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Analysis of a Fractional-Order Quadratic Functional Integro-Differential Equation with Nonlocal Fractional-Order Integro-Differential Condition

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Abstract: Here, we center on the solvability of a fractional-order quadratic functional integro-differential equation with a nonlocal fractional-order integro-differential condition in the class of continuous functions. The maximal and minimal solutions will be discussed. The continuous dependence of the solutions on a few parameters will be examined. Finally, the problems of conjugate orders and integer orders, and some other problems and remarks will be discussed and presented.

Keywords: Caputo fractional derivative; fractional-order integro-differential equations; conjugate-order integro-differential equations; Schauder fixed-point theorem; maximal and minimal solutions; continuous dependence

MSC: 26A33; 34K45; 47G10

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1. Introduction

The solvability of nonlocal problems of fractional orders has been discussed by many authors; see, for example, refs. [1–9].

Many existence and uniqueness results of differential equations and inclusions with nonlocal boundary conditions have been investigated by some researchers and using different techniques (see [4–8] and the references therein).

Some existence results under mixed Lipschitz and Caratheodory conditions have been discussed for functional integro-differential equations of fractional orders, and the existence of some properties of the solution has been proved in [6].

Subashini et al. [10] discussed the existence of a nonlocal fractional integro-differential problem of the Hilfer type by applying the Monch fixed-point theorem and techniques of measure of noncompactness.

An investigation described the applicability of the a priori estimate method on a nonlocal, nonlinear fractional differential equation for which the weak solution and the unique solution exist [11].

Motivated by the above results, we discuss the solvability of the nonlocal problem of the fractional-order quadratic functional integro-differential equation

$$\frac{dx}{d\mathfrak{r}}(\mathfrak{r}) = f_1\Big(\mathfrak{r}, D^{\alpha}x(\mathfrak{r}) \times \mathfrak{I}^{\beta} f_2(\mathfrak{r}, D^{\gamma}x(\mathfrak{r}))\Big), \qquad \alpha, \beta, \gamma \in (0,1], \ \ \mathfrak{r} \in (0,1] \tag{1}$$

with the nonlocal fractional-order integro-differential condition

$$x(0) + \int_0^1 h(\varsigma, x(\varsigma), D^{\delta}x(\varsigma))d\varsigma = x_0, \qquad \delta \in (0,1], \tag{2}$$

where $\mathfrak{I}^{\beta}$ is the fractional Riemann–Liouville integral of order β and D^{α} is a Caputo fractional derivative of order $\alpha \in (0,1]$.

We investigate the existence of a solution to problem (1) and (2) in $C(I)$ of all continuous functions on $I = [0,1]$. Moreover, we establish some features and characteristics of these solutions and obtain some particular cases and remarks.

Now, we recall the following definitions.

Definition 1 ([12]). *Let $\mathfrak{h} \in L^1(I)$, $\beta \in R^+$. The Riemann–Liouville fractional integral of $\mathfrak{h}$ of order β is given by*

$$\mathfrak{I}^{\beta}\mathfrak{h}(\mathfrak{r}) = \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \mathfrak{s})^{\beta-1}}{\Gamma(\beta)} \mathfrak{h}(\mathfrak{s})\, d\mathfrak{s},$$

Definition 2. *Caputo, M. Fractional derivative D_a^{α} of order $\alpha \in (0,1]$ of $\mathfrak{h}(\mathfrak{r}) \in AC(I)$ is given by (see [12])*

$$D^{\alpha}\, \mathfrak{h}(\mathfrak{r}) = \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \mathfrak{s})^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{d\mathfrak{s}} \mathfrak{h}(\mathfrak{s})\, d\mathfrak{s} = \mathfrak{I}^{1-\alpha} \frac{d}{d\mathfrak{r}} \mathfrak{h}(\mathfrak{r})$$

2. Existence of Solution

We take into account the following assumptions:

(i) $f_1 : I \times R \to R$ is measurable in $\mathfrak{r} \in I$ $\forall y \in R$ and continuous in $y \in R$ $\forall \mathfrak{r} \in I$. Furthermore, $\exists$ a bounded measurable function $a_1 : I \to R$ and a constant $b_1 > 0$ where

$$|f_1(\mathfrak{r}, y)| \leq |a_1(\mathfrak{r})| + b_1|y| \leq a_1^* + b_1|y|, \quad a_1^* = \sup_{\mathfrak{r}\in I} \mathfrak{I}^{1-\alpha}|a_1(\mathfrak{r})|.$$

(ii) $f_2 : I \times R \to R$ is measurable in $\mathfrak{r} \in I$ $\forall y \in R$ and continuous in $y \in R$ $\forall \mathfrak{r} \in I$. Furthermore, $\exists$ a bounded measurable function $a_2 : I \to R$ and a constant $b_2 > 0$ where

$$|f_2(\mathfrak{r}, y)| \leq |a_2(\mathfrak{r})| + b_2|y| \leq a_2^* + b_2|y|, \quad a_2^* = \sup_{\mathfrak{r}\in I} \mathfrak{I}^{\beta}|a_2(\mathfrak{r})|.$$

(iii) r_1 is a positive root of the following equation:

$$a_1^* + \left(\frac{b_1 a_2^*}{\Gamma(2-\alpha)} - 1\right) r_1 + \frac{b_1 b_2 r_1^2}{\Gamma(2-\alpha)\Gamma(\alpha+\beta-\gamma+1)} = 0. \tag{3}$$

(iv) $h : I \times R \times R \longrightarrow R$ is measurable in $\mathfrak{r}$ for all $x, y \in R$ and continuous in x, y for $\mathfrak{r} \in I$, and there exists a bounded measurable function $a_3 : I \to R$ and a constant $b_3 > 0$ where

$$|h(\mathfrak{r}, x, y)| \leq |a_3(\mathfrak{r})| + b_3(|x| + |y|)$$

where

$$\sup_{\mathfrak{r}\in I} \int_0^t |a_3(\varsigma)| d\varsigma \leq N.$$

(v) $\|x_0\| > N + \frac{b_3 r_1}{\Gamma(\alpha-\delta+1)} - \frac{r_1}{\Gamma(\alpha+1)}$

Now, we have to prove the following auxiliary result.

Lemma 1. *The solution to the problem (1) and (2) is given in the form, if it exists,*

$$x(\mathfrak{r}) = x_0 - \int_0^1 h(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma + \mathfrak{I}^{\alpha} y(\mathfrak{r}), \quad \delta,\ \gamma \leq \alpha. \tag{4}$$

where y satisfies the equation

$$y(\mathfrak{r}) = \mathfrak{I}^{1-\alpha} f_1\left(\mathfrak{r}, y(\mathfrak{r}) \times \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma) d\sigma) d\theta\right) \tag{5}$$

and the two systems (1) and (2), and (4) and (5) are equivalent such that $\delta,\ \gamma \leq \alpha$.

