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Sugeno Integral Based on Overlap Function and Its Application to Fuzzy Quantifiers and Multi-Attribute Decision-Making

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Abstract: The overlap function is an important class of aggregation function that is closely related to the continuous triangular norm. It has important applications in information fusion, image processing, information classification, intelligent decision-making, etc. The usual multi-attribute decision-making (MADM) is to select the decision object that performs well on all attributes (indicators), which is quite demanding. The MADM based on fuzzy quantifiers is to select the decision object that performs well on a certain proportion or quantification (such as most, many, more than half, etc.) of attributes. Therefore, it is necessary to study how to express and calculate fuzzy quantifiers such as most, many, etc. In this paper, the Sugeno integral based on the overlap function (called the O-Sugeno integral) is used as a new information fusion tool, and some related properties are studied. Then, the truth value of a linguistic quantified proposition can be estimated by using the O-Sugeno integral, and the O-Sugeno integral semantics of fuzzy quantifiers is proposed. Finally, the MADM method based on the O-Sugeno integral semantics of fuzzy quantifiers is proposed and the feasibility of our method is verified by several illustrative examples such as the logistics park location problem.

Keywords: overlap function; sugeno integral; fuzzy quantifier; multi-attribute decision-making

MSC: 03B52; 03E72; 90B50

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Citation: Mao, X.; Temuer, C.; Zhou, H. Sugeno Integral Based on Overlap Function and Its Application to Fuzzy Quantifiers and Multi-Attribute Decision-Making. *Axioms* **2023**, *12*, 734. https://doi.org/10.3390/axioms12080734

Academic Editors: Ta-Chung Chu and Wei-Chang Yeh

Received: 6 June 2023
Revised: 20 July 2023
Accepted: 25 July 2023
Published: 27 July 2023

1. Introduction

The triangular norm (t-norm) first appeared in Menger's paper "Statistical metrics" in 1942, which proposed t-norm as a natural generalization of triangular inequalities in classical metric spaces [1]. A t-norm is an aggregation operator that satisfies commutativity, monotonicity, and associativity, and has the identity element 1. From the mathematical structure, t-norms and t-conorms are pairs of dual operators. T-norms play an important role as the general fuzzy "and" operator in the fuzzy logic community. In order to apply more widely, researchers proposed many generalized forms of t-norms and t-conorms, such as t-seminorms [2], pseudo-t-norms [3], t-operators [4], uninorms [5], semiuninorms [6], etc. Recently, many scholars still studied the extension structure of t-norms. For example, Dan proposed a universal way to study t-semi(co)norms and semiuninorms in terms of behavior operations [7]. For partially defined binary operations in practical problems, Borzooei et al. introduced a partial t-norm on a bounded lattice [8]. Zhang et al. further investigated the partial residual implications of partial t-norms and partial residuated lattices [9].

The measures discussed in classical measure theory are additive, and they are abstractions of real-world concepts such as length, area, volume and weight. However, additivity cannot be satisfied in many practical situations, e.g., the work efficiency of two people in cooperation is often greater or less than the combined work efficiency of two people. In 1971, Shilkret introduced the maxitive measure in place of the usual additive measure, and then proposed the integral with respect to a maxitive measure in [10]. The concepts of fuzzy measures and fuzzy integrals (also called Sugeno integrals) were first introduced by Sugeno in his doctoral thesis in 1974 [11]. Fuzzy measure is a class of set functions using weaker monotonicity instead of additivity. Fuzzy measures have been widely used in different scenarios and can be described as similar concepts such as importance, reliability, and satisfaction. Sugeno integrals replace the addition and multiplication of Lebesgue integrals with the maximum and minimum operators, respectively. These operations have limitations. Subsequently, many scholars further generalized Sugeno's integral theory based on other operators. For example, Garcia and Alvarez defined semi-normed fuzzy integrals and semiconorm fuzzy integrals and pointed out that Sugeno integrals are a special case [2]. Dudois et al. introduced Sugeno-like qualitative integrals and qualitative co-integrals defined in terms of fuzzy conjunctions and implications, respectively [12]. In 2010, Klement et al. Proposed the framework covering generalizations of Sugeno integrals, in which the role of multiplication is played by semicopulas [13]. Note that the multiplication of Shilkret integrals is still the standard product. Jin et al. introduced the concept of weak universal integrals based on semicopulas, which are generalizations of Sugeno integrals and Shilkret integrals [14]. Mihailovi and Pap defined Sugeno integrals based on set functions that have the properties of absolute monotony and sign stability [15]. In recent years, many scholars have been interested in Sugeno integrals and their generalizations, such as [16–20].

Eslami et al. pointed out that t-norms are not suitable for solving natural interpretations of language words [21]. In addition, Fodor and Keresztfalvi proposed non-associative conjunctions are very effective in generalized inference patterns [22]. Bustince et al., in 2010, introduced the concept of overlap functions as a special class of bivariate continuous aggregation functions, which are closely related to continuous t-norms [23]. Subsequently, scholars deeply studied the theory of overlap functions and their application, such as [24–29]. Overlap functions are mainly used for image processing, classification problems, decision analysis, and intelligent information fusion, in which the associative law is not strongly required. In order to be applied in more fields, overlap functions were generalized in various ways, including general overlap functions [30], Archimedean overlap functions [31], quasi-overlap functions [32], pseudo-overlap functions [33], semi-overlap functions [34], and interval-valued pseudo-overlap functions [35].

The MADM requires comprehensively considering multiple attributes through the aggregation function, and gives the optimal choice or sorts the schemes. The usual MADM is to select the decision object that performs well on all attributes (indicators), which is quite demanding. Attribute weight values can reflect the importance of attributes. In the MADM problem with known attribute weights, the decision-maker considers all the attributes together by means of an aggregation function and gives the optimal choice or ranking of decision objects. However, different attribute weights will affect the decision results and it is difficult to obtain the optimal attribute weights. The MADM problem with unknown attribute weights has been studied by many researchers from different perspectives. For example, a new MADM method based on rough sets and fuzzy measures was proposed by Wang et al. [36]. Because decision objects that perform well on all attributes are difficult to be selected out, many scholars have begun to study the MADM based on fuzzy quantifiers. The MADM problem based on fuzzy quantifiers selects the decision object that performs well on a certain proportion or quantification (such as most, many, more than half, etc.) of attributes. Therefore, it is necessary to study how to express and calculate fuzzy quantifiers such as most, many, etc.

In 1983, Zadeh first used the term "fuzzy quantifier" and described a method for quantifying fuzzy sets [37]. Zadeh treated fuzzy quantifiers as fuzzy numbers, and linguistic quantified propositions correspond to fuzzy sets defined by linguistic predicates. Zadeh obtained the truth value of a quantification proposition by calculating the cardinality of the fuzzy set. In 1988, Yager, an American scholar, proposed the method of evaluating a linguistic quantified proposition based on the ordered weighted average (OWA) operators [38]. Recently, Dvorak et al. proposed the notion of fuzzy quantifiers over fuzzy domains and investigated relevant semantic properties [39]. Medina et al. further investigated the properties of generalized quantifiers and defined the semantics of multi-adjoint logic programs [40]. In 2006, Ying, a Chinese scholar, proposed a method for modeling linguistic statements involving fuzzy quantifiers in natural language, in which fuzzy measures can be used to represent fuzzy quantifiers and Sugeno integrals can be used to calculate the truth value of a quantified statement [41]. Zhang et al. studied fuzzy quantifiers and their integral semantics based on the Sugeno integral with t-norm, and successfully applied them to the problem [42].

For wider application, we generalized the t-norm-based Sugeno integral in [42] by replacing t-norms with overlap function, which is non-associative and continuous. The Sugeno integral based on overlap functions (O-Sugeno integral) is proposed as a new information fusion tool, and its related properties are studied. Then, the O-Sugeno integral is used to deal with fuzzy quantifiers and the O-Sugeno integral semantics of fuzzy quantifiers is proposed. In fuzzy quantifier integral semantics, fuzzy measures are usually used to represent fuzzy quantifiers, and O-Sugeno integrals are used to calculate the truth value of a quantified proposition. Finally, a novel MADM method is proposed based on the O-Sugeno integral semantics of fuzzy quantifiers. The method is used to solve the fuzzy quantifiers-based MADM problems.

2. Preliminaries

We briefly review the basic definitions and conclusions that are used in our discussion of overlap functions, fuzzy quantifiers, and Sugeno integrals.

