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Solutions for Some Specific Mathematical Physics Problems Issued from Modeling Real Phenomena: Part 2

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Abstract: This paper brings together methods to solve and/or characterize solutions of some problems of mathematical physics equations involving p-Laplacian and p-pseudo-Laplacian. Using the widely debated results of surjectivity or variational approaches, one may obtain or characterize weak solutions for Dirichlet or Newmann problems for these important operators. The relevance of these operators and the possibility to be involved in the modeling of an important class of real phenomena is once again revealed by their applications. The use of certain variational methods facilitates the complete solution of the problem using appropriate numerical methods and computational algorithms. Some theoretical results are involved to complete the solutions for a sequence of models issued from real phenomena drawing.

Keywords: modeling real phenomena; p-Laplacian; p-pseudo-Laplacian; surjectivity methods; variational methods; Dirichlet problem; Neumann problem

MSC: 35A01; 35A15; 35J35; 35J40; 47J30

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Citation: Meghea, I. Solutions for Some Specific Mathematical Physics Problems Issued from Modeling Real Phenomena: Part 2. *Axioms* **2023**, *12*, 726. https://doi.org/10.3390/axioms12080726

Academic Editor: Hans J. Haubold

Received: 2 June 2023
Revised: 15 July 2023
Accepted: 22 July 2023
Published: 26 July 2023

1. Introduction

This paper is devoted to obtaining some solutions for special problems of mathematical physics equations involving p-Laplacian and p-pseudo-Laplacian for models issued from real phenomena. This is still a current topic that is very widely debated, as is proven in papers [1–4] considering the p-Laplacian where the accent falls on the applications and the importance of the models. Positive solutions for boundary value problems with p-Laplacian are obtained and discussed in [5–7].

Global weak solutions for evolution problems with logarithmic nonlinearities involving the p-pseudo-Laplacian are presented in [7–10], and those with p-Laplacian are studied in [11]. The existence of periodic solution [12], radial symmetry [13], symmetry [14], or principal eigenvalues [15] for fractional p-Laplacian, minimizers [16], and Picone type identities for p-Laplacian [17] or p-pseudo-Laplacian [18], regularity, and multiplicity results [19] represent interesting and topical issues that have important implications.

This paper is the requisite continuation of the work under the same title, part one [20]. The theoretical framework is constructed and correspondingly extended in [20] in order to understand the entire abstract background, while the next step towards the real problems is made by applying the main results to propose solutions and/or their characterizations for real phenomena models. In this work, problems for partial differential equations involving the p-Laplacian and p-pseudo-Laplacian are studied considering Dirichlet problems derived from glaciology, nonlinear elastic membrane with p-Laplace or p-pseudo-Laplace equation, vibration of a nonhomogeneous membrane fixed along the boundary, and pseudo torsion problems are solved using propositions from [20] obtained *via* surjectivity results. Fredholm alternative type results are used to solve applications for nonlinear elastic membrane with p-Laplacian and p-pseudo-Laplacian. Surjectivity to different homogeneity degrees also provides some statements which are involved here to provide the formulation of the existence of the solution for nonlinear elastic membrane for both operators, but under different conditions. Critical points and weak solutions for elliptic type equations prove existing propositions applied for real models issued from glaciology, nonlinear elastic membrane, and the pseudo torsion problem. Critical points for nondifferentiable functionals provided the appropriate frame to characterize the solution in modeling the thermal transfer, and also some of the problems mentioned above. Other results obtained through the Ekeland variational principle and some conditions of the Palais-Smale type provide solving methods for some problems from glaciology, injection mold filling, thermal transfer, or the pseudo torsion problem. Finally, the last frame, which originated in a perturbed variational principle, was provided to intervene in the study of a generalized Helle-Show flow of a power-law fluid, and also for the pseudo torsion problem. For some of the problems derived from real phenomena modeling presented in this work, the author proposed different solving methods to those reported in the paper [21]. Starting from theoretical results based on a namely version of Mountain Pass Theorem, the study [21] aimed to gain a better understanding of the passage from the high abstract frame to characterize the solution, with designing new models for special physical phenomena being the final goal. Some other applications for problems involving the p-Laplacian having the theoretical background in a perturbed variational principle have been presented by the author in [22].

For these problems that resulted from the mathematical modeling, we came up with original solving methods following the abstract frame from [20]. The novelty of this work stems from these original approaches, as it represents one of the preliminary studies for future developments to find a mathematical model (there are none available) to be applied for describing diffusion phenomena involved in micro-emulsification of disperse systems in tight connection with surface properties at the interface in self-organized systems.

This paper is organized in the following manner: the propositions or the theorems which are directly related to the discussed problems are briefly remembered, followed by the model derived from the real phenomenon considered. For any theoretical detail, the paper [20] can be consulted.

2. Surjectivity for the Operators $\lambda J_\phi - S$. Applications to Partial Differential Equations

2.1. Theoretical Results

We start this section displaying the (final) theoretical results widely presented in [20] which are directly involved in solving some types of mathematical physics problems and characterization of their solutions for a few models drawing real phenomena.

The role of surjectivity in solving problems of the following types was highlighted in [20], and one can cite recent papers such as [23] where new surjectivity (based on Ekeland variational principle) results are discussed and involved in applications for Dirichlet problems for p-Laplacian.

Statement for the Problem

$$(*)\begin{cases} -\lambda\, \Delta_{\mathrm{p}}\, u = f(\cdot\,,\ u(\cdot)) + h, x \in \Omega, \lambda \in \mathbf{R} \\ u \mid \partial\Omega = 0 \end{cases}$$

Proposition 1. *Let Ω be an open bounded set of C^1class from $\mathbf{R}^{\mathrm{N}}$, $N \geq 2$, $p \in (1, +\infty)$, h from $W^{-1,p'}(\Omega)$ and f: $\Omega \times \mathbf{R} \to \mathbf{R}$ a Carathéodory function with the properties*

1^0 $f(x, -s) = -f(x, s)$ $\forall s$ *from* $\mathbf{R}$, $\forall x$ *from* Ω,

2^0 $\mid f(x, s) \mid \leq c_1|s|^{q-1} + \beta(x)$ $\forall s$ *from* $\mathbf{R}$, $\forall x$ *from* $\Omega \setminus A$, $\mu(A) = 0$,

where $c_1 \geq 0$, $q \in (1, p)$, $\beta \in L^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Then, for any $\lambda \neq 0$, *the problem* $(*)$ *has solution in* $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$.

