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Countably Generated Algebras of Analytic Functions on Banach Spaces

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Abstract: In the paper, we study various countably generated algebras of entire analytic functions on complex Banach spaces and their homomorphisms. Countably generated algebras often appear as algebras of symmetric analytic functions on Banach spaces with respect to a group of symmetries, and are interesting for their possible applications. Some conditions of the existence of topological isomorphisms between such algebras are obtained. We construct a class of countably generated algebras, where all normalized algebraic bases are equivalent. On the other hand, we find non-isomorphic classes of such algebras. In addition, we establish the conditions of the hypercyclicity of derivations in countably generated algebras of entire analytic functions of the bounded type. We use methods from the theory of analytic functions of several variables, the theory of commutative Fréchet algebras, and the theory of linear dynamical systems.

Keywords: algebras of analytic functions on Banach spaces; spectra of algebras; hypercyclic operator; analytic mappings of unbounded type

MSC: 46E50; 46G20; 47A16

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Citation: Novosad, Z.; Vasylyshyn, S.; Zagorodnyuk, A. Countably Generated Algebras of Analytic Functions on Banach Spaces. *Axioms* **2023**, *12*, 798. https://doi.org/10.3390/axioms12080798

Academic Editor: Sidney A. Morris

Received: 21 July 2023
Revised: 11 August 2023
Accepted: 16 August 2023
Published: 18 August 2023

1. Introduction

Let X be a complex Banach space and $H(X)$ the algebra of all entire functions on X. This algebra is locally multiplicatively convex (locally m-convex) with respect to the topology of the uniform convergence on compact subsets of X. It is known (see, e.g., [1,2]) that if X has a Schauder basis, the spectrum of $H(X)$ consists of the point evaluation functionals. Consequently, every homomorphism of $H(X)$ can be represented as a composition operator with an analytic map. In applications we have to deal with some particular subalgebras of $H(X)$. The subalgebra of $H(X)$ comprising functions which are bounded on bounded subsets (functions of bounded type), $H_b(X)$, has been studied by many authors [3–6] and is always a proper subalgebra if X is infinite-dimensional (see [7], p. 157). Spectra and homomorphisms of $H_b(X)$ can be explicitly described for some special cases of Banach spaces (e.g., if $X = c_0$ or X is the Tsirelson space [1,8,9]), while in the general case, the spectrum of $H_b(X)$ may have a very complicated structure [3,6]. Thus, it is reasonable to consider some "smaller" subalgebras of analytic functions to obtain a complete and explicit description of their homomorphisms. There are a number of papers related to algebras, generated by linear functionals on X [4,10–14]. Another approach is related to the investigation of subalgebras of analytic functions that are invariant with respect to a group (or a semigroup) of operators on the underlying space. In [15,16], the authors considered algebras of symmetric analytic functions (with respect to the group of permutations of basis vectors) on ℓ_p. These investigations were continued in [17,18] and in [19,20] for the case $X = L_p$. Various results in this direction for different subalgebras were obtained in [21–32].

In many cases, algebras of symmetric analytic functions with respect to a symmetric group are generated by a countable family of homogeneous polynomials. So, it makes sense to study algebras generated by a sequence of algebraically independent polynomials (countably generated algebras) in the general case. Such algebras were investigated in [33–35] and in [36] for the finite-dimensional case. In particular, spectra have been described for some special countably generated algebras of analytic functions on Banach spaces c_0 and ℓ_p, including their homomorphisms, and some operations have been constructed on the spectra. In the presented paper, we continue these investigations for abstract countably generated algebras of analytic functions on Banach spaces.

Finitely generated algebras of polynomials and their spectra form a typical object in the classical invariant theory. For the countably generated case, it is important to work with algebras of analytic functions, and depending on the topology of the space of polynomials, we will have different algebras of analytic functions. The first question arising here is as follows: *How many different countably generated algebras of analytic functions do we have?* In other words, under which conditions can we guarantee that two given countably generated algebras of analytic functions are isomorphic? This question leads to another about the equivalence of algebraic bases and is connected to the theory on the automatic continuity of homomorphisms of Fréchet algebras. In this paper, we describe some conditions when two countably generated algebras are isomorphic and find many examples of non-isomorphic pairs of such algebras. Also, we construct a countably generated algebra, where all normalized algebraic bases of homogeneous polynomials are equivalent. In addition, we consider natural operators of derivations on countably generated algebras of analytic functions and find the conditions for which these operators are hypercyclic.

