q\mathtt{w}_1^q\ln\frac{\mathtt{w}_2}{\mathtt{w}_1}-\mathtt{w}_1^q)|Q'(\mathtt{w}_1)|^q+\left(\mathtt{w}_1^q+q\mathtt{w}_2^q\ln\frac{\mathtt{w}_2}{\mathtt{w}_1}-\mathtt{w}_2^q\right)|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}},$$

if $\alpha>0$ and $\frac{1}{p}+\frac{1}{q}=1$, and

$$\left|\left(\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\right)\Big[{}_HJ^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2)+{}_HJ^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big]-\Big[{}_HJ^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+{}_HJ^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\Big]\right|$$
$$\leq \frac{\|g\|_\infty \ln^{\alpha+1-\frac{2}{q}}\left(\frac{\mathtt{w}_2}{\mathtt{w}_1}\right)}{q^{\frac{2}{q}}\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\right]^{\frac{1}{p}}$$
$$\times\left[(\mathtt{w}_2^q-q\mathtt{w}_1^q\ln\frac{\mathtt{w}_2}{\mathtt{w}_1}-\mathtt{w}_1^q)|Q'(\mathtt{w}_1)|^q+\left(\mathtt{w}_1^q+q\mathtt{w}_2^q\ln\frac{\mathtt{w}_2}{\mathtt{w}_1}-\mathtt{w}_2^q\right)|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}},$$

if $0<\alpha\leq 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Definition 22 ([72]). *The function $Q:\mathbb{I}\subseteq\mathbb{R}^+\to\mathbb{R}^+$ is said to be GA-s-convex (geometric-arithmetically convex) on $\mathbb{I}$, if, for every $x,y\in\mathbb{I}$ and $t\in[0,1]$, we have*

$$Q(x^ty^{1-t})\leq t^sQ(x)+(1-t)^sQ(y).$$

We will use the notations

$$L(t)=\mathtt{w}_1^tG^{1-t},\;\; U(t)=\mathtt{w}_2^tG^{1-t},\;\; G=G(\mathtt{w}_1,\mathtt{w}_2)=\sqrt{\mathtt{w}_1\mathtt{w}_2}.$$

Theorem 87 ([73]). *Let $Q:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° and $|Q'|$ is GA-convex on $[\mathtt{w}_1,\mathtt{w}_2]$ with $\mathtt{w}_1<\mathtt{w}_2$. If $g:[\mathtt{w}_1,\mathtt{w}_2]\to[0,\infty)$ is a continuous positive mapping and geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$ (i.e., $g\left(\frac{\mathtt{w}_1\mathtt{w}_2}{x}\right)=g(x)$ holds for all $x\in[\mathtt{w}_1,\mathtt{w}_2]$), then*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2}{2}\Big[\,{}_HJ^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2)+\,{}_HJ^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big]-\Big[\,{}_HJ^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+\,{}_HJ^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\Big]\right|$$

$$\leq \frac{(\ln \mathtt{w}_2 - \ln \mathtt{w}_1)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}\|g\|_\infty\Big[Z_1(\alpha)|Q'(\mathtt{w}_1)| + Z_2(\alpha)|Q'(G)| + Z_3(\alpha)|Q'(\mathtt{w}_2)|\Big],$$

where

$$\begin{aligned} Z_1(\alpha) &= \int_0^1 [(1+t)^\alpha - (1-t)^\alpha] t\mathtt{w}_1^t G^{1-t} dt, \\ Z_2(\alpha) &= \int_0^1 (1-t)[(1+t)^\alpha - (1-t)^\alpha][\mathtt{w}_1^t G^{1-t} + \mathtt{w}_2^t G^{1-t}] dt, \\ Z_3(\alpha) &= \int_0^1 [(1+t)^\alpha - (1-t)^\alpha] t\mathtt{w}_2^t G^{1-t} dt. \end{aligned}$$

Theorem 88 ([73]). *Let $Q : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° and $|Q'|^q$ is GA-convex on $[\mathtt{w}_1, \mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $q \geq 1$. If g is as in Theorem 87, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2}{2}\Big[\,{}_HJ^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2) + {}_HJ^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\Big] - \Big[\,{}_HJ^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2) + {}_HJ^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\Big]\right| \\ \leq\; & \frac{(\ln \mathtt{w}_2 - \ln \mathtt{w}_1)^{\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}\|g\|_\infty\Big(\frac{2^{\alpha+2}-2^2}{\alpha+1}\Big)^{1-\frac{1}{q}} \\ & \times\Big[Z_1(\alpha,q)|Q'(\mathtt{w}_1)|^q + Z_2(\alpha,q)|Q'(G)|^q + Z_3(\alpha,q)|Q'(\mathtt{w}_2)|^q\Big]^{\frac{1}{q}}, \end{aligned}$$

where $\alpha > 0$ and

$$\begin{aligned} Z_1(\alpha,q) &= \int_0^1 [(1+t)^\alpha - (1-t)^\alpha] t\mathtt{w}_1^{qt} G^{q(1-t)} dt, \\ Z_2(\alpha,q) &= \int_0^1 (1-t)[(1+t)^\alpha - (1-t)^\alpha][\mathtt{w}_1^{qt} G^{q(1-t)} + \mathtt{w}_2^t q G^{q(1-t)}] dt, \\ Z_3(\alpha,q) &= \int_0^1 [(1+t)^\alpha - (1-t)^\alpha] t\mathtt{w}_2^{qt} G^{q(1-t)} dt. \end{aligned}$$

Theorem 89 ([74]). *Let $Q : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1, \mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is GA-s-convex on $[\mathtt{w}_1, \mathtt{w}_2]$, $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|Q(\sqrt{\mathtt{w}_1\mathtt{w}_2})\Big[{}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}g(\mathtt{w}_1) + {}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}g(\mathtt{w}_2)\Big] - \Big[{}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}(Qg)(\mathtt{w}_1) + {}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}(Qg)(\mathtt{w}_2)\Big]\right| \\ \leq\; & \frac{\|g\|_\infty\Big(\ln\frac{\mathtt{w}_2}{\mathtt{w}_1}\Big)^{\alpha+1}}{\Gamma(\alpha+1)}[H_1(\alpha)|Q'(\mathtt{w}_1)| + H_2(\alpha)Q'(\mathtt{w}_2)|], \quad \alpha > 0, \end{aligned}$$

where

$$\begin{aligned} H_1(\alpha) &= \int_0^{1/2} u^\alpha(1-u)^s(\mathtt{w}_1^{1-u}\mathtt{w}_2^u)du + \int_{1/2}^1 (1-u)^{\alpha+s}(\mathtt{w}_1^{1-u}\mathtt{w}_2^u)du, \\ H_2(\alpha) &= \int_0^{1/2} u^{\alpha+s}(\mathtt{w}_1^{1-u}\mathtt{w}_2^u)du + \int_{1/2}^1 (1-u)^\alpha u^s(\mathtt{w}_1^{1-u}\mathtt{w}_2^u)du. \end{aligned}$$

Theorem 90 ([74]). *Let $Q : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1, \mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q$ is GA-s-convex on $[\mathtt{w}_1, \mathtt{w}_2]$, $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$, then the fractional inequalities are given as:*

$$\left|Q(\sqrt{\mathtt{w}_1\mathtt{w}_2})\Big[{}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}g(\mathtt{w}_1) + {}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}g(\mathtt{w}_2)\Big] - \Big[{}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}(Qg)(\mathtt{w}_1) + {}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}(Qg)(\mathtt{w}_2)\Big]\right|$$

$$\leq \frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{\mathtt{w}_2}{\mathtt{w}_1}\right)}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)}(\alpha+1)^{1-\frac{1}{q}}\Gamma(\alpha+1)}\left\{\left[H_3(\alpha)|Q'(\mathtt{w}_1)|^q + H_4(\alpha)|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left. + \left[H_5(\alpha)|Q'(\mathtt{w}_1)|^q + H_6(\alpha)|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}}\right\}, \ \alpha > 0,$$

where

$$H_3(\alpha) = \int_0^{1/2} u^\alpha (1-u)^s (\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du, \quad H_4(\alpha) = \int_0^{1/2} u^{\alpha+s}(\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du$$
$$H_5(\alpha) = \int_{1/2}^1 (1-u)^{\alpha+s}(\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du, \quad H_6(\alpha) = \int_{1/2}^1 (1-u)^\alpha u^s (\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du.$$

Theorem 91 ([74]). *Let Q and g be as in Theorem 90. If $|Q'|^q, q > 1$ is GA-s-convex on $[\mathtt{w}_1, \mathtt{w}_2]$, then the fractional inequality is given as:*

$$\left|Q(\sqrt{\mathtt{w}_1\mathtt{w}_2})\left[{}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}g(\mathtt{w}_1) + {}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}g(\mathtt{w}_2)\right] - \left[{}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}(Qg)(\mathtt{w}_1) + {}_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}(Qg)(\mathtt{w}_2)\right]\right|$$
$$\leq \frac{\|g\|_\infty \alpha \ln^{\alpha+1-\frac{1}{q}}\left(\frac{\mathtt{w}_2}{\mathtt{w}_1}\right)}{2^{\frac{\alpha p+1}{p}}(\alpha p+1)^{\frac{1}{p}} q^{\frac{1}{q}}\Gamma(\alpha+1)}\left\{\left[H_7|Q'(\mathtt{w}_1)|^q + H_8|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}}\right.$$
$$\left. + \left[H_9|Q'(\mathtt{w}_1)|^q + H_{10}|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}}\right\},$$

with $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ and

$$H_7 = \int_0^{1/2} (1-u)^s (\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du, \quad H_8 = \int_0^{1/2} u^s (\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du,$$
$$H_9 = \int_{1/2}^1 (1-u)^s (\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du, \quad H_{10} = \int_{1/2}^1 u^s (\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du.$$

An important generalization of Hadamard fractional integrals is the Hadamard k-fractional integral operators.

Definition 23 ([75]). *Let $Q \in L_1[\mathtt{w}_1, \mathtt{w}_2]$. The left-sided and right-sided Hadamard k-fractional integrals of order $\alpha \in \mathbb{R}^+$ and $k, \mathtt{w}_1 \in \mathbb{R}^+$ of function Q are defined by*

$$({}^k_HJ^\alpha_{\mathtt{w}_1+}Q)(x) = \frac{1}{k\Gamma_k(\alpha)}\int_{\mathtt{w}_1}^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1} Q(t)\frac{dt}{t}, \ 0 < \mathtt{w}_1 < x \leq \mathtt{w}_2,$$

and

$$({}^k_HJ^\alpha_{\mathtt{w}_2-}Q)(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^{\mathtt{w}_2} \left(\ln\frac{t}{x}\right)^{\frac{\alpha}{k}-1} Q(t)\frac{dt}{t}, \ 0 < \mathtt{w}_1 \leq x < \mathtt{w}_2,$$

where $\Gamma_k(\alpha)$ is the k-Gamma function defined by $\Gamma_k(\alpha) = \int_0^\infty s^{\alpha-1}e^{-\frac{s^k}{k}}ds$.

Theorem 92 ([76]). *Assume that $Q : \mathbb{I} \subseteq (0, \infty) \to \mathbb{R}$ is a differentiable function on I° such that $Q' \in L_1[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1, \mathtt{w}_2 \in \mathbb{I}$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $\alpha > 0$. If $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is a continuous, positive function, geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$, and $Q'^q, q \geq 1$ is GA-s-convex on $[\mathtt{w}_1, \mathtt{w}_2]$, then the fractional inequality is given as:*

$$Q(\sqrt{\mathtt{w}_1\mathtt{w}_2})\left[({}^k_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}}(Q(\mathtt{w}_1)) + ({}^k_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}}(Q(\mathtt{w}_2))\right]$$
$$-\left[({}^k_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}}(Qg)(\mathtt{w}_1)) + ({}^k_HJ^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}}(Qg)(\mathtt{w}_2))\right]$$

$$\leq \frac{(\ln \mathtt{w}_2 - \ln \mathtt{w}_1)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty \Big\{ \Big[\frac{\sqrt{\mathtt{w}_1\mathtt{w}_2}\left(\frac{\alpha}{k}+1\right)+\mathtt{w}_1}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} \Big]^{1-\frac{1}{q}}$$
$$\times \Big(\mathbb{B}_1(k,\alpha,s)|Q'(\mathtt{w}_1)|^q + \mathbb{B}_2(k,\alpha,s)|Q'(\mathtt{w}_2)|^q\Big)^{\frac{1}{q}}$$
$$+ \Big[\frac{\sqrt{\mathtt{w}_1\mathtt{w}_2}\left(\frac{\alpha}{k}+1\right)+\mathtt{w}_2}{\left(\frac{\alpha}{k}+1\right)\left(\frac{\alpha}{k}+2\right)} \Big]^{1-\frac{1}{q}} \Big(\mathbb{B}_3(k,\alpha,s)|Q'(\mathtt{w}_1)|^q + \mathbb{B}_4(k,\alpha,s)|Q'(\mathtt{w}_2)|^q\Big)^{\frac{1}{q}} \Big\},$$

where

$$\mathbb{B}_1(k,\alpha,s) = \frac{\sqrt{\mathtt{w}_1\mathtt{w}_2}-\mathtt{w}_1}{\frac{\alpha}{k}+2}\,{}_2F_1\Big(-s,\frac{\alpha}{k}+2;\frac{\alpha}{k}+3;\frac{1}{2}\Big) + \frac{\mathtt{w}_1}{\frac{\alpha}{k}+1}\,{}_2F_1\Big(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{2}\Big),$$
$$\mathbb{B}_2(k,\alpha,s) = \frac{\sqrt{\mathtt{w}_1\mathtt{w}_2}-\mathtt{w}_1}{2^s\frac{\alpha}{k}+s+2} + \frac{\mathtt{w}_1}{2^s\left(\frac{\alpha}{k}+s+1\right)},$$
$$\mathbb{B}_3(k,\alpha,s) = \frac{\sqrt{\mathtt{w}_1\mathtt{w}_2}-\mathtt{w}_2}{2^s\frac{\alpha}{k}+s+2} + \frac{\mathtt{w}_2}{2^s\left(\frac{\alpha}{k}+s+1\right)},$$
$$\mathbb{B}_4(k,\alpha,s) = \frac{\sqrt{\mathtt{w}_1\mathtt{w}_2}-\mathtt{w}_2}{\frac{\alpha}{k}+2}\,{}_2F_1\Big(-s,\frac{\alpha}{k}+2;\frac{\alpha}{k}+3;\frac{1}{2}\Big) + \frac{\mathtt{w}_2}{\frac{\alpha}{k}+1}\,{}_2F_1\Big(-s,\frac{\alpha}{k}+1;\frac{\alpha}{k}+2;\frac{1}{2}\Big).$$