Proposition 2. *Let Ω be an open bounded set of C^1 class from $\mathbf{R}^N$, $N \geq 2$, $p \in (1, +\infty)$, h from $W^{-1,p'}(\Omega)$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ Carathéodory function having the properties*

1^0 $f(x, -s) = -f(x, s)$ $\forall x$ *from* Ω, $\forall s$ *from* $\mathbf{R}$,
2^0 $| f(x, s) | \leq c_1|s|^{p-1} + \beta(x)$ $\forall s$ *from* $\mathbf{R}$, $\forall x$ *from* $\Omega \setminus A$, $\mu(A) = 0$,

where $c_1 \geq 0$, $\beta \in L^{p'}(\Omega), \frac{1}{p} + \frac{1}{p'} = 1$.

Finally, let $i : W_0^{1,p}(\Omega) \to L^p(\Omega)$ *be linear compact embedding. Then, for any* λ, *if*

$$|\lambda| > c_1 \lambda_1^{-1},\ \lambda_1 := \inf\left\{ \frac{\left|\left|u\right|\right|_{1,p}^{p}}{\left|\left|i(u)\right|\right|_{0,p}^{p}} : u \in W_0^{1,p}(\Omega)\backslash\{0\} \right\},$$

the problem $(*)$ *has solution in* $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$.

Statement for the Problem

$$(**)\begin{cases} -\lambda\,\Delta_p^s u = f(\cdot, u(\cdot)) + h,\ x \in \Omega,\ \lambda \in \mathbf{R} \\ u \mid \partial\Omega \ = \ 0 \end{cases}$$

Proposition 3. *Let Ω be an open bounded of C^1 class set from $\mathbf{R}^N$, $N \geq 2$, $p \in [2, +\infty)$, h from $W^{-1,p'}(\Omega)$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ Carathéodory function with the properties*

1^0 $f(x, -s) = -f(x, s)$ $\forall x$ *from* Ω, $\forall s$ *from* $\mathbf{R}$,
2^0 $| f(x, s) | \leq c_1|s|^{q-1} + \beta(x)$ $\forall s$ *from* $\mathbf{R}$, $\forall x$ *from* $\Omega \setminus A$, $\mu(A) = 0$,

where $c_1 \geq 0$, $q \in (1, p)$, $\beta \in L^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Then, for any $\lambda \neq 0$, *the problem* $(**)$ *has solution in* $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$.

Proposition 4. *Let Ω be an open bounded set of C^1class from $\mathbf{R}^N$, $N \geq 2$, $p \in [2, +\infty)$, h from $W^{-1,p'}(\Omega)$ and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ Carathéodory function having the properties*

1^0 $f(x, -s) = -f(x, s)$ $\forall x$ *from* Ω, $\forall s$ *from* $\mathbf{R}$,
2^0 $| f(x, s)| \leq c_1|s|^{p-1} + \beta(x)$ $\forall s$ *from* $\mathbf{R}$, $\forall x$ *from* $\Omega \setminus A$, $\mu(A) = 0$,

where $c_1 \geq 0$, $\beta \in L^{p'}(\Omega), \frac{1}{p} + \frac{1}{p'} = 1$.

Finally, let $i : W_0^{1,p}(\Omega) \to L^p(\Omega)$ *be linear compact embedding. Then, for any* λ, *if*

$$|\lambda| > c_1 \lambda_1^{-1},\ \lambda_1 := \inf\left\{ \frac{\| u \|_{1,p}^{p}}{\left|\left|i(u)\right|\right|_{0,p}^{p}} : u \in W_0^{1,p}(\Omega)\backslash\{0\} \right\},$$

the problem $(**)$ *has solution in* $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$.

2.2. Applications in Solving some Real Models

Apply propositions 1, 2, 3, and 4 to solve problems issued from real phenomena mo-deling.

2.2.1. Application in Glaciology

Elliptic partial differential equations with the principal part $\Delta_p u$ are applied in physics, for instance, the description of phenomena in glaciology as the sliding of glaciers, see [24–26]. In [27], a Dirichlet problem is formulated:

$$-\Delta_p u = 1 \text{on} \Omega,$$

$$u = \varphi \text{ on } \partial\Omega,$$

with $p = 1+\frac{1}{g}$, g being the Glen exponent, where $0 < g < +\infty$, hence $0 < p < +\infty$ and, for a real glacier, $g \geq 3$, and usually taken, $g = 3$, so $p = \frac{4}{3}$. Replacing u by $v = u - \varphi$ on the boundary of Ω in the sense of the trace and $u \equiv v$ on the interior of Ω, one can apply Proposition 1 in the particular case $f \equiv 0$, $h \equiv 1$, $\lambda = 1$ (hence $\lambda \neq 0$) to give another proof for the existence of the solution of the above problem.

Novel and interesting applications, completely solved *via* special variational and numerical methods, can be consulted in [28].

2.2.2. Nonlinear Elastic Membrane

I. To describe a nonlinear elastic membrane under the load f, we can use the mathematical model:

$$-\Delta_p u = f \text{on} \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

The solution u stands for the deformation of the membrane from the rest position [29,30]. The nonlinearity of the p-Laplacian is often used to reflect the impact of non-ideal material to the usual vibrating homogeneous elastic membrane modeled by Laplace operator. In this case, the deformation energy is given by $\int_\Omega |\nabla u|^{\mathrm{p}} dx$. Therefore, the greatest lower bound on $W_0^{1,p}(\Omega)\backslash\{0\}$, λ_1, of the Rayleigh quotients from the above Proposition 2 is usually referred to as the principal frequency of the vibrating nonlinear elastic membrane.