In Section 2, we introduce basic definitions and recall preliminary results on polynomials and analytic functions on Banach spaces. In Section 3, we consider homomorphisms on algebras of analytic functions on X that are invariant with respect to a semigroup of operators. Section 4 is devoted to the general case of algebras, generated by a sequence of homogeneous polynomials. We find some conditions when two such algebras are isomorphic and construct several examples of different countably generated algebras. In Section 5, we consider a special case of a countably generated algebra where all algebraic bases of homogeneous polynomials are equivalent. In Section 6, we investigate the hypercyclicity of derivations and operators constructed by derivations in countably generated algebras.

For general information on polynomials and analytic functions, we refer the readers to [7,37], and for information on hypercyclic operators, to [38,39]. Basic results of the Gelfand theory and its relationship with algebras of analytic functions on Banach spaces can be found in [37].

2. Definitions and Preliminary Results

Let X and Y be complex Banach spaces. A mapping $P_n\colon X \to Y$ is a continuous *n-homogeneous polynomial* if there exists a continuous n-linear map $B_n\colon X^n \to Y$ such that $P_n(x) = B_n(x, \ldots, x)$. A map $P\colon X \to Y$ is said to be a *continuous polynomial of degree n* if $P = P_0 + P_1 + \cdots + P_n$, where $P_0 \in Y$, P_k are k-homogeneous polynomials and $P_n \neq 0$. For example, any continuous linear functional ϕ on X is a 1-homogeneous continuous polynomial with values in the set of complex numbers $\mathbb{C}$, and ϕ^n is an n-homogeneous continuous polynomial, while $\phi^n + 1$, $n \geq 1$ is a continuous polynomial of degree n, which is not homogeneous. The space of all continuous polynomials from X into Y is denoted by $\mathcal{P}(X, Y)$. We will use notations $\mathcal{P}(X)$ for $\mathcal{P}(X, \mathbb{C})$ and $\mathcal{P}(^n X)$ for the space of continuous n-homogeneous complex-valued polynomials. The space $\mathcal{P}(X)$ is an algebra with pointwise multiplication. This algebra is locally m-convex with respect to the metrizable topology generated by the following countable family of norms

$$\|P\|_r = \sup_{\|x\| \leq r} |P(x)|, \quad r \in \mathbb{Q}_+.$$

The completion $H_b(X)$ of $\mathcal{P}(X)$ in this topology consists of all analytic functions on X which are bounded on every bounded subset of X (so-called *functions of bounded type*). Let us recall that a function $f\colon X \to \mathbb{C}$ is *analytic (or holomorphic)* if it is continuous and the restriction of f to every finite-dimensional subspace is an analytic function. Every analytic function on X can be represented by its Taylor's series

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \quad f_n \in \mathcal{P}(^n X).$$

We denote by $H(X)$ the algebra of all analytic functions on X. $H_b(X)$ is a proper subalgebra of $H(X)$ whenever X is infinite-dimensional ([7], p. 157). Elements in $H(X) \setminus H_b(X)$ are called *analytic functions of unbounded type*. The behavior of analytic functions of unbounded type was studied in [40,41].