Definition 1. *Binary mapping O:* $[0, 1]^2 \to [0, 1]$ *is called an overlap function if it satisfies the following requirements: for any* $x, y \in [0, 1]$*:*

(*i*) *O is commutative, that is,* $O(x, y) = O(y, x)$*;*
(*ii*) $O(x, y) = 0$ *if and only if* $x\,y = 0$*;*
(*iii*) $O(x, y) = 1$ *if and only if* $x\,y = 1$*;*
(*iv*) *O is non-decreasing; and*
(*v*) *O is an continuous function [23].*

Definition 2. *Overlap function O is inflationary if it satisfies the condition* $O(x, 1) \geq x$*, and is deflationary if it satisfies* $O(x, 1) \leq x$*; and has unit element 1 if* $O(x, 1) = x$ *holds for each* $x \in [0, 1]$ *[43].*

Example 1.

(1) *The binary function is defined by*

$$O(x, y) = xy\frac{x+y}{2}$$

where x, y are two arbitrary element on the unit interval. Then it is an overlap function that does not have associativity and 1 is not a unit element, therefore, it is not a continuous t-norm.

(2) *The binary function is defined by*

$$O(x, y) = min(x^p, y^p)$$

for every $x, y \in [0, 1]$ *and* $p > 0$*. Then it is an overlap function and is deflationary if* $p > 1$ *and inflationary if* $0 < p < 1$*, and has neutral element 1 if* $p = 1$*.*

In natural languages, many "vague" words are used to express quantity, such as "several", "a few", "quite a few", "most", "many", "very many", "not many", "not very many", "approximately eight", "frequently", etc. These linguistic components used to represent inexact amounts are called fuzzy quantifiers [37].

Definition 3. *A fuzzy quantifier includes two items:*
For arbitrary non-empty set X, a Borel field $\wp_X$ *over X; and*

a selection function

$$Q\colon (X, \wp_X) \text{ a } Q(X, \wp_X) \in M(X, \wp X)$$

of the truth class {M(X, ℘ X): (X, ℘ X) is a measurable space} [41].

For convenience, the selection function $Q(X, \wp_X)$ is usually abbreviated as Q_X when the Borel field does not need to be specifically indicated. Given X as a discourse domain, if E represents individuals in X that have a specific attribute A, then $Q_X(E)$ is seen as the truth value of the linguistic quantified statement "Q Xs are As".

Example 2.
The quantifier "at least five" is defined as follows:

$$at\ least\ five_X(E) = \begin{cases} 1, \ if\ |E| \geq 5, \\ 0, \ otherwise. \end{cases}$$

where the domain X is any nonempty set, and E is any subset of X. Then the quantifier "at least five" is a crisp quantifier because at least five$_X$*(E)* $\in$ *{0, 1}.*

As is well known, ∀ and ∃ are also crisp quantifiers. The following example gives three typical fuzzy quantifiers.

Example 3.
The terms "many", "most", and "almost all" are often used to indicate inexact amounts in natural language, and are defined based on the following fuzzy measures [41]:

$$many_X(E) = \frac{|E|}{|X|}, most_X(E) = \left(\frac{|E|}{|X|}\right)^{3/2}, almost_X(E) = \left(\frac{|E|}{|X|}\right)^2,$$

for every non-empty set X and any subset E of X, where ∣E∣ *represents the cardinality of E.*

Definition 4: Suppose *$(X, 2^X, m)$ is a fuzzy measure space. If h: X → [0, 1] is a measurable function, then the Sugeno integral of h over A ∈ ℘ is defined as follows [11]:*

$$\int_A h \circ m = \sup_{F \in 2^X} \min[\inf_{x \in F} h(x), m(A \cap F)]$$

Theorem 1. *Given (X, ℘, m) as a fuzzy measure space, for any ℘ measurable function h: X → [0, 1], we have*

$$\int_A h \circ m = \sup_{\lambda \in [0,1]} \min[\lambda, m(A \cap h_\lambda)]$$

where h_λ = {x∈X: h(x) ≥ λ} for every λ ∈ [0, 1] [11].

In particular, $\int_A h \circ m$ will be abbreviated as $\int h \circ m$ whenever $A = X$.

3. Sugeno Integrals Based on Overlap Functions

Definition 5. *Given (X, ℘, m) as a fuzzy measure space and O: $[0, 1]^2 \to [0, 1]$ as an overlap function, if h: X → [0, 1] is a ℘ measurable function, then the Sugeno integral based on the overlap function (O-Sugeno integral) of h over A ∈ ℘ is defined by*

$$\int_A^{(OS)} h \circ m = \sup_{\lambda \in [0,1]} O[\lambda, m(A \cap h_\lambda)]$$

where h_λ = {x∈X: h(x) ≥ λ} for every λ ∈ [0, 1].

When the Borel field in measurable space is the power set of the underlying set, the O-Sugeno integral can be simplified.

Theorem 2. *Assume (X, $\wp$, m) is a fuzzy measure space and O: $[0, 1]^2 \to [0, 1]$ is an overlap function. If $\wp = 2^X$, then for any $\wp$ measurable function h: $X \to [0, 1]$ and any subset A of X, we have*

$$\int_A^{(OS)} h \circ m = \sup_{F \subseteq X} O[\inf_{x \in F} h(x), m(A \cap F)]$$

where $h_\lambda = \{x \in X: h(x) \geq \lambda\}$ for every $\lambda \in [0, 1]$.

Proof of Theorem 2.

(1) $\forall F \subseteq X$, Let $\lambda' = \inf_{x \in F} h(x)$.
If $\lambda' = 0$, then $h_{\lambda'} = X$, so $F \subseteq h_{\lambda'}$;
If $\lambda' > 0$, then $\forall x \in F$, $h(x) \geq \lambda'$, so $F \subseteq h_{\lambda'}$.
Hence, we have

$$O[\lambda', m(A \cap h_{\lambda'})] \geq O[\inf_{x \in F} h(x), m(A \cap F)]$$

Further, we obtain

$$\begin{aligned} \int_A^{(TS)} h \circ m &= \sup_{\lambda \in [0,1]} O[\lambda, m(A \cap h_\lambda)] \\ &\geq \sup_{F \subseteq X} O[\lambda', m(A \cap h_{\lambda'})], \ (\lambda' = \inf_{x \in F} h(x)) \\ &\geq \sup_{F \subseteq X} O[\inf_{x \in F} h(x), m(A \cap F)] \end{aligned}$$

(2) $\forall \lambda \in [0, 1]$, Let $F' = h_\lambda$, then $\forall x \in F'$, $h(x) \geq \lambda$, so $\inf_{x \in F'} h(x) \geq \lambda$.

Hence, we have

$$O[\inf_{x \in F'} h(x), m(A \cap h_\lambda)] \geq O[\lambda, m(A \cap h_\lambda)]$$

Further, we obtain

$$\begin{aligned} \int_A^{(OS)} h \circ m &= \sup_{\lambda \in [0,1]} O[\lambda, m(A \cap h_\lambda)] \\ &\leq \sup_{\lambda \in [0,1]} O[\inf_{x \in F'} h(x), m(A \cap F')], \ (F' = h_\lambda) \\ &\leq \sup_{F \subseteq X} O[\inf_{x \in F} h(x), m(A \cap F)] \end{aligned}$$

In summary, we can get

$$\int_A^{(OS)} h \circ m = \sup_{F \subseteq X} O[\inf_{x \in F} h(x), m(A \cap F)]$$

□.

In the case where the domain is finite, the O-Sugeno integral over it can be further simplified.

Theorem 3. *Given domain $X = \{x_1, \ldots, x_n\}$ as a finite set, and $\wp = 2^X$, (X, $\wp$, m) as a fuzzy measure space, and O: $[0, 1]^2 \to [0, 1]$ as an overlap function, if a $\wp$ measurable function h: $X \to [0, 1]$ such* *that $h(x_i) \leq h(x_{i+1})$, for $1 \leq i \leq n-1$ (if not, rearrange $h(x_i)$, $1 \leq i \leq n$). Then the O-Sugeno integral of h over A is further simplified as follows:*

$$\int_A^{(OS)} h \circ m = \max_{i=1}^{n} O[h(x_i), m(A \cap X_i)]$$

where A is any subset of X and $X_i = \{x_j\colon i \leq j \leq n\}$, $1 \leq i \leq n$.