Consider the problem:

$$-\lambda\Delta_p u = |u|^{p-2}u,\ x \in \Omega,\ \lambda \in \mathbf{R}$$
$$u\,|\,\partial\Omega = 0.$$

This can be considered as $(*)$ with $f(\cdot, u(\cdot)) = |u|^{p-2}u$ and $h \equiv 0$. f having this form fulfills the two conditions − properties of Carathéodory function from Proposition 2 taking $c_1 = 1$ and $\beta \equiv 0$ in 2^0. Therefore, for any λ, e.g., $|\lambda| \geq \lambda_1^{-1}$, the last problem has a solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p\prime}(\Omega)$.

II. In the paper [31], a model was proposed for the vibration of a nonhomogeneous membrane which is fixed along the boundary. Several materials (with different densities) were investigated there, following the location of these materials inside Ω by studying the first mode in the vibration of the membrane. Ω is a bounded smooth domain in $\mathbf{R}^{\mathrm{N}}$ and g is a Lebesgue measurable function verifying the condition $0 \leq g(x) \leq H$, $\forall x \in \Omega$, where H is a positive constant, $g \not\equiv 0$ and H. g can be replaced by any Lebesgue measurable function, equal to it almost everywhere. Consider the eigenvalue Dirichlet problem:

$$-\Delta_p u = \lambda g(x)|u|^{p-1}u,\ x \in \Omega,$$
$$u\,|\,\partial\Omega = 0.$$

To superpose our theory, it is sufficient to replace λ by $\frac{1}{\lambda}$ since this is made with the equation: $-\lambda\Delta_p u = f(\cdot, u(\cdot)) + h(\cdot)$. Take in Proposition 2: $f(x, s) = g(x)|s|^{p-1}$, so $|f(x,s)| = |g(x)||s|^{p-1} \leq H|s|^{p-1}$, hence 2^0 is fulfilled with $h \equiv 0$, $c_1 = H$, and, for any λ with $\left|\frac{1}{\lambda}\right| \geq H\lambda_1^{-1}$, equivalent $|\lambda| \leq \frac{\lambda_1}{H}$, $\lambda_1 := \inf\left\{\frac{\left|\left|u\right|\right|_{1,p}^{p}}{\left|\left|i(u)\right|\right|_{0,p}^{p}} : u \in W_0^{1,p}(\Omega)\backslash\{0\}\right\}$, the above problem has a solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p\prime}(\Omega)$. in the sense of $W^{-1,p\prime}(\Omega)$.

III. Using Proposition 2, we can propose a proof for the existence of the solution of the nonlinear problem of elastic membrane under the load $f + h$, in the general case when f is a Carathéodory function which fulfills the conditions 1^0 and 2^0, h from $W^{-1,p'}(\Omega)$ and λ such that

$$|\lambda| > c_1 \lambda_1^{-1},\ \lambda_1 := \inf\left\{ \frac{\left|\left|u\right|\right|_{1,p}^{p}}{\left|\left|i(u)\right|\right|_{0,p}^{p}} : u \in W_0^{1,p}(\Omega)\backslash\{0\} \right\}$$

from the cited result.

The nonlinear elastic membrane treated with variational methods is another actual research field, as the papers [32–35] prove.

2.2.3. The Pseudo Torsion Problem

In [36], we found the pseudo torsion problem:

$$-\Delta_p^s u = 1$$

$$u \mid \partial\Omega = 0$$

for which we applied Proposition 3 in the particular case $f \equiv 0$, $h \equiv 1$, $\lambda = 1$ (hence $\lambda \neq 0$) to provide another proof for the existence of the solution of the above problem.

The torsion problem has been treated as a Dirichlet eigenvalue problem in [37], with the bounded reaction (free term of the fractional differential equation) involved in a Dirichlet problem in [38], and the asymptotic behavior for solutions of some torsional-creep type problems is studied in [39].

2.2.4. Nonlinear Elastic Membrane with p-Pseudo-Laplacian

I. In [40], the expression $\int_{\Omega} \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^{p} dx$ was proposed for the deformation energy of the membrane and was woven out of elastic strings in a rectangular form. The phenomenon can be modelled with a Dirichlet problem for the p-pseudo-Laplacian:

$$-\lambda \Delta_p^s u = |u|^{p-2} u,\ x \in \Omega,\ \lambda \in \mathbf{R}$$

$$u \mid \partial\Omega = 0.$$

Under the above notation, $\frac{1}{\lambda}$ is the eigenvalue for such a problem. One can apply Proposition 4, taking $f(\cdot, u(\cdot)) = |u|^{p-2}u$, $h \equiv 0$ and with $c_1 = 1$ and $\beta \equiv 0$ in 2^0, to obtain that the last exposed problem has a solution in $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$, for any λ, with

$$\left|\lambda\right| > \lambda_1^{-1},\ \lambda_1 := \inf\left\{ \frac{\| u \|_{1,p}^{p}}{\left|\left|i(u)\right|\right|_{0,p}^{p}} : u \in W_0^{1,p}(\Omega)\backslash\{0\} \right\}.$$

II. Take in **I** the load $f + h$, in the general case when f is a Carathéodory function, which fulfills the conditions 1^0 and 2^0 from Proposition 4, h from $W^{-1,p'}(\Omega)$ and λ e.g., $|\lambda| > c_1 \lambda_1^{-1}$,

$$\lambda_1 := \inf\left\{ \frac{\| u \|_{1,p}^{p}}{\left|\left|i(u)\right|\right|_{0,p}^{p}} : u \in W_0^{1,p}(\Omega)\backslash\{0\} \right\}$$

from the cited result.

Nonlinear elastic membrane is one of the wide subjects which also benefits the eigenvalues and eigenfunctions. An extended theoretical study exploited to model such a kind of phenomenon is presented in [41] by the extension through the numerical simulation.

