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Classical Solutions for the Generalized Korteweg-de Vries Equation

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Abstract: The Korteweg-de Vries equation models the formation of solitary waves in the context of shallow water in a channel. In Equation (1), f or $p = 2$ and $p = 3$ (Korteweg-de Vries equations (KdV)) and (modified Korteweg-de Vries (mKdV) respectively), these equations have many applications in Physics. (gKdV) is a Hamiltonian system. In this article we investigate the generalized Korteweg-de Vries (gKdV) Equation (3). A new topological approach is applied to prove the existence of at least one classical solution. The arguments are based upon recent theoretical results.

Keywords: gKdV equation; classical solution; existence

1. Introduction

If p is integer, the Korteweg-de Vries Equation [1] is as follows

$$\partial_t u + (u_{xx} + |u|^p)_x = 0. \quad (1)$$

It is particularly very important as a prototypical example of an exactly solvable non-linear system (that is, completely integrable infinite dimensional system). The generalized Korteweg-de Vries equation (gKdV) is a Hamiltonian system. In particular, three quantities are conserved, at least formally

$$\begin{aligned} \int u dx &= \int u_0 dx, \\ \int u^2 dx &= \int u_0^2 dx, & (L^2 - \text{mass}) \end{aligned} \quad (2)$$

$$E(u) = \frac{1}{2} \int u_x^2 dx - \frac{1}{p+1} \int |u|^{p+1} dx = E(0), \quad (\text{Energy})$$

The natural energy space for the study of this equation is therefore H^1 . Note however that the first conservation law is little used, because it is not a signed quantity, and moreover it is not in the energy space. Moreover, the equation admits a scale invariance: if u is a solution of (gKdV), we have

$$u_\lambda(x, t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^3 t).$$



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Note that

$$\begin{aligned}\int u_\lambda &= \lambda^{(3-p)/(p-1)} \int u, \\ \int u_\lambda^2 &= \lambda^{(5-p)/(p-1)} \int u^2, \\ E(u_\lambda) &= \lambda^{(p+3)/(p-1)} E(u).\end{aligned}$$

Especially for $p = 5$, $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$ and the equation is L^2 -critical for the invariance of scale is (cKdV)

$$\partial_t u + (u_{xx} + u^5)_x = 0.$$

The stability of these solutions was investigated in [2], whereas asymptotic stability has been studied in [3,4].

In this paper, we investigate the Cauchy problem for the generalized Korteweg-de Vries equation

$$\begin{aligned}\partial_t u + \partial_x^3 u + u^k \partial_x u &= 0, \quad t \in [0, \infty), x \in \mathbb{R}, \\ u(t=0) &= u_0(x), \quad x \in \mathbb{R},\end{aligned}\tag{3}$$

under the next hypothesis

(Hyp1) $k \geq 0$, $u_0 \in C^3(\mathbb{R})$, $0 \leq u_0 \leq B$ on $\mathbb{R}$ for some $B > 0$.

Let us now suppose

(Hyp2) There exist a function $h \in C([0, \infty) \times \mathbb{R})$, $0 < h$ on $(0, \infty) \times (\mathbb{R} \setminus \{0\})$, $h(0, x) = h(t, 0) = 0$, $t \in [0, \infty)$, $x \in \mathbb{R}$, and $A > 0$ such that

$$6 \cdot 2^4 (t^2 + t + 1) (1 + |x| + x^2 + |x|^3) \int_0^t \left| \int_0^x h(t_2, x_2) dx_2 \right| dt_2 \leq A, \tag{4}$$

$$t \in [0, \infty), x \in \mathbb{R},$$

and

(Hyp3) $AB_1 < B$, $B_1 = \max\{2B, B + B^{k+1}\}$.

In the last section we will give examples for g , A , B and B_1 that satisfy (Hyp2) and (Hyp3).

The aim of this paper is to investigate the initial value problem Equation (3) for existence of global classical solutions.