Proof of Theorem 3. For any $\lambda \in [0, 1]$, it holds that

$$h_\lambda = \begin{cases} X_1 = X, 0 \leq \lambda \leq h(x_1) \\ X_2, h(x_1) < \lambda \leq h(x_2) \\ X_3, h(x_2) < \lambda \leq h(x_3) \\ \cdots \\ X_n, h(x_{n-1}) < \lambda \leq h(x_n) \\ \Phi, h(x_n) < \lambda \end{cases}$$

So, we obtain

$$\begin{aligned} \int_A^{(OS)} h \circ m &= \sup_{\lambda \in [0,1]} O[\lambda, m(A \cap h_\lambda)] \\ &= \max_{i=1}^{n+1} \left\{ \sup_{\lambda \in [h(x_{i-1}), h(x_i)]} O[\lambda, m(A \cap h_\lambda)] \right\} (h(x_0) = 0, h(x_{n+1}) = 1) \\ &= \max_{i=1}^{n+1} \left\{ \sup_{\lambda \in [h(x_i), h(x_{i+1})]} O[\lambda, m(A \cap X_i)] \right\} (X_{n+1} = \Phi) \\ &= \max_{i=1}^{n+1} \{O[h(x_i), m(A \cap X_i)]\} \\ &= \max_{i=1}^{n} \{O[h(x_{i+1}), m(A \cap X_{i+1})]\} \vee O[1, 0] \\ &= \max_{i=1}^{n} \{O[h(x_{i+1}), m(A \cap X_{i+1})]\} \end{aligned}$$

□.

Theorem 4. *Given (X, ℘, m) as a fuzzy measure space and ℘ = 2^X and O: $[0, 1]^2 \to [0, 1]$ as an overlap function, for arbitrary ℘ measurable functions h, h_1, and h_2 and arbitrary subset A of X, the following conclusion is established:*

(1) *If $h_1 \leq h_2$ (i.e., for any $x \in X$, $h_1(x) \leq h_2(x)$), then it holds that*

$$\int_A^{(OS)} h_1 \circ m \leq \int_A^{(OS)} h_2 \circ m$$

(2) *If m(A) = 0, then it holds that*

$$\int_A^{(OS)} h \circ m = 0$$

(3) *If a constant $c \in [0, 1]$, then it holds that*

$$\int_A^{(OS)} c \circ m = O[c,\ m(A)]$$

(4) *If a constant $c \in [0, 1]$, and for any $x \in X$, max(c, h)(x) = max{c, h(x)}, then it holds that*

$$\int_A^{(OS)} \max(c, h) \circ m = \max\left(\int_A^{(OS)} c \circ m,\ \int_A^{(OS)} h \circ m \right)$$

(5) *If $A_1 \subseteq A_2$, then it holds that*

$$\int_{A_1}^{(OS)} h \circ m \leq \int_{A_2}^{(OS)} h \circ m$$

(*6*)

$$\int_{A}^{(OS)} \max(h_1, h_2) \circ m \geq \max\left(\int_{A}^{(OS)} h_1 \circ m, \int_{A}^{(OS)} h_2 \circ m\right)$$

Proof of Theorem 4.

(1) For any $\lambda \in [0, 1]$, it holds that for any $x \in X$, $\lambda \leq h_1(x) \leq h_2(x)$.

Then,

$$h_{1\lambda} = \{x \in X: h_1(x) \geq \lambda\} \subseteq \{x \in X: h_2(x) \geq \lambda\} = h_{2\lambda}$$

So,

$$m(A \cap h_{1\lambda}) \leq m(A \cap h_{2\lambda}).$$

Then, we have

$$O(\lambda, m(A \cap h_{1\lambda})) \leq O(\lambda, m(A \cap h_{2\lambda}))$$

Furthermore, we obtain

$$\sup_{\lambda \in [0,1]} O(\lambda, m(A \cap h_{1\lambda})) \leq \sup_{\lambda \in [0,1]} O(\lambda, m(A \cap h_{2\lambda}))$$

that is,

$$\int_{A}^{(OS)} h_1 \circ m \leq \int_{A}^{(OS)} h_2 \circ m$$

(2) From $m(A) = 0$, we have $m(A \cap h_\lambda) = 0$.

Thus, it holds that

$$\int_{A}^{(OS)} h \circ m = \sup_{\lambda \in [0,1]} O[\lambda, m(A \cap h_\lambda)] = \sup_{\lambda \in [0,1]} O[\lambda, 0] = 0$$

(3) For any $x \in X$, we define $h(x) = c$.

If $\lambda \leq c$, then $h_\lambda = X$. So,

$$O[\lambda, m(A \cap h_\lambda)] = O[\lambda, m(A \cap X)] = O[\lambda, m(A)].$$

If $\lambda > c$, then $h_\lambda = \Phi$. So,

$$O[\lambda, m(A \cap \Phi)] = O[\lambda, m(\Phi)] = O[\lambda, 0] = 0$$

Hence, we can obtain

$$\begin{aligned} \int_{A}^{(OS)} h \circ m &= \sup_{\lambda \in [0,1]} O[\lambda, m(A \cap h_\lambda)] \\ &= \max\left(\sup_{\lambda \in [0,c]} O[\lambda, m(A \cap h_\lambda)], \sup_{\lambda \in [c,1]} O[\lambda, m(A \cap h_\lambda)]\right) \\ &= \max\left(\sup_{\lambda \in [0,c]} O[\lambda, m(A)], \sup_{\lambda \in [c,1]} O[\lambda, m(\Phi)]\right) \\ &= \max(O[c, m(A)], 0) \\ &= O[c, m(A)] \end{aligned}$$

(4) If $\lambda \leq c$, then for every $x \in X$,

$$\max(c, h)(x) = \max(c, h(x)) \geq c \geq \lambda,$$

that is, $\max(c, h)_\lambda = X$.

Hence,

$$O[\lambda, m(A \cap \max(c, h)_\lambda)] = O[\lambda, m(A \cap X)] = O[\lambda, m(A)].$$

If $\lambda > c$, then

$$\{x \in X: \max(c, h(x)) \geq \lambda\} = \{x \in X: h(x) \geq \lambda\},$$

that is, $\max(c, h)_\lambda = h_\lambda$.

Hence,

$$O[\lambda, m(A \cap \max(c, h)_\lambda)] = O[\lambda, m(A \cap h_\lambda)].$$

Furthermore, we obtain

$$\begin{aligned} &\sup_{\lambda\in[0,1]} O[\lambda, m(A \cap \max(c,h)_\lambda)] \\ &= \max\left(\sup_{\lambda\in[0,c]} O[\lambda, m(A \cap \max(c,h)_\lambda)], \sup_{\lambda\in[c,1]} O[\lambda, m(A \cap \max(c,h)_\lambda)]\right) \\ &= \max\left(O[c, m(A)], \sup_{\lambda\in[c,1]} O[\lambda, m(A \cap h_\lambda)]\right) \end{aligned}$$

And because

$$\sup_{\lambda\in[0,c]} O[\lambda, m(A \cap h_\lambda)] \leq \sup_{\lambda\in[0,c]} O[\lambda, m(A)] \leq O[c, m(A)]$$

we obtain

$$\begin{aligned} \int_A^{(OS)} \max(c,h) \circ m &= \sup_{\lambda\in[0,1]} O[\lambda, m(A \cap \max(c,h)_\lambda)] \\ &= \max\left(O[c, m(A)], \sup_{\lambda\in[0,1]} O[\lambda, m(A \cap h_\lambda)]\right) \\ &= \max\left(\int_A^{(OS)} c \circ m, \int_A^{(OS)} h \circ m\right) \end{aligned}$$

(5) For any $\lambda \in [0, 1]$ and $A_1 \subseteq A_2$, we have

$$(A_1 \cap h_\lambda) \subseteq (A_2 \cap h_\lambda), \text{ then } m(A_1 \cap h_\lambda) \leq m(A_2 \cap h_\lambda).$$

Thus, we can obtain

$$\sup_{\lambda\in[0,1]} O[\lambda, m(A_1 \cap h_\lambda)] \leq \sup_{\lambda\in[0,1]} O[\lambda, m(A_2 \cap h_\lambda)]$$

that is,

$$\int_{A_1}^{(OS)} h \circ m \leq \int_{A_2}^{(OS)} h \circ m$$

(6) For any $x \in X$, it holds that

$$\max(h_1, h_2)(x) = \max(h_1(x), h_2(x)).$$

So, for any $\lambda \in [0, 1]$, we have

$$\max(h_1, h_2)(x) \geq h_1(x) \geq \lambda \text{ and } \max(h_1, h_2)(x) \geq h_2(x) \geq \lambda.$$

Then,

$$h_{1\lambda} \subseteq \max(h_1, h_2)_\lambda \text{ and } h_{2\lambda} \subseteq \max(h_1, h_2)_\lambda.$$

Therefore, we get

$$\mathrm{m}(A \cap h_{1\lambda}) \leq m(A \cap \max(h_1, h_2)_\lambda) \text{ and } m(A \cap h_{2\lambda}) \leq m(A \cap \max(h_1, h_2)_\lambda).$$

Furthermore, we obtain

$$\int_A^{(OS)} \max(h_1, h_2) \circ m \geq \int_A^{(OS)} h_1 \circ m, \int_A^{(OS)} \max(h_1, h_2) \circ m \geq \int_A^{(OS)} h_2 \circ m$$

that is,

$$\int_A^{(Os)} \max(h_1, h_2) \circ m \geq \max\left(\int_A^{(OS)} h_1 \circ m, \int_A^{(OS)} h_2 \circ m\right)$$

□.