Theorem 93 ([76]). *Assume that $Q : \mathbb{I} \subseteq (0,\infty) \to \mathbb{R}$ is a differentiable function on I° such that $Q' \in L_1[\mathtt{w}_1,\mathtt{w}_2]$, where $\mathtt{w}_1,\mathtt{w}_2 \in \mathbb{I}$ with $\mathtt{w}_1 < \mathtt{w}_2$ and $\alpha > 0$. If $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is a continuous, positive function, geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$, and $Q'^q, q\,1$ is GA-s-convex on $[\mathtt{w}_1,\mathtt{w}_2]$, then the following k-Hadamard fractional integral inequality with $k,\alpha > 0$ holds:*

$$Q(\sqrt{\mathtt{w}_1\mathtt{w}_2}\Big[({}^k_H J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}}(Q(\mathtt{w}_1)) + ({}^k_H J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}}(Q(\mathtt{w}_2))\Big]$$
$$-\Big[({}^k_H J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}}(Qg)(\mathtt{w}_1)) + ({}^k_H J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}}(Qg)(\mathtt{w}_2))\Big]$$
$$\leq \frac{(\ln \mathtt{w}_2 - \ln \mathtt{w}_1)^{1+\frac{\alpha}{k}}}{2^{1+\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\|g\|_\infty\Big(\frac{kq-k}{(k+\alpha)q-k}\Big)^{1-\frac{1}{q}}\Big\{\Big[\mathbb{C}_1(q,s)|Q'(\mathtt{w}_1)|^q + \mathbb{C}_2(q,s)|Q'(\mathtt{w}_2)|^q\Big]^{\frac{1}{q}}$$
$$+\Big[\mathbb{C}_3(q,s)|Q'(\mathtt{w}_1)|^q + \mathbb{C}_4(q,s)|Q'(\mathtt{w}_2)|^q\Big]^{\frac{1}{q}}\Big\},$$

where

$$\mathbb{C}_1(k,\alpha,s) = \frac{1}{2}(\mathtt{w}_1^{\frac{q}{2}}\mathtt{w}_2^{\frac{q}{2}} - \mathtt{w}_1^q)\,{}_2F_1\Big(-s,2;3;\frac{1}{2}\Big) + \frac{2\mathtt{w}_1^q}{s+1}\Big(1-\frac{1}{2^{s+1}}\Big),$$
$$\mathbb{C}_2(k,\alpha,s) = \frac{\mathtt{w}_1^{\frac{q}{2}}\mathtt{w}_2^{\frac{q}{2}}}{2^s(s+2)} + \frac{\mathtt{w}_1^q}{2^s(s+1)(s+2)},$$
$$\mathbb{C}_3(k,\alpha,s) = \frac{\mathtt{w}_1^{\frac{q}{2}}\mathtt{w}_2^{\frac{q}{2}}}{2^s(s+2)} + \frac{\mathtt{w}_2^q}{2^s(s+1)(s+2)},$$
$$\mathbb{C}_4(k,\alpha,s) = \frac{1}{2}(\mathtt{w}_1^{\frac{q}{2}}\mathtt{w}_2^{\frac{q}{2}} - \mathtt{w}_2^q)\,{}_2F_1\Big(-s,2;3;\frac{1}{2}\Big) + \frac{2\mathtt{w}_2^q}{s+1}\Big(1-\frac{1}{2^{s+1}}\Big).$$

Fejér-Type Fractional Integral Inequalities for Quasi-Geometrically Convex Functions

Definition 24 ([77]). *A function $Q : I \subset (0,\infty) \to \mathbb{R}$ is said to be quasi-geometrically convex on I if*

$$Q(x^t y^{1-t}) \leq \sup\{Q(x), Q(y)\}$$

for all $x, y \in I$ and $t \in [0,1]$.

We give some Fejér-type fractional integral inequalities for quasi-geometrically convex functions.

Theorem 94 ([78]). *Let $Q : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$, where $\mathtt{w}_1,\mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is quasi-geometrically convex on $[\mathtt{w}_1,\mathtt{w}_2]$, $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$, then the fractional inequality is given as:*

$$\begin{aligned}&\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|\\ \le\;&\frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{\mathtt{w}_2}{\mathtt{w}_1}\right)}{\Gamma(\alpha+1)}\sup\{|Q'(\mathtt{w}_1)|,|Q'(\mathtt{w}_2)|\}\int_0^{1/2}[(1-u)^\alpha-u^\alpha][\mathtt{w}_1^{1-u}\mathtt{w}_2^u+\mathtt{w}_1^u\mathtt{w}_2^{1-u}]du,\end{aligned}$$

where $\alpha > 0$.

Theorem 95 ([78]). *Let $Q : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$, where $\mathtt{w}_1,\mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q, q \ge 1$ is quasi-geometrically convex on $[\mathtt{w}_1,\mathtt{w}_2]$, $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$, then the fractional inequality is given as:*

$$\begin{aligned}&\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|\\ \le\;&\frac{\|g\|_\infty \ln^{\alpha+1}\left(\frac{\mathtt{w}_2}{\mathtt{w}_1}\right)}{\Gamma(\alpha+1)}\left[\left(1-\frac{1}{2^\alpha}\right)\left(\frac{2}{\alpha+1}\right)\right]^{1-\frac{1}{q}}\left[\sup\{|Q'(\mathtt{w}_1)|^q,|Q'(\mathtt{w}_2)|^q\}\right]^{\frac{1}{q}}\\ &\times\left(\left[\int_0^{1/2}((1-u)^\alpha-u^\alpha)[(\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q+(\mathtt{w}_1^u\mathtt{w}_2^{1-u})^q]du\right],\quad \alpha>0.\end{aligned}$$

Theorem 96 ([78]). *Under the assumptions of Theorem 95, where $|Q'|^q, q > 1$ is quasi-geometrically convex on $[\mathtt{w}_1,\mathtt{w}_2]$, the fractional inequality is given as:*

$$\begin{aligned}&\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|\\ \le\;&\frac{\|g\|_\infty \ln^{\alpha+1-\frac{1}{q}}\left(\frac{\mathtt{w}_2}{\mathtt{w}_1}\right)}{q^{\frac{1}{q}}\Gamma(\alpha+1)}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\\ &\times(\mathtt{w}_2^q-\mathtt{w}_1^q)^{\frac{1}{q}}\left[\sup\{|Q'(\mathtt{w}_1)|^q,|Q'(\mathtt{w}_2)|^q\}\right]^{\frac{1}{q}},\quad \alpha>0\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^\alpha_{\mathtt{w}_1+}g(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]-\left[J^\alpha_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+J^\alpha_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right]\right|\\ \le\;&\frac{\|g\|_\infty \ln^{\alpha+1-\frac{1}{q}}\left(\frac{\mathtt{w}_2}{\mathtt{w}_1}\right)}{q^{\frac{1}{q}}\Gamma(\alpha+1)}\left[\frac{1}{\alpha p+1}\right]^{\frac{1}{p}}(\mathtt{w}_2^q-\mathtt{w}_1^q)^{\frac{1}{q}}\left[\sup\{|Q'(\mathtt{w}_1)|^q,|Q'(\mathtt{w}_2)|^q\}\right]^{\frac{1}{q}},\end{aligned}$$

with $0 < \alpha \le 1$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Theorem 97 ([79]). *Let $Q : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1,\mathtt{w}_2]$, where $\mathtt{w}_1,\mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is quasi-geometrically convex on $[\mathtt{w}_1,\mathtt{w}_2]$, $g : [\mathtt{w}_1,\mathtt{w}_2] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$, then the fractional inequality is given as:*

$$\left|Q(\sqrt{\mathtt{w}_1\mathtt{w}_2})\left[J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}g(\mathtt{w}_1)+J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}g(\mathtt{w}_2)\right]-\left[J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}(Qg)(\mathtt{w}_1)+J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}(Qg)(\mathtt{w}_2)\right]\right|$$

$$\leq \frac{\|g\|_\infty \left(\ln \frac{\mathtt{w}_2}{\mathtt{w}_1}\right)^{\alpha+1}}{\Gamma(\alpha+1)} \sup\{|Q'(\mathtt{w}_1)|, |Q'(\mathtt{w}_2)|\} \int_0^1 u^\alpha [\mathtt{w}_1^{1-u}\mathtt{w}_2^u + \mathtt{w}_1^u\mathtt{w}_2^{1-u}]du, \quad \alpha > 0.$$

Theorem 98 ([79]). *Let $Q : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, such that $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$, where $\mathtt{w}_1, \mathtt{w}_2 \in I$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q$ is quasi-geometrically convex on $[\mathtt{w}_1, \mathtt{w}_2]$, $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{\mathtt{w}_1\mathtt{w}_2}$, then the fractional inequality is given as:*

$$\begin{aligned} &\left| Q(\sqrt{\mathtt{w}_1\mathtt{w}_2}) \left[J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-} g(\mathtt{w}_1) + J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+} g(\mathtt{w}_2) \right] - \left[J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}(Qg)(\mathtt{w}_1) + J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}(Qg)(\mathtt{w}_2) \right] \right| \\ \leq\; & \frac{\|g\|_\infty \left(\ln^{\alpha+1} \frac{\mathtt{w}_2}{\mathtt{w}_1} \right) (\alpha+1)^{\frac{1}{q}-1}}{2^{(\alpha+1)\left(1-\frac{1}{q}\right)} \Gamma(\alpha+1)} \sup\{|Q'(\mathtt{w}_1)|^q, |Q'(\mathtt{w}_2)|^q\} \\ & \times \left(\left[\int_0^{1/2} u^\alpha (\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du \right]^{\frac{1}{q}} + \left[\int_{1/2}^1 (1-u)^\alpha (\mathtt{w}_1^{1-u}\mathtt{w}_2^u)^q du \right]^{\frac{1}{q}} \right), \quad \alpha > 0. \end{aligned}$$

Theorem 99 ([79]). *Let Q and g be as in Theorem 98. If $|Q'|^q, q > 1$ is quasi-geometrically convex on $[\mathtt{w}_1, \mathtt{w}_2]$, then the fractional inequality is given as:*

$$\begin{aligned} &\left| Q(\sqrt{\mathtt{w}_1\mathtt{w}_2}) \left[J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-} g(\mathtt{w}_1) + J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+} g(\mathtt{w}_2) \right] - \left[J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}-}(Qg)(\mathtt{w}_1) + J^\alpha_{\sqrt{\mathtt{w}_1\mathtt{w}_2}+}(Qg)(\mathtt{w}_2) \right] \right| \\ \leq\; & \frac{\|g\|_\infty \alpha \ln^{\alpha+1-\frac{1}{q}} \left(\frac{\mathtt{w}_2}{\mathtt{w}_1} \right)}{2^{\frac{\alpha p+1}{p}} (\alpha p+1)^{\frac{1}{p}} q^{\frac{1}{q}} \Gamma(\alpha+1)} \left[\sup\{|Q'(\mathtt{w}_1)|^q, |Q'(\mathtt{w}_2)|^q\} \right]^{\frac{1}{q}} \\ & \times \left[\left[\left(\frac{\mathtt{w}_2}{\mathtt{w}_1} \right)^{\frac{q}{2}} - 1 \right]^{\frac{1}{q}} + \left[\left(\frac{\mathtt{w}_2}{\mathtt{w}_1} \right)^q - \left(\frac{\mathtt{w}_2}{\mathtt{w}_1} \right)^{\frac{q}{2}} \right]^{\frac{1}{q}} \right], \end{aligned}$$

with $\alpha > 0$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

8. Fejér-Type Fractional Integral Inequalities via Raina Integral

In [80], Raina introduced a class of functions formally defined by

$$\mathcal{F}^\sigma_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k, \quad \rho, \lambda > 0, \quad |x| < \infty,$$

where the coefficients $\sigma(k)$ ($k \in \mathbb{N} = \mathbb{N} \cup \{0\}$) are a bounded sequence of real positive numbers. In [81], the following left-sided and right-sided fractional integral operators are defined, respectively, as:

$$(J^\sigma_{\rho,\lambda,\mathtt{w}_1+,w}\phi)(x) = \int_{\mathtt{w}_1}^x (x-t)^{\lambda-1} \mathcal{F}^\sigma_{\rho,\lambda}[w(x-t)^\rho]\phi(t)dt, \quad x > \mathtt{w}_1 > 0,$$

$$(J^\sigma_{\rho,\lambda,\mathtt{w}_2-,w}\phi)(x) = \int_x^{\mathtt{w}_2} (t-x)^{\lambda-1} \mathcal{F}^\sigma_{\rho,\lambda}[w(t-x)^\rho]\phi(t)dt, \quad 0 < x < \mathtt{w}_2.$$

Now, we present Fejér-type fractional integral inequalities via the Raina integral.