Proof. x is a solution of (1) and (2). Then, we obtain

$$D^{\alpha} x(\mathfrak{r}) = \mathfrak{I}^{1-\alpha} \frac{dx}{d\mathfrak{r}} = \mathfrak{I}^{1-\alpha} f_1(\mathfrak{r}, D^{\alpha} x(\mathfrak{r}) \times \mathfrak{I}^{\beta} f_2(\mathfrak{r}, \mathfrak{I}^{\alpha-\gamma} y(\mathfrak{r})).$$

We take $D^{\alpha} x(\mathfrak{r}) = y(\mathfrak{r})$; thus,

$$y(\mathfrak{r}) = \mathfrak{I}^{1-\alpha} f_1\left(\mathfrak{r}, y(\mathfrak{r}) \times \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma) d\sigma) d\theta\right)$$

and

$$x(\mathfrak{r}) = x(0) + \mathfrak{I}^{\alpha} y(\mathfrak{r}). \tag{6}$$

Now,

$$\begin{aligned} \mathfrak{I}^{\alpha-\delta} y(\mathfrak{r}) &= \mathfrak{I}^{\alpha-\delta} D^{\alpha} x(\mathfrak{r}) = \mathfrak{I}^{\alpha-\delta} \mathfrak{I}^{1-\alpha} \frac{dx}{d\mathfrak{r}} \\ &= \mathfrak{I}^{1-\delta} \frac{dx}{d\mathfrak{r}} = D^{\delta} x(\mathfrak{r}), \end{aligned}$$

then from (2) and (6), we obtain (4).
By letting $y(\mathfrak{r}) = D^{\alpha} x(\mathfrak{r})$ in (4), we obtain (2), and in (5), we obtain

$$\mathfrak{I}^{1-\alpha} \frac{d}{d\mathfrak{r}} x(\mathfrak{r}) = \mathfrak{I}^{1-\alpha} f_1\left(\mathfrak{r}, D^{\alpha} x(\mathfrak{r}) \times \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, D^{\gamma} x(\theta) d\theta\right).$$

Operating by $\mathfrak{I}^{\alpha}$, then by $\frac{d}{d\mathfrak{r}}$, respectively, we obtain (1). This proves the equivalence of the two systems (1) and (2), and (4) and (5). □

Theorem 1. *Let us assume that the conditions (i)–(iv) hold; then, problem (1) and (2) has a solution in* $C(I)$.

Proof. We characterize a closed ball Q_{r_1} and an operator F_1 as

$$Q_{r_1} = \{y \in C(I) : \|y\| \leq r_1\},\ \ r_1 = a_1^* + \frac{1}{\Gamma(2-\alpha)}(r_1 b_1 a_2^* + \frac{b_1 b_2 r_1^2}{\Gamma(\alpha+\beta-\gamma+1)})$$

and

$$F_1 y(\mathfrak{r}) = \mathfrak{I}^{1-\alpha} f_1\left(\mathfrak{r}, y(\mathfrak{r}) \times \int_0^{\mathfrak{r}} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma) d\sigma) d\theta\right) d\varsigma.$$

For $y \in Q_{r_1}$, we can obtain

$$\begin{aligned}
|y(\mathfrak{r})| &= \mathfrak{I}^{1-\alpha}|f_1\Big(\mathfrak{r}, y(\mathfrak{r}) \times \mathfrak{I}^{\beta} f_2(\mathfrak{r}, \mathfrak{I}^{\alpha-\gamma} y(\mathfrak{r}))\Big)| \\
&\leq \mathfrak{I}^{1-\alpha}\Big[|a_1(\mathfrak{r})| + b_1|y(\mathfrak{r}) \times \mathfrak{I}^{\beta} f_2(\mathfrak{r}, \mathfrak{I}^{\alpha-\gamma} y(\mathfrak{r}))|\Big] \\
&\leq \mathfrak{I}^{1-\alpha}\Big[|a_1(\mathfrak{r})| + b_1\|y\|(a_2^* + b_2\mathfrak{I}^{\alpha+\beta-\gamma}\|y\|)\Big] \\
&\leq \mathfrak{I}^{1-\alpha}\Big[|a_1(\mathfrak{r})| + b_1 r_1(a_2^* + \frac{r_1 b_2}{\Gamma(\alpha+\beta-\gamma+1)})\Big],
\end{aligned}$$

$$\mathfrak{I}^{1-\alpha}|f_1(\mathfrak{r}, y(\mathfrak{r}) \times \mathfrak{I}^{\beta} f_2(\mathfrak{r}, \mathfrak{I}^{\alpha-\gamma} y(\mathfrak{r}))| \leq a_1^* + \frac{1}{\Gamma(2-\alpha)}(r_1 b_1 a_2^* + \frac{r_1^2 b_1 b_2}{\Gamma(\alpha+\beta-\gamma+1)})$$

and we deduce that

$$\begin{aligned}
|F_1 y(\mathfrak{r})| &= \Big|\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta\Big)d\varsigma\Big| \\
&\leq \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)}\Big(a_1 + b_1|y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta|\Big)d\varsigma \\
&\leq \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)}\Big(a_1 + b_1\|y\| \times (a_2^* + \frac{b_2\|y\|}{\Gamma(\alpha+\beta-\gamma+1)}\Big)d\varsigma \\
&\leq a_1^* + \frac{1}{\Gamma(2-\alpha)}(r_1 b_1 a_2^* + \frac{b_1 b_2 r_1^2}{\Gamma(\alpha+\beta-\gamma+1)}) = r_1.
\end{aligned}$$

So, we have

$$\|F_1 y\| \leq a_1^* + \frac{1}{\Gamma(2-\alpha)}(r_1 b_1 a_2^* + \frac{b_1 b_2 r_1^2}{\Gamma(\alpha+\beta-\gamma+1)}) = r_1.$$