Example 4. *Consider the decision-making problem of a commercial housing purchase. After preliminary screening, the buyer needs to choose one of two properties. Assume that the evaluation of the property mainly considers the attributes geographical location, floor, and orientation, which are recorded as s_1, s_2, and s_3. Let the attribute set X = {s_1, s_2, s_3}. The importance of each attribute is determined by experts and house buyers as follows:*

$$m(\Phi) = 0, m(\{s_1\}) = 0.7, m(\{s_2\}) = 0.5, m(\{s_3\}) = 0.4, m(\{s_1, s_2\}) = 0.9, m(\{s_1, s_3\}) = 0.6, m(\{s_2, s_3\}) = 0.8, m(\{s_1, s_2, s_3\}) = 1$$

The buyer rates the two properties and the three attributes as follows:

First property: $h_1(\{s_1\}) = 0.9$, $h_1(\{s_2\}) = 0.8$, $h_1(\{s_3\}) = 0.5$; second property: $h_2(\{s_1\}) = 0.6$, $h_2(\{s_2\}) = 0.9$, $h_2(\{s_3\}) = 0.7$.

Taking the importance of the attribute in the property evaluation as a measure of the attribute set, it easy to see that it is non-additive. The h_1 and h_2 scores of the two properties are regarded as functions of the property set X. The overlap function is defined by

$$O(x, y) = x^2y^2, \text{ for any } x, y \in [0, 1].$$

Then the buyer's composite score for the first property can be calculated by the O-Sugeno integral of h_1 over X, as follows:

$$\begin{aligned} \int^{(OS)} h_1 \circ m &= \max_{i=1}^{3} O[h_1(x_i), m(X_i)] \\ &= \max\{O(h_1(x_3),\ m(X)),\ O(h_1(x_2),\ m(\{x_1,\ x_2\})),\ O(h_1(x_1),\ m(\{x_1\}))\} \\ &= \max\{0.5^2 \times 1^2,\ 0.8^2 \times 0.9^2,\ 0.9^2 \times 0.7^2\} \\ &= 0.5184 \end{aligned}$$

The buyer's composite score for the second property can be calculated by the O-Sugeno integral of h_2 over X, as follows:

$$\begin{aligned} \int^{(OS)} h_2 \circ m &= \max_{i=1}^{3} O[h_2(x_i), m(X_i)] \\ &= \max\{O(h_2(x_1),\ m(X)),\ O(h_2(x_3),\ m(\{x_2,\ x_3\})),\ O(h_2(x_2),\ m(\{x_2\}))\} \\ &= \max\{0.6^2 \times 1^2,\ 0.7^2 \times 0.6^2,\ 0.9^2 \times 0.5^2\} \\ &= 0.36 \end{aligned}$$

This shows that the buyer has a higher comprehensive score for the first property, so he should buy the first property.

4. O-Sugeno Integral Semantics of Fuzzy Quantifiers

For the sake of completeness, we recall several concepts of a first-order logical language L_q with fuzzy quantifiers.

Definition 6. *A first order logical language L_q contains the following:*

(i) *An enumerable set of individual variables: x_0, x_1, x_2;*
(ii) *A set of predicate symbols: $F = \cup_{n=0}^{\infty} F_n$, where F_n indicates the set of all n-place predicate symbols for every $n \geq 0$, assuming that $\cup_{n=0}^{\infty} F_n \neq \Phi$;*
(iii) *Propositional connectors: ~ and ∧; and*
(iv) *Parentheses: ( ) [41].*

The following definition gives the syntax of language L_q:

Definition 7. *The minimum set of symbol strings is called set Wff of well-formed formula if the following conditions are satisfied:*

(i) *For every $n \geq 0$, if F is an n-place predicate symbol and $y_1, \ldots, y_n$ are individual variables, then $F(y_1, \ldots, y_n)$ is a well-formed formula;*
(ii) *If Q is a quantifier, x is an individual variable, and φ is a well-formed formula, then (Qx) φ is also a well-formed formula; and*
(iii) *If φ, φ_1, and φ_2 are all well-formed formulas, then ~φ, φ_1, and $\wedge\varphi_2$ are also well-formed formulas [41].*

The following definitions give the semantics of language L_q:

Definition 8. *The following items comprise an interpretation I of the logic language:*

(i) *A measurable space (X, ℘), which is called the domain of the interpretation;*
(ii) *For every $n \geq 0$, there exists an element x_i^I in X corresponding to the individual variable x_i; and*
(iii) *For every $n \geq 0$ and any $F \in F_n$, there exists a $\wp^n$-measurable function F^I: $X^n \rightarrow [0, 1]$ [41].*

Definition 9. *Assume that I is an interpretation. Then the truth value $T_I(\varphi)$ of formula φ under I based on O-Sugeno integrals is defined recursively as follows:*

(i) *If $\varphi = F(y_1, \ldots, y_n)$, then*

$$T^I(\varphi) = F({y_1}^I, \ldots, {y_n}^I).$$

(ii) *If φ = (Qx) y, then*

$$T_I(\varphi) = \int^{(OS)} T_{I\{\cdot/x\}}(\psi) \circ Q_X$$

where X is the domain of I, $T_{I\{./x\}}$: X → [0, 1] is a mapping such that

$$T_{I\{./x\}}(\varphi)(u) = T_{I\{u/x\}}(\varphi), \text{ for every } u \in X,$$

and I {u/x} is the interpretation which is different from I only in the assignment of the individual variable x, that is,

$$y^{I\{u/x\}} = y^I \text{ and } x^{I\{u/x\}} = u, \text{ for every } x, y \in X \text{ and } x \neq y.$$

(iii) *If φ = ~ ψ, then*

$$T_I(\varphi) = 1 - T_I(\psi),$$

and if $\varphi = \varphi_1 \wedge \varphi_2$*, then*

$$T_I(\varphi) = \min\{T_I(\varphi_1), T_I(\varphi_2)\} = \int^{(OS)} T_{I\{\cdot/x\}}(\varphi) \circ Q_X$$

Proposition 1. *For any quantifier Q and for any formula* $\varphi \in Wff$*, if O is an overlap function with unit element 1 and I is an interpretation with the domain of a single point set X = {u}, then*

$$T_I\,((Qx)\varphi) = T_I\,(\varphi).$$

Proof of Proposition 1.

$$\begin{aligned} T_I((Qx)\varphi) &= \textstyle\int^{(OS)} T_{I\{\cdot/x\}}(\varphi) \circ Q_X \\ &= \sup_{F\subseteq X} O[\inf_{u\in F} T_{I\{u/x\}}(\varphi), Q_X(F)] \\ &= O[\inf_{u\in\varphi} T_{I\{u/x\}}(\varphi), Q_X(\varphi)] \vee O[T_{I\{u/x\}}(\varphi), Q_X(\{u\})] \\ &= O[\inf_{u\in\varphi} T_{I\{u/x\}}(\varphi),\ 0)] \vee O[T_{I\{u/x\}}(\varphi),\ 1)] \\ &= O[T_{I\{u/x\}}(\varphi),\ 1)] \\ &= T_{I\{u/x\}}(\varphi) \\ &= T_I(\varphi) \end{aligned}$$

□.

The above proposition states that quantification degenerates in a single point domain.

Proposition 2. *For any quantifier Q and for any formula* $\varphi \in Wff$*, if O is an overlap function with unit element 1, then for any interpretation I with domain X*

(1) $T_I((\exists x)\varphi) = \sup_{u\in X} T_{I\{u/x\}}(\varphi)$ *and*

(2) $T_I((\forall x)\varphi) = \inf_{u\in X} T_{I\{u/x\}}(\varphi)$.

Proof of Proposition 2.