Theorem 100 ([82]). *Let $Q : I \to \mathbb{R}$ be a differentiable mapping on I° and $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ with$\mathtt{w}_1 < \mathtt{w}_2$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ continuous. If $|Q'|$ is convex on $[\mathtt{w}_1, \mathtt{w}_2]$, then the fractional inequality is given as:*

$$\begin{aligned} &\Bigg| Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2} \right) \left[J^\sigma_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+,w} g(\mathtt{w}_2) + J^\sigma_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-,w} g(\mathtt{w}_1) \right] \\ &- \left[J^\sigma_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+,w}(Qg)(\mathtt{w}_2) + J^\sigma_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-,w}(Qg)(\mathtt{w}_1) \right] \\ \leq\; & (\mathtt{w}_2-\mathtt{w}_1)^\lambda \|g\|_\infty \mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(\mathtt{w}_2-\mathtt{w}_1)^\rho(|Q'(\mathtt{w}_1)| + |Q'(\mathtt{w}_2)|), \end{aligned}$$

where $\lambda > 0$ *and* $\sigma_1(k) = \sigma(k)\dfrac{1}{2^{\lambda+\rho k+1}(\lambda+\rho k+1)}$.

Theorem 101 ([82]). *Let Q and g be as in Theorem 100. If* $|Q'|^q$ *is convex on* $[\mathtt{w}_1, \mathtt{w}_2]$, $q \geq 1$, *then the fractional inequality is given as:*

$$\begin{aligned}
&\Bigg|Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^{\sigma}_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+,w}g(\mathtt{w}_2) + J^{\sigma}_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-,w}g(\mathtt{w}_1)\right] \\
&-\left[J^{\sigma}_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+,w}(Qg)(\mathtt{w}_2) + J^{\sigma}_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-,w}(Qg)(\mathtt{w}_1)\right]\Bigg| \\
\leq\ & \|g\|_\infty\left((\mathtt{w}_2-\mathtt{w}_1)^{\lambda+1}\right)^{\frac{1}{q}}\left((\mathtt{w}_2-\mathtt{w}_1)^{\lambda+1}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}[|w|(\mathtt{w}_2-\mathtt{w}_1)^{\rho}]\right)^{1-\frac{1}{q}} \\
&\times\left\{\left[\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(\mathtt{w}_2-\mathtt{w}_1)^{\rho}]|Q'(\mathtt{w}_1)|^q + \mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|w|(\mathtt{w}_2-\mathtt{w}_1)^{\rho}]|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[|w|(\mathtt{w}_2-\mathtt{w}_1)^{\rho}]|Q'(\mathtt{w}_1)|^q + \mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[|w|(\mathtt{w}_2-\mathtt{w}_1)^{\rho}]|Q'(\mathtt{w}_2)|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

where $\lambda > 0$, σ_1 *is defined in Theorem 100 and*

$$\begin{aligned}
\sigma_2(k) &= \sigma(k)\frac{\lambda+\rho k+3}{(\lambda+\rho k+1)(\lambda+\rho k+2)2^{\lambda+\rho k+2}}, \\
\sigma_3(k) &= \sigma(k)\frac{1}{(\lambda+\rho k+2)2^{\lambda+\rho k+2}}.
\end{aligned}$$

Theorem 102 ([82]). *Let Q and g be as in Theorem 100. If* $|Q'|^q$ *is convex on* $[\mathtt{w}_1, \mathtt{w}_2]$, $q \geq 1$, *then the fractional inequality is given as:*

$$\begin{aligned}
&\Bigg|Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[J^{\sigma}_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+,w}g(\mathtt{w}_2) + J^{\sigma}_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-,w}g(\mathtt{w}_1)\right] \\
&-\left[J^{\sigma}_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}+,w}(Qg)(\mathtt{w}_2) + J^{\sigma}_{\rho,\lambda,\frac{\mathtt{w}_1+\mathtt{w}_2}{2}-,w}(Qg)(\mathtt{w}_1)\right]\Bigg| \\
\leq\ & \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\lambda+1}\|g\|_\infty}{2^{\frac{3}{q}}}\left(\mathcal{F}^{\sigma_4}_{\rho,\lambda+1}[|w|(\mathtt{w}_2-\mathtt{w}_1)^{\rho}]\right)^{\frac{1}{p}} \\
&\times\left\{[3|Q'(\mathtt{w}_1)|^q + |Q'(\mathtt{w}_2)|^q]^{\frac{1}{q}} + [|Q'(\mathtt{w}_1)|^q + 3|Q'(\mathtt{w}_2)|^q]^{\frac{1}{q}}\right\},
\end{aligned}$$

where $\lambda > 0$, $\sigma_4(k) = \sigma(k)\dfrac{1}{2^{\lambda+\rho k+\frac{1}{p}}(\lambda p+\rho k p+1)^{\frac{1}{p}}}$ *and* $\dfrac{1}{p}+\dfrac{1}{q} = 1$.

Theorem 103 ([83]). *Let* $Q : I \to \mathbb{R}$ *be a differentiable mapping on* I° *and* $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ *with* $\mathtt{w}_1 < \mathtt{w}_2$ *and* $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *continuous and symmetric to* $\dfrac{\mathtt{w}_1+\mathtt{w}_2}{2}$. *If* $|Q'|$ *is convex on* $[\mathtt{w}_1, \mathtt{w}_2]$, *then the fractional inequality is given as:*

$$\begin{aligned}
&\Bigg|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[J^{\sigma}_{\rho,\lambda,\mathtt{w}_1+,w}g(\mathtt{w}_2) + J^{\sigma}_{\rho,\lambda,\mathtt{w}_2-,w}g(\mathtt{w}_1)\right] \\
&-\left[J^{\sigma}_{\rho,\lambda,\mathtt{w}_1+,w}(Qg)(\mathtt{w}_2) + J^{\sigma}_{\rho,\lambda,\mathtt{w}_2-,w}(Qg)(\mathtt{w}_1)\right]\Bigg| \\
\leq\ & (\mathtt{w}_2-\mathtt{w}_1)^{\lambda+1}\|g\|_\infty\mathcal{F}^{\sigma_1}_{\rho,\lambda}[|w|(\mathtt{w}_2-\mathtt{w}_1)^{\rho}(|Q'(\mathtt{w}_1)| + |Q'(\mathtt{w}_2)|),
\end{aligned}$$

where $\lambda > 0$ *and* $\sigma_1(k) = \sigma(k)\dfrac{1}{(\lambda+\rho k)(\lambda+\rho k+1)}\Big(1-\dfrac{1}{2^{\lambda+\rho k}}\Big)$.

Theorem 104 ([83]). *Let Q and g be as in Theorem 103. If $|Q'|^q, q>1$ is convex on $[\mathtt{w}_1,\mathtt{w}_2]$, then the fractional inequality is given as:*

$$
\begin{aligned}
&\Bigg|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^{\sigma}_{\rho,\lambda,\mathtt{w}_1+,w}g(\mathtt{w}_2)+J^{\sigma}_{\rho,\lambda,\mathtt{w}_2-,w}g(\mathtt{w}_1)\Big] \\
&-\Big[J^{\sigma}_{\rho,\lambda,\mathtt{w}_1+,w}(Qg)(\mathtt{w}_2)+J^{\sigma}_{\rho,\lambda,\mathtt{w}_2-,w}(Qg)(\mathtt{w}_1)\Big]\Bigg| \\
\leq\ &\frac{2\|g\|_\infty(\mathtt{w}_2-\mathtt{w}_1)^{\lambda+1}}{(\mathtt{w}_2-\mathtt{w}_1)^{\frac{1}{q}}}\mathcal{F}^{\sigma_1}_{\rho,\lambda}[|w|(\mathtt{w}_2-\mathtt{w}_1)^{\rho}]\Big(\frac{|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q}{2}\Big)^{\frac{1}{q}},
\end{aligned}
$$

where $\lambda > 0$, $\dfrac{1}{p}+\dfrac{1}{q}=1$ *and* $\sigma_1(k) = \sigma(k)\dfrac{1}{(\lambda+\rho k)(\lambda+\rho k+1)}\Big(1-\dfrac{1}{2^{\lambda+\rho k}}\Big)$.

Theorem 105 ([83]). *Under the assumptions of Theorem 103, the fractional inequality is given as:*

$$
\begin{aligned}
&\Bigg|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\Big[J^{\sigma}_{\rho,\lambda,\mathtt{w}_1+,w}g(\mathtt{w}_2)+J^{\sigma}_{\rho,\lambda,\mathtt{w}_2-,w}g(\mathtt{w}_1)\Big] \\
&-\Big[J^{\sigma}_{\rho,\lambda,\mathtt{w}_1+,w}(Qg)(\mathtt{w}_2)+J^{\sigma}_{\rho,\lambda,\mathtt{w}_2-,w}(Qg)(\mathtt{w}_1)\Big]\Bigg| \\
\leq\ &\|g\|_\infty(\mathtt{w}_2-\mathtt{w}_1)^{\lambda+1}\mathcal{F}^{\sigma_1}_{\rho,\lambda}[|w|(\mathtt{w}_2-\mathtt{w}_1)^{\rho}]\Big(\frac{|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q}{2}\Big)^{\frac{1}{q}},
\end{aligned}
$$

where $\lambda > 0$, $\dfrac{1}{p}+\dfrac{1}{q}=1$ *and* $\sigma_1(k) = \sigma(k)\dfrac{1}{\lambda+\rho k}\Big[\dfrac{2}{(\lambda+\rho k)p+1}\Big(1-\dfrac{1}{2^{\lambda+\rho k}}\Big)\Big]^{\frac{1}{p}}$.

9. Fejér-Type Fractional Integral Inequalities via Conformable Integrals

Definition 25 ([84]). *The left and right fractional conformable integrals of order $\beta\in\mathbb{C}$, $\Re(\beta)>0$, are defined by*

$$
\begin{aligned}
{}^{\beta}J^{\alpha}_{\mathtt{w}_1}Q(x) &= \frac{1}{\Gamma(\beta)}\int_{\mathtt{w}_1}^{x}\Big(\frac{(x-\mathtt{w}_1)^{\alpha}-(t-\mathtt{w}_1)^{\alpha}}{\alpha}\Big)^{\beta}\frac{Q(t)}{(t-\mathtt{w}_1)^{1-\alpha}}dt, \\
{}^{\beta}J^{\alpha}_{\mathtt{w}_2}Q(x) &= \frac{1}{\Gamma(\beta)}\int_{x}^{\mathtt{w}_2}\Big(\frac{(\mathtt{w}_2-x)^{\alpha}-(\mathtt{w}_2-t)^{\alpha}}{\alpha}\Big)^{\beta}\frac{Q(t)}{(\mathtt{w}_2-t)^{1-\alpha}}dt.
\end{aligned}
$$

Definition 26 ([85]). *We say that the function $Q:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ is symmetrized convex on the interval $[\mathtt{w}_1,\mathtt{w}_2]$ if the symmetrical transform $\frac{1}{2}[Q(t)+Q(\mathtt{w}_1+\mathtt{w}_2-t)]$, $t\in[\mathtt{w}_1,\mathtt{w}_2]$ is convex on $[\mathtt{w}_1,\mathtt{w}_2]$.*

Definition 27 ([86]). *Let I be a non-empty interval on $\mathbb{R}$. Then, a function $Q:I\to\mathbb{R}$ is called Wright-quasi-convex on I if*

$$\frac{1}{2}\Big[Q(tx+(1-t)y+Q((1-t)x+ty)\Big]\leq\max\{Q(x),Q(y)\}$$

for all $x,y\in I$ and $t\in[0,1]$.

Definition 28 ([86]). *Let I and J be intervals on $\mathbb{R}$ with $(0,1)\subseteq J$. Also, let $Q:I\to[0,\infty)$ be a function and $h:J\to(0,\infty)$ a function with $h\not\equiv0$. Then, Q is called h-convex on I if*

$$Q(tx+(1-t)y\leq h(t)Q(x)+h(1-t)Q(y)$$

for all $x, y \in I$ *and* $t \in [0,1]$.