Theorem 1. Suppose (Hyp1) and (Hyp2). Then the initial value problem Equation (3) has at least one solution $u \in C^1([0, \infty), C^3(\mathbb{R}))$.

Theorem 2. Suppose (Hyp1), (Hyp2) and (Hyp3). Then the initial value problem Equation (3) has at least one nonnegative solution $u \in C^1([0, \infty), C^3(\mathbb{R}))$.

The present paper is marshaled as follows. In the second Section 2, we state some useful auxiliary results and needed tools. In Section 3, we present and prove some needed results. In Section 4 we prove the main Theorem 1 and the second main result Theorem 2 will be shown in Section 5. The last Section 6 will be an example illustrating our main results.

2. Preliminary Results

The first continuation theorems applicable to nonlinear problems were due to Leray and Schauder (1934) [5] (Theorem 10.3.10). This result is the most famous and most general result of the continuation theorems (see [5] pages 28,29). In [6] (1955), Scheafer formulated a special case of Leray-Schauder continuation theorem in the form of an alternative, and

proves it as a consequence of Schauder fixed point theorem. In this paper, we will use some nonlinear alternatives, in one hand, to develop a new fixed point theorem and in another hand to study the existence of solutions for Problem Equation (3). In what follows we recall these alternatives.

Proposition 1. (Leray-Schauder nonlinear alternative [7]) Let $C \subset E$ be a convex, closed subset in a Banach space E , $0 \in V \subset C$ where V is an open set. Let $f: \overline{V} \rightarrow C$ be a compact, continuous map. Then

- (a) either f has a fixed point in $\overline{V}$,
- (b) or there exist $x \in \partial V$, and $\lambda \in (0, 1)$ such that $x = \lambda f(x)$.

As a consequence, we obtain

Proposition 2. (Schafer's Theorem or Leray-Schauder alternative, [8], p124 or [5], p 29) Let E be a Banach space and $f: E \rightarrow E$ be completely continuous map. Then,

- (a) either f has a fixed point in E ,
- (b) or for any $\lambda \in (0, 1)$, the set $\{x \in E : x = \lambda f(x)\}$ is unbounded.

Another version of Scheafer's Theorem is given by:

Proposition 3. (Scheafer's Theorem [6]) Let E be a Banach space and $f: E \rightarrow E$ be completely continuous map. Then

- (a) either there exists for each $\lambda \in [0, 1]$ one small $x \in E$ such that $x = \lambda f(x)$,
- (b) or the set $\{x \in E : x = \lambda f(x), 0 < \lambda < 1\}$ is bounded in E .

The following theorem will be used to prove Theorems 1 and 2.

Theorem 3. Let E be a Banach space, Z a closed, convex subset of E ,

$$V = \{x \in Z : \|x\| < R\},$$

with $R > 0$. Consider two operators W and G , where

$$Wx = \varepsilon x, x \in \overline{V},$$

for $\varepsilon \in \mathbb{R}$, and $G: \overline{V} \rightarrow E$ be such that

- (i) $I - G: \overline{V} \rightarrow Z$ continuous, compact and
- (ii) $\{x \in Z : x = \operatorname{sgn}(\varepsilon)\lambda(I - G)x, \|x\| = R\} = \emptyset$, for any $\lambda \in (0, \frac{1}{|\varepsilon|})$, where $\operatorname{sgn}(\varepsilon)$ is the signum of ε .