(1)

$$\begin{aligned} T_I((\exists x)\varphi) &= \textstyle\int^{OS} T_{I\{u/x\}}(\varphi) \circ \exists_X \\ &= \sup_{F\subseteq X} O[\inf_{u\in F} T_{I\{u/x\}}(\varphi), \exists_X(F)] \\ &= O[\inf_{u\in\varphi} T_{I\{u/x\}}(\varphi), \exists_X(\varphi)] \vee \sup_{\varphi\neq F\subseteq X} O[\inf_{u\in F} T_{I\{u/x\}}(\varphi), \exists_X(F)] \\ &= O[\inf_{u\in\varphi} T_{I\{u/x\}}(\varphi), 0)] \vee \sup_{\varphi\neq F\subseteq X} O[\inf_{u\in F} T_{I\{u/x\}}(\varphi),\ 1)] \\ &= \sup_{\varphi\neq F\subseteq X} O[\inf_{u\in F} T_{I\{u/x\}}(\varphi),\ 1] \\ &= \sup_{\varphi\neq F\subseteq X} \inf_{u\in F} T_{I\{u/x\}}(\varphi) \\ &= \sup_{u\in X} T_{I\{u/x\}}(\varphi) \end{aligned}$$

(2)

$$\begin{aligned} T_I((\forall x)\varphi) &= \int^{OS} T_{I\{u/x\}}(\varphi) \circ \exists_X \\ &= \sup_{F \subseteq X} O[\inf_{u \in F} T_{I\{u/x\}}(\varphi), \forall_X(F)] \\ &= O[\inf_{u \in X} T_{I\{u/x\}}(\varphi), \forall_X(X)] \vee \sup_{F \subset X} O[\inf_{u \in F} T_{I\{u/x\}}(\varphi), \forall_X(F)] \\ &= O[\inf_{u \in X} T_{I\{u/x\}}(\varphi),\ 1] \vee \sup_{F \subset X} O[\inf_{u \in F} T_{I\{u/x\}}(\varphi),\ 0] \\ &= O[\inf_{u \in X} T_{I\{u/x\}}(\varphi),\ 1] \\ &= \inf_{u \in X} T_{I\{u/x\}}(\varphi) \end{aligned}$$

□.

The above proposition shows that for the two quantifiers ∀ and ∃, the method of calculating the truth value of a quantified proposition based on O-Sugeno integrals corresponds to the standard method, which shows that the O-Sugeno integral semantics of fuzzy quantifiers is reasonable.

Example 5. *We consider a comprehensive evaluation of students' health status (see Example 43 in [41]). Assuming X is a set consisting of 10 students, X = {s_1, s_2, . . ., s_{10}}, and H is a linguistic predicate, "to be healthy". The health evaluation of these students is shown in Table 1.*

Table 1. Health condition of 10 students.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}
$H(x)$	0.95	0.1	0.73	1	0.84	0.7	0.67	0.9	1	0.81

Next, we choose the fuzzy quantifier Q = "most" to describe the overall health status of this group of students.

Let $QxH(x)$ be the proposition "Most students are healthy" and I be the interpretation given in Table 1, then $T_I((Qx)H(x))$ represents the truth value of $QxH(x)$ under I calculated by the O-Sugeno integral. According to Theorem 2 in Section 3 and Definition 4 in Section 4, we have

$$T_I((Qx)H(x)) = \int^{(OS)} T_{I\{\cdot/x\}}(H(x)) \circ Q_X = \max_{i=1}^{10} O[h(x_i), Q_X(X_i)]$$

where $h(x_i)$ is the result of rearranging the possible values of $H(x)$ in non-decreasing order, and $X_i = \{x_j: i \le j \le 10\}$ for $1 \le i \le 10$. Table 2 presents the rearranged truth values $h(x_i)$ for $1 \le i \le 10$.

Table 2. Rearranged truth values.

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
$h(x_i)$	0.1	0.67	0.7	0.73	0.81	0.84	0.9	0.95	1	1

According to the definition of the quantifier "most" in Example 3, we calculate the fuzzy measures of X_i as follows:

$$Q_X(X_i) = (|X_i| / |X|)^{3/2} = [(11 - i)/10]^{3/2}, \text{ for } 1 \le i \le 10$$

Using the O-Sugeno integral in which the overlap function $O(x, y) = x^2y^2$ for every $x, y \in [0, 1]$, the truth value of $QxH(x)$ is calculated as follows:

$$T_I((Qx)H(x)) = \max_{i=1}^{10} O[h(x_i), Q_X(X_i)]$$
$$= 0.01 \vee 0.327 \vee 0.251 \vee 0.183 \vee 0.142 \vee 0.088 \vee 0.052 \vee 0.024 \vee 0.008 \vee 0.001$$
$$= 0.327$$

If the overlap function by $O(x, y) = \min(\sqrt{x}, \sqrt{y})$ for any $x, y \in [0, 1]$, the truth value of $QxH(x)$ is calculated as follows:

$$T_I((Qx)H(x)) = \max_{i=1}^{10} O[h(x_i), Q_X(X_i)]$$
$$= 0.316 \vee 0.819 \vee 0.837 \vee 0.765 \vee 0.682 \vee 0.595 \vee 0.503 \vee 0.405 \vee 0.299 \vee 0.178$$
$$= 0.837$$

The above example shows that choosing different overlap functions to calculate the true value of the proposition "Most students are healthy" under the interpretation will lead to different results. In decision-making problems based on preference relationships, different overlap functions can reflect different fuzzy preferences, which provide multiple choices for decision-makers (they can manifest the preference relationship by choosing different overlap functions).

Example 6. *We consider a comprehensive evaluation of the weather conditions for a week (see Example 42 in [41]). Let X be a set consisting of 7 days, X = {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}. And let P_1 and P_2 represent respectively the linguistic predicates "to be cloudy" and "to be cold". The respective weather conditions of the week are indicated in Table 3. Suppose Q is a fuzzy quantifier, "most", then the formula $\varphi = (Qx)\psi = (Qx)\ (P_1(x) \wedge\sim P_2(x))$ represents "many days (in this week) are cloudy but not cold".*

Table 3. Truth values of linguistic predicates P_1 and P_2.

	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
${P_1}^I$	0.1	0	0.5	0.8	0.6	1	0.2
${P_2}^I$	1	0.9	0.4	0.7	0.3	0.4	0

With interpretation I and truth values P_1 and P_2 given in Table 3, then $T_I(\varphi) = T_I((Qx)\ (P_1(x) \wedge\sim P_2(x)))$ represents the truth value of $\varphi = (Qx)\ (P_1(x) \wedge\sim P_2(x))$ under interpretation I about the O-Sugeno integral. According to Theorem 2 in Section 3 and Definition 4 in Section 4, we have

$$T_I(\varphi) = \int_A^{(OS)} T_{I\{\cdot/x\}}[P_1^I(x) \wedge \sim P_2^I(x)] \circ Q_X = \max_{i1}^{7} O[h(x_i), Q_X(X_i)]$$

where $h(x_i)$ is the result of the rearranged possible values of $T_I\ (P_1(x) \wedge\sim P_2(x))$ in non-decreasing order, and $X_i = \{x_j: i \leq j \leq 7\}$ for $1 \leq i \leq 7$. Table 4 presents the rearranged truth values $h(x_i)$ for $1 \leq i \leq 7$.

Table 4. Rearranged truth values.

	1	2	3	4	5	6	7
$h(x_i)$	0	0	0.2	0.3	0.5	0.6	0.6

According to the definition of the quantifier "most" in Example 3, fuzzy measures about the fuzzy quantifier of X_i are calculated as follows:

$$Q_X(X_1) = 1,\ Q_X(X_2) = 6/7,\ Q_X(X_3) = 5/7,\ Q_X(X_4) = 4/7,\ Q_X(X_5) = 3/7,\ Q_X(X_6) = 2/7,\ Q_X(X_7) = 1/7.$$

The overlap function be defined as $O(x, y) = \min(\sqrt{x}, \sqrt{y})$ for any $x, y \in [0, 1]$, and by using the O-Sugeno integral, the truth value of $(Qx)(P_1(x) \wedge \sim P_2(x))$ is calculated as follows:

$$\begin{aligned} T_I(\varphi) &= \max_{i=1}^{7} O[h(x_i), Q_X(X_i)] \\ &= 0 \vee 0 \vee 0.447 \vee 0.548 \vee 0.655 \vee 0.535 \vee 0.378 \\ &= 0.655 \end{aligned}$$

5. Applying Integral Semantics of Fuzzy Quantifiers to MADM

The MADM based on fuzzy quantifiers is to select the decision object that performs well on a certain proportion or quantification (such as most, many, more that half, etc.) of attributes. In this section, we propose a MADM method based on O-Sugeno integral semantics of fuzzy quantifiers to solve the MADM problem involving fuzzy quantifiers. The specific process is described as follows.

The basic representations are as follows: $S = \{s_1, s_2, \ldots, s_m\}$ is a set of m decision objects (also known as feasible alternatives), $G = \{g_1, g_2, \ldots, g_n\}$ is a set of n evaluation indicators (also called attributes), and Q represents the fuzzy quantifiers such as most, many, more than half, etc.

Step 1: Calculate the truth values of linguistic predicates under the interpretations and rearrange them to obtain the standardized truth values.

The performance of each decision-making object on the attributes is regarded as an interpretation I. For any $x \in G$, $\varphi(x)$ means that the predicate meets the requirements of attribute x. For each decision-making object $s \in S$, we compute the truth value of the linguistic predicate $\varphi(x)$ under interpretation I, and then rearrange all of them to get $h(x_i)$, where for $1 \leq i \leq n-1$, $h(x_i) \leq h(x_{i+1})$.