This proves that $F_1 : Q_{r_1} \to Q_{r_1}$ and the family $\{F_1 y\}$ is uniformly bounded on Q_{r_1}. Let $y \in Q_{r_1}$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in I$, where $\mathfrak{r}_2 > \mathfrak{r}_1$ and $|\ \mathfrak{r}_1 - \mathfrak{r}_2\ | \leq \delta$; therefore,

$$\begin{aligned}
&|F_1 y(\mathfrak{r}_2) - F_1 y(\mathfrak{r}_1)| = \\
&\Big|\int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta\Big)ds \\
-& \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(\varsigma, y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta\Big)d\varsigma\Big| \\
\leq& \Big|\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f\Big(\varsigma, y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta\Big)d\varsigma \\
+& \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta\Big)d\varsigma \\
-& \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta\Big)d\varsigma\Big| \\
\leq& \Big|\int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} - \frac{(\mathfrak{r}_1-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)}(a_1^* + r_1 b_1 a_2^* + \frac{b_1 b_2 r_1^2}{\Gamma(\alpha+\beta-\gamma+1)})d\varsigma \\
+& \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)}(a_1 + \frac{b_1 a_2 r_1}{\Gamma(1+\beta)} + \frac{b_1 b_2 r_1^2}{\Gamma(1+\beta)\Gamma(1+\alpha-\gamma)})d\varsigma\Big| \\
\leq& \int_0^{\mathfrak{r}_1} |\frac{(\mathfrak{r}_2-\varsigma)^{\alpha} - (\mathfrak{r}_1-\varsigma)^{\alpha}}{\Gamma(1-\alpha)(\mathfrak{r}_1-\varsigma)^{\alpha}(\mathfrak{r}_2-\varsigma)^{\alpha}}|(a_1^* + r_1 b_1 a_2^* + \frac{b_1 b_2 r_1^2}{\Gamma(\alpha+\beta-\gamma+1)})ds \\
+& \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} |\frac{(\mathfrak{r}_2-\varsigma)^{\alpha}}{\Gamma(1-\alpha)(\mathfrak{r}_2-\varsigma)^{\alpha}}|(a_1^* + r_1 b_1 a_2^* + \frac{b_1 b_2 r_1^2}{\Gamma(\alpha+\beta-\gamma+1)})d\varsigma.
\end{aligned}$$

This shows that family $\{F_1 y\}$ is equi-continuous on Q_{r_1}, and by the Arzela–Ascoli result [13], mapping F_1 is relatively compact.
Let $\{y_n\} \subset Q_{r_1}$ be such that $y_n \to y$; then,

$$F_1 y_n(\mathfrak{r}) = \Im^{1-\alpha} f_1\Big(\mathfrak{r}, y_n(\mathfrak{r}) \times \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_n(\sigma) d\sigma) d\theta\Big),$$

$$\begin{aligned}
\lim_{n\to\infty} F_1 y_n(\mathfrak{r}) &= \lim_{n\to\infty} \Im^{1-\alpha} f_1\Big(\mathfrak{r}, y_n(\mathfrak{r}) \times \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_n(\sigma) d\sigma) d\theta\Big) \\
&= \Im^{1-\alpha} f_1\Big(\mathfrak{r}, y(\mathfrak{r}) \times \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} \lim_{n\to\infty} y_n(\sigma) d\sigma) d\theta\Big) d\varsigma \\
&= \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma) d\sigma) d\theta\Big) d\varsigma \\
&= F_1 y(\mathfrak{r}).
\end{aligned}$$

This yields that $F_1 y_n(\mathfrak{r}) \to F_1 y(\mathfrak{r})$.

Thus, mapping F_1 is continuous, and by applying the Schauder fixed-point theorem [13], $y \in Q_{r_1} \subset C(I)$ is a solution of (5).

To prove that $x \in C(I)$ exists and satisfies Equation (4), we prove the next result. □

Theorem 2. *Let us suppose that (i)–(iv) are satisfied; then, for a fixed function y (given by Theorem 1), $\exists x \in C(I)$ and satisfies (4).*

Proof. Let Q_{r_2} be the closed ball

$$Q_{r_2} = \{x \in C(I) : \|x\| \le r_2\}, \ \ r_2 = \frac{\|x_0\| - N - \frac{b_3 r_1}{\Gamma(\alpha-\delta+1)} + \frac{r_1}{\Gamma(\alpha+1)}}{(1+b_3)}$$

and define operator F_2

$$F_2 x(\mathfrak{r}) = x_0 - \int_0^1 h(\varsigma, x(\varsigma), \Im^{\alpha-\delta} y(\varsigma)) d\varsigma + \Im^{\alpha} y(\mathfrak{r}).$$

Let $x \in Q_{r_2}$; then,

$$\begin{aligned}
|F_2 x(\mathfrak{r})| &= \left| x_0 - \int_0^1 h(\varsigma, x(\varsigma), \Im^{\alpha-\delta} y(\varsigma)) d\varsigma + \Im^{\alpha} y(\mathfrak{r}) \right| \\
&\le |x_0| - \int_0^1 |h(\varsigma, x(\varsigma), \Im^{\alpha-\delta} y(\varsigma))| d\varsigma + \Im^{\alpha} |y(\mathfrak{r})| \\
&\le |x_0| - \int_0^1 \Big(|a_3| + b_3 (|x(\varsigma)| + \frac{\|y\|}{\Gamma(\alpha-\delta+1)}) \Big) d\varsigma + \frac{\|y\|}{\Gamma(\alpha+1)} \\
&\le |x_0| - \int_0^1 |a_3(\varsigma)| d\varsigma - b_3 \int_0^1 |x(\varsigma)| d\varsigma - \frac{b_3 r_1}{\Gamma(\alpha-\delta+1)} \int_0^1 d\varsigma + \frac{r_1}{\Gamma(\alpha+1)} \\
&\le \|x_0\| - N - b_3 r_2 - \frac{b_3 r_1}{\Gamma(\alpha-\delta+1)} + \frac{r_1}{\Gamma(\alpha+1)} = r_2
\end{aligned}$$

and

$$\|F_2 x\| \le |x_0| - N - b_3 r_2 - \frac{b_3 r_1}{\Gamma(\alpha-\delta+1)} + \frac{r_1}{\Gamma(\alpha+1)} = r_2$$

Similarly as performed above, $F_2 : Q_{r_2} \to Q_{r_2}$, and $\{F_2 x\}$ is uniformly bounded on Q_{r_2}.
Let $x \in Q_{r_2}$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in I$; moreover, $\mathfrak{r}_2 > \mathfrak{r}_1$ and $| \mathfrak{r}_1 - \mathfrak{r}_2 | \le \delta$. Then,

$$\begin{aligned}|F_2x(\mathfrak{r}_2)-F_2x(\mathfrak{r}_1)| &= \left|x_0-\int_0^1 h(\varsigma,x(\varsigma),\mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma+\mathfrak{I}^{\alpha}y(\mathfrak{r}_2)\right.\\ &- \left. x_0+\int_0^1 h(\varsigma,x(\varsigma),\mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma-\mathfrak{I}^{\alpha}y(\mathfrak{r}_1)\right|\\ &\leq \int_0^{\mathfrak{r}_2}|f_1(\varsigma,y(\varsigma))\times \mathfrak{I}^{\beta}f_2(\mathfrak{r},\mathfrak{I}^{\alpha-\gamma}y(\varsigma))|d\varsigma\\ &- \int_0^{\mathfrak{r}_1}|f_1(\varsigma,y(\varsigma))\times \mathfrak{I}^{\beta}f_2(\mathfrak{r},\mathfrak{I}^{\alpha-\gamma}y(\varsigma))|d\varsigma\\ &\leq \int_{\mathfrak{r}_1}^{\mathfrak{r}_2}|f_1(\varsigma,y(\varsigma))\times \mathfrak{I}^{\beta}f_2(\mathfrak{r},\mathfrak{I}^{\alpha-\gamma}y(\varsigma))|d\varsigma.\end{aligned}$$