Definition 29 ([86]). *Let h be the function in Definition 28. A function* $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *is called h-symmetrized convex on the interval* $[\mathtt{w}_1, \mathtt{w}_2]$ *if the symmetrical transform* $\frac{1}{2}[Q(t) + Q(\mathtt{w}_1 + \mathtt{w}_2 - t)]$, $t \in [\mathtt{w}_1, \mathtt{w}_2]$ *is h-convex on* $[\mathtt{w}_1, \mathtt{w}_2]$.

We present certain Fejér-type fractional integral inequalities involving the above defined fractional integral operators.

Theorem 106 ([86]). *Let* $\beta \in \mathbb{C}$, $\Re(\beta) > 0$. *Also, let* $[\mathtt{w}_1, \mathtt{w}_2]$, $\mathtt{w}_1 < \mathtt{w}_2$ *be an interval on* $\mathbb{R}$, $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{C}$ *a symmetrized convex and integrable function and* $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *be integrable and symmetric to* $\dfrac{\mathtt{w}_1 + \mathtt{w}_2}{2}$. *Then,*

$$Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta}{2(x-\mathtt{w}_1)^{\alpha\beta}\|g\|_{\min}}\left[{}^\beta J^\alpha_{\mathtt{w}_1} Q(x) + {}^\beta J^\alpha_{\mathtt{w}_2} Q(\mathtt{w}_1+\mathtt{w}_2-x)\right], \quad (\mathtt{w}_1 < x \le \mathtt{w}_2);$$
$$\frac{\Gamma(\beta+1)\alpha^\beta}{2(x-\mathtt{w}_1)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{\mathtt{w}_1} Q(x) + {}^\beta J^\alpha_{\mathtt{w}_2} Q(\mathtt{w}_1+\mathtt{w}_2-x)\right] \le \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}, \quad (\mathtt{w}_1 < x \le \mathtt{w}_2);$$
$$Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta}{2(\mathtt{w}_2-x)^{\alpha\beta}\|g\|_{\min}}\left[{}^\beta J^\alpha_{\mathtt{w}_1} Q(\mathtt{w}_1+\mathtt{w}_2-x) + {}^\beta J^\alpha_{\mathtt{w}_2} Q(x)\right], \quad (\mathtt{w}_1 \le x < \mathtt{w}_2);$$
$$\frac{\Gamma(\beta+1)\alpha^\beta}{2(\mathtt{w}_2-x)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{\mathtt{w}_1} Q(\mathtt{w}_1+\mathtt{w}_2-x) + {}^\beta J^\alpha_{\mathtt{w}_2} Q(x)\right] \le \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}, \quad (\mathtt{w}_1 \le x < \mathtt{w}_2).$$

Theorem 107 ([86]). *Assume that the conditions of Theorem 106 hold. Then*

$$Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta}{2(x-\mathtt{w}_1)^{\alpha\beta}\|g\|_{\min}}\left[{}^\beta J^\alpha_{x} Q(\mathtt{w}_1) + {}^\beta J^\alpha_{\mathtt{w}_1+\mathtt{w}_2-x} Q(\mathtt{w}_2)\right], \quad (\mathtt{w}_1 < x \le \mathtt{w}_2);$$
$$\frac{\Gamma(\beta+1)\alpha^\beta}{2(x-\mathtt{w}_1)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{x} Q(\mathtt{w}_1) + {}^\beta J^\alpha_{\mathtt{w}_1+\mathtt{w}_2-x} Q(\mathtt{w}_2)\right] \le \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}, \quad (\mathtt{w}_1 < x \le \mathtt{w}_2);$$
$$Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta}{2(\mathtt{w}_2-x)^{\alpha\beta}\|g\|_{\min}}\left[{}^\beta J^\alpha_{\mathtt{w}_1+\mathtt{w}_2-x} Q(\mathtt{w}_1) + {}^\beta J^\alpha_{x} Q(\mathtt{w}_2)\right], \quad (\mathtt{w}_1 \le x < \mathtt{w}_2);$$
$$\frac{\Gamma(\beta+1)\alpha^\beta}{2(\mathtt{w}_2-x)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{\mathtt{w}_1+\mathtt{w}_2-x} Q(\mathtt{w}_1) + {}^\beta J^\alpha_{x} Q(\mathtt{w}_2)\right] \le \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}, \quad (\mathtt{w}_1 \le x < \mathtt{w}_2).$$

Theorem 108 ([86]). *Let* $\beta \in \mathbb{C}$, $\Re(\beta) > 0$. *Also, let* $[\mathtt{w}_1, \mathtt{w}_2]$, $\mathtt{w}_1 < \mathtt{w}_2$ *be an interval on* $\mathbb{R}$, $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{C}$ *be an integrable function. Also,* $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *is Wright-quasi-convex and integrable on* $[\mathtt{w}_1, \mathtt{w}_2]$, *and* $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *is integrable and symmetric to* $\dfrac{\mathtt{w}_1 + \mathtt{w}_2}{2}$. *Then,*

$$\frac{\Gamma(\beta+1)\alpha^\beta}{2(x-\mathtt{w}_1)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{\mathtt{w}_1}(Qg)(x) + {}^\beta J^\alpha_{\mathtt{w}_2}(Qg)(\mathtt{w}_1+\mathtt{w}_2-x)\right] \le \max\{Q(\mathtt{w}_1), Q(\mathtt{w}_2)\};$$
$$\frac{\Gamma(\beta+1)\alpha^\beta}{2(\mathtt{w}_2-\mathtt{w}_1)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{\mathtt{w}_1}(Qg)(\mathtt{w}_2) + {}^\beta J^\alpha_{\mathtt{w}_2} Q(\mathtt{w}_1)\right] \le \max\{Q(\mathtt{w}_1), Q(\mathtt{w}_2)\};$$
$$\frac{\Gamma(\beta+1)\alpha^\beta 2^{\alpha\beta-1}}{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{\mathtt{w}_1}(Q)\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right) + {}^\beta J^\alpha_{\mathtt{w}_2}(Qg)\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\right] \max\{Q(\mathtt{w}_1), Q(\mathtt{w}_2)\}.$$

Theorem 109 ([86]). *Assume that the conditions of Theorem 108 hold. Then*

$$\frac{\Gamma(\beta+1)\alpha^\beta}{2(x-\mathtt{w}_1)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{x}(Qg)(\mathtt{w}_1) + {}^\beta J^\alpha_{\mathtt{w}_1+\mathtt{w}_2-x}(Qg)(\mathtt{w}_2)\right] \le \max\{Q(\mathtt{w}_1), Q(\mathtt{w}_2)\}; \ \mathtt{w}_1 < x \le \mathtt{w}_2;$$
$$\frac{\Gamma(\beta+1)\alpha^\beta 2^{\alpha\beta-1}}{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)} Q(\mathtt{w}_1) + {}^\beta J^\alpha_{\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)} Q(\mathtt{w}_2)\right] \max\{Q(\mathtt{w}_1), Q(\mathtt{w}_2)\}.$$

Theorem 110 ([86]). *Assume that the function $Q : [\mathtt{w}_1, \mathtt{w}_2] \to [0, \infty)$ is h-symmetrized convex on the interval $[\mathtt{w}_1, \mathtt{w}_2]$ with h being integrable on $[0, 1]$ and Q is integrable on $[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be integrable and symmetric to $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$. Then, we have*

$$Q\left(\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}\right) \leq \frac{h\left(\frac{1}{2}\right)\Gamma(\beta+1)\alpha^\beta}{(x-\mathtt{w}_1)^{\alpha\beta}\|g\|_{\min}}\left[{}^\beta J^\alpha_{\mathtt{w}_1}(Qg)(x) + {}^\beta J^\alpha_{\mathtt{w}_2} Q(\mathtt{w}_1 + \mathtt{w}_2 - x)\right],$$

$$\frac{\Gamma(\beta)\alpha^{\beta-1}}{2(x-\mathtt{w}_1)^{\alpha\beta}\|g\|_\infty}\left[{}^\beta J^\alpha_{\mathtt{w}_1} Q(x) + {}^\beta J^\alpha_{\mathtt{w}_2} Q(\mathtt{w}_1 + \mathtt{w}_2 - x)\right]$$
$$\leq \frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2}\int_0^1 (1-s^\alpha)^{\beta-1}s^{1-\alpha}\left[h\left(1 - \frac{x-\mathtt{w}_1}{\mathtt{w}_2 - \mathtt{w}_1}s\right) + h\left(\frac{x-\mathtt{w}_1}{\mathtt{w}_2-\mathtt{w}_1}s\right)\right]ds.$$

In the sequence, we give some more Fejér-Type fractional integral inequalities via the fractional conformable integral operators for p-convex functions.

Definition 30 ([87]). *A function $Q : \mathbb{I} \subset (0, \infty) \to \mathbb{R}$ is said to be p-convex, if*

$$Q\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq tQ(x) + (1-t)Q(y),$$

for all $x, y \in \mathbb{I}$ and $t \in [0, 1]$.

Theorem 111 ([88]). *Assume that $Q : \mathbb{I} \subset (0, \infty) \to \mathbb{R}$ is a p-convex function, $p \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$ and $\mathtt{w}_1, \mathtt{w}_2 \in \mathbb{I}$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is non-negative, integrable and p-symmetric with respect to $\left[\frac{\mathtt{w}_1^p + \mathtt{w}_2^p}{2}\right]^{\frac{1}{p}}$, then, the fractional inequalities are given as:*

$$Q\left(\left[\frac{\mathtt{w}_1^p + \mathtt{w}_2^p}{2}\right]^{\frac{1}{p}}\right)\left[{}^\beta J^\alpha_{\mathtt{w}_1^p+}(g \circ h)(\mathtt{w}_2^p) + {}^\beta J^\alpha_{\mathtt{w}_2^p-}(g \circ h)(\mathtt{w}_1^p)\right]$$
$$\leq \left[{}^\beta J^\alpha_{\mathtt{w}_1^p+}(Qg \circ h)(\mathtt{w}_2^p) + {}^\beta J^\alpha_{\mathtt{w}_2^p-}(Qg \circ h)(\mathtt{w}_1^p)\right]$$
$$\leq \frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2}\left[{}^\beta J^\alpha_{\mathtt{w}_1^p+}(g \circ h)(\mathtt{w}_2^p) + {}^\beta J^\alpha_{\mathtt{w}_2^p-}(g \circ h)(\mathtt{w}_1^p)\right], \quad p > 0,$$

with $h(x) = x^{\frac{1}{p}}$, $x \in [\mathtt{w}_1^p, \mathtt{w}_2^p]$, and

$$Q\left(\left[\frac{\mathtt{w}_1^p + \mathtt{w}_2^p}{2}\right]^{\frac{1}{p}}\right)\left[{}^\beta J^\alpha_{\mathtt{w}_2^p+}(g \circ h)(\mathtt{w}_1^p) + {}^\beta J^\alpha_{\mathtt{w}_1^p-}(g \circ h)(\mathtt{w}_2^p)\right]$$
$$\leq \left[{}^\beta J^\alpha_{\mathtt{w}_2^p+}(Qg \circ h)(\mathtt{w}_1^p) + {}^\beta J^\alpha_{\mathtt{w}_1^p-}(Qg \circ h)(\mathtt{w}_2^p)\right]$$
$$\leq \frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2}\left[{}^\beta J^\alpha_{\mathtt{w}_2^p+}(g \circ h)(\mathtt{w}_1^p) + {}^\beta J^\alpha_{\mathtt{w}_1^p-}(g \circ h)(\mathtt{w}_2^p)\right], \quad p < 0,$$

with $h(x) = x^{\frac{1}{p}}$, $x \in [\mathtt{w}_2^p, \mathtt{w}_1^p]$.

Definition 31 ([89]). *Let $\alpha \in (n, n+1]$. Then, the left- and right-sided conformable fractional integrals of order $\alpha > 0$ are given by*

$$\mathbb{I}^\alpha_{\mathtt{w}_1} Q(t) = \frac{1}{n!}\int_{\mathtt{w}_1}^x (t-x)^n (x-\mathtt{w}_1)^{\beta-1} Q(x)dx,$$
$$\mathbb{I}^\alpha_{\mathtt{w}_2} Q(t) = \frac{1}{n!}\int_x^{\mathtt{w}_2} (x-t)^n (\mathtt{w}_2 - x)^{\beta-1} Q(x)dx.$$

Now, we give Fejér-type fractional integral inequalities for convex functions via the conformable fractional integral defined above.

Theorem 112 ([90]). *Assume that $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is a convex function, with $\mathtt{w}_1 < \mathtt{w}_2$. If $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$, then the fractional inequalities are given as:*

$$\begin{aligned} Q\Big(\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}\Big)\Big[\mathbb{I}^{\alpha}_{\mathtt{w}_1} g(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_1} g(\mathtt{w}_1)\Big] &\leq \Big[\mathbb{I}^{\alpha}_{\mathtt{w}_1}(Qg)(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_1}(Qg)(\mathtt{w}_1)\Big] \\ &\leq \frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2}\Big[\mathbb{I}^{\alpha}_{\mathtt{w}_1} g(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_1}(g \circ h)(\mathtt{w}_1)\Big], \end{aligned}$$

for $\alpha \in (n, n+1]$.