Step 2: Calculate fuzzy measures about the fuzzy quantifier.

According to the semantic analysis of the fuzzy quantifier, we can calculate a family of fuzzy measures $Q_X(X_i)$, where for $1 \leq i \leq n$, $X_i = \{x_j: i \leq j \leq n\}$.

Step 3: Calculate the truth value of the proposition for each decision object based on O-Sugeno integral semantics.

We consider the proposition $(Qx)\varphi(x)$ = "A decision object meets the requirements of attributes with the fuzzy proportion Q". Based on the O-Sugeno integral semantics of fuzzy quantifiers, we calculate the truth values $D(s_i)$ of proposition $(Qx)\varphi(x)$ under its interpretation for each decision-making object $s_i \in S$:

$$D(s) = T_I((Qx)\varphi(x)) = \int^{(OS)} s \circ Q_X = \max_{i=1}^{n} O[h(x_i), Q_X(X_i)]$$

for any $s \in G$

Step 4: Obtain the optimal object by ranking the truth values of decision objects.

Example 7. *Decision-making problem for selecting excellent students. The best of three high school students will be recommended to enter a well-known university based on their mathematics, physics, biology, chemistry, and literature grades. The relevant data in Table 5 show the grades for each student in each course.*

Table 5. Performance of three students.

	Mathematics	Physics	Biology	Chemistry	Literature
s_1	0.75	0.85	0.95	0.90	0.86
s_2	0.85	0.92	0.91	0.95	0.86
s_3	0.92	0.87	0.90	0.89	0.91

In order to obtain a comprehensive evaluation of each student, we consider the proposition "(student) has performed well in almost all courses". The domain is indicated as X = {mathematics, physics, biology, chemistry, literature}, the fuzzy quantifier is Q = "almost all", and the predicate is $\varphi(x)$ = "(student) performed well on x" for each student, then the proposition "(student) performs well in almost all courses" is expressed as the logic formula $(Qx)\varphi(x)$.

Each student's performance in the five courses is considered as an interpretation I. For each student, we rearrange the truth values of the linguistic predicate $\varphi(x)$ under its interpretation I to get $h(x_i)$ for $1 \le i \le 5$. Table 6 presents the rearranged truth values.

Table 6. Rearranged truth values.

	1	**2**	**3**	**4**	**5**
s_1 $(h_1(x_i))$	0.75	0.85	0.86	0.9	0.95
s_2 $(h_2(x_i))$	0.85	0.86	0.91	0.92	0.95
s_3 $(h_3(x_i))$	0.87	0.89	0.90	0.91	0.92

According to the definition of "almost all" in Example 3, we can calculate fuzzy measures of $X_i = \{x_j: i \le j \le 5\}$ for $1 \le i \le 5$ about the fuzzy quantifier as follows:

$$Q_X(X_1) = 1,\ Q_X(X_2) = 0.64,\ Q_X(X_3) = 0.36,\ Q_X(X_4) = 0.16,\ Q_X(X_5) = 0.04.$$

The overlap function is defined by $O(x, y) = x\,y\,(x + y)/2$ for any $x, y \in [0, 1]$, then the true value $(Qx)\varphi(x)$ of each student under interpretation I based on the O-Sugeno integral is calculated as follows:

$$\begin{aligned}
&D(s_1) = T_I((Qx)\varphi(x)) = \int^{(OS)} s_1 \circ Q_X = \max_{i=1}^{5} O[h_1(x_i), Q_X(X_i)]\\
&= 0.75 \times 1 \times (0.75 + 1)/2 \vee 0.85 \times 0.64 \times (0.85 + 0.64)/2 \vee 0.86 \times 0.36 \times (0.86 + 0.36)/2\\
&\vee 0.9 \times 0.16 \times (0.9 + 0.16)/2 \vee 0.95 \times 0.04 \times (0.95 + 0.04)/2\\
&= 0.656 \vee 0.405 \vee 0.189 \vee 0.076 \vee 0.019\\
&= 0.656.
\end{aligned}$$

$$\begin{aligned}
&D(s_2) = T_I((Qx)\varphi(x)) = \int^{(OS)} s_2 \circ Q_X = \max_{i=1}^{5} O[h_2(x_i), Q_X(X_i)]\\
&= 0.85 \times 1 \times (0.85 + 1)/2 \vee 0.86 \times 0.64 \times (0.86 + 0.64)/2 \vee 0.91 \times 0.36 \times (0.91 + 0.36)/2\\
&\vee 0.92 \times 0.16 \times (0.92 + 0.16)/2 \vee 0.95 \times 0.04 \times (0.95 + 0.04)/2\\
&= 0.786 \vee 0.413 \vee 0.208 \vee 0.079 \vee 0.019\\
&= 0.786.
\end{aligned}$$

$$\begin{aligned}
&D(s_3) = T_I((Qx)\varphi(x)) = \int^{(OS)} s_3 \circ Q_X = \max_{i=1}^{5} O[h_3(x_i), Q_X(X_i)]\\
&= 0.87 \times 1 \times (0.87 + 1)/2 \vee 0.89 \times 0.64 \times (0.89 + 0.64)/2 \vee 0.9 \times 0.36 \times (0.9 + 0.36)/2\\
&\vee 0.91 \times 0.16 \times (0.91 + 0.16)/2 \vee 0.92 \times 0.04 \times (0.92 + 0.04)/2\\
&= 0.813 \vee 0.436 \vee 0.204 \vee 0.078 \vee 0.018\\
&= 0.813.
\end{aligned}$$

The maximum value $D(s_3)$ can be obtained by ranking these true values, so student s_3 is the best student.

Example 8. *Decision-making problem about supplier selection. A factory needs to choose a supplier for an important raw material, and the decision-maker intends to select from four alternative suppliers, which are represented as s_1, s_2, s_3, and s_4. The decision-maker evaluates these suppliers in four aspects, which are called decision attributes: product price, product quality, service level, and reputation. The specific data are given in Table 7.*

Table 7. Attribute values of four suppliers.

	Product Price	Product Quality	Service Level	Reputation
s_1	0.95	0.71	0.85	0.8
s_2	0.8	0.76	0.92	0.83
s_3	0.85	0.81	0.7	0.86
s_4	0.76	0.9	0.75	0.84

In order to obtain a comprehensive evaluation of each supplier, we consider the proposition "(supplier) meets the requirements for most attributes". Domain X is indicated as X = {product price, product quality, service level, reputation}, the fuzzy quantifier is Q = "most", and the predicate is φ (x) = "(supplier) meets the requirement of x" for each $x \in X$, then the proposition "(supplier) meets the requirements for most attributes" is expressed as the logic formula $(Qx)\varphi(x)$.

Each supplier's performance on four attributes is considered as an interpretation I. For each supplier, we rearrange the truth values of the linguistic predicate $\varphi(x)$ under its interpretation I to get $h(x_i)$ for $1 \leq i \leq 4$. Table 8 presents the rearranged truth values.

Table 8. Rearranged truth values.

	1	2	3	4
s_1 ($h_1(x_i)$)	0.71	0.8	0.85	0.95
s_2 ($h_2(x_i)$)	0.76	0.8	0.83	0.92
s_3 ($h_3(x_i)$)	0.7	0.81	0.85	0.86
s_4 ($h_4(x_i)$)	0.75	0.76	0.84	0.9

According to the definition of "most" in Example 3, we can calculate fuzzy measures of $X_i = \{x_j: i \leq j \leq 4\}$ for $1 \leq i \leq 4$ about the fuzzy quantifier as follows:

$$Q_X(X_1) = 1,\ Q_X(X_2) = (3/4)^{3/2} \approx 0.650,\ Q_X(X_3) = (1/2)^{3/2} \approx 0.354,\ Q_X(X_4) = (1/4)^{3/2} \approx 0.125.$$