Thus, family $\{F_2x\}$ is equicontinuous on Q_{r_2}, and operator F_2 is relatively compact.
For $\{x_n\}\subset Q_{r_2}$, and $x_n\to y$; then,

$$F_2x_n(\mathfrak{r}) = x_0-\int_0^1 h(\varsigma,x(\varsigma),\mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma+\mathfrak{I}^{\alpha}y(\mathfrak{r})$$

and

$$\lim_{n\to\infty}F_2x_n(\mathfrak{r}) = \lim_{n\to\infty}\left(x_0-\int_0^1 h(\varsigma,x(\varsigma),\mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma+\mathfrak{I}^{\alpha}y(\mathfrak{r})\right)$$

Using the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}F_2x_n(\mathfrak{r}) &= x_0-\int_0^1 h(\varsigma,\lim_{n\to\infty}x(\varsigma),\mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma+\mathfrak{I}^{\alpha}y(\mathfrak{r})\\ &= x_0-\int_0^1 h(\varsigma,x(\varsigma),\mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma+\mathfrak{I}^{\alpha}y(\mathfrak{r})=F_2x(\mathfrak{r})\end{aligned}$$

This means that $F_2x_n(\mathfrak{r})\to F_2x(\mathfrak{r})$. Hence, operator F_2 is continuous, and there exists $x\in Q_{r_2}\subset C(I)$ that satisfies (4).
Therefore, we prove the existence of a continuous solution x of Equation (4). □

2.1. Conjugate-Order Problem

For $\gamma=1-\alpha,\ \alpha\in[\frac{1}{2},1)$ in problem (1) and (2), we have the nonlocal problem of conjugate-orders $(\alpha,1-\alpha)$

$$\frac{dx}{d\mathfrak{r}}(\mathfrak{r})=f_1\left(\mathfrak{r},D^{\alpha}x(\mathfrak{r})\times \mathfrak{I}^{\beta}f_2(\mathfrak{r},D^{1-\alpha}x(\mathfrak{r}))\right),\qquad \alpha,\beta\in(0,1],\ \ \mathfrak{r}\in(0,1] \tag{7}$$

with nonlocal fractional-order integro-differential condition (2).
By putting $D^{\alpha}x(\mathfrak{r})=y(\mathfrak{r})$, we can prove that nonlocal problem (7) with condition (2) is equivalent to the functional integral equation

$$y(\mathfrak{r})=\mathfrak{I}^{1-\alpha}f_1\left(\mathfrak{r},y(\mathfrak{r})\times \mathfrak{I}^{\beta}f_2(\mathfrak{r},\mathfrak{I}^{2\alpha-1}y(\mathfrak{r}))\right),\ \ \alpha\geq\frac{1}{2} \tag{8}$$

with (4). Then, by applying Theorems 1 and 2, we can deduce the solvability of nonlocal problem (7) and (2) in $C(I)$.

2.2. Absolutely Continuous Solution

By taking $y(\mathfrak{r})=\frac{dx}{d\mathfrak{r}}(\mathfrak{r})$, (1) and (2) reduce to the fractional-order quadratic integral equation

$$y(\mathfrak{r})=f_1\left(\mathfrak{r},\mathfrak{I}^{1-\alpha}y(\mathfrak{r})\times \mathfrak{I}^{\beta}f_2(\mathfrak{r},\mathfrak{I}^{1-\gamma}y(\mathfrak{r}))\right),\ \ \mathfrak{r}\in(0,1] \tag{9}$$

and the nonlocal integro-differential condition

$$x(0) \ + \ \int_0^1 h(\varsigma, x_0 + \int_0^\varsigma y(s)ds, \Im^{1-\delta} y(\varsigma))d\varsigma = \ x_0. \tag{10}$$

In the same way as in Lemma 1, we can prove the equivalence between (1) and (2), and system (9) and

$$x(\mathfrak{r}) = x_0 - \int_0^1 h(\varsigma, x(\varsigma), y(\varsigma))d\varsigma + \int_0^t y(s)ds \tag{11}$$

With the same technique as in [14], the existence of solution $y \in L^1(I)$ of (9) can be proved. So, the existence of an absolutely continuous solution

$$x(\mathfrak{r}) = x_0 - \int_0^1 h(\varsigma, x(\varsigma), \Im^{1-\delta} y(\varsigma))d\varsigma + \int_0^t y(s)ds \in AC(I) \tag{12}$$

of problem (1) and (2) can be proved.

2.3. Integer-Order Problem

By letting $\alpha, \ \beta, \ \gamma \to 1$ in problem (1) and (2), we have the integer-order equation

$$\frac{dx}{d\mathfrak{r}}(\mathfrak{r}) = f_1\Big(\mathfrak{r}, \frac{d}{d\mathfrak{r}}x(\mathfrak{r}) \times \int_0^\mathfrak{r} f_2(\varsigma, \frac{d}{d\varsigma}x(\varsigma))d\varsigma\Big), \ \ \mathfrak{r} \in (0,1] \tag{13}$$

with the nonlocal integro-differential condition

$$x(0) \ + \ \int_0^1 h(\varsigma, x(\varsigma), \frac{d}{d\varsigma}x(\varsigma))d\varsigma = \ x_0. \tag{14}$$

In the same way as in Lemma 1, we can prove the equivalence between (13) and (14), and the system

$$y(\mathfrak{r}) = f_1\Big(\mathfrak{r}, y(\mathfrak{r}) \times \int_0^\mathfrak{r} f_2(\varsigma, y(\varsigma))d\varsigma\Big), \ \ \mathfrak{r} \in (0,1] \tag{15}$$

and

$$x(\mathfrak{r}) = x_0 - \int_0^1 h(\varsigma, x(\varsigma), y(\varsigma))d\varsigma + \int_0^t y(s)ds \tag{16}$$

and as in [14], the existence of solution $y \in L^1(I)$ of (15) can be proved. Consequently, $x \in AC(I)$ of integer-order problem (13) and (14) can be proved.