Theorem 113 ([90]). *Assume that $Q : I \to \mathbb{R}$ is a differentiable function on I°, and $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is convex on $[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and symmetric with respect to $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$, then the fractional inequalities are given as:*

$$\begin{aligned} &\frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2}\Big[\mathbb{I}^{\alpha}_{\mathtt{w}_1} g(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_1} g(\mathtt{w}_1)\Big] - \Big[\mathbb{I}^{\alpha}_{\mathtt{w}_1}(Qg)(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_1}(Qg)(\mathtt{w}_1)\Big] \\ \leq\ &\frac{(\mathtt{w}_2 - \mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{n!}\Big(|Q'(\mathtt{w}_1)| + |Q'(\mathtt{w}_2)|\Big) \\ &\times\Big[\frac{1}{2}B(n+1, \alpha-n) + B_{1/2}(\alpha-n+1, n+1) + B_{1/2}(n+2, \alpha-n)\Big], \end{aligned}$$

for $\alpha \in (n, n+1]$.

Theorem 114 ([90]). *Assume that $Q : I \to \mathbb{R}$ is a differentiable function on I°, and $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q$ is convex on $[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and symmetric with respect to $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$, then the fractional inequalities are given as:*

$$\begin{aligned} &\frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2}\Big[\mathbb{I}^{\alpha}_{\mathtt{w}_1} g(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_1} g(\mathtt{w}_1)\Big] - \Big[\mathbb{I}^{\alpha}_{\mathtt{w}_1}(Qg)(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_1}(Qg)(\mathtt{w}_1)\Big] \\ \leq\ &\frac{(\mathtt{w}_2 - \mathtt{w}_1)^{\alpha+1}\|g\|_\infty 2^{\frac{1}{p}}}{n!}\Big(\frac{|Q'(\mathtt{w}_1)|^q + |Q'(\mathtt{w}_2)|^q}{2}\Big)^{\frac{1}{q}} \\ &\times\Big(\int_0^{1/2} B_t(n+1, \alpha-n)^p - B_{1-t}(n+1, \alpha-n)]dt\Big)^{\frac{1}{p}}, \end{aligned}$$

for $\alpha \in (n, n+1]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 115 ([90]). *Assume that $Q : I \to \mathbb{R}$ is a differentiable function on I°, and $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q, q > 1$ is convex on $[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and symmetric with respect to $\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}$, then the fractional inequalities are given as:*

$$\begin{aligned} &\left|\frac{Q(\mathtt{w}_1) + Q(\mathtt{w}_2)}{2}\Big[\mathbb{I}^{\alpha}_{\mathtt{w}_1} g(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_1} g(\mathtt{w}_1)\Big] - \Big[\mathbb{I}^{\alpha}_{\mathtt{w}_1}(Qg)(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_1}(Qg)(\mathtt{w}_1)\Big]\right| \\ \leq\ &\frac{2(\mathtt{w}_2 - \mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{n!}\Big(\frac{|Q'(\mathtt{w}_1)|^q + |Q'(\mathtt{w}_2)|^q}{2}\Big)^{\frac{1}{q}} \\ &\times\Big(\int_0^{1}\Big(|B_t(n+1, \alpha-n) - B_{1-t}(n+1, \alpha-n)|\Big)dt\Big)^{1-\frac{1}{q}} \\ &\times\Big(\frac{1}{2}B(n+1, \alpha-n) + B_{1/2}(\alpha-n+1, n+1) + B_{1/2}(n+2, \alpha-n)\Big)^{\frac{1}{q}}, \end{aligned}$$

for $\alpha \in (n, n+1]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 116 ([91]). *Assume that $Q : I \to \mathbb{R}$ is a differentiable function on I°, and $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|$ is convex on $[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and symmetric with respect to $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$, then the fractional inequalities are given as:*

$$\left| Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[\mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} g(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} g(\mathtt{w}_1)\right] - \left[\mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} (Qg)(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} (Qg)(\mathtt{w}_1)\right] \right|$$
$$\leq \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{n!}\left(\frac{|Q'(\mathtt{w}_1)| + |Q'(\mathtt{w}_2)|}{2^{\alpha+1}}\right)[B(n+1, \alpha-n) - B(n+2, \alpha-n)],$$

for $\alpha \in (n, n+1], n = 0, 1, 2, \ldots,$.

Theorem 117 ([91]). *Assume that $Q : I \to \mathbb{R}$ is a differentiable function on I°, and $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q$ is convex on $[\mathtt{w}_1, \mathtt{w}_2]$ for $q > 1$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and symmetric with respect to $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$, then the fractional inequalities are given as:*

$$\left| Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[\mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} g(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} g(\mathtt{w}_1)\right] - \left[\mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} (Qg)(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} (Qg)(\mathtt{w}_1)\right] \right|$$
$$\leq \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^\alpha n!}\left[\left(\int_0^{1/2}[B_{2t}(n+1,\alpha-n)]^p dt\right)^{1/p}\left(\frac{3|Q'(\mathtt{w}_1)|^q + |Q'(\mathtt{w}_2)|^q}{8}\right)^{1/q}\right.$$
$$\left. + \left[\left(\int_{1/2}^{1}[B_{2-2t}(n+1,\alpha-n)]^p dt\right)^{1/p}\left(\frac{|Q'(\mathtt{w}_1)|^q + 3|Q'(\mathtt{w}_2)|^q}{8}\right)^{1/q}\right],$$

for $\alpha \in (n, n+1], n = 0, 1, 2, \ldots,$ and $1/p + 1/q = 1$.

Theorem 118 ([91]). *Assume that $Q : I \to \mathbb{R}$ is a differentiable function on I°, and $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$ with $\mathtt{w}_1 < \mathtt{w}_2$. If $|Q'|^q$ is convex on $[\mathtt{w}_1, \mathtt{w}_2]$ for $q > 1$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ is continuous and symmetric with respect to $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$, then the fractional inequalities are given as:*

$$\left| Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[\mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} g(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} g(\mathtt{w}_1)\right] - \left[\mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} (Qg)(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\frac{\mathtt{w}_1+\mathtt{w}_2}{2}} (Qg)(\mathtt{w}_1)\right] \right|$$
$$\leq \frac{(\mathtt{w}_2-\mathtt{w}_1)^{\alpha+1}\|g\|_\infty}{2^{\alpha-\frac{\alpha}{q}} n!}\left[\left(\int_0^{1/2}[B_{2t}(n+1,\alpha-n)]^p dt\right)^{1-1/q}\right.$$
$$\times\left(\frac{|Q'(\mathtt{w}_1)|}{2^{\alpha+1}}\left[\frac{3}{4}B(n+1,\alpha-n) - B(n+2,\alpha-n) + \frac{1}{4}B(n+3,\alpha-n)\right]\right.$$
$$\left. + \frac{|Q'(\mathtt{w}_2)|}{2^{\alpha+3}}[B(n+1,\alpha-n) - B(n+3,\alpha-n)]\right)^{1/q} + \left(\int_{1/2}^{1} B_{2-2t}(n+1,\alpha-n)dt\right)^{1-1/q}$$
$$\times\left(\frac{|Q'(\mathtt{w}_1)|}{2^{\alpha+1}}\left[-\frac{3}{4}B(n+1,\alpha-n) + B(n+1,\alpha-n) - \frac{1}{4}B(n+3,\alpha-n)\right]\right.$$
$$+ \frac{|Q'(\mathtt{w}_2)|}{2^{\alpha+1}}\left[-\frac{1}{4}B(n+1,\alpha-n) + B(n+1,\alpha-n) - B(n+2,\alpha-n)\right.$$
$$\left.\left. + \frac{1}{4}B(n+3,\alpha-n)\right)^{1/q}\right],$$

for $\alpha \in (n, n+1], n = 0, 1, 2, \ldots,$.

Theorem 119 ([92]). *Assume that $Q : [\mathtt{w}_1, \mathtt{w}_2] \subset (0, \infty) \to \mathbb{R}$ is a p-convex function, $p \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$ and $\mathtt{w}_1 < \mathtt{w}_2$. If $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$ and $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R} \setminus \{0\}$ is non-negative, integrable and p-symmetric with respect to $\left[\frac{\mathtt{w}_1^p + \mathtt{w}_2^p}{2}\right]^{\frac{1}{p}}$, then the fractional inequalities are given as:*

$$Q\left(\left[\frac{\mathtt{w}_1^p+\mathtt{w}_2^p}{2}\right]^{\frac{1}{p}}\right)\left[{}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_1^p+}(g\circ h)(\mathtt{w}_2^p) + {}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_2^p-}(g\circ h)(\mathtt{w}_1^p)\right]$$

$$\leq \left[{}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_1^p+}(Qg\circ h)(\mathtt{w}_2^p) + {}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_2^p-}(Qg\circ h)(\mathtt{w}_1^p)\right]$$
$$\leq \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[{}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_1^p+}(g\circ h)(\mathtt{w}_2^p) + {}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_2^p-}(g\circ h)(\mathtt{w}_1^p)\right], \quad p>0,$$

with $h(x) = x^{\frac{1}{p}},\ x \in [\mathtt{w}_1^p, \mathtt{w}_2^p]$, *and*

$$Q\left(\left[\frac{\mathtt{w}_1^p+\mathtt{w}_2^p}{2}\right]^{\frac{1}{p}}\right)\left[{}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_2^p+}(g\circ h)(\mathtt{w}_1^p) + {}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_1^p-}(g\circ h)(\mathtt{w}_2^p)\right]$$
$$\leq \left[{}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_2^p+}(Qg\circ h)(\mathtt{w}_1^p) + {}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_1^p-}(Qg\circ h)(\mathtt{w}_2^p)\right]$$
$$\leq \frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[{}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_2^p+}(g\circ h)(\mathtt{w}_1^p) + {}^{\beta}\mathbb{I}^{\alpha}_{\mathtt{w}_1^p-}(g\circ h)(\mathtt{w}_2^p)\right], \quad p<0,$$

with $h(x) = x^{\frac{1}{p}},\ x \in [\mathtt{w}_2^p, \mathtt{w}_1^p]$,

Definition 32 ([93]). *Let $k : (0,1) \to \mathbb{R}$ be a given function. Then, the subset D of a real linear space X will be called k-convex if $k(t)x + k(1-t)y \in D$ for all $x, y \in D$ and $t \in (0,1)$.*

Let $k, h : (0,1) \to \mathbb{R}$ be two given functions and suppose that $D \subset X$ be a k-convex set. Then, a function $Q : D \to \mathbb{R}$ is (k,h)-convex, if for all $x, y \in D$ and $t \in (0,1)$,

$$Q(k(t)x + k(1-t)y) \leq h(t)Q(x) + h(1-t)Q(y).$$

Theorem 120 ([94]). *Let $Q : D \to \mathbb{R}$ be a (k,h)-convex function with $h(1/2) > 0$. Assume that $\mathtt{w}_1 < \mathtt{w}_2$ such that $[\mathtt{w}_1, \mathtt{w}_2] \subset D$ and let $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ be a non-negative function which is symmetric with respect to $\frac{\mathtt{w}_1+\mathtt{w}_2}{2}$. Then, the fractional inequalities are given as:*

$$\frac{Q(k(1/2)(\mathtt{w}_1+\mathtt{w}_2))}{2h(1/2)}\left[\mathbb{I}^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\right] \leq \left[\mathbb{I}^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2) + \mathbb{I}^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right],$$

for $\alpha > 0$.

10. Fejér-Type Fractional Integral Inequalities via Non-Conformable Fractional Integral

Definition 33 ([95]). *For each $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$ and $0 < \mathtt{w}_1 < \mathtt{w}_2$, non-conformable fractional integral operator is then given b y*

$$_{N_3}J^{\alpha}_{u}Q(x) = \int_u^x \wp^{-\alpha}Q(\wp)d\wp,$$

for every $x, u \in [\mathtt{w}_1, \mathtt{w}_2]$ and $\alpha \in \mathbb{R}$.

Definition 34 ([95]). *For each function $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$, then left and right non-conformable fractional integral operators are stated by*

$$_{N_3}J^{\alpha}_{\mathtt{w}_1^+}Q(x) = \int_{\mathtt{w}_1}^x (x-\wp)^{-\alpha}Q(\wp)d\wp,$$

$$_{N_3}J^{\alpha}_{\mathtt{w}_2^-}Q(x) = \int_x^{\mathtt{w}_2} (\wp-x)^{-\alpha}Q(\wp)d\wp,$$

for every $x \in [\mathtt{w}_1, \mathtt{w}_2]$ and $\alpha \in \mathbb{R}$.