Then there exists $x^* \in \overline{V}$ such that

$$Wx^* + Gx^* = x^*. \quad (5)$$

Proof. We have that the operator $\frac{1}{\varepsilon}(I - G): \overline{V} \rightarrow Z$ is continuous and compact. Suppose that $\exists x_0 \in \partial V$ and $\mu_0 \in (0, 1)$ such that

$$x_0 = \mu_0 \frac{1}{\varepsilon}(I - G)x_0,$$

that is

$$x_0 = \operatorname{sgn}(\varepsilon) \frac{\mu_0}{|\varepsilon|} (I - G)x_0.$$

This contradicts the condition (ii). From Leray-Schauder nonlinear alternative, it follows that there exists $x^* \in \bar{V}$ so that

$$x^* = \frac{1}{\varepsilon}(I - G)x^* \quad (6)$$

or

$$\varepsilon x^* + Gx^* = x^*, \quad (7)$$

or

$$Wx^* + Gx^* = x^*. \quad (8)$$

□

3. Auxiliary Results

Let $X = \mathcal{C}^1([0, \infty), \mathcal{C}^3(\mathbb{R}))$ be endowed with the norm

$$\begin{aligned} \|u\| = \max\{ & \sup_{t \in [0, \infty), x \in \mathbb{R}} |u|, \sup_{t \in [0, \infty), x \in \mathbb{R}} |u_t|, \\ & \sup_{t \in [0, \infty), x \in \mathbb{R}} |u_x|, \sup_{t \in [0, \infty), x \in \mathbb{R}} |u_{xx}|, \\ & \sup_{t \in [0, \infty), x \in \mathbb{R}} |u_{xxx}| \}, \end{aligned} \quad (9)$$

exists. $\forall u \in X$, we define

$$G_1(u) = u - u_0(x) + \int_0^t \left(\partial_x^3 u(s, x) + (u(s, x))^k \partial_x u(s, x) \right) ds, \quad (10)$$

$t \in [0, \infty), x \in \mathbb{R}$.

Lemma 1. Suppose (Hyp1) holds. If $u \in X$ satisfies

$$G_1(u) = 0, \quad t \in [0, \infty), x \in \mathbb{R}, \quad (11)$$

then u is solution of Equation (3).

Proof. We have

$$\begin{aligned} G_1(u) &= u - u_0(x) + \int_0^t \left(\partial_x^3 u(s, x) + (u(s, x))^k \partial_x u(s, x) \right) ds \\ &= 0, \end{aligned} \quad (12)$$

$t \in [0, \infty), x \in \mathbb{R}$, which we differentiate with respect to t and we have

$$0 = \partial_t u + (u)^k \partial_x u, \quad t \in [0, \infty), x \in \mathbb{R}. \quad (13)$$

We put $t = 0$ in Equation (12) and we obtain

$$0 = u(t = 0) - u_0(x), \quad x \in \mathbb{R}. \quad (14)$$

Then, the function u is solution to the initial value problem Equation (3), which completes our proof. □

Lemma 2. Suppose (Hyp1) holds. If $u \in X$ and $B \geq \|u\|$, then we have

$$|G_1(u)| \leq B_1(t + 1), \quad t \in [0, \infty), x \in \mathbb{R}. \quad (15)$$

Proof. We have

$$\begin{aligned}
 |G_1(u)| &= \left| u - u_0(x) + \int_0^t \left(\partial_x^3 u(s, x) + (u(s, x))^k \partial_x u(s, x) \right) ds \right| \\
 &\leq |u| + |u_0(x)| + \int_0^t \left(|\partial_x^3 u(s, x)| + |u(s, x)|^k |\partial_x u(s, x)| \right) ds \\
 &\leq 2B + \int_0^t (B + B^{k+1}) ds \\
 &= 2B + (B + B^{k+1})t \\
 &\leq B_1(t + 1), \quad t \in [0, \infty), x \in \mathbb{R}.
 \end{aligned} \tag{16}$$

This completes the proof. $\square$

For $u \in X$, define the operator

$$G_2(u) = \int_0^t \int_0^x (t - t_2)(x - x_2)^3 h(t_2, x_2) G_1(u)(t_2, x_2) dx_2 dt_2, \tag{17}$$

$t \in [0, \infty), x \in \mathbb{R}$.