3. Applications for Results of the Fredholm Alternative Type for Operators $\lambda J_{\phi} - S$

3.1. Results

Proposition 5. *Let p be from* $(1, +\infty)$ *and* $\lambda \neq 0$. *If*

$$\lambda(-\Delta_p u) = |u|^{p-2} u$$

does not have a nonzero solution in $W_0^{1,p}(\Omega)$, *then, for any h from* $W^{-1,p'}(\Omega)$, *the equation*

$$\lambda(-\Delta_p u) = |u|^{p-2} u + h$$

has a solution in $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$.

Proposition 6. *In the statement of Proposition 5, if* $p \in [2, +\infty)$, *then* $-\Delta_p$ *can be replaced by* $-\Delta_p^s$.

Proposition 7. *Let p be from* $(1, +\infty)$ *and* $\lambda \neq 0$. *If*

$$\lambda(-\Delta_p u) = N_f u$$

has no nonzero solution in $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$, *then, for any h from* $W^{-1,p'}(\Omega)$, *the equation:*

$$\lambda(-\Delta_p u) = f(\cdot, u(\cdot)) + h, \ x \in \Omega$$

has a solution in $W_0^{1,p}(\Omega)$ *in the sense of* $W^{-1,p'}(\Omega)$.

Statements of the Fredholm alternative type represent an important theoretical instrument to solve fractional evolution problems having a certain kind of boundary conditions as the paper [42] provides, and they can also be involved in optimization and evaluation of the process performance, as the recent work [43] proves.

3.2. Applications for Real Phenomena

3.2.1. Nonlinear Elastic Membrane

Take the nonlinear elastic membrane under the load f, for which the mathematical model can be considered:

$$-\Delta_p u = f \text{on} \Omega$$

$$u = 0 \text{ on } \partial\Omega.$$

In the case of the homogeneous problem:

$$-\lambda \Delta_p u = |u|^{p-2} u, \ x \in \Omega, \ \lambda \in \mathbf{R}$$

$$u \mid \partial\Omega = 0.$$

has no nonzero solution. By applying Proposition 5, for any h from $W^{-1,p'}(\Omega)$, the pro-blem:

$$\lambda(-\Delta_p u) = |u|^{p-2} u + h, \ x \in \Omega, \ \lambda \in \mathbf{R}$$

$$u \mid \partial\Omega = 0.$$

has a solution in the sense of $W^{-1,p'}(\Omega)$.

Remark. For the load f being a Carathéodory function, one can obtain a similar result using Proposition 7.

3.2.2. Nonlinear Elastic Membrane with p-Pseudo-Laplacian

When the membrane is woven out of elastic strings in a rectangular form, then the phenomenon can be modelled with a Dirichlet problem for the p-pseudo-Laplacian and the load f has another form as in Section 2.2.4, and if the homogeneous problem:

$$-\lambda\, \Delta_p^s u = |u|^{p-2}u,\ x \in \Omega,\ \lambda \in \mathbf{R}$$

$$u \mid \partial\Omega = 0.$$

has no nonzero solution, therefore, applying Proposition 6 for $p \in [2, +\infty)$,

$$-\lambda\, \Delta_p^s u = |u|^{p-2}u + h,\ x \in \Omega,\ \lambda \in \mathbf{R}$$

$$u \mid \partial\Omega = 0$$

has a solution in the sense of $W^{-1,p\prime}(\Omega)$, for any h from $W^{-1,p\prime}(\Omega)$.

4. Problems Solved Using Surjectivity to Different Homogeneity Degrees

4.1. Theoretical Results

Proposition 8. *Under the above conditions, for any $\lambda \neq 0$ and for any h from $W^{-1,p\prime}(\Omega)$, there exists u_0 in $W_0^{1,p}(\Omega)$ such that*

$$\lambda(-\,\Delta_p)u_0 = (i' \text{ o } N \text{ o } i)u_0 + h$$

respectively,

$$\lambda(-\Delta_p^s)u_0 = (i' \text{ o } N \text{ o } i)u_0 + h.$$

Proposition 9. *Under the above conditions, for any $\lambda \neq 0$ and for any h in $W^{-1,p\prime}(\Omega)$, there exists u_0 in $W_0^{1,p}(\Omega)$ such that*

$$\lambda(-\,\Delta_p)u_0 = \Big(i' \text{ o } N_f \ \text{ o } i)u_0 + h$$

respectively,

$$\lambda(-\Delta_p^s)u_0 = \Big(i' \text{ o } N_f \ \text{ o } i)u_0 + h.$$

The association of homogeneity and fractional problems remains a very current topic, as the rich development of results from the paper [44] demonstrates.

4.2. Applications to Real Phenomena Models

4.2.1. Nonlinear Elastic Membrane

I. Use Proposition 8 to prove the existence of the solution for the problem of a nonlinear elastic membrane under the load $f + h$, where we can use the mathematical model:

$$\lambda(-\,\Delta_p\, u) = |u|^{q-2}\, u + h \text{on} \Omega$$

$$u = 0 \text{ on } \partial\,\Omega,$$

with $q \in (1, p), \frac{1}{q} + \frac{1}{q'} = 1$, $\forall\lambda \neq 0$ and $\forall h$ in $W^{-1,p\prime}(\Omega)$.

II. Similarly, for the problem:

$$\lambda(-\,\Delta_p)u = f + h \text{on} \Omega$$

$$u = 0 \text{ on } \partial\Omega,$$

with $f : \Omega \times \mathbf{R} \to \mathbf{R}$ an odd Carathéodory function and $(q - 1)$ - homogeneous in the second variable, which verifies the growth condition:

$$|f(x,s)| \leq c_1|s|^{q-1} + \beta(x)\forall s \text{ in } \mathbf{R},\ \forall x \text{ in } \Omega\backslash A, \mu(A) = 0,$$

where $c_1 \geq 0$, $\beta \in L^{q'}(\Omega)$ by applying Proposition 9.