Theorem 121 ([96]). *Suppose $Q : [\mathtt{w}_1, \mathtt{w}_1 + \Phi(\mathtt{w}_2, \mathtt{w}_1)] \to \mathbb{R}$ is an h–preinvex function, condition-C for Φ holds and $\Phi(\mathtt{w}_2, \mathtt{w}_1) > 0$, $h(\frac{1}{2}) > 0$ and $\mathcal{F} : [\mathtt{w}_1, \mathtt{w}_1 + \Phi(\mathtt{w}_2, \mathtt{w}_1)] \to \mathbb{R}$, $\mathcal{F} \geq 0$ is symmetric with respect to $\mathtt{w}_1 + \frac{1}{2}\Phi(\mathtt{w}_2, \mathtt{w}_1)$ and $\alpha \leq -1$. Then,*

$$\frac{Q\left(\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)\right)}{2h\left(\frac{1}{2}\right)\Phi(\mathtt{w}_2,\mathtt{w}_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)^-}\mathcal{F}(\mathtt{w}_1)+{}_{N_3}J^{\alpha}_{\mathtt{w}_1^+}\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\right]$$
$$\leq\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)^-}Q(\mathtt{w}_1)\mathcal{F}(\mathtt{w}_1)+{}_{N_3}J^{\alpha}_{\mathtt{w}_1^+}Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\right]$$
$$\leq[Q(\mathtt{w}_1)+Q(\mathtt{w}_2)]\int_0^1\vartheta^{-\alpha}[h(\vartheta)+h(1-\vartheta)]\mathcal{F}(\mathtt{w}_1+\vartheta\Phi(\mathtt{w}_2,\mathtt{w}_1))d\vartheta.$$

Theorem 122 ([96])**.** *Let $\mathcal{H}\subseteq\mathbb{R}$ be an open invex subset with respect to $\Phi:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ and $\mathtt{w}_1,\mathtt{w}_2\in\mathcal{H}$ with $\Phi(\mathtt{w}_2,\mathtt{w}_1)>0$. Suppose that $Q:\mathcal{H}\to\mathbb{R}$ is a differentiable mapping on $\mathcal{H}$ and $\mathcal{F}:\mathcal{H}\to[0,\infty)$ is differentiable and symmetric to $\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)$. If $|Q'|$ is h-preinvex on $\mathcal{H}$ and $\alpha\leq-1$, then*

$$\left|\frac{2\alpha}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{-2\alpha}}\left({}_{N_3}J^{2\alpha-1}_{\mathtt{w}_1^+}\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+{}_{N_3}J^{2\alpha-1}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)\right)\right.$$
$$-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{1-2\alpha}}\left({}_{N_3}J^{2\alpha}_{\mathtt{w}_1^+}\mathcal{F}'(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+{}_{N_3}J^{2\alpha}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}'(\mathtt{w}_1)Q(a)\right)$$
$$\left.-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}[\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)-Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))]\right|$$
$$\leq[|Q'(\mathtt{w}_1)|+|Q'(\mathtt{w}_2)|]\cdot\int_0^1\vartheta^{-2\alpha}\mathcal{F}(\mathtt{w}_1+\vartheta\Phi(\mathtt{w}_2,\mathtt{w}_1))[h(\vartheta)+h(1-\vartheta)]d\vartheta.$$

Theorem 123 ([96])**.** *Let $\mathcal{H}\subseteq\mathcal{R}$ be an open invex subset with respect to $\Phi:\mathcal{H}\times\mathcal{H}\to\mathcal{R}$ and $\mathtt{w}_1,\mathtt{w}_2\in\mathcal{H}$ with $\Phi(\mathtt{w}_2,\mathtt{w}_1)>0$. Suppose that $Q:\mathcal{H}\to\mathcal{R}$ is a differentiable mapping on $\mathcal{H}$ and $\mathcal{F}:\mathcal{H}\to[0,\infty)$ is differentiable and symmetric to $\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)$. If $|Q'|^q$, $q\geq1$, is h-preinvex on $\mathcal{H}$ and $\alpha\leq-1$, then one has:*

$$\left|\frac{2\alpha}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{-2\alpha}}\left({}_{N_3}J^{2\alpha-1}_{\mathtt{w}_1^+}\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+{}_{N_3}J^{2\alpha-1}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)\right)\right.$$
$$-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{1-2\alpha}}\left({}_{N_3}J^{2\alpha}_{\mathtt{w}_1^+}\mathcal{F}'(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+{}_{N_3}J^{2\alpha}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}'(\mathtt{w}_1)Q(a)\right)$$
$$\left.-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}[\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)-Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))]\right|$$
$$\leq\left(\frac{2}{1-2\alpha}\right)^{1-\frac{1}{q}}\left(\left[|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q\right]\int_0^1\vartheta^{-2\alpha}[\mathcal{F}(\mathtt{w}_1+\vartheta\Phi(\mathtt{w}_2,\mathtt{w}_1))]^q[h(\vartheta)+h(1-\vartheta)]d\vartheta\right)^{\frac{1}{q}}.$$

Theorem 124 ([96])**.** *Let $\mathcal{H}\subseteq\mathbb{R}$ be an open invex subset with respect to $\Phi:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ and $\mathtt{w}_1,\mathtt{w}_2\in\mathbb{H}$ with $\Phi(\mathtt{w}_2,\mathtt{w}_1)>0$. Suppose that $\mathcal{H}:\mathcal{H}\to\mathbb{R}$ is a differentiable mapping on $\mathcal{H}$ and $\mathcal{F}:\mathcal{H}\to[0,\infty)$ is differentiable and symmetric to $\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)$. If $|Q'|^q,q>1$, is h-preinvex on $\mathcal{H}$ and $\alpha\leq-1$, then*

$$\left|\frac{2\alpha}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{-2\alpha}}\left({}_{N_3}J^{2\alpha-1}_{\mathtt{w}_1^+}\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+{}_{N_3}J^{2\alpha-1}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)\right)\right.$$
$$-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)^{1-2\alpha}}\left({}_{N_3}J^{2\alpha}_{\mathtt{w}_1^+}\mathcal{F}'(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))+{}_{N_3}J^{2\alpha}_{(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))^-}\mathcal{F}'(\mathtt{w}_1)Q(\mathtt{w}_1)\right)$$
$$\left.-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}[\mathcal{F}(\mathtt{w}_1)Q(\mathtt{w}_1)-Q(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))\mathcal{F}(\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1))]\right|$$
$$\leq\frac{2}{(1-2\alpha\rho)^{\frac{1}{p}}}\left(\left[|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q\right]\cdot\int_0^1[\mathcal{F}(\mathtt{w}_1+\vartheta\Phi(\mathtt{w}_2,\mathtt{w}_1))]^q h(\vartheta)d\vartheta\right)^{\frac{1}{q}},$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1.$

The next theorems include Fejér-type integral inequalities via a non-conformable fractional integral operator for generalized (m,h)-preinvex functions.

Definition 35 ([97]). *Assume that $\Phi : \mathbb{X}\times\mathbb{X}\times(0,1]\to\mathbb{R}$, $\mathbb{X}\subset\mathbb{R}$ and $h:[0,1]\to\mathbb{R}$. Then, Q is said to be generalized (m,h)-preinvex function if*

$$Q(m\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\le h(\wp)Q(\mathtt{w}_2)+mh(1-\wp)Q(\mathtt{w}_1), \tag{3}$$

for all $\mathtt{w}_1,\mathtt{w}_2\in\mathbb{X}$ and $\wp\in[0,1]$.

Theorem 125 ([98]). *Suppose $Q:[m\mathtt{w}_1, m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m)]\to\mathbb{R}$ is an (m,h)-preinvex function, Condition C for Φ holds, $\Phi(\mathtt{w}_2,\mathtt{w}_1,m)>0$, $h(\frac{1}{2})>0$, and $\mathcal{F}:[m\mathtt{w}_1, m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m)]\to\mathbb{R}$, $\mathcal{F}\ge 0$ is symmetric with respect to $m\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1,m)$. Then*

$$\begin{aligned}
&\frac{Q\left(m\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1,m)\right)}{2h\left(\frac{1}{2}\right)\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^{1-\alpha}}\left[{}_{N_3}J^{\alpha}_{m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^-}\mathcal{F}(m\mathtt{w}_1)+{}_{N_3}J^{\alpha}_{m\mathtt{w}_1{}^+}\mathcal{F}(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\right]\\
&\le\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^{1-\alpha}}\Big[{}_{N_3}J^{\alpha}_{m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^-}Q(m\mathtt{w}_1)\mathcal{F}(m\mathtt{w}_1)\\
&\quad+{}_{N_3}J^{\alpha}_{m\mathtt{w}_1{}^+}Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\mathcal{F}(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\Big]\\
&\le[Q(m\mathtt{w}_1)+Q(\mathtt{w}_2)]\int_0^1\wp^{-\alpha}[h(\wp)+h(1-\wp)]\mathcal{F}(m\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1,m))d\wp.
\end{aligned}$$

Theorem 126 ([98]). *Assume that $\Phi:\mathbb{X}\times\mathbb{X}\to\mathbb{R}$ and $\mathbb{X}\subseteq\mathbb{R}$, such that $\mathbb{X}$ is an open m-invex subset with respect to Φ and $\mathtt{w}_1,\mathtt{w}_2\in\mathbb{X}$ with $\Phi(\mathtt{w}_2,\mathtt{w}_1,m)>0$. Suppose that $Q:\mathbb{X}\to\mathbb{R}$ is a differentiable mapping on $\mathbb{X}$ and $\mathcal{F}:\mathbb{X}\to[0,\infty)$ is differentiable and symmetric to $m\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1,m)$. If $|Q'|$ is generalized (m,h)-preinvex on $\mathbb{X}$, then*

$$\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\mathtt{w}_1^+}\mathcal{F}(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\\
&\quad+{}_{N_3}J^{2\alpha-1}_{(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))^-}\mathcal{F}(m\mathtt{w}_1)Q(m\mathtt{w}_1)\Big)\\
&\quad-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^{1-2\alpha}}\Big({}_{N_3}J^{2\alpha}_{m\mathtt{w}_1^+}\mathcal{F}'(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\\
&\quad+{}_{N_3}J^{2\alpha}_{(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))^-}\mathcal{F}'(m\mathtt{w}_1)Q(m\mathtt{w}_1)\Big)\\
&\quad-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)}[\mathcal{F}(m\mathtt{w}_1)Q(m\mathtt{w}_1)-Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\mathcal{F}(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))]\Bigg|\\
&\le\left[m|Q'(\mathtt{w}_1)|+|Q'(\mathtt{w}_2)|\right]\cdot\int_0^1\wp^{-2\alpha}\mathcal{F}(m\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1,m))[h(\wp)+h(1-\wp)]d\wp.
\end{aligned}$$

Theorem 127 ([98]). *Assume that $\mathbb{X}$ and $\mathcal{F}$ are defined as in Theorem 122. If $|Q'|^q$—where $q>1$—is generalized (h,m)-preinvex on $\mathbb{X}$, then one has*

$$\begin{aligned}
&\Bigg|\frac{2\alpha}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\mathtt{w}_1^+}\mathcal{F}(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\\
&\quad+{}_{N_3}J^{2\alpha-1}_{(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))^-}\mathcal{F}(m\mathtt{w}_1)Q(m\mathtt{w}_1)\Big)\\
&\quad-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^{1-2\alpha}}\Big({}_{N_3}J^{2\alpha}_{m\mathtt{w}_1^+}\mathcal{F}'(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))
\end{aligned}$$

$$+_{N_3}J^{2\alpha}_{(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))^-}\mathcal{F}'(m\mathtt{w}_1)Q(m\mathtt{w}_1)\Big)$$
$$-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)}[\mathcal{F}(m\mathtt{w}_1)Q(m\mathtt{w}_1)-Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\mathcal{F}(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))]\Big|$$
$$\leq\left(\frac{2}{1-2\alpha}\right)^{1-\frac{1}{q}}\left(\left[m|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q\right]\right.$$
$$\left.\times\int_0^1\wp^{-2\alpha}[\mathcal{F}(m\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1,m))]^q[h(\wp)+h(1-\wp)]d\wp\right)^{\frac{1}{q}}.$$

Theorem 128 ([98]). *Assume that $\mathbb{X}$ and $\mathcal{F}$ are defined as in Theorem 122. If $|Q'|^q$—where $q>1$—is generalized (h,m)-preinvex on $\mathbb{H}$, then*

$$\Bigg|\frac{2\alpha}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^{-2\alpha}}\Big({}_{N_3}J^{2\alpha-1}_{m\mathtt{w}_1^+}\mathcal{F}(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))$$
$$+_{N_3}J^{2\alpha-1}_{(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))^-}\mathcal{F}(m\mathtt{w}_1)Q(m\mathtt{w}_1)\Big)$$
$$-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)^{1-2\alpha}}\Big({}_{N_3}J^{2\alpha}_{m\mathtt{w}_1^+}\mathcal{F}'(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))$$
$$+_{N_3}J^{2\alpha}_{(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))^-}\mathcal{F}'(m\mathtt{w}_1)Q(m\mathtt{w}_1)\Big)$$
$$-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1,m)}[\mathcal{F}(m\mathtt{w}_1)Q(m\mathtt{w}_1)-Q(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))\mathcal{F}(m\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1,m))]\Bigg|$$
$$\leq\frac{2}{(1-2\alpha\rho)^{\frac{1}{p}}}\left(\left[m|Q'(\mathtt{w}_1)|^q+|Q'(\mathtt{w}_2)|^q\right]\cdot\int_0^1[\mathcal{F}(m\mathtt{w}_1+\wp\Phi(\mathtt{w}_2,\mathtt{w}_1,m))]^qh(\wp)d\wp\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

11. Fejér-Type Fractional Integral Inequalities via Katugampola Fractional Integral

Definition 36 ([99]). *Let $[\mathtt{w}_1,\mathtt{w}_2]\subset\mathbb{R}$ be a finite interval. Then, the left- and right-side Katugampola fractional integrals of order $\alpha>0$ of $Q\in X^p_c(\mathtt{w}_1,\mathtt{w}_2)$ are defined by*

$${}^\rho I^\alpha_{\mathtt{w}_1+}Q(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mathtt{w}_1}^x\frac{t^{\rho-1}}{(x^\rho-t^\rho)^{1-\alpha}}Q(t)dt \ \text{ and } \ {}^\rho I^\alpha_{\mathtt{w}_2-}Q(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^{\mathtt{w}_2}\frac{t^{\rho-1}}{(t^\rho-x^\rho)^{1-\alpha}}Q(t)dt,$$

with $\mathtt{w}_1<x<\mathtt{w}_2$ and $\rho>0$, if the integrals exist. Here, $X^p_c(\mathtt{w}_1,\mathtt{w}_2)$, $c\in\mathbb{R}$, $1\leq p\leq\infty$ denote the space of those complex-valued Lebesque measurable functions Q on $[\mathtt{w}_1,\mathtt{w}_2]$ for which $\|Q\|_{X^p_c}<\infty$, where $\|Q\|_{X^p_c}=\left(\int_{\mathtt{w}_1}^{\mathtt{w}_2}|t^cQ(t)|^p\frac{dt}{t}\right)^{1/p}<\infty$ for $1\leq p<\infty$ and $\|Q\|_{X^p_c}=\operatorname{ess\,sup}_{\mathtt{w}_1\leq t\leq\mathtt{w}_2}[t^c|Q(t)|]$, if $p=\infty$.