3. Some Characteristics of the Solution

3.1. Maximal and Minimal Solutions

Lemma 2. *Let us assume that the conditions of Theorem 1 hold. Let $x, \ y \in C(I)$ satisfy*

$$x(\mathfrak{r}) \ \ \leq \ \ \int_0^\mathfrak{r} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, x(\varsigma) \times \int_0^\varsigma \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^\theta \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} x(\sigma)d\sigma)d\theta\Big)d\varsigma,$$

$$y(\mathfrak{r}) \ \ \geq \ \ \int_0^\mathfrak{r} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, y(\varsigma) \times \int_0^\varsigma \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^\theta \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta\Big)d\varsigma$$

such that one of them is strict. Let us assume that f_1 and f_2 are monotonic non-decreasing in the second argument; thus,

$$x(\mathfrak{r}) \ \ < \ \ y(\mathfrak{r}), \ \ \ \mathfrak{r} > 0. \tag{17}$$

Proof. Let us assume that result (17) is not true; then, $\exists t_1$ where
$x(\mathfrak{r}_1) = y(\mathfrak{r}_1), \ \ \mathfrak{r}_1 > 0, \ \ x(\mathfrak{r}) < y(\mathfrak{r}) \ \ \ 0 < \mathfrak{r} < \mathfrak{r}_1.$
From the monotonicity of f_1 and f_2, we obtain

$$\begin{aligned} x(\mathfrak{r}_1) &\leq \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\left(\varsigma, x(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} x(\sigma)d\sigma)d\theta\right)d\varsigma \\ &< \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\left(\varsigma, y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta\right)d\varsigma \\ &= y(\mathfrak{r}_1). \end{aligned}$$

Hence $x(\mathfrak{r}_1) < y(\mathfrak{r}_1)$, which contradicts that $x(\mathfrak{r}_1) = y(t_1)$; then,
$x(\mathfrak{r}) < y(\mathfrak{r})$, $\mathfrak{r} \in I$. □

Theorem 3. *Let us assume that the conditions of Theorem 1 hold. For monotonic non-decreasing functions f_1 and f_2 in the second argument, problem (1) and (2) has maximal and minimal solutions.*

Proof. To investigate the existence of the maximal solution of Equation (5), we take $\epsilon > 0$ and consider the following integral equations:

$$y_\epsilon(\mathfrak{r}) = \Im^{1-\alpha} f_{1\epsilon}\left(\mathfrak{r}, y_\epsilon(\mathfrak{r}) \times \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\theta)^{\beta-1}}{\Gamma(\beta)} f_{2\epsilon}(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_\epsilon(\sigma)d\sigma)d\theta\right), \quad (18)$$

where

$$f_{1\epsilon}(\mathfrak{r}, y_\epsilon(\mathfrak{r})) = f_1(\mathfrak{r}, y_\epsilon(\mathfrak{r})) + \epsilon$$
$$f_{2\epsilon}(\mathfrak{r}, y_\epsilon(\mathfrak{r})) = f_2(\mathfrak{r}, y_\epsilon(\mathfrak{r})) + \epsilon.$$

Obviously, $f_{1\epsilon}$ and $f_{2\epsilon}$ satisfy the assumptions of Theorem 1; according to Theorem 1, Equation (18) has a continuous solution y_ϵ and

$$\begin{aligned} y_\epsilon(\mathfrak{r}) &= \\ &\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} \\ &\times\left[\epsilon + f_1\left(\varsigma, y_\epsilon(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} \left\{\epsilon + f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_\epsilon(\sigma)d\sigma)d\theta\right)\right\}d\varsigma\right]. \end{aligned}$$

Next, we take $\epsilon_1, \epsilon_2 > 0$ such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$; then,

$$\begin{aligned} y_{\epsilon_1}(\mathfrak{r}) &= \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} \left[\epsilon_1 + f_1\left(\varsigma, y_{\epsilon_1}(\varsigma)\right.\right. \\ &\quad \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} \left\{\epsilon_1 + f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_{\epsilon_1}(\sigma)d\sigma)d\theta\right)\right\}d\varsigma\Big] \\ &> \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} \left[\epsilon_2 + f_1\left(\varsigma, y_{\epsilon_1}(\varsigma)\right.\right. \\ &\quad \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} \left\{\epsilon_2 + f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_{\epsilon_1}(\sigma)d\sigma)d\theta\right)\right\}d\varsigma\Big] \quad (19) \end{aligned}$$

$$y_{\epsilon_2}(\mathfrak{r}) = \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} \left[\epsilon_2 + f_1\left(\varsigma, y_{\epsilon_2}(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} \left\{\epsilon_2 + f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_{\epsilon_2}(\sigma)d\sigma)d\theta\right)\right\}d\varsigma\right] \quad (20)$$

By applying Lemma 2 on estimates (19) and (20), we obtain

$$y_{\epsilon_2}(\mathfrak{r}) < y_{\epsilon_1}(\mathfrak{r}), \ \mathfrak{r} \in I.$$

As proved in Theorem 1, $\{y_\epsilon(\mathfrak{r})\}$ is equicontinuous and uniformly bounded on I; then, $\{y_\epsilon\}$ is relatively compact, and there exists a decreasing sequence ϵ_n such that $\epsilon_n \to 0$, $n \to \infty$ and $\lim_{n\to\infty} y_{\epsilon_n}(\mathfrak{r})$ exists uniformly on I. Let

$$\lim_{n\to\infty} y_{\epsilon_n}(\mathfrak{r}) = q(\mathfrak{r}).$$

Then,

$$\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, y_{\epsilon_n}(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_{\epsilon_n}(\sigma)d\sigma)d\theta\Big)d\varsigma) \to \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, q(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma)-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} q(\sigma)d\sigma)d\theta\Big)d\varsigma,$$

then,

$$\begin{aligned} q(\mathfrak{r}) &= \lim_{n\to\infty} y_{\epsilon_n}(\mathfrak{r}) \\ &= \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, q(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma)-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} q(\sigma)d\sigma)d\theta\Big)d\varsigma \end{aligned}$$

which yields that $q(\mathfrak{r})$ is a solution of (5).
Now, if each $y(\mathfrak{r})$ is a solution of (5), then

$$\begin{aligned} y_\epsilon(\mathfrak{r}) &= \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} \Big[\epsilon + f_1\Big(\varsigma, y_\epsilon(\varsigma) \\ &\quad \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} \Big\{\epsilon + f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_\epsilon(\sigma)d\sigma)d\theta\Big)\Big\}d\varsigma\Big] \\ &> \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, y_\epsilon(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_\epsilon(\sigma)d\sigma)d\theta\Big)d\varsigma \end{aligned} \tag{21}$$

and

$$y(\mathfrak{r}) = \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f_1\Big(\varsigma, y(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y(\sigma)d\sigma)d\theta\Big)d\varsigma \tag{22}$$

Using Lemma 2 on estimates (21) and (22), we obtain

$$y(\mathfrak{r}) < y_\epsilon(\mathfrak{r}),\ \mathfrak{r} \in I.$$

The uniqueness of the maximal solution yields that $y_\epsilon(\mathfrak{r}) \to q(\mathfrak{r})$ uniformly on I as $\epsilon \to 0$.
Hence, q is the maximal solution of (5).
Similarly, we study the existence of the minimal solution of problem (1) and (2) □