4.2.2. Nonlinear Elastic Membrane with p-Pseudo-Laplacian

I. Demonstrate the existence of the solution from $W_0^{1,p}(\Omega)$ in the sense of $W^{-1,p'}(\Omega)$ for the problem:

$$\lambda(-\Delta_p^s u) = |u|^{q-2}u + h,\ x \in \Omega,\ \lambda \in \mathbf{R},$$

$$u \mid \partial\Omega = 0,$$

with $q \in (1, p), \frac{1}{q} + \frac{1}{q'} = 1, \lambda \neq 0$ and h in $W^{-1,p'}(\Omega)$ by applying Proposition 8.

II. For a similar problem:

$$\lambda(-\Delta_p^s u) = f + h,\ x \in \Omega,\ \lambda \in \mathbf{R},$$

$$u \mid \partial\Omega = 0,$$

with $f : \Omega \times \mathbf{R} \to \mathbf{R}$ an odd Carathéodory function and $(q - 1)$ - homogeneous in the second variable, which verifies the growth condition:

$$|f(x,s)| \leq c_1|s|^{q-1} + \beta(x)\forall s \text{ in } \mathbf{R},\ \forall x \text{ in } \Omega\backslash A, \mu(A) = 0,$$

where $c_1 \geq 0$, $\beta \in L^{q'}(\Omega)$ and we have the existence of the solution by applying Proposition 9.

5. Problems Having Weak Solutions Starting from Ekeland Variational Principle

There are three groups of results. The first two are based on critical point notion and the last one, which is of a different kind, will be involved in the construction of the solutions of some mathematical physics problems. Many methods use critical points theory to find or characterize solutions for fractional boundary problems as in [45]. Both highly theoretical and extremely applicative approaches involving critical points and fractional calculus were performed in [46] to draw solutions for several real models and obtain the concrete solution *via* specific numerical methods.

5.1. Critical Points and Weak Solutions for Elliptic Type Equations — Applications

5.1.1. Theoretical Support

Let Ω be an open bounded nonempty set in $\mathbf{R}^N$ and let $f : \Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function with the growth condition:

$$|\ f\ (x,\ s)| \leq c|s|^{p-1} + b(x), \quad (1)$$

where $c > 0$, $2 \leq p \leq \frac{2N}{N-2}$ when $N \geq 3$ and $2 \leq p < +\infty$ when $N = 1, 2$, and where $b \in L^q(\Omega)$,

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Consider the problems:

$$(*)\begin{cases} -\Delta_p u = f(x,u),\ x \in \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

and

$$(**)\begin{cases} -\Delta_{\mathrm{p}}^{\mathrm{s}} u = f(x,u),\ x \in \Omega \\ u = 0 \ \text{ on } \partial\Omega\,. \end{cases}$$

Proposition 10. *Let Ω be an open bounded of C^1 class set in $\mathbf{R}^{\mathrm{N}}$, $N \geq 3$, $f : \Omega \times \mathbf{R} \to \mathbf{R}$ a Carathéodory function and u_1, u_2 from $W_0^{1,p}(\Omega)$ bounded weak subsolution and weak supersolution of $(*)$, respectively, with $u_1(x) \leq u_2(x)$ a.e. on Ω. Suppose that f verifies* (1) *and there is* $\rho > 0$ *such that the function g*: $g(x, s) = f(x, s) + \rho s$ *is strictly increasing in s on* $[\inf u_1(\Omega), \sup u_2(\Omega)]$. *Then there is a weak solution $\overline{u}$ of $(*)$ in $W_0^{1,p}(\Omega)$ with the property*

$$u_1(x) \leq \overline{u}(x) \leq u_2(x) \ a.e.on\ \Omega$$

Proposition 11. *Let Ω be an open bounded of C^1 class set in $\mathbf{R}^{\mathrm{N}}$, $N \geq 3$* and *$f : \Omega \times \mathbf{R} \to \mathbf{R}$ a Carathéodory function and u_1, u_2 from $W_0^{1,p}(\Omega)$ bounded weak subsolution and weak supersolution of $(**)$, respectively, with $u_1(x) \leq u_2(x)$ a.e. on Ω. Suppose that f verifies* (1) *and there is* $\rho > 0$ *such that the function g*: $g(x, s) = f(x, s) + \rho s$ *is strictly increasing in s on* $[\inf u_1(\Omega), \sup u_2(\Omega)]$. *Additionally, there is a weak solution $\overline{u}$ of $(**)$ in $W_0^{1,p}(\Omega)$ with the property*:

$$u_1(x) \leq \overline{u}(x) \leq u_2(x) \ a.e.on\ \Omega.$$

5.1.2. Applications to Real Phenomena

I. *Application in Glaciology*

Consider once again the problem presented in Section 2.2.1:

$$-\Delta_p u = 1 \text{ on } \Omega,$$

$$u = 0 \text{ on } \partial\Omega,$$

and remark that we can characterize the weak solution for this using Proposition 10. Observe that all the conditions from the cited result are fulfilled here by combining it, for instance, with the above example.

II. *Nonlinear Elastic Membrane*

Take once again the problem describing a nonlinear elastic membrane under the load *f*, for which we use the mathematical model:

$$-\Delta_p u = f \text{ on } \Omega$$

$$u = 0 \text{ on } \partial\Omega.$$

When the free term of the equation fulfills the conditions from Proposition 10, one obtains a characterization of the solution of the considered problem.

III. *The Pseudo Torsion Problem*

Consider once again the pseudo torsion problem discussed in Section 2.2.3:

$$-\Delta_p^s u = 1 \text{ on } \Omega,$$

$$u \mid \partial\Omega = 0.$$

By applying Proposition 11, one obtains a characterization of the weak solution of this (remark that the conditions from the cited result are fulfilled combining, for instance, with the above example).

IV. *Nonlinear Elastic Membrane with p-Pseudo-Laplacian*

When the phenomenon can be modelled with a Dirichlet problem for the *p*-pseudo-Laplacian:

$$-\Delta_p^s u = |u|^{p-2}u,\ x \in \Omega,$$

$$u \mid \partial\Omega = 0,$$

by applying Proposition 11, one gives a characterization of the solution for the above pro-blem.