Fejér-type fractional integral inequalities via Katugampola fractional integral are given in the next theorems.

Theorem 129 ([100]). *Let $f:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ be convex function with $\mathtt{w}_1<\mathtt{w}_2$ and $f\in L[\mathtt{w}_1,\mathtt{w}_2]$. Then, $Q(x)$ is also convex and $Q\in L[\mathtt{w}_1,\mathtt{w}_2]$. If $F:[\mathtt{w}_1,\mathtt{w}_2]\to\mathbb{R}$ is non-negative and integrable function, then*

$$Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right)\left[{}^\rho I^\alpha_{\mathtt{w}_1+}F(\mathtt{w}_2)+{}^\rho I^\alpha_{\mathtt{w}_2-}F(\mathtt{w}_1)\right]\leq\left[{}^\rho I^\alpha_{\mathtt{w}_1+}(QF)(\mathtt{w}_2)+{}^\rho I^\alpha_{\mathtt{w}_2-}(QF)(\mathtt{w}_1)\right]$$
$$\leq\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[{}^\rho I^\alpha_{\mathtt{w}_1+}F(\mathtt{w}_2)+{}^\rho I^\alpha_{\mathtt{w}_2-}F(\mathtt{w}_1)\right],$$

with $\alpha > 0$ *and* $\rho > 0$.

Theorem 130 ([100]). *Let* $f : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *be differentiable function on* $(\mathtt{w}_1, \mathtt{w}_2)$ *and* $f' \in L[\mathtt{w}_1, \mathtt{w}_2]$ *with* $\mathtt{w}_1 < \mathtt{w}_2$. *Then* $Q(x)$ *is also differentiable and* $Q' \in L[\mathtt{w}_1, \mathtt{w}_2]$. *If* $Q(x)$ *is convex and* $F : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *is a continuous function, then*

$$\left|\frac{Q(\mathtt{w}_1)+Q(\mathtt{w}_2)}{2}\left[{}^{\rho}I^{\alpha}_{\mathtt{w}_1+}F(\mathtt{w}_2)+{}^{\rho}I^{\alpha}_{\mathtt{w}_2-}F(\mathtt{w}_1)\right]-\left[{}^{\rho}I^{\alpha}_{\mathtt{w}_1+}(QF)(\mathtt{w}_2)+{}^{\rho}I^{\alpha}_{\mathtt{w}_2-}(QF)(\mathtt{w}_1)\right]\right|$$
$$\leq \frac{(\mathtt{w}_2-\mathtt{w}_1)||F||_\infty}{\rho^\alpha\Gamma(\alpha+1)}(|Q'(\mathtt{w}_1)+|Q'(\mathtt{w}_2)|)\int_0^1|K(t)dt|$$

with $\alpha > 0$ *and* $\rho > 0$, *where* $||F||_\infty = \sup_{t\in[\mathtt{w}_1,\mathtt{w}_2]}|F(x)|$ *and* $K(t) = [((1-t)\mathtt{w}_1 + \mathtt{w}_2 t)^\rho - \mathtt{w}_1{}^\rho]^\alpha$ $-[\mathtt{w}_2{}^\rho - ((1-t)\mathtt{w}_1 + \mathtt{w}_2 t)^\rho]^\alpha$.

In the following, we present a Fejér-type fractional integral inequality for (k,h)-convex functions via a Katugampola fractional integral.

Theorem 131 ([101]). *Let* $Q : D \to \mathbb{R}$ *be a* (k,h)*-convex function with* $h(1/2) > 0$. *Assume that* $0 < \mathtt{w}_1 < \mathtt{w}_2$ *such that* $[\mathtt{w}_1, \mathtt{w}_2] \subset D$ *and* $Q \in L[\mathtt{w}_1, \mathtt{w}_2]$. *If* $Q : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *is a non-negative function which is symmetric with respect to* $\dfrac{\mathtt{w}_1+\mathtt{w}_2}{2}$ *and* $g : [\mathtt{w}_1, \mathtt{w}_2] \to \mathbb{R}$ *is non-negative and integrable, then*

$$\frac{Q(k(1/2)(\mathtt{w}_1+\mathtt{w}_2))}{h(1/2)}\left[{}^{\rho}I^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_2)+{}^{\rho}I^{\alpha}_{\mathtt{w}_2-}g(\mathtt{w}_1)\right]\leq\left[{}^{\rho}I^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_2)+{}^{\rho}I^{\alpha}_{\mathtt{w}_2-}(Qg)(\mathtt{w}_1)\right],$$

for $\alpha > 0$ *and* $\rho > 0$.

12. Fejér-Type Fractional Integral Inequalities for Invex Functions

Fejér-Type Fractional Integral Inequalities for (Φ_1, Φ_2)*-Convex Functions*

Definition 37 ([102]). *A set* $I \subseteq \mathbb{R}$ *is invex with respect to a real bifunction* $\Phi : I \times I \to \mathbb{R}$, *if*

$$x, y \in I, \ \ \lambda \in [0,1] \Rightarrow y + \lambda\Phi(x,y) \in I.$$

Definition 38 ([102]). *Let* $I \subseteq \mathbb{R}$ *be an invex set with respect to* $\Phi_1 : I \times I \to \mathbb{R}$. *Consider* $Q : I \to \mathbb{R}$ *and* $\Phi_2 : Q(I) \times Q(I) \to \mathbb{R}$. *The function* Q *is said to be* (Φ_1, Φ_2)*-convex if*

$$Q(x + \lambda\Phi_1(y,x)) \leq Q(x) + \lambda\Phi_2(Q(y), Q(x)),$$

for all $x, y \in I$ *and* $\lambda \in [0,1]$.

Fejér-type fractional integral inequalities for (Φ_1, Φ_2)-convex functions are presented in the following theorems.

Theorem 132 ([102]). *Let* $I \subseteq \mathbb{R}$ *be an invex set with respect to* Φ_1 *such that*

$$\Phi_1(x_2 + t_2\Phi_1(x_1,x_2), x_2 + t_1\Phi_1(x_1,x_2)) = (t_2 - t_1)\Phi_1(x_1,x_2),$$

for all $x_1, x_2 \in I$ *and* $t_1, t_2 \in [0,1]$. *Also, let* $Q : I \to \mathbb{R}$ *be an* (Φ_1, Φ_2)*-convex function, where* Φ_2 *is an integrable bifunction on* $Q(I) \times Q(I)$. *For any* $\mathtt{w}_1, \mathtt{w}_2 \in I$ *with* $\Phi_1(\mathtt{w}_2, \mathtt{w}_1) > 0$, *suppose that* $Q \in L_1[\mathtt{w}_1, \mathtt{w}_1 + \Phi_1(\mathtt{w}_2, \mathtt{w}_1)]$ *and the function* $g : [\mathtt{w}_1, \mathtt{w}_1 + \Phi_1(\mathtt{w}_2, \mathtt{w}_1)] \to \mathbb{R}^+$ *are integrable and symmetric to* $\mathtt{w}_1 + (1/2)\Phi_1(\mathtt{w}_2, \mathtt{w}_1)$. *Then,*

$$Q\left(\frac{2\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1)}{2}\right)\left[J^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1))+J^{\alpha}_{(\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1))-}g(\mathtt{w}_1)\right]$$

$$-\frac{1}{2\Gamma(\alpha)}\int_{\mathtt{w}_1}^{\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1)}\Big[(x-\mathtt{w}_1)^{\alpha-1}+(\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1)-x)^{\alpha-1}\Big]$$
$$\times\Phi_2(Q(x),Q(2\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1)-x))g(x)dx$$
$$\leq \Big[J^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1))+J^{\alpha}_{(\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1))-}(Qg)(\mathtt{w}_1)\Big].$$

Theorem 133 ([102]). *Let $I\subseteq\mathbb{R}$ be an invex set with respect to Φ_1 let $Q:I\to\mathbb{R}$ be an (Φ_1,Φ_2)-convex function, where Φ_2 is an integrable bifunction on $Q(I)\times Q(I)$. For any $\mathtt{w}_1,\mathtt{w}_2\in I$ with $\Phi_1(\mathtt{w}_2,\mathtt{w}_1)>0$, suppose that $g:[\mathtt{w}_1,\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1)]\to\mathbb{R}^+$ is integrable and symmetric to $\mathtt{w}_1+(1/2)\Phi_1(\mathtt{w}_2,\mathtt{w}_1)$ and $Q\in L_1[\mathtt{w}_1,\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1)]$. Then*

$$\Big[J^{\alpha}_{\mathtt{w}_1+}(Qg)(\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1))+J^{\alpha}_{(\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1))-}(Qg)(\mathtt{w}_1)\Big]$$
$$\leq \left(\frac{2Q(\mathtt{w}_1)+\Phi_2(Q(\mathtt{w}_2),Q(\mathtt{w}_1))}{2}\right)\Big[J^{\alpha}_{\mathtt{w}_1+}g(\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1))+J^{\alpha}_{(\mathtt{w}_1+\Phi_1(\mathtt{w}_2,\mathtt{w}_1))-}g(\mathtt{w}_1)\Big].$$

Definition 39 ([103]). *Let $K\subseteq\mathbb{R}^n$ be an invex set with respect to $\Phi:K\times K\to\mathbb{R}^n$. A function $Q:K\to\mathbb{R}$ is said to be (s,m)-preinvex with respect to Φ for every $x,y\in K$, $t\in[0,1]$ and $m\in(0,1]$, if*

$$Q(\mathtt{w}_1+t\Phi(\mathtt{w}_2,\mathtt{w}_1))\leq m(1-t)^sQ\Big(\frac{\mathtt{w}_1}{m}\Big)+t^sQ(\mathtt{w}_2).$$

Theorem 134 ([103]). *Let $K\subseteq\mathbb{R}^n$ be an open invex subset with respect to $\Phi:K\times K\to\mathbb{R}$ and $\mathtt{w}_1,\mathtt{w}_2\in K$ with $\mathtt{w}_1<\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)$ where $\Phi(\mathtt{w}_2,\mathtt{w}_1)\neq 0$. Suppose $Q:K\to\mathbb{R}$ is a differentiable mapping on K° such that $Q'\in L_1[\mathtt{w}_1,\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)]$. If $w:[\mathtt{w}_1,\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)]\to[0,\infty)$ is a continuous mapping and symmetric to $\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)$, and Q' is (s,m)-preinvex on K, then*

$$\left|\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}\int_{\mathtt{w}_1}^{\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)}Q(x)w(x)dx-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}Q\Big(\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)\Big)\int_{\mathtt{w}_1}^{\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)}w(x)dx\right|$$
$$\leq \frac{\|w\|_\infty}{(s+1)(s+2)}\Big[m\Big|Q'\Big(\frac{\mathtt{w}_1}{m}\Big)\Big|+|Q'(\mathtt{w}_2)|\Big]\Big(1-\frac{1}{2^{s+1}}\Big).$$

Theorem 135 ([103]). *Assume that Q is as in Theorem 134 and $|Q'|^q, q>1$ is (s,m)-preinvex on K, then fractional integral inequality is given as:*

$$\left|\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}\int_{\mathtt{w}_1}^{\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)}Q(x)w(x)dx-\frac{1}{\Phi(\mathtt{w}_2,\mathtt{w}_1)}Q\Big(\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)\Big)\int_{\mathtt{w}_1}^{\mathtt{w}_1+\Phi(\mathtt{w}_2,\mathtt{w}_1)}w(x)dx\right|$$
$$\leq \Phi(\mathtt{w}_2,\mathtt{w}_1)\left(\frac{1}{(\Phi(\mathtt{w}_2,\mathtt{w}_1))^2}\int_{\mathtt{w}_1}^{\mathtt{w}_1+\frac{1}{2}\Phi(\mathtt{w}_2,\mathtt{w}_1)}\Big[\frac{\Phi(\mathtt{w}_2,\mathtt{w}_1)}{2}-(x-\mathtt{w}_1)\Big]w^p(x)dx\right)^{\frac{1}{p}}$$
$$\times\Bigg[\left(m\Big|Q'\Big(\frac{\mathtt{w}_1}{m}\Big)\Big|^q\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}+|Q'(\mathtt{w}_2)|^q\frac{1}{2^{s+2}(s+2)}\right)^{\frac{1}{q}}$$
$$+\left(m\Big|Q'\Big(\frac{\mathtt{w}_1}{m}\Big)\Big|^q\frac{1}{2^{s+2}(s+2)}+|Q'(\mathtt{w}_2)|^q\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}\right)^{\frac{1}{q}}\Bigg],$$

where $\frac{1}{p}+\frac{1}{q}=1.$

13. Fejér-Type Fractional Integral Inequalities via (p,q)-Calculus

We first give the basic concepts of (p,q)-calculus which will be used in our work. Throughout this section, we let $I=[\mathtt{w}_1,\mathtt{w}_2]\subseteq\mathbb{R}$, $I^0=(\mathtt{w}_1,\mathtt{w}_2)$, and $0<q<p\leq 1$ be constants.