3.2. Existence of Unique Solution

Let us assume the following conditions:

$(i)^*$ $f_1, f_2 : I \times R \to R$ are measurable in $\mathfrak{r} \in I$ $\forall x \in R$ and satisfy the Lipschitz condition,

$$|f_1(\mathfrak{r}, x) - f_1(\mathfrak{r}, y)| \le L_1|x-y|,\ \mathfrak{r} \in I,\ x, y \in R. \tag{23}$$

$$|f_2(\mathfrak{r}, x) - f_2(\mathfrak{r}, y)| \le L_2|x-y|,\ \mathfrak{r} \in I,\ x, y \in R. \tag{24}$$

Assumption (iv) implies that

$$|f_1(\mathfrak{r}, y)| \le |f_1(\mathfrak{r}, 0)| + L_1|y|$$

and

$$|f_1(\mathfrak{r},y)| \leq a_1 + L_1|y|, \quad \textit{such that } \; a_1 = \sup_{\mathfrak{r}\in I} |f_1(\mathfrak{r},0)|.$$

Furthermore,

$$|f_2(\mathfrak{r},y)| \leq |f_2(\mathfrak{r},0)| + L_2|y|$$

and

$$|f_2(\mathfrak{r},y)| \leq a_2 + L_2|y|, \quad \textit{where } \; a_2 = \sup_{\mathfrak{r}\in I} |f_2(\mathfrak{r},0)|.$$

$(ii)^*$ $h : I \times R \times R \to R$ is measurable in $\mathfrak{r} \in I$ $\forall x, y \in R$ and satisfies the Lipschitz condition,

$$|h(\mathfrak{r},x_1,y_1) - h(\mathfrak{r},x_2,y_2)| \leq L_3(|x_1 - x_2| + |y_1 - y_2|) \tag{25}$$

with Lipschitz condition $L_3 > 0$.

Theorem 4. *Let us assume that (iii), (i*) are verified. If*

$$\frac{2b_1b_2r_1}{\Gamma(2-\alpha)\Gamma(\alpha+\beta-\gamma+1)} + \frac{b_1a_2^*}{\Gamma(2-\alpha)} < 1. \tag{26}$$

then the solution of (5) is unique.

Proof. Let us consider that assumptions (iii), (i*) are satisfied; then, Theorem 1 is verified, and the solution of Equation (5) exists. Let y_1, y_2 be two solutions of (5), where

$$\begin{aligned}
|y_2(\mathfrak{r}) - y_1(\mathfrak{r})| &= \Bigg| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f\Bigg(\varsigma, y_2(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_2(\sigma)d\sigma)d\theta\Bigg)d\varsigma \\
&- \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} f\Bigg(\varsigma, y_1(\varsigma) \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} y_1(\sigma)d\sigma)d\theta\Bigg)d\varsigma\Bigg| \\
&\leq b_1 \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} \Bigg(|y_2(\varsigma)| \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} |y_2(\sigma)|d\sigma)d\theta \\
&- |y_1(\varsigma)| \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} |y_1(\sigma)|d\sigma)d\theta\Bigg)d\varsigma \\
&\leq b_1 \int_0^{t} \frac{(t-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} \Bigg(|y_2(\varsigma)| \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} |y_2(\sigma)|d\sigma)d\theta \\
&- |y_2(\varsigma)| \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} |y_1(\sigma)|d\sigma)d\theta \\
&+ |y_2(\varsigma)| \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} |y_1(\sigma)|d\sigma)d\theta \\
&- |y_1(\varsigma)| \times \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} f_2(\theta, \int_0^{\theta} \frac{(\theta-\sigma)^{(\alpha-\gamma)-1}}{\Gamma(\alpha-\gamma)} |y_1(\sigma)|d\sigma)d\theta\Bigg)d\varsigma \\
&\leq b_1 \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\varsigma)^{-\alpha}}{\Gamma(1-\alpha)} \Bigg(\|y_2\| \times b_2 \int_0^{\varsigma} \frac{(\varsigma-\theta)^{\beta-1}}{\Gamma(\beta)} \int_0^{\theta} \frac{(\theta-\sigma)^{\alpha-1}}{\Gamma(\alpha)} |y_2(\sigma) - y_1(\sigma)|d\sigma d\theta \\
&+ \|y_2 - y_1\| \times (a_2^* + \frac{b_2r_1}{\Gamma(\beta+\alpha-\gamma+1)})\Bigg)d\varsigma \\
&\leq \frac{b_1}{\Gamma(2-\alpha)} \Bigg(\|y_2 - y_1\| \frac{b_2r_1}{\Gamma(\beta+\alpha-\gamma+1)} \\
&+ \|y_2 - y_1\|(a_2^* + \frac{b_2r_1}{\Gamma(\beta+\alpha-\gamma+1)})\Bigg).
\end{aligned}$$

Hence,

$$\|y_2 - y_1\| \left(1 - \left(\frac{2b_1b_2r_1}{\Gamma(2-\alpha)\Gamma(\beta+\alpha-\gamma+1)} + \frac{b_1a_2^*}{\Gamma(2-\alpha)}\right)\right) \leq 0.$$

which implies the uniqueness of the solution of (5). □

Corollary 1. *Let us suppose that Theorem (4) is verified. If $L_3 < 1$, then for every solution $y \in C(I)$ of Equation (5), there exists a unique solution x of (4).*

Proof. Let $y \in C(I)$ satisfy Equation (5) and x_1, x_2 satisfy Equation (4); then, we have

$$\begin{aligned}
|x_2(\mathfrak{r}) - x_1(\mathfrak{r})| &= \Big| x_0 - \int_0^1 h(\varsigma, x_2(\varsigma), \mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma + \mathfrak{I}^{\alpha}y(\mathfrak{r}) \\
&- x_0 + \int_0^1 h(\varsigma, x_1(\varsigma), \mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma - \mathfrak{I}^{\alpha}y(\mathfrak{r}) \Big| \\
&\leq \int_0^1 \Big(|h(\varsigma, x_2(\varsigma), \mathfrak{I}^{\alpha-\delta}y(\varsigma))| - |h(\varsigma, x_1(\varsigma), \mathfrak{I}^{\alpha-\delta}y(\varsigma))| \Big) d\varsigma \\
&\leq L_3 |x_2(\mathfrak{r}) - x_1(\mathfrak{r})|,
\end{aligned}$$

then,

$$\|x_2 - x_1\|(1 - L_3) \leq 0 \;\rightarrow\; x_2 = x_1.$$

□

3.3. Continuous Dependency Results

Definition 3. *The unique solution $x \in C(I)$ of (4) depends continuously on x_0, if $\forall \epsilon > 0$, $\exists\, \delta > 0$, where*

$$|x_0 - x_0^*| \leq \delta \quad \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* satisfies

$$x^*(\mathfrak{r}) = x_0^* - \int_0^1 h(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma + \mathfrak{I}^{\alpha}y(\mathfrak{r}).$$