Lemma 3. Suppose (Hyp1) and (Hyp2) hold. If $u \in X$ satisfies

$$G_2(u) = c, \quad t \in [0, \infty), x \in \mathbb{R}, \tag{18}$$

for some constant c , then u is solution of Equation (3).

Proof. We differentiate two times in t and four times in x the Equation (18) to get

$$h(t, x) G_1(u) = 0, \quad t \in [0, \infty), x \in \mathbb{R}. \tag{19}$$

Then,

$$G_1(u) = 0, \quad t \in (0, \infty), x \in (\mathbb{R} \setminus \{0\}). \tag{20}$$

As $G_1(u)(\cdot, \cdot)$ is a continuous on $[0, \infty) \times \mathbb{R}$, we have

$$\begin{aligned}
 0 &= G_1(u)(0, x) \\
 &= \lim_{t \rightarrow 0} G_1(u) \\
 &= \lim_{x \rightarrow 0} G_1(u) \\
 &= G_1(u)(t, 0), \quad t \in [0, \infty), x \in \mathbb{R}.
 \end{aligned} \tag{21}$$

Therefore

$$G_1(u) = 0, \quad t \in [0, \infty), x \in \mathbb{R}. \tag{22}$$

By using Lemma 1, the desired result is obtained. $\square$

Lemma 4. Suppose (Hyp1) and (Hyp2) hold. If $u \in X$, $\|u\| \leq B$, then

$$\|G_2(u)\| \leq AB_1. \tag{23}$$

Proof. The next inequality

$$(z + w)^q \leq 2^q (z^q + w^q), \quad w, z, q \geq 0,$$

will be used. We have

$$\begin{aligned}
 |G_2(u)| &= \left| \int_0^t \int_0^x (t-t_2)(x-x_2)^3 h(t_2, x_2) G_1(u)(t_2, x_2) dx_2 dt_2 \right| \\
 &\leq \int_0^t \left| \int_0^x (t-t_2)|x-x_2|^3 h(t_2, x_2) |G_1(u)(t_2, x_2)| dx_2 \right| dt_2 \\
 &\leq B_1 \int_0^t \left| \int_0^x (t-t_2)(t_2+1)|x-x_2|^3 h(t_2, x_2) dx_2 \right| dt_2 \quad (24) \\
 &\leq B_1 t(t+1)2^4 |x|^3 \int_0^t \left| \int_0^x h(t_2, x_2) dx_2 \right| dt_2 \\
 &\leq AB_1, \quad t \in [0, \infty), x \in \mathbb{R},
 \end{aligned}$$

and

$$\begin{aligned}
 |\partial_t G_2(u)| &= \left| \int_0^t \int_0^x (x-x_2)^3 h(t_2, x_2) G_1(u)(t_2, x_2) dx_2 dt_2 \right| \\
 &\leq \int_0^t \left| \int_0^x |x-x_2|^3 h(t_2, x_2) |G_1(u)(t_2, x_2)| dx_2 \right| dt_2 \\
 &\leq B_1 \int_0^t \left| \int_0^x (t_2+1)|x-x_2|^3 h(t_2, x_2) dx_2 \right| dt_2 \quad (25) \\
 &\leq B_1 (t+1)2^4 |x|^3 \int_0^t \left| \int_0^x h(t_2, x_2) dx_2 \right| dt_2 \\
 &\leq AB_1, \quad t \in [0, \infty), x \in \mathbb{R},
 \end{aligned}$$

and

$$\begin{aligned}
 |\partial_x G_2(u)| &= 3 \left| \int_0^t \int_0^x (t-t_2)(x-x_2)^2 h(t_2, x_2) G_1(u)(t_2, x_2) dx_2 dt_2 \right| \\
 &\leq 3 \int_0^t \left| \int_0^x (t-t_2)(x-x_2)^2 h(t_2, x_2) |G_1(u)(t_2, x_2)| dx_2 \right| dt_2 \\
 &\leq 3B_1 \int_0^t \left| \int_0^x (t-t_2)(t_2+1)(x-x_2)^2 h(t_2, x_2) dx_2 \right| dt_2 \quad (26) \\
 &\leq B_1 t(t+1)2^3 |x|^2 \int_0^t \left| \int_0^x h(t_2, x_2) dx_2 \right| dt_2 \\
 &\leq AB_1, \quad t \in [0, \infty), x \in \mathbb{R},
 \end{aligned}$$