Definition 40 ([104,105]). *Let $Q : I \to \mathbb{R}$ be a continuous function. The (p,q)-derivative of the function Q on $[\mathtt{w}_1, \mathtt{w}_2]$ at x is defined by*

$$ {}_{\mathtt{w}_1}D_{p,q}Q(x) = \frac{Q(px + (1-p)\mathtt{w}_1) - Q(qx + (1-q)\mathtt{w}_1)}{(p-q)(x-\mathtt{w}_1)}, \qquad x \neq \mathtt{w}_1. $$

Since Q is continuous function, we have ${}_{\mathtt{w}_1}D_{p,q}Q(\mathtt{w}_1) = \lim_{x \to \mathtt{w}_1} {}_{\mathtt{w}_1}D_{p,q}Q(x)$.

We say that Q is (p,q)-differentiable on I provided that ${}_{\mathtt{w}_1}D_{p,q}Q(x)$ exists for all $x \in I$. In Definition 40, if $a = 0$, then ${}_0D_{p,q}Q = D_{p,q}f$, where $D_{p,q}f$ is defined by:

$$ D_{p,q}Q(x) = \frac{Q(px) - Q(qx)}{(p-q)x}, \quad x \neq 0. $$

Furthermore, if $p = 1$, then ${}_{\mathtt{w}_1}D_{p,q}Q = {}_{\mathtt{w}_1}D_qQ$ which is the q-derivative of the function Q.

Definition 41 ([104,105]). *If $Q : I \to \mathbb{R}$ is a continuous function, then the (p,q)-integral of the function Q for $x \in I$ is defined by*

$$ \int_{\mathtt{w}_1}^{x} Q(t)_{\mathtt{w}_1}d_{p,q}t = (p-q)(x-\mathtt{w}_1)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} Q\left(\frac{q^n}{p^{n+1}}x + \left(1 - \frac{q^n}{p^{n+1}}\right)\mathtt{w}_1\right). $$

Moreover, if $c \in (\mathtt{w}_1, x)$, then the (p,q)-integral is defined by

$$ \int_{c}^{x} Q(t)_{\mathtt{w}_1}d_{p,q}t = \int_{\mathtt{w}_1}^{x} Q(t)_{\mathtt{w}_1}d_{p,q}t - \int_{\mathtt{w}_1}^{c} Q(t)_{\mathtt{w}_1}d_{p,q}t. $$

We say that Q is (p,q)-integrable on I provided that $\int_{\mathtt{w}_1}^{x} Q(t)_{\mathtt{w}_1}d_{p,q}t$ exists for all $x \in I$. Note that, if $\mathtt{w}_1 = 0$ and $p = 1$, then Definition 41 reduces to the q-integral of function Q.

Next, we give some Fejér-type fractional integral inequalities by using the (p,q)-integral.

Theorem 136 ([106]). *If $Q : I \to \mathbb{R}$ is a twice (p,q)-differentiable function such that there exist real constants m and M so that $m \leq Q'' \leq M$, then*

$$ \begin{aligned} \frac{mpq^2(\mathtt{w}_2-\mathtt{w}_1)^2}{(p+q)(p^2+pq+q^2)} &\leq [Q(\mathtt{w}_1) + Q(\mathtt{w}_2)] - \frac{1}{p(\mathtt{w}_2-\mathtt{w}_1)} \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} [Q(x) + Q(\mathtt{w}_1+\mathtt{w}_2-x)]\ {}_{\mathtt{w}_1}d_{p,q}x \\ &\leq \frac{Mpq^2(\mathtt{w}_2-\mathtt{w}_1)^2}{(p+q)(p^2+pq+q^2)}, \end{aligned} $$

and

$$ \begin{aligned} &\frac{m(\mathtt{w}_2-\mathtt{w}_1)^2}{4}\left[1 - \frac{4p}{p+q} + \frac{4p^2}{p^2+pq+q^2}\right] \\ &\quad \leq \frac{1}{p(\mathtt{w}_2-\mathtt{w}_1)} \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} [Q(x) + Q(\mathtt{w}_1+\mathtt{w}_2-x)]\ {}_{\mathtt{w}_1}d_{p,q}x - 2Q\left(\frac{\mathtt{w}_1+\mathtt{w}_2}{2}\right) \\ &\quad \leq \frac{M(\mathtt{w}_2-\mathtt{w}_1)^2}{4}\left[1 - \frac{4p}{p+q} + \frac{4p^2}{p^2+pq+q^2}\right]. \end{aligned} $$

Theorem 137 ([106]). *Let $Q : I \to \mathbb{R}$ be a twice (p,q)-differentiable function such that there exist real constants m and M so that $m \leq Q'' \leq M$. If $g : I \to \mathbb{R}$ is non-negative, (p,q)-integrable on I, and symmetric about $x = (\mathtt{w}_1+\mathtt{w}_2)/2$, then*

$$ m\int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} (x-\mathtt{w}_1)(\mathtt{w}_2-x)g(x)\ {}_{\mathtt{w}_1}d_{p,q}x $$

$$\leq [Q(\mathtt{w}_1) + Q(\mathtt{w}_2)] \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} g(x)\ {}_{\mathtt{w}_1}d_{p,q}x$$
$$- \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} [Q(x) + Q(\mathtt{w}_1 + \mathtt{w}_2 - x)]g(x)\ {}_{\mathtt{w}_1}d_{p,q}x$$
$$\leq M \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} (x - \mathtt{w}_1)(\mathtt{w}_2 - x)g(x)\ {}_{\mathtt{w}_1}d_{p,q}x,$$

and

$$\frac{m}{4} \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} (2x - a - b)^2 g(x)\ {}_{\mathtt{w}_1}d_{p,q}x$$
$$\leq \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} [Q(x) + Q(\mathtt{w}_1 + \mathtt{w}_2 - x)]g(x)\ {}_{\mathtt{w}_1}d_{p,q}x$$
$$- 2Q\left(\frac{\mathtt{w}_1 + \mathtt{w}_2}{2}\right) \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} g(x)\ {}_{\mathtt{w}_1}d_{p,q}x$$
$$\leq \frac{M}{4} \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} (2x - a - b)^2 g(x)\ {}_{\mathtt{w}_1}d_{p,q}x.$$

Theorem 138 ([106]). *If $Q : I \to \mathbb{R}$ is a twice (p,q)-differentiable function with ${}_{\mathtt{w}_1}D^2_{p,q}Q$ (p,q)-integrable on I and $m \leq {}_{\mathtt{w}_1}D^2_{p,q}Q \leq M$, then*

$$\frac{mp^3q^2(\mathtt{w}_2 - \mathtt{w}_1)^3}{(p+q)(p^2+pq+q^2)} \leq \frac{q}{p}(\mathtt{w}_2 - \mathtt{w}_1)Q_{pq}(\mathtt{w}_2, \mathtt{w}_1) + (\mathtt{w}_2 - \mathtt{w}_1)Q(\mathtt{w}_1)$$
$$- \left(\frac{p+q}{p^2}\right) \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} f_{q^2}(x, \mathtt{w}_1)\ {}_{\mathtt{w}_1}d_{p,q}x$$
$$\leq \frac{Mp^3q^2(\mathtt{w}_2 - \mathtt{w}_1)^3}{(p+q)(p^2+pq+q^2)},$$

and

$$\frac{m(p^4 + 2p^3q + pq^3)(\mathtt{w}_2 - \mathtt{w}_1)^3}{(p+q)(p^2+pq+q^2)}$$
$$\leq (\mathtt{w}_2 - \mathtt{w}_1)^2 \left({}_{\mathtt{w}_1}D_{p,q}Q_p(\mathtt{w}_2, \mathtt{w}_1) - {}_{\mathtt{w}_1}D_{p,q}Q(\mathtt{w}_1)\right) - \frac{2q}{p^2}(\mathtt{w}_2 - \mathtt{w}_1)Q_q(\mathtt{w}_2, \mathtt{w}_1)$$
$$- \frac{2}{p}(\mathtt{w}_2 - \mathtt{w}_1)Q(\mathtt{w}_1) + 2\left(\frac{p+q}{p^3}\right) \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} Q_{q^2}(x, \mathtt{w}_1)\ {}_{\mathtt{w}_1}d_{p,q}x$$
$$\leq \frac{M(p^4 + 2p^3q + pq^3)(\mathtt{w}_2 - \mathtt{w}_1)^3}{(p+q)(p^2+pq+q^2)}.$$

Theorem 139 ([106]). *If $Q : I \to \mathbb{R}$ is a twice (p,q)-differentiable function with ${}_{\mathtt{w}_1}D^2_{p,q}Q$ (p,q)-integrable on I and $m \leq {}_{\mathtt{w}_1}D^2_{p,q}Q \leq M$, then*

$$\left| \left(\frac{q}{p^2}(\mathtt{w}_2 - \mathtt{w}_1)Q_{pq}(\mathtt{w}_2, \mathtt{w}_1) + \frac{1}{p}(\mathtt{w}_2 - \mathtt{w}_1)Q(\mathtt{w}_1) - \left(\frac{p+q}{p^3}\right) \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} Q_{q^2}(x, \mathtt{w}_1)\ {}_{\mathtt{w}_1}d_{p,q}x\right) \right.$$
$$\left. - \frac{pq^2(\mathtt{w}_2 - \mathtt{w}_1)^2({}_{\mathtt{w}_1}D_{p,q}Q_p(\mathtt{w}_2, \mathtt{w}_1) - {}_{\mathtt{w}_1}D_{p,q}Q(\mathtt{w}_1))}{(p+q)(p^2+pq+q^2)} \right| \leq \frac{p(\mathtt{w}_2 - \mathtt{w}_1)^3}{16}(M - m).$$

Theorem 140 ([106]). *If $Q : I \to \mathbb{R}$ is a twice (p,q)-differentiable function with ${}_{\mathtt{w}_1}D^2_{p,q}Q$ (p,q)-integrable on I and $m \leq {}_{\mathtt{w}_1}D^2_{p,q}Q \leq M$, then*

$$\frac{mp^5q(\mathtt{w}_2 - \mathtt{w}_1)^3}{(p+q)^2(p^2+pq+q^2)} \leq \frac{p(\mathtt{w}_2 - \mathtt{w}_1)(qQ_p(\mathtt{w}_2, \mathtt{w}_1) + pQ(\mathtt{w}_1))}{p+q} - \int_{\mathtt{w}_1}^{p\mathtt{w}_2+(1-p)\mathtt{w}_1} Q_q(x, \mathtt{w}_1)\ {}_{\mathtt{w}_1}d_{p,q}x$$

$$\leq \frac{Mp^5q(\mathtt{w}_2-\mathtt{w}_1)^3}{(p+q)^2(p^2+pq+q^2)}.$$

14. Conclusions

Fractional calculus has significant importance and several uses in applied mathematics. Researchers and mathematicians in the theory of inequalities have employed fractional calculus operators to investigate and explore specific estimations and developments. Fractional integral operators are remarkable and meaningful applicable tools for generalizing classical integral inequalities. Due to its many applications and uses, fractional calculus has captured the attention and excitement of many mathematicians. Considering its utility in the mathematical modeling of various complicated and non-local nonlinear systems, fractional calculus has developed to be a crucial field of investigation. It is critical while exploring optimization problems since it has a variety of advantageous inequalities.

The main intention and aim of this review paper were to deliver an in-depth and current assessment of Fejér-type integral inequalities. We offered numerous results of fractional Fejér-type integral inequality via convexity.

We anticipate that this study will inspire as well as offer a forum for scholars pursuing Fejér-type inequalities to acquire information about prior research on the topic before drawing fresh conclusions. This review paper's future research is promising and might motivate numerous additional studies.

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