Furthermore, the right-hand term f of the last equation is as in Proposition 11, then one obtains a more general characterization of the solution of such a problem.

5.2. Applications of Critical Points for Nondifferentiable Functionals

5.2.1. Theoretical Results

Let Ω be a bounded domain of $\mathbf{R}^N$ with the smooth boundary $\partial\Omega$ (topological boundary). Consider the nonlinear boundary value problems $(*)$ and $(**)$ from Section 5.1.2 above, where $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a measurable function with *subcritical growth*, i.e.,

$$\text{(I)} \;\; |f\,(x,\ s)| \leq \;\; a + b|s|^{\sigma} \;\; \forall s \in \mathbf{R}, x \in \Omega \; a.e.,$$

where $a, b > 0$, $0 \leq \sigma < \frac{N+2}{N-2}$ for $N > 2$ and $\sigma \in [0, +\infty)$ for $N = 1$ or $N = 2$.

Set:

$$\underline{f}(x,\ t) = \underline{\lim}_{s \to t} f\,(x,\ s), \overline{f}(x,\ t) = \overline{\lim}_{s \to t} f(x,\ s)$$

Suppose

$$\text{(II)} \underline{f}, \overline{f} :\ \Omega \times \mathbf{R} \to \mathbf{R} \text{ are measurable with respect to } x.$$

Associate to $(*)$ the locally Lipschitz functional Φ: $X \to \mathbf{R}$,

$$\Phi(u) = \frac{1}{p}\left|\left|u\right|\right|_{1,p}^{p} - \int_{\Omega} F(x,u)\, dx,\ u\ \in X,$$

and associate to $(**)$

$$\Phi(u) = \frac{1}{p}\| u \|_{1,p}^{p} - \int_{\Omega} F(x,u)\, dx,\ u\ \in X,$$

where $F(x, s) = \int_0^s f(x,t)dx$.

Proposition 12. *If* (I) *and* (II) *are verified, every critical point of* Φ *is a solution for* $(*)$ *and* $(**)$, *respectively*.

5.2.2. Applications − Real Phenomena Modeling

I. We propose here an immediate application to characterize the solution of the modeling given in [47] for thermal transfer. Among cryogenic fluids used in industrial or laboratory applications, helium II offers remarkable properties. However, its behavior, both mechanical and thermal, appears particularly complex. Regarding the heat transfer at the heart of this fluid and through the interface with a solid, the two phenomenological laws have been used in [47] in order to obtain a mathematical model which should make it possible to optimally ensure the desired temperature in the problems associated with cooling by helium II; this goal is particularly crucial for multifilamentary superconductors.

The mathematical model proposed in the last cited work aims to determine u: $\Omega \to (0, c)$ (the value of c is experimentally obtained as 2.4844×10^{-3}) such that:

$$-\Delta_p u = r \text{ on } \Omega,$$

where r is a source term for a given $p \geq 1$, and the second time the value $p = \frac{4}{3}$ is established there, together with a boundary condition of Dirichlet type $u = \varphi$ on $\partial\Omega$ in the sense of the trace. If $\varphi \equiv 0$, one can replace u by $v = u - \varphi$. When the conditions from Proposition 12 are fulfilled, conclude that one can characterize the solution of this problem to be a critical point for the corresponding Φ functional.

II. One can give characterizations of the solutions using this kind of definition for the Dirichlet problems issued from the already-presented problems of the movement of the glacier, nonlinear elastic membrane, pseudo torsion problem, or nonlinear elastic membrane with p-pseudo-Laplacian.

5.3. Other Solutions Starting from Ekeland Principle

5.3.1. Theoretical Results

Definition 1. u from $W^{2,p}$ (Ω), $p > 1$ is *solution* of $(*)$ and $(**)$, respectively, from this section if $Bu = 0$ on $\partial\Omega$ in the sense of trace (meaning introduced above) and

$$-\Delta_p u(x) \in \left[\underline{f}(x, u(x)), \overline{f}(x, u(x))\right] \text{ in } \Omega \text{ a.e.} \tag{2}$$

and

$$-\Delta_p^s u(x) \in \left[\underline{f}(x, u(x)), \overline{f}(x, u(x))\right] \text{ in } \Omega \text{ a.e.} \tag{3}$$

respectively.

Proposition 13. *Nonlinear Neumann problems*

$$(N) \begin{cases} -\Delta_p u = g(u) + h(x), x \in \Omega \\ Bu = 0 \text{ on } \partial\Omega, \end{cases}$$

and

$$(N') \begin{cases} -\Delta_p^s u = g(u) + h(x), x \in \Omega \\ Bu = 0 \text{ on } \partial\Omega, \end{cases}$$

respectively, with the conditions

(III) g: $\mathbf{R} \to \mathbf{R}$ *bounded measurable T-periodic,*

$$\int_0^T g(s)ds = 0 \text{ and}$$

(IV) h *bounded measurable,*

$$\int_\Omega h dx = 0,$$

have a solution in $W^{1,p}(\Omega)$ *in the sense of* (2) and (3) respectively.

5.3.2. Related Applications

I. We can apply Proposition 13 with $g \equiv 1$ and $h \equiv 0$ to prove that there exists a solution for the velocity problem under the assumption of solid friction, see [26],

$$-\Delta_p u = \text{sgn } u \text{ on } \Omega,$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n} = \begin{cases} 0, \text{ on } \Gamma_0 \\ -f, \text{ on } \Gamma_1 \end{cases},$$

where $p = 1 + \frac{1}{n}$, $1 < p < 2$, u in $W^{2,p}(\Omega)$.

This problem appears when the flow of glaciers is studied [26], i.e., to determine the velocity under the Weertman hypothesis:

$$-\Delta_p u = \text{sgn } u \text{ on } \Omega,$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n}=0 \text{ on } \Gamma_0,$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial n}+c_1|u|^{\pi-2}\text{u}=0 \text{ on } \Gamma_1,$$

where $p = 1 + \frac{1}{n}$ and $\pi = 1 + \frac{2}{n+1}$, $c_1 > 0$ is a constant, and n is the exponent from the Glen law.