and

$$\begin{aligned}
 |\partial_{xx}G_2(u)| &= 6 \left| \int_0^t \int_0^x (t-t_2)(x-x_2)h(t_2, x_2)G_1(u)(t_2, x_2)dx_2dt_2 \right| \\
 &\leq 6 \int_0^t \left| \int_0^x (t-t_2)|x-x_2|h(t_2, x_2)|G_1(u)(t_2, x_2)|dx_2 \right| dt_2 \\
 &\leq 6B_1 \int_0^t \left| \int_0^x (t-t_2)(t_2+1)|x-x_2|h(t_2, x_2)dx_2 \right| dt_2 \quad (27) \\
 &\leq B_1 t(t+1)12|x| \int_0^t \left| \int_0^x h(t_2, x_2)dx_2 \right| dt_2 \\
 &\leq AB_1, \quad t \in [0, \infty), x \in \mathbb{R},
 \end{aligned}$$

and

$$\begin{aligned}
 |\partial_{xxx}G_2(u)| &= 6 \left| \int_0^t \int_0^x (t-t_2)h(t_2, x_2)G_1(u)(t_2, x_2)dx_2dt_2 \right| \\
 &\leq 6 \int_0^t \left| \int_0^x (t-t_2)h(t_2, x_2)|G_1(u)(t_2, x_2)|dx_2 \right| dt_2 \\
 &\leq 6B_1 \int_0^t \left| \int_0^x (t-t_2)(t_2+1)h(t_2, x_2)dx_2 \right| dt_2 \quad (28) \\
 &\leq B_1 t(t+1)6 \int_0^t \left| \int_0^x h(t_2, x_2)dx_2 \right| dt_2 \\
 &\leq AB_1, \quad t \in [0, \infty), x \in \mathbb{R}.
 \end{aligned}$$

Thus,

$$\|G_2(u)\| \leq AB_1. \quad (29)$$

This completes the proof. $\square$

4. Proof of Theorem 1

Below, assume that the hypotheses (Hyp1) and (Hyp2) are satisfied. Let $\tilde{Z}$ denote the set of all equi-continuous families in X with respect to $\|\cdot\|$. Let also, $Z = \bar{\tilde{Z}}$ be the closure of $\tilde{Z}$,

$$V = \{u \in Z : \|u\| < B\}. \quad (30)$$

For $u \in \bar{V}$ and $\epsilon > 0$, define the operators

$$\begin{aligned}
 W(u) &= \epsilon u, \\
 G(u) &= u - \epsilon u - \epsilon G_2(u), \quad t \in [0, \infty), x \in \mathbb{R}. \quad (31)
 \end{aligned}$$

For $u \in \overline{V}$, we have

$$\begin{aligned}\|(I - G)(u)\| &= \|\epsilon u + \epsilon G_2(u)\| \\ &\leq \epsilon \|u\| + \epsilon \|G_2(u)\| \\ &\leq \epsilon B + \epsilon AB_1.\end{aligned}\tag{32}$$