A nonzero continuous complex valued homomorphism of a Fréchet algebra is a *character* of this algebra. The set of characters of a given algebra is called the *spectrum* of this algebra. The spectrum of $H_b(X)$ is denoted by $M_b(X)$. For every $\varphi \in M_b(X)$ we denote by φ_n its restriction to the Banach space of n-homogeneous continuous polynomials $\mathcal{P}(^n X)$. Then, the *radius function* $R(\varphi)$ is defined as the infimum of $r > 0$ such that φ is continuous with respect to the norm of uniform convergence in $H_b(X)$ on the ball of radius r centered at the origin. According to [3],

$$R(\varphi) = \limsup_{n\to\infty} \|\varphi_n\|^{\frac{1}{n}} < \infty. \tag{1}$$

Conversely, if a sequence of functionals φ_n on spaces $\mathcal{P}(^n X)$ satisfies (1), then there exists a continuous linear functional φ on $H_b(X)$ such that

$$\varphi(f) = \sum_{n=0}^{\infty} \varphi_n(f_n),$$

where $f = \sum_{n=0}^{\infty} f_n$ is the Taylor series expansion of $f \in H_b(X)$ [3]. In [33], it is proved that Formula (1) is true for any subalgebra of $H_b(X)$. For a given $f \in H_b(X)$, we denote by $\widehat{f}$ the Gelfand transform of f, that is, $\widehat{f}(\varphi) = \varphi(f)$, $\varphi \in M_b(X)$.

A continuous linear operator $T\colon E \to E$ acting on a separable Fréchet space E is called *hypercyclic* if there is a vector $x \in E$ for which the orbit under T,

$$\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \dots\}$$

is dense in E. It is well known that any separable infinite-dimensional Fréchet space admits a hypercyclic operator. According to the Birkhoff result [42], for every complex number $a \neq 0$, the operator $T_a\colon f(x) \mapsto f(x+a)$ is hypercyclic on the space $H(\mathbb{C})$. A similar result for the space $H(\mathbb{C}^n)$ was obtained in [43]. The hypercyclicity of translation operators for analytic functions on Banach spaces was considered in [44–46].

3. Subalgebras of Polynomials and Semigroups of Symmetry

Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$. Let $\mathcal{S}_0$ be a set of affine continuous mappings $\nu\colon X \to X$ such that $P(\nu(x)) = P(x)$ for every $P \in \mathcal{P}_0(X)$ and $x \in X$. We will always suppose that $\mathcal{S}_0$ contains the identity operator and is closed with respect to the operation of composition. Thus, $\mathcal{S}_0$ is a unital semigroup which is called a *semigroup of symmetry* of $\mathcal{P}_0(X)$, and polynomials that are invariant with respect to the action $P \mapsto P \circ \nu$ for any $\nu \in \mathcal{S}_0$ are called $\mathcal{S}_0$*-symmetric*. A subalgebra $\mathcal{P}_0(X)$ is said to be $\mathcal{S}_0$*-complete* if every $\mathcal{S}_0$-symmetric continuous polynomial belongs to $\mathcal{P}_0(X)$. Clearly, the set of all $\mathcal{S}_0$-symmetric continuous polynomials forms a $\mathcal{S}_0$-complete subalgebra in $\mathcal{P}(X)$.

Let us consider the natural partial order on the set of all semigroups of symmetry of $\mathcal{P}_0(X)$ with respect to the set theoretical inclusion. It is easy to see that if a polynomial $P \in \mathcal{P}_0(X)$ is $\mathcal{S}_0$-symmetric and $\mathcal{S}_0 \supset \mathcal{S}_0'$, then P is $\mathcal{S}_0'$-symmetric. Using standard arguments involving Zorn's lemma, we have the following proposition.

Proposition 1. *For every subalgebra $\mathcal{P}_0(X) \subset \mathcal{P}(X)$ there exists a maximal semigroup of symmetry. The maximal semigroup is unique.*

Proof. Let
$$\mathcal{S}_0^1 \subset \mathcal{S}_0^2 \subset \cdots$$
be a chain of semigroups of symmetry of $\mathcal{P}_0(X)$. Then, their union is a semigroup of symmetry of $\mathcal{P}_0(X)$. Thus, by Zorn's lemma, the partially ordered set of all semigroups of symmetry of $\mathcal{P}_0(X)$ has a maximal element. Let $\mathcal{S}_0'$ and $\mathcal{S}_0''$ be two maximal elements. Clearly,
$$\mathcal{S}_0'\mathcal{S}_0'' = \{\eta = \nu\mu \colon \nu \in \mathcal{S}_0',\ \mu \in \mathcal{S}_0''\}$$
is a semigroup of symmetry of $\mathcal{P}_0(X)$ and since $\mathcal{S}_0''$ contains the identity operator E, it follows $\mathcal{S}_0' \subset \mathcal{S}_0'\mathcal{S}_0''$ because $\nu = \nu E \in \mathcal{S}_0'\mathcal{S}_0''$ for every $\nu \in \mathcal{S}_0'$. From the maximality of $\mathcal{S}_0'$ we have that $\mathcal{S}_0' = \mathcal{S}_0'\mathcal{S}_0''$. By the same reason, $\mathcal{S}_0'' = \mathcal{S}_0'\mathcal{S}_0''$. Hence, $\mathcal{S}_0'' = \mathcal{S}_0'$. □