Let the overlap function O: $[0, 1]^2 \rightarrow [0, 1]$ be defined as

$$O(x, y) = \frac{1}{1 - xy + 1/xy}$$

for any $x, y \in [0, 1]$, then the true value $(Qx)\varphi(x)$ of each supplier under its interpretation I based on the O-Sugeno integral is calculated as follows:

$$\begin{aligned} D(s_1) &= T_I((Qx)\varphi(x)) = \int^{(OS)} s_1 \circ Q_X = \max_{i=1}^{4} O[h_1(x_i), Q_X(X_i)] \\ &= \tfrac{1}{1-0.71+1/0.71} \vee \tfrac{1}{1-0.8\times0.65+1/(0.8\times0.65)} \\ &\vee \tfrac{1}{1-0.85\times0.354+1/(0.85\times0.354)} \vee \tfrac{1}{1-0.95\times0.125+1/(0.95\times0.125)} \\ &= 0.589 \vee 0.416 \vee 0.248 \vee 0.108 \\ &= 0.589. \end{aligned}$$

$$\begin{aligned} D(s_2) &= T_I((Qx)\varphi(x)) = \int^{(OS)} s_2 \circ Q_X = \max_{i=1}^{4} O[h_2(x_i), Q_X(X_i)] \\ &= \tfrac{1}{1-0.76+1/0.76} \vee \tfrac{1}{1-0.8\times0.65+1/(0.8\times0.65)} \\ &\vee \tfrac{1}{1-0.83\times0.354+1/(0.83\times0.354)} \vee \tfrac{1}{1-0.92\times0.125+1/(0.92\times0.125)} \\ &= 0.643 \vee 0.416 \vee 0.243 \vee 0.104 \\ &= 0.643. \end{aligned}$$

$$D(s_3) = T_I(Qx)\varphi(x)) = \int^{(OS)} s_3 \circ Q_X = \max_{i=1}^{4} O[h_3(x_i), Q_X(X_i)]$$
$$= \frac{1}{1-0.7+1/0.7} \vee \frac{1}{1-0.81\times0.65+1/(0.81\times0.65)}$$
$$\vee \frac{1}{1-0.85\times0.354+1/(0.85\times0.354)} \vee \frac{1}{1-0.86\times0.125+1/(0.86\times0.125)}$$
$$= 0.579 \vee 0.421 \vee 0.248 \vee 0.098$$
$$= 0.579.$$

$$D(s_4) = T_I(Qx)\varphi(x)) = \int^{(OS)} s_4 \circ Q_X = \max_{i=1}^{4} O[h_4(x_i), Q_X(X_i)]$$
$$= \frac{1}{1-0.75+1/0.75} \vee \frac{1}{1-0.76\times0.65+1/(0.76\times0.65)}$$
$$\vee \frac{1}{1-0.84\times0.354+1/(0.84\times0.354)} \vee \frac{1}{1-0.9\times0.125+1/(0.9\times0.125)}$$
$$= 0.632 \vee 0.395 \vee 0.246 \vee 0.102$$
$$= 0.632.$$

Therefore, the evaluation shows that supplier s_2 has the highest score, thus supplier s_2 should be selected.

Example 9. *Decision-making problem about logistics park location. A city wants to build a logistics park, and the decision-maker plans to choose from eight alternatives, which are represented as s_i, for $1 \leq i \leq 8$. The decision-maker evaluates these alternatives in 12 aspects, which are called decision attributes: urban support, traffic conditions, geological environment, land price, urban traffic improvement, convenient delivery, surrounding facilities, neighboring enterprises, talent attraction, logistics development space, prospect of environmental development, and predicted economic development. The specific data are given in Table 9.*

Table 9. Evaluation of 12 attributes.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
Urban support	0.912	0.97	0.824	0.706	0.964	0.556	0.656	0.734
Traffic conditions	0.9	0.846	0.786	0.93	0.824	0.972	0.738	0.892
Geological environment	0.89	0.876	0.93	0.824	0.772	0.932	0.936	0.814
Land price	0.69	0.574	0.856	0.712	0.592	0.93	0.726	0.794
Urban traffic improvement	0.7	0.624	0.858	0.89	0.652	0.978	0.972	0.904
Convenient delivery	0.85	0.864	0.904	0.774	0.902	0.606	0.596	0.912
Surrounding facilities	0.648	0.774	0.912	0.842	0.804	0.67	0.806	0.796
Neighboring enterprises	0.806	0.828	0.912	0.774	0.812	0.604	0.772	0.804
Talent attraction	0.846	0.972	0.826	0.774	0.962	0.604	0.796	0.806
Logistics development space	0.796	0.712	0.912	0.804	0.806	0.608	0.778	0.952
Prospect of environmental development	0.792	0.774	0.956	0.796	0.846	0.734	0.752	0.846
Predicted economic development	0.808	0.808	0.816	0.842	0.792	0.774	0.804	0.912

In order to obtain a comprehensive evaluation of each alternative, we consider the proposition "(alternative) meets the requirements of most attributes". Domain X is indicated as X = {urban support, traffic conditions, geological environment, land price, urban traffic improvement, convenient delivery, surrounding facilities, neighboring enterprises, talent attraction, logistics development space, prospect of environmental development, predicted economic development}, the fuzzy quantifier is Q = "most", and the predicate is φ (x) = "(alternative) meets the requirement of x" for each $x \in X$, then the proposition "(alternative) meets the requirements of most attributes" is expressed as the logic formula $(Qx)\varphi(x)$.

The performance of each alternative on 12 attributes is considered as an interpretation I. For each alternative, we rearrange the truth values of the linguistic predicate $\varphi(x)$ under its interpretation I to get $h(x_i)$ for $1 \leq i \leq 12$. Table 10 presents the rearranged truth values.

Table 10. Rearranged truth values.

	s_1 $(h_1(x_i))$	s_2 $(h_2(x_i))$	s_3 $(h_3(x_i))$	s_4 $(h_4(x_i))$	s_5 $(h_5(x_i))$	s_6 $(h_6(x_i))$	s_7 $(h_7(x_i))$	s_8 $(h_8(x_i))$
1	0.648	0.574	0.786	0.706	0.592	0.556	0.596	0.734
2	0.69	0.624	0.816	0.712	0.652	0.604	0.656	0.794
3	0.7	0.712	0.824	0.774	0.772	0.604	0.726	0.796
4	0.792	0.774	0.826	0.774	0.792	0.606	0.738	0.804
5	0.796	0.774	0.856	0.774	0.804	0.608	0.752	0.806
6	0.806	0.808	0.858	0.796	0.806	0.67	0.772	0.814
7	0.808	0.828	0.904	0.804	0.812	0.734	0.778	0.846
8	0.846	0.846	0.912	0.824	0.824	0.774	0.796	0.892
9	0.85	0.864	0.912	0.842	0.846	0.93	0.804	0.904
10	0.89	0.876	0.912	0.842	0.902	0.932	0.806	0.912
11	0.9	0.97	0.93	0.89	0.962	0.972	0.936	0.912
12	0.912	0.972	0.956	0.93	0.964	0.978	0.972	0.952

According to the definition of "most" in Example 3, we can calculate fuzzy measures of $X_i = \{x_j: i \leq j \leq 12\}$ for $1 \leq i \leq 12$ about the fuzzy quantifier as follows:

$$Q_X(X_i) = (\mid X_i \mid / \mid X \mid)^{3/2} = [(13 - i)/12]^{3/2}, \text{ for } 1 \leq i \leq 12.$$

After the calculation, we can obtain:

$$Q_X(X_1) = 1,\ Q_X(X_2) \approx 0.878,\ Q_X(X_3) \approx 0.761,\ Q_X(X_4) \approx 0.650,\ Q_X(X_5) \approx 0.544,\ Q_X(X_6) \approx 0.446,$$

$$Q_X(X_7) \approx 0.354,\ Q_X(X_8) \approx 0.269,\ Q_X(X_9) \approx 0.192,\ Q_X(X_{10}) = 0.125,\ Q_X(X_{11}) \approx 0.068,\ Q_X(X_{12}) \approx 0.024.$$

The overlap function is defined as

$$O(x,y) = \min(\sqrt{x}, \sqrt{y})$$

for any $x, y \in [0, 1]$, then the true value $(Qx)\varphi(x)$ of each supplier under its interpretation I based on the O-Sugeno integral is calculated as follows:

$$D(s_j) = T_I((Qx)\varphi(x)) = \int^{(OS)} s_j \circ Q_X = \max_{i=1}^{12} O[h_j(x_i), Q_X(X_i)]$$

for $1 \leq j \leq 8$.

After the calculation, we obtain the comprehensive evaluation values of all alternatives, as shown in Table 11.

Table 11. Comprehensive evaluation values of eight alternatives.

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
Evaluation value	0.837	0.844	0.903	0.872	0.872	0.778	0.852	0.891

Therefore, the evaluation shows that alternative s_3 has the highest score, thus alternative s_3 should be selected.