Thus, $G : \overline{V} \rightarrow X$ is continuous and $(I - G)(\overline{V})$ resides in a compact subset of Z . Now, suppose that there is a $u \in Z$ so that $\|u\| = B$ and

$$u = \lambda(I - G)(u)$$

or

$$u = \lambda \epsilon (I + G_2)(u),\tag{33}$$

for some $\lambda \in (0, \frac{1}{\epsilon})$. Note that $(Z, \|\cdot\|)$ is a Banach space. Assume that the set

$$\mathcal{A} = \{u \in Z : u = \mu(I + G_2)(u), \quad 0 < \mu < 1\}\tag{34}$$

is bounded. By Equation (33), it follows that the set $\mathcal{A}$ is not empty set. Then, by the Schaefer's Theorem, it follows that there is a $u^* \in Z$ such that

$$u^* = (I + G_2)(u^*),\tag{35}$$

or

$$G_2(u^*) = 0,$$

i.e., u^* is solution to Equation (3). Assume that the set $\mathcal{A}$ is unbounded. Then, by Schaefer's Theorem, it follows that the equation

$$u = \mu(I + G_2)(u), \quad u \in Z,$$

has at least one small solution $u^* \in Z$ for any $\mu \in [0, 1]$. In particular, for $\mu = 1$, there is a $u^* \in Z$ such that Equation (35) holds and then it is solution to Equation (3). Let now,

$$\{u \in Z : u = \lambda_1(I - G)(u), \quad \|u\| = B\} = \emptyset$$

$\forall \lambda_1 \in (0, \frac{1}{\epsilon})$. Then, by Theorem 3, the operator $W + G$ has a fixed point $u^* \in Z$. Then

$$\begin{aligned}u^* &= W(u^*) + G(u^*) \\ &= \epsilon u^* + u^* \\ &\quad - \epsilon u^* - \epsilon G_2(u^*), \quad t \in [0, \infty), x \in \mathbb{R},\end{aligned}\tag{36}$$

immediately after which

$$G_2(u^*) = 0, \quad t \in [0, \infty), x \in \mathbb{R}.$$

Then, u^* is solution to the problem Equation (3). The proof is now completed.

5. Proof of Theorem 2

Below, assume that the hypotheses (Hyp1), (Hyp2) and (Hyp3) are satisfied. Let $\tilde{Z}$ denote the set of all equi-continuous families in X with respect to $\|\cdot\|$. Let also, $\mathbb{Z} = \overline{\tilde{Z}}$ be the closure of $\tilde{Z}$, so $(\mathbb{Z}, \|\cdot\|)$ is a Banach space. Denote

$$Z = \{u \in \mathbb{Z} : u \geq 0\}.$$

We have that Z is a closed, convex subset in $\mathbb{Z}$. Let

$$\Omega = \{u \in Z : \|u\| < B\}. \quad (37)$$

Note that $\overline{\Omega}$ is a compact set in Z . For $u \in \overline{\Omega}$ and $\epsilon > 0$, define the operators

$$\begin{aligned} W(u) &= -\epsilon u, \\ G(u) &= u + \epsilon u + \epsilon G_2(u) - 7\epsilon B, \end{aligned} \quad (38)$$

$t \in [0, \infty)$, $x \in \mathbb{R}$. For $u \in \overline{\Omega}$, we have

$$\begin{aligned} \|(I - G)(u)\| &= \|- \epsilon u - \epsilon G_2(u) + 7\epsilon B\| \\ &\leq \epsilon \|u\| + \epsilon \|G_2(u)\| + 7\epsilon B \\ &\leq \epsilon (8B + AB_1). \end{aligned} \quad (39)$$

Thus, $I - G : \overline{\Omega} \rightarrow Z$ is continuous and $(I - G)(\overline{\Omega})$ resides in a compact subset of Z .

Let us suppose that there is a $u \in \overline{\Omega}$ so that $\|u\| = B$ and

$$u = -\lambda(I - G)(u) \quad (40)$$

for some $\lambda \in (0, \frac{1}{\epsilon})$. Hence, we find

$$\frac{1}{\lambda}u = \epsilon u + \epsilon G_2(u) - 7\epsilon B. \quad (41)$$

From the assumption (Hyp3), we get $\|G_2(u)\| \leq B$.