Let us consider two relations of equivalence on X associated with $\mathcal{P}_0(X)$ and a semigroup of symmetry $\mathcal{S}_0$ of $\mathcal{P}_0(X)$, respectively. For any pair $x, y \in X$, we say that $x \sim y$ if $P(x) = P(y)$ for every $P \in \mathcal{P}_0(X)$, and $x \approx y$ if there are $\nu, \mu \in \mathcal{S}_0$ such that $\nu(x) = \mu(y)$. It is easy to check that $x \approx y$ implies $x \sim y$, but the converse statement is not true in general. We will use the following notations for the classes of equivalence
$$[x] = \{y \in X \colon y \sim x\} \quad \text{and} \quad [[x]] = \{y \in X \colon y \approx x\}.$$

Let us denote by $H_{b0}(X)$ the closure of $\mathcal{P}_0(X)$ in $H_b(X)$. In other words, $H_{b0}(X)$ is the minimal Fréchet subalgebra of $H_b(X)$ containing $\mathcal{P}_0(X)$. Also, we denote by $H_0(X)$ the minimal subalgebra of $H(X)$ containing $\mathcal{P}_0(X)$. If $\Phi \colon X \to X$ is an analytic map, then the composition operator $C_\Phi \colon H(X) \to H(X)$, $C_\Phi(f) = f \circ \Phi$ is a continuous homomorphism. If Φ is of bounded type, then the restriction of C_Φ to $H_b(X)$ is a continuous homomorphism from $H_b(X)$ to itself (see, e.g., [3]). The following theorem generalizes Theorem 2.2 in [45].

Theorem 1. *Let $\mathcal{P}_0(X)$ be a subalgebra of $\mathcal{P}(X)$ and $\mathcal{S}_0$ a semigroup of symmetry of $\mathcal{P}_0(X)$, and Φ an analytic map from X to itself. If Φ is such that $\Phi(x) \approx \Phi(y)$ whenever $x \approx y$ and $\mathcal{P}_0(X)$ is $\mathcal{S}_0$-complete, then C_Φ is a continuous homomorphism from $H_0(X)$ to $H_0(X)$. If, moreover, Φ is of bounded type, then the restriction of C_Φ to $H_{b0}(X)$ is a continuous homomorphism from $H_{b0}(X)$ to itself.*

Proof. The restriction of C_Φ to $H_0(X)$ is a continuous homomorphism from $H_0(X)$ to $H(X)$. By the definition of Φ, $g = f \circ \Phi$ is $\mathcal{S}_0$-symmetric for every $f \in H_0(X)$. Let
$$g(x) = \sum_{n=0}^{\infty} g_n(x)$$
be the Taylor series expansion of g. Since $\mathcal{S}_0$ consists of affine maps, $\deg g_n(\nu(x)) = n$ and
$$g(\nu(x)) = \sum_{n=0}^{\infty} g_n(\nu(x)) = \sum_{n=0}^{\infty} g_n(x)$$
implies $g_n(\nu(x)) = g_n(x)$, that is, all g_n are $\mathcal{S}_0$-symmetric. But $\mathcal{P}_0(X)$ is $\mathcal{S}_0$-complete and so all g_n are in $\mathcal{P}_0(X)$. Thus, $g \in H_0(X)$. For the case $H_{b0}(X)$ if Φ is of bounded type, the proof is quite similar. □