II. Involve Proposition 13 to give another solution for the problem studied in [48]. According to [42], the physical problem is as follows: the polymer is injected, over a period of time, $t_0 < t < t_1$ at some point $x_1 \in \Omega$. It is not necessary to have the domain Ω simply connected. Some notations: Ω_t = the part of Ω which is filled by fluid at time t; φ = pressure; v = fluid velocity (averaged over $-h \le z \le h$); $\Gamma_0 = \partial\Omega_t \cap \Omega$, and $\Gamma_1 = \partial\Omega_t \cap \partial\Omega$. Γ_0 is the flow front. It is assumed that $\partial\Omega$ is solid (except for air vents). The equation for φ is:

$$\Delta_p \varphi = 0 \text{ on } \Omega_t \setminus \{x_1\},$$

where $p = \frac{1}{n} + 1$, n is a material constant. Further, φ = constant (chosen 0) on Γ_0 and $\frac{\partial\varphi}{\partial n} = 0$ on Γ_1. It follows that the fluid front Γ_0 meets $\partial\Omega$ at right angles (provided that φ is smooth up to the boundary and $\nabla\varphi \neq 0$). Also note that φ must have a singularity at x_1. In this approach, the development with time of the domain Ω_t filled by fluid is controlled by φ, which is determined from an elliptic partial differential equation. Note that, in this physically oriented description, all considered curves, functions, and vector fields are assumed "smooth", such that each of the crucial expressions has a well-defined pointwise meaning. Following [48], the mathematical problem means to obtain the solution φ of the instantaneous flow problem which can be obtained as the solution φ^* of a convex extremum problem by an appropriate and rather obvious choice of that problem. Taking $g \equiv 1$ and $h \equiv 0$, Ω_t instead of Ω in (N), we are placed under the conditions of Proposition 13 and give, *via* this result, another proof for the existence of the solution for the mentioned problem.

III. We can add here the example of the solution of the modeling given in [47] for thermal transfer described at Section 5.2.2 with the same equation under Neumann boundary conditions (flux imposed): $|\nabla u|^{p-2}\frac{\partial u}{\partial n} = \psi$ on $\partial\Omega$. Proposition 13 establishes the existence of the solution.

IV. Consider the pseudo torsion problem:

$$-\Delta_p^s u = \text{sgn}\, u \text{ on } \Omega,$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0$$

for which we applied Proposition 13 in order to establish an existence result.

6. Weak Solutions Using a Perturbed Variational Principle

6.1. Theoretical Results

Let Ω be an open bounded set of C^1 class in $\mathbf{R}^N$, $N \ge 3$. Consider the problems $(*)$ and $(**)$ from Section 5.1.1, where $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function with the growth condition

$$|f\ (x,\ s)| \le c|s|^{p-1} + b(x), c > 0,\ 2 \le p \le \frac{2N}{N-2},\ b \in L^{p\prime}(\Omega), \frac{1}{p} + \frac{1}{p\prime} = 1.$$

The functionals φ: $W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\varphi(u) = \int_\Omega \left[\frac{1}{p}\left(|u|^p + \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^p\right) - F(x,u(x))\right]dx$$

and

$$\varphi(u) = \int_{\Omega}\left(\frac{1}{p}\sum_{i=1}^{N}\left|\frac{\partial u \partial}{x_i}\right|^p - F(x, u(x))\right)dx,$$

with $F(x, s) : = \int_0^s f(x,t)\,dt$, are of Fréchet C^1 class and their critical points are the weak solutions of the problems (∗) and (∗∗), respectively.

Problem (∗). Let λ_1 be the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ with a homogeneous boun-dary condition. We have (see, for instance, in the Section 2)

$$\lambda_1 = \inf\left\{\frac{\left|\left|u\right|\right|_{1,p}^p}{\left|\left|i(u)\right|\right|_{0,p}^p} : u \in W_0^{1,p}(\Omega)\backslash\{0\}\right\} \text{ (the Rayleigh-Ritz quotient).}$$

Proposition 14. *Under the above assumptions and, in addition, the growth condition*

$$F(x,\ s) \leq c_1\,\frac{s^p}{p} + \alpha(x)s,$$

with $0 < c_1 < \lambda_1$, $\alpha \in L^{q'}(\Omega)$ *for some* $2 \leq q \leq \frac{2N}{N-2}$ *and* $f\ (x, -s) = -f\ (x, s)$, $\forall x$ *from* Ω, *the following assertions hold*:

(*i*) *The set of functions h from* $W^{-1,p'}(\Omega)$, *having the property that the functional* φ_h: $W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\varphi_h(u) = \frac{1}{p}\ \left|\left|u\right|\right|_p^p - \int_{\Omega}(F(x,u(x)) + h(u(x)))\,dx$$

has in only one point an attained minimum, includes a G_δ *set everywhere dense;*

(*ii*) *The set of functions h from* $W^{-1,p'}(\Omega)$, *having the property*

$$\text{the problem}\begin{cases} -\Delta_p u = f(x,u) + h(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \text{has solutions,}$$

includes a G_δ *set everywhere dense;*

(*iii*) *Moreover, if* $s \to f\ (x, s)$ *is increasing, then the set of functions h from* $W^{-1,p'}(\Omega)$, having *the property*

$$\text{the problem}\begin{cases} -\Delta_p u = f(x,u) + h(u) \ \textit{in}\ \Omega \\ u = 0 \ \textit{on}\ \partial\Omega \end{cases} \text{has a unique solution,}$$

includes a G_δ *set everywhere dense.*

Problem (∗∗). Let λ_1 be the first eigenvalue of $-\Delta_p^s$ in $W_0^{1,p}(\Omega)$ with a homogeneous boundary condition. We have (see in Section 2)

$$\lambda_1 = \inf\left\{\frac{\|u\|_{1,p}^p}{\left|\left|i(u)\right|\right|_{0,p}^p} : u \in W_0^{1,p}(\Omega)\backslash\{0\}\right\} \text{ (the Rayleigh-Ritz quotient).}$$