Theorem 5. *Let assumption (iv) hold; then, the unique solution of (4) depends continuously on x_0.*

Proof. Let us assume that the two functions $x(\mathfrak{r})$ and $x^*(\mathfrak{r})$ satisfy (4); then,

$$\begin{aligned}
|x(\mathfrak{r}) - x^*(\mathfrak{r})| &= \Big| x_0 - \int_0^1 h(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta}y(\varsigma))ds + \mathfrak{I}^{\alpha}y(\mathfrak{r}) \\
&- x_0^* + \int_0^1 h(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta}y(\varsigma))d\varsigma - \mathfrak{I}^{\alpha}y(\mathfrak{r}) \Big| \\
&\leq |x_0 - x_0^*| + \int_0^1 \Big| h(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta}y(\varsigma)) - h(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta}y(\varsigma)) \Big| d\varsigma \\
&\leq |x_0 - x_0^*| + L_3 |x(\varsigma) - x^*(\varsigma)| \\
&\leq \delta + L_3 \|x - x^*\|.
\end{aligned}$$

Hence,

$$\|x - x^*\|(1 - L_3) \leq \delta$$

and

$$\|x - x^*\| \leq \frac{\delta}{1 - L_3} = \epsilon.$$

□

Definition 4. *The unique solution $x \in C(I)$ of Equation (4) depends continuously on y; if $\forall \epsilon > 0$, $\exists\, \delta > 0$, where*

$$|y - y^*| \leq \delta \quad \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* satisfies the integral equation

$$x^*(\mathfrak{r}) = x_0^* - \int_0^1 h(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma + \mathfrak{I}^{\alpha} y(\mathfrak{r}).$$

Theorem 6. *Let assumption (iv) hold; then, the unique solution of the integral Equation (4) depends continuously on y.*

Proof. Let us assume that the two functions $x(\mathfrak{r})$ and $x^*(\mathfrak{r})$ satisfy Equation (4); then,

$$\begin{aligned}
|x(\mathfrak{r}) - x^*(\mathfrak{r})| &= \Big| x_0 - \int_0^1 h(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma + \mathfrak{I}^{\alpha} y(\mathfrak{r}) \\
&- x_0^* + \int_0^1 h(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma - \mathfrak{I}^{\alpha} y(\mathfrak{r}) \Big| \\
&= \Big| \int_0^1 (h(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) - h(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma))) d\varsigma \\
&+ \mathfrak{I}^{\alpha}(y(\mathfrak{r}) - y^*(\mathfrak{r})) \Big| \\
&\leq L_3 |x(\varsigma) - x^*(\varsigma)| + L_3 \mathfrak{I}^{\alpha-\delta} |y(\varsigma) - y^*(\varsigma)| + \mathfrak{I}^{\alpha} |y(\mathfrak{r}) - y^*(\mathfrak{r})| \\
&\leq L_3 \|x - x^*\| + \frac{L_3 \delta}{\Gamma(\alpha - \delta + 1)} + \frac{\delta}{\Gamma(\alpha + 1)}.
\end{aligned}$$

Hence,

$$\|x - x^*\|(1 - L_3) \leq \frac{L_3 \delta}{\Gamma(\alpha - \delta + 1)} + \frac{\delta}{\Gamma(\alpha + 1)}$$

$$\|x - x^*\| \leq \frac{\frac{L_3 \delta}{\Gamma(\alpha-\delta+1)} + \frac{\delta}{\Gamma(\alpha+1)}}{(1 - L_3)} = \epsilon.$$

□

Definition 5. *The unique solution $x \in C(I)$ of Equation (4) depends continuously on h; if $\forall \epsilon > 0$, $\exists\, \delta > 0$, where*

$$|h(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) - h^*(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma))| \leq \delta \quad \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* satisfies the integral equation

$$x^*(\mathfrak{r}) = x_0 - \int_0^1 h^*(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma + \mathfrak{I}^{\alpha} y(\mathfrak{r}).$$

Theorem 7. *Let assumption (iv) hold; then, the unique solution of (4) depends continuously on h.*

Proof. Let us assume that the two functions $x(\mathfrak{r})$ and $x^*(\mathfrak{r})$ satisfy Equation (4); then,

$$\begin{aligned}
|x(\mathfrak{r}) - x^*(\mathfrak{r})| &= \Big| x_0 - \int_0^1 h(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) ds + \mathfrak{I}^{\alpha} y(\mathfrak{r}) \\
&- x_0 + \int_0^1 h^*(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma - \mathfrak{I}^{\alpha} y(\mathfrak{r}) \Big| \\
&= \Big| \int_0^1 h(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma - \int_0^1 h^*(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma \\
&+ \int_0^1 h^*(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma - \int_0^1 h^*(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) d\varsigma \Big| \\
&= \int_0^1 \Big| h(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) - \int_0^1 h^*(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) \Big| d\varsigma \\
&+ \int_0^1 \Big| h^*(\varsigma, x(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) - \int_0^1 h^*(\varsigma, x^*(\varsigma), \mathfrak{I}^{\alpha-\delta} y(\varsigma)) \Big| d\varsigma \\
&\leq \delta + L_3 |x(\varsigma) - x^*(\varsigma)| \\
&\leq \delta + L_3 \|x - x^*\|.
\end{aligned}$$

Hence,

$$\|x - x^*\|(1 - L_3) \leq \delta$$

so,

$$\|x - x^*\| \leq \frac{\delta}{1 - L_3} = \epsilon.$$

□

Example 1. *We give the quadratic functional integro-differential equation*

$$\frac{dx}{d\mathfrak{r}}(\mathfrak{r}) = (\frac{\mathfrak{r}}{2})^2 + \frac{1}{2}(D^{\frac{3}{4}} x(\mathfrak{r}) \times \mathfrak{I}^{\frac{1}{2}}(\frac{\sqrt{\mathfrak{r}}}{3} + \frac{1}{2} D^{\frac{1}{5}} x(\varsigma))). \quad \mathfrak{r} \in (0,1] \tag{27}$$

with the nonlocal integro-differential condition

$$x(0) - 1 = \int_0^1 (\frac{\mathfrak{r}}{2})^3 + \frac{1}{2}(x(\mathfrak{r}) + D^{\frac{1}{2}} x(\varsigma)) d\varsigma. \tag{28}$$