For every $x \in X$, the functional $\delta_x \colon f \mapsto f(x)$ is a continuous complex homomorphism (character) of $H(X)$ and $H_b(X)$ [3] and of any closed subalgebras of $H(X)$ and $H_b(X)$ as well. Note that $x \sim y$ with respect to the equivalence induced by $\mathcal{P}_0(X)$ if and only if $\delta_x = \delta_y$ act as functionals on $H_0(X)$. In other words, the quotient set $X/\sim$ can be considered as a subset of the set of characters of $H_{b0}(X)$ (or $H_0(X)$) with respect to the embedding $[x] \mapsto \delta_x$. Suppose that all characters of $H_{b0}(X)$ are of the form δ_x, $x \in X$. We do not know the answer to the following question: *Is every homomorphism $\Psi\colon H_{b0}(X) \to H_{b0}(X)$ of the form $\Psi = C_\Phi$ for some analytic map of bounded type $\Phi\colon X \to X$?*

Let us recall that a sequence of polynomials $\mathbf{P} = \{P_1, \ldots, P_n, \ldots\}$ on X is *algebraically dependent* if there exists a number $m \in \mathbb{N}$ and a nonzero polynomial q of m complex variables such that

$$q\big(P_1(x), \ldots, P_m(x)\big) = 0 \quad \text{for every} \quad x \in X.$$

If a sequence of polynomials is not algebraically dependent then it is *algebraically independent*. The sequence $\mathbf{P}$ is *generating* for $\mathcal{P}_0(X)$ if every polynomial in $\mathcal{P}_0(X)$ can be represented as a finite algebraic combination of polynomials in $\mathbf{P}$. If $\mathbf{P}$ is generating and algebraically independent, then it is called an *algebraic basis* of $\mathcal{P}_0(X)$. The algebra $\mathcal{P}_0(X)$ is *countably generated* if it has a generating sequence of polynomials. If $\mathcal{P}_0(X)$ is countably generated by a generating sequence $\mathbf{P}$, which is not a basis, then there are *algebraic dependencies* on $\mathcal{P}_0(X)$, that is, the family of nonzero polynomials $q_i(t_1, \ldots, t_{m_i})$ of several but finite numbers of variables such that

$$q_i\big(P_1(x), \ldots, P_{m_i}(x)\big) \equiv 0.$$

Since the number of finite subsets of a countable set is countable, the set of algebraic dependencies is finite or countable.

The following proposition shows that any homomorphism of a countably generated algebra of polynomials can be defined on a solely algebraic basis.

Proposition 2. *Let $\mathcal{P}_0(X)$ be countably generated by a generating sequence* $\mathbf{P}$ *and $\mathcal{A}$ be a complex algebra. Then, any homomorphism $\Psi\colon \mathcal{P}_0(X) \to \mathcal{A}$ is completely defined by its values on polynomials in* $\mathbf{P}$. *That is, $\Psi = \Psi'$ if and only if $\Psi(P) = \Psi'(P)$ for every $P \in \mathbf{P}$. If $q_i(t_1, \ldots, t_{m_i})$ are algebraic dependencies on $\mathcal{P}_0(X)$, then*

$$q_i\big(\Psi(P_1), \ldots, \Psi(P_{m_i})\big) = 0.$$

Corollary 1. *Let $\mathcal{P}_0(X)$ be countably generated by a generating sequence* $\mathbf{P}$ *and $\mathcal{A}$ be a complex topological algebra. Then, any homomorphism $\Psi\colon H_{b0}(X) \to \mathcal{A}$ is completely defined by its values on polynomials in* $\mathbf{P}$.

Proof. Since $\mathcal{P}_0(X)$ is dense in $H_{b0}(X)$, every continuous map on $H_{b0}(X)$ is completely defined by its values on $\mathcal{P}_0(X)$, and we can apply Proposition 2. □

Let us consider the following basic examples of algebras $\mathcal{P}_0(X)$ and the corresponding algebras of analytic functions.

Example 1. *Let $\mathcal{P}_0(X) = \mathcal{P}(X)$. Then, any semigroup of symmetry is trivial (consists of the identity operator). The set of continuous complex homomorphisms