Proposition 15. *Under the above assumptions and, in addition, the growth condition*

$$F(x,\ s) \leq c_1\frac{s^p}{p} + \alpha(x)s,$$

with $0 < c_1 < \lambda_1$, $\alpha \in L^{q'}(\Omega)$ *for some* $2 \leq q \leq \frac{2N}{N-2}$ *and* $f\ (x, -s) = -f\ (x, s)$, $\forall x$ *from* Ω, *the following assertions hold.*

(*i*) *The set of functions h from* $W^{-1,p'}(\Omega)$, *having the property that the functional* φ_h: $W_0^{1,p}(\Omega) \to \mathbf{R}$,

$$\varphi_h(u) = \frac{1}{p}\| u \|_p^p - \int_\Omega (F(x,u(x)) + h(u(x)))\,dx$$

has in only one point an attained minimum, includes a G_δ *set everywhere dense;*

(*ii*) *The set of functions h from* $W^{-1,p'}(\Omega)$, *having the property*

$$\text{theproblem}\begin{cases} -\Delta_p^s u = f(x,u) + h(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \text{has solutions,}$$

includes a G_δ *set everywhere dense;*

(*iii*) *Moreover, if s* $\to$ *f (x, s) is increasing, then the set of functions h from* $W^{-1,p'}(\Omega)$, *having the property*

$$\text{theproblem}\begin{cases} -\Delta_p^s u = f(x,u) + h(u) \textit{ in } \Omega \\ u = 0 \textit{ on } \partial\Omega \end{cases} \text{has a uniquesolutions,}$$

includes a G_δ *set everywhere dense.*

Both unconstrained and constrained minimization problems for the energy functional, involving a special kind of Dirichlet problem with *p*-Laplacian, have been studied in [49], where the existence and the uniqueness of some types of solution have been obtained. One of the most important characteristics of the application of variational principles is the possibility to conduct the solving of the problem until the visualization of the result of the real-world model *via* an appropriate numerical method as the work [50] successfully sustains.

6.2. Applications

I. As the first application, we propose characterizing the solution of the Dirichlet problem which models the compression molding of polymers. This means to study a generalized Helle-Show flow of a power-law fluid which leads to the *p*-Poisson equation for the instantaneous pressure in the fluid. Therefore, this pressure *u* is the solution of:

$$-\Delta_p u = 1 \text{ on } \Omega,$$

$$u = 0 \text{ on } \partial\Omega,$$

where $p = \frac{1}{n} + 1$, *n* being the power-law index of the polymer, a typical value going over the interval (0.3, 0.5) [51,52]. Observe that the conditions from Proposition 6, (*iii*) are fulfilled when we take $f \equiv 0$, as in the equation above. One can consider $h \equiv 1$ to obtain that the pressure Dirichlet problem has a unique solution using the cited result. One can also apply Proposition 14, (*ii*) or (*iii*), considering the same *f* and *h* fulfilling either the conditions from (*ii*) or those from (*iii*) to prove either the existence only or the existence and the uniqueness of the minimum for the corresponding functional for pressure, and hence one can discuss about its dual problem which has as solution (maximum) the flow function [53].

II. We can also prove the existence and the uniqueness of the solution of the pseudo torsion problem:

$$-\Delta_p^s u = 1$$

$$u \mid \partial\Omega = 0,$$

for which we apply Proposition 15. For the right-hand member of the equation of the form $f + h$, as in Proposition 15 (*ii*) or (*iii*), we can obtain either the existence only or the existence and the uniqueness of the solution.

7. Conclusions

In this paper, some methods to solve different problems issued from real phenomena models have been presented. The theoretical results widely developed and debated in the paper under the same name, part one, are applied in order to draw the solutions and to characterize them. This is the natural step towards an appropriate numerical method, hence the validation of the model.

The novelty of this work consists in the presentation of detailed applications invol-ving the obtained theory in solving mathematical physics problems describing real phenomena. We proposed solving methods and characterization of the solutions for problems issued from glaciology, nonlinear elastic membrane either with p-Laplacian or with p-pseudo-Laplacian and for the pseudo torsion problem, those starting from some surjectivity statements for a special kind of operators, or *via* theorems of Fredholm alternative types, or surjectivity at different homogeneity degrees.

Weak solutions starting from the Ekeland variational principle, applied to critical points and weak solutions for elliptic type equations, critical points for nondifferentiable functionals applied for glaciology problems, pseudo torsion problems, nonlinear elastic membrane with p-Laplacian and p-pseudo-Laplacian, thermal transfer models, Neumann problems issued from injection mold filling, and other Neumann problems with p-pseudo-Laplacian have been presented. In the end, using a result obtained *via* a perturbed variational principle, the solution for a Dirichlet problem studying a generalized Helle-Show flow of a power-law fluid and the existence and the uniqueness of the solution for a pseudo torsion problem are obtained.

We are particularly interested in these applications, and this work is a necessary study in our future developments, as our final goal is to obtain a mathematical model for a specific process involving transfer phenomena for targeted environmental engineering applications. The role of the reactor (in nanofabrication) in nano-liquid-liquid dispersed systems is played by micro- or nano-droplets, and the determinant parameters are related to surface phenomena as a result of special intermolecular forces at interface. In this context, some innovative modeling mathematical methods have to be proposed and tested in order to properly simulate the physical-chemical interactions and processes specific to nanofabrication. However, one may stress that there is no model available to be applied for describing diffusion phenomena involved in the micro-emulsification of disperse systems in connection with surface properties at the interface in self-organized systems. Moreover, such disperse systems can be successfully applied for the removal of organic or inorganic pollutants from wastewater, and this is the subject of future research.

Funding: This research was funded by the University POLITEHNICA of Bucharest.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

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