Hence, we have $u \geq 0$ and $G_2(u) - B \leq 0$ on $[0, \infty) \times \mathbb{R}$, whereupon

$$\begin{aligned} 0 \leq \left(\frac{1}{\lambda} - \epsilon\right)u &= \epsilon G_2(u) - \epsilon B - 6\epsilon B \\ &< 0, \end{aligned}$$

which is contradicts our claim. Then, from Theorem 3, it follows that the operator $W + G$ has a fixed point $u^* \in \Omega$. Then

$$\begin{aligned} u^* &= W(u^*) + G(u^*) \\ &= \epsilon u^* + u^* \\ &\quad - \epsilon u^* + \epsilon G_2(u^*) - 7\epsilon B, \quad t \in [0, \infty), x \in \mathbb{R}, \end{aligned} \quad (42)$$

immediately after which

$$G_2(u^*) = 7B, \quad t \in [0, \infty), x \in \mathbb{R}.$$

Then, u^* is a nonnegative bounded solution to the problem Equation (3). This completes the proof.

6. An Example

Here, we shall illustrate our two main results. For $k = 2$, $B = 10$, we have

$$B_1 = \max\{20, 10 + 10^3\} = 10 + 10^3 \quad (43)$$

and

$$AB_1 = \frac{1}{10^9} \cdot (10 + 10^3) < B, \quad (44)$$

i.e., (Hyp3) holds.

$$h(s) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \quad (45)$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned} \quad (46)$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s \rightarrow \pm\infty} (s^2 + s + 1)h(s) < \infty, \\ -\infty &< \lim_{s \rightarrow \pm\infty} (s^2 + s + 1)l(s) < \infty. \end{aligned} \quad (47)$$

Hence, there exists a positive constant C_1 so that

$$(s^2 + s + 1 + s^3) \left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}} \right) \leq C_1,$$

$s \in \mathbb{R}$. Note that $\lim_{s \rightarrow \pm 1} l(s) = \frac{\pi}{2}$ and by [9] (p. 707, Integral 79), we have

$$\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}} \log \frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1 - z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1 + s^{44})(s^2 + s + 1)^2}, \quad s \in \mathbb{R},$$

and

$$g_1(t, x) = Q(t)Q(x), \quad t \in [0, \infty), \quad x \in \mathbb{R}.$$

Then there exists a constant $C_2 > 0$ such that

$$6 \cdot 2^4 (t^2 + t + 1) (1 + |x| + x^2 + |x|^3) \int_0^t \left| \int_0^x g_1(t_2, x_2) dx_2 \right| dt_2 \leq C_2, \quad t \in [0, \infty), x \in \mathbb{R}.$$

Let

$$h(t, x) = \frac{A}{C_2} g_1(t, x), \quad t \in [0, \infty), x \in \mathbb{R}. \quad (48)$$

Then

$$6 \cdot 2^4 (t^2 + t + 1) (1 + |x| + x^2 + |x|^3) \int_0^t \left| \int_0^x h(t_2, x_2) dx_2 \right| dt_2 \leq A, \quad t \in [0, \infty), x \in \mathbb{R},$$

i.e., (Hyp2) holds. Therefore for the IVP

$$\begin{aligned} \partial_t u + \partial_x^3 u + u^2 \partial_x u &= 0, \quad t \in [0, \infty), x \in \mathbb{R}, \\ u(0, x) &= \frac{1}{1+x^2}, \quad x \in \mathbb{R}, \end{aligned} \quad (49)$$

are fulfilled all conditions of Theorems 1 and 2.

7. Conclusions

This paper concerning the problem of existence of solutions of the generalized Korteweg-de Vries equation. The considered work represents a variant of classical question about the structure of solutions of partial derivative system. It adds more to previous results. The obtained theorems are very interesting, and the model is important and finds applications, such as physical, chemical, biological, thermal and economics. Here, a new topological approach is applied to prove the existence of at least one classical solution. The arguments are based upon recent several axiomatic theoretical results. Our results are illustrated by example.

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