Thus,

$$f_1(\mathfrak{r}, D^{\alpha} x(\mathfrak{r}) \times \mathfrak{I}^{\beta} f_2(\mathfrak{r}, D^{\gamma} x(\mathfrak{r}))) = (\frac{\mathfrak{r}}{2})^2 + \frac{1}{2}(D^{\frac{1}{2}} x(\mathfrak{r}) \times \mathfrak{I}^{\frac{1}{2}}(\frac{\sqrt{\mathfrak{r}}}{3} + \frac{1}{2} D^{\frac{1}{5}} x(\varsigma))) \quad \mathfrak{r} \in I,$$
$$f_2(\mathfrak{r}, D^{\gamma} x(\mathfrak{r})) = \frac{\sqrt{\mathfrak{r}}}{3} + \frac{1}{2} D^{\frac{1}{5}} x(\varsigma) \quad \mathfrak{r} \in I.$$

Obviously, Theorem 1 holds, when $\mathfrak{r} = 1$,

$$a_1^* = 0.196136, \ a_2^* = \frac{1}{3}, \ a_3^* = \frac{1}{8}, \ b_1 = b_2 = b_3 = \frac{1}{2}, \ \delta = \frac{1}{2}, \ \alpha = \frac{3}{4}, \ \beta = \frac{1}{2}, \ \gamma = \frac{1}{5} \ \textit{and} \ x_0 = 1.$$

From (3), we have

$$\frac{41}{64} r_1^2 - \frac{19}{24} r_1 + 0.19613558245703772 = 0,$$

then, $r_1 = 0.342894$ *or* $r_1 = 0.892878$ *is a positive number. Then, problem (27) and (28) has at least one solution.*

Example 2. *Given the problem of conjugate order*

$$\frac{dx}{d\mathfrak{r}}(\mathfrak{r}) = (\frac{\mathfrak{r}}{2})^2 + \frac{1}{2}(D^{\frac{1}{2}}x(\mathfrak{r}) \times \Im^{\frac{1}{2}}(\frac{\sqrt{\mathfrak{r}}}{3} + \frac{1}{2}D^{\frac{1}{2}}x(\varsigma))). \quad \mathfrak{r} \in (0,1] \tag{29}$$

with the nonlocal integro-differential condition

$$x(0) - 1 = \int_0^1 (\frac{\mathfrak{r}}{2})^3 + \frac{1}{2}(x(\mathfrak{r}) + D^{\frac{1}{2}}x(\varsigma))d\varsigma. \tag{30}$$

we have

$$\frac{9}{16}r_1^2 - \frac{3}{4}r_1 + 0.19613558245703772 = 0,$$

then, $r_1 = 0.4211280954182888$ *or* $r_1 = 0.6899830156928223.$

4. Conclusions

Integral and differential equations contribute significantly to mathematical analysis and have numerous applications to issues in the real world. It has a wide range of uses in mechanics, population dynamics, mathematical biology, engineering, mathematical physics and other fields [1–3,8,15–18].

In this study, we establish the solvability of nonlocal problem (1) and (2). The existence of solution $x \in C(I)$ of problem (1) and (2) is investigated, and the existence of the maximal and minimal solutions of problem (1) and (2) is proved. Furthermore, some continuous dependency results of solution x on fractional-order derivative $y(t)$, on parameter x_0 and on function h are also proved.

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References

1. Banaś, J.; Lecko, M.; El-Sayed, W.G. Existence theorems for some quadratic integral equations. *J. Math. Anal. Appl.* **1998**, *222*, 276–285. [CrossRef]
2. Banaś, J.; Zajac, T. A new approach to the theory of functional integral equations of fractional order. *J. Math. Anal. Appl.* **2011**, *375*, 375–387. [CrossRef]
3. Banaś, J.; Caballero, J.; Rocha, J.; Sadarangani, K. Monotonic solutions of a class of quadratic integral equations of Volterra type. *Comput. Math. Appl.* **2005**, *49*, 943–952. [CrossRef]
4. Bader, R.; Papageorgiou, N.S. Nonlinear multivalued boundary value problems, Discussiones Mathematicae Differential Inclusions. *Control Optim.* **2001**, *21*, 127–148.
5. Boucherif, A.; Precup, R. On the nonlocal initial value problem for first order differential equations. *Fixed Point Theory* **2003**, *4*, 205–212.
6. El Borai, M.; Abbas, M.I. On Some Integro-Differential Equations of Fractional Orders Involving Caratheodory Nonlinearities. *Int. J. Mod. Math.* **2007**, *2*, 41–52.
7. Kamenskii, M.; Obukhovskii, V.; Petrosyan, G.; Yao, J. Existence and approximation of solutions to nonlocal boundary value problems for fractional differential inclusions. *Fixed Point Theory Appl.* **2019**, *2019*, 2. [CrossRef]
8. Zeng, S.; Rădulescu, V.D.; Winkert, P. Double phase implicit Obstacle problems with convection and multivalued mixed boundary value conditions. *SIAM J. Math. Anal.* **2022**, *54*, 1–25. [CrossRef]
9. Sumelka, W. Fractional viscoplasticity. *Mech. Res. Commun.* **2014**, *56*, 31–36. [CrossRef]

10. Subashini, R.; Jothimani, K.; Nisar, K.S.; Ravichandran, C. New results on nonlocal functional integrodifferential equations via Hilfer fractional derivative. *Alex. Eng. J.* **2020**, *59*, 2891–2899. [CrossRef]
11. Aicha, S.; Merad, A. Solvability of nonlinear fractional integro-differential equation with nonlocal condition. *Arab. J. Math. Sci.*, **2023** , *29*, 172–190. [CrossRef]
12. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
13. Curtain, R.F.; Pritchard, A.J. *Functional Analysis in Modern Applied Mathematics*; Academic Press: Cambridge, MA, USA, 1977.
14. El-Sayed, A.M.A.; Hashem, H.G.; Al-Issa, S.M. An implicit hybrid delay functional integral equation: Existence of integrable solutions and continuous dependence. *Mathematics* **2021**, *9*, 3234. [CrossRef]
15. El-Sayed, A.M.A.; Hashem, H.H.G.; Ziada, E.A.A. Picard and Adomian decomposition methods for a quadratic integral equation of fractional order. *Comput. Appl. Math.* **2014**, *33*, 95–109. [CrossRef]
16. Failla, G.; Zingales, M. Advanced materials modeling via fractional calculus: Challenges and perspectives. *Phil. Trans. R. Soc.* **2020**, *378*, 20200050. [CrossRef] [PubMed]
17. Fierro, R.; Martínez, C.; Morales, C.H. Caratheodory selections for multi-valued mappings. *Nonlinear Anal.* **2006**, *64*, 1229–1235. [CrossRef]
18. Srivastava, H.M.; El-Sayed, A.M.A.; Gaafar, F.M. A Class of nonlinear boundary value problems for an arbitrary fractional-order differential equation with the Riemann-Stieltjes functional integral and infinite-point boundary conditions. *Symmetry* **2018**, *10*, 508. [CrossRef]