Example 10. *Decision-making problem about the purchase of a new energy car. A customer is going to buy a new energy car. After preliminary screening, the customer has four alternatives. These alternatives are represented as s_i, for $1 \leq i \leq 4$. In order to purchase a satisfactory car, the customer browsed the comments of each alternative on various network platforms and evaluated them from seven aspects (attributes): appearance, interior, space, comfort, power, operation difficulty,* *and cost performance. Through text sentiment analysis, all evaluation information is converted into specific data, as shown in Table 12.*

Table 12. Evaluation of 7 attributes.

	s_1	s_2	s_3	s_4
Appearance	0.8149	0.7320	0.8352	0.6786
Interior	0.6890	0.7302	0.7056	0.6810
Space	0.5969	0.3858	0.2555	0.3183
Comfort	0.7058	0.6030	0.7398	0.6429
Power	0.5708	0.5227	0.6259	0.4488
Operation difficulty	0.6632	0.6041	0.4893	0.4579
Cost performance	0.6765	0.4597	0.6123	0.4090

In order to obtain a comprehensive evaluation of each alternative, we consider the proposition of "(alternative) meets the requirements for almost all attributes". Domain X is indicated as X = {appearance, interior, space, comfort, power, operation difficulty, cost performance}, the fuzzy quantifier is Q = "almost all", and the predicate is $\phi(x)$ = "(alternative) meets the requirement of x" for each $x \in X$, then the proposition "(alternative) meets the requirements of almost all attributes" is expressed as the logic formula $(Qx)\phi(x)$.

The performance of each alternative on seven attributes is considered as an interpretation I. For each alternative, we rearrange the truth values of the linguistic predicate $\phi(x)$ under its interpretation I to get $h(x_i)$ for $1 \leq i \leq 7$. Table 13 presents the rearranged truth values.

Table 13. Rearranged truth values.

	s_1 ($h_1(x_i)$)	s_2 ($h_2(x_i)$)	s_3 ($h_3(x_i)$)	s_4 ($h_4(x_i)$)
1	0.5708	0.3858	0.2555	0.3183
2	0.5969	0.4597	0.4893	0.4090
3	0.6632	0.5227	0.6123	0.4488
4	0.6765	0.6030	0.6259	0.4579
5	0.6890	0.6041	0.7056	0.6429
6	0.7058	0.7302	0.7398	0.6786
7	0.8149	0.7320	0.8352	0.6810

According to the definition of "almost all" in Example 3, we can calculate fuzzy measures of $X_i = \{x_j: 1 \leq i \leq 7\}$ for $1 \leq i \leq 7$ about fuzzy quantifier as follows:

$$Q_X(X_i) = (|X_i| / |X|)^2 = [(8 - i)/7]^2, \text{ for } 1 \leq i \leq 7.$$

After the calculation, we can obtain:

$$Q_X(X_1) = 1,\ Q_X(X_2) = 0.735,\ Q_X(X_3) = 0.510,\ Q_X(X_4) = 0.327,\ Q_X(X_5) = 0.184,\ Q_X(X_6) = 0.082,\ Q_X(X_7) = 0.020$$

The overlap function is defined as

$$O(x, y) = \min(\sqrt{x}, \sqrt{y}),$$

for any $x, y \in [0, 1]$, then the truth value $(Qx)\phi(x)$ of each alternative under its interpretation I based on the O-Sugeno integral is calculated as follows:

$$D(s_j) = T_I((Qx)\varphi(x)) = \int^{OS} s_j \circ Q_X = \max_{i=1}^{7} O[h_j(x_i), Q_X(X_i)]$$

for $1 \leq j \leq 4$.

After the calculation, we obtain the comprehensive evaluation values of all alternatives, as shown in Table 14.

Table 14. Comprehensive evaluation values of four alternatives.

	s_1	s_2	s_3	s_4
Evaluation value	0.773	0.714	0.714	0.670

Therefore, the evaluation shows that alternatives s_1 has the highest score, thus alternatives s_1 should be selected.

Example 11. *Decision-making problem about red wine selection. There are currently four types of red wines. In order to select the optimal one, the components (attributes) of each wine needs to be measured and evaluated, including fixed acidity, volatile acidity, citric acid, residual sugar, chlorides, free sulfur dioxide, total sulfur dioxide, sulfate and alcohol. The specific data are revealed in Table 15. (data from open source datasets website).*

Table 15. Evaluation of nine components.

	s_1	s_2	s_3	s_4
Fixed acidity	0.9689	0.9222	0.8210	0.8327
Volatile acidity	0.4055	0.9843	0.7323	0.7323
Citric acid	0.6648	0.5369	0.8750	0.7983
Residual sugar	0.8147	0.6207	0.8922	0.9914
Chlorides	0.7727	0.4513	0.8312	0.7581
Free sulfur dioxide	0.8285	0.7531	0.3430	0.9456
Total sulfur dioxide	0.7143	0.8937	0.6246	0.9867
Sulfate	0.6903	0.8148	0.8726	0.9580
Alcohol	0.7607	0.7855	0.8020	0.8682

In order to obtain a comprehensive evaluation of each alternative, we consider the proposition of "(alternative) meets the requirements for almost all attributes". Domain X is indicated as X = {fixed acidity, volatile acidity, citric acid, residual sugar, chlorides, free sulfur dioxide, total sulfur dioxide, sulfate, alcohol}, the fuzzy quantifier is Q = "almost all", and the predicate is $\phi(x)$ = "(alternative) meets the requirement of x" for each $x \in X$, then the proposition "(alternative) meets the requirements for almost all attributes" is expressed as the logic formula $(Qx)\phi(x)$.

The performance of each alternative on nine attributes is considered as an interpretation I. For each alternative, we rearrange the truth values of the linguistic predicate $\phi(x)$ under its interpretation I to get $h(x_i)$ for $1 \leq i \leq 4$. Table 16 presents the rearranged truth values.

Table 16. Rearranged truth values.

	s_1 $(h_1(x_i))$	s_2 $(h_2(x_i))$	s_3 $(h_3(x_i))$	s_4 $(h_4(x_i))$
Fixed acidity	0.4055	0.4513	0.3430	0.7323
Volatile acidity	0.6648	0.5369	0.6246	0.7581
Citric acid	0.6903	0.6207	0.7323	0.7983
Residual sugar	0.7143	0.7531	0.8020	0.8327
Chlorides	0.7607	0.7855	0.8210	0.8682
Free sulfur dioxide	0.7727	0.8148	0.8312	0.9456
Total sulfur dioxide	0.8147	0.8937	0.8726	0.9580
Sulfate	0.8285	0.9222	0.8750	0.9867
Alcohol	0.9689	0.9843	0.8922	0.9914

According to the definition of "almost all" in Example 3, we can calculate fuzzy measures of $X_i = \{x_j: 1 \leq i \leq 9\}$ for $1 \leq i \leq 9$ about the fuzzy quantifier as follows:

$$Q_X(X_i) = (|X_i|/|X|)^2 = [(9-i)/9]^2, \text{ for } 1 \leq i \leq 9$$

After the calculation, we can obtain:

$$Q_X(X_1) = 1, Q_X(X_2) = 64/81, Q_X(X_3) = 49/81, Q_X(X_4) = 4/9, Q_X(X_5) = 25/81, Q_X(X_6) = 16/81, Q_X(X_7) = 1/9, Q_X(X_8) = 4/81, Q_X(X_9) = 1/81.$$

The overlap function is defined as

$$O(x, y) = \min(\sqrt{x}, \sqrt{y}),$$

for any x, $y \in [0, 1]$, then the truth value $(Qx)\phi(x)$ of each supplier under its interpretation I based on the O-Sugeno integral is calculated as follows:

$$D(s_j) = T_I((Qx)\varphi(x)) = \int^{OS} s_j \circ Q_X = \max_{i=1}^{9} O[h_j(x_i), Q_X(X_i)]$$

for $1 \leq j \leq 4$.

After the calculation, we obtain the comprehensive evaluation values of all alternatives, as shown in Table 17.

Table 17. Comprehensive evaluation values of nine alternatives.

	s_1	s_2	s_3	s_4
Evaluation value	0.815	0.778	0.790	0.871

Therefore, the evaluation shows that alternatives s_4 has the highest score, and the alternatives s_4 should be selected.

6. Conclusions

In this study, we proposed O-Sugeno integrals and studied their basic properties. Since overlap functions can be non-associative, the range of applications of O-Sugeno integrals is greatly expanded. Fuzzy quantifiers can be quantified by fuzzy measures, and linguistic quantifier propositions containing fuzzy quantifiers can be calculated their truth values using O-Sugeno integrals. Then, we researched the O-Sugeno integral semantics of fuzzy quantifiers. Finally, we proposed a MADM method based on O-Sugeno integral semantics of fuzzy quantifiers to solve the MADM problem involving fuzzy quantifier-based.

In future work, we will introduce Choquet integrals based on overlap functions and apply them to MADM problems involving fuzzy quantifiers.

Author Contributions: Writing—original draft preparation, X.M.; writing—review and editing, C.T.; writing—inspection and modification, H.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Natural Science Foundation of Zhejiang Province, grant number LY20A010012, and by the Zhejiang Provincial Soft Science Research Project, grant number 2022C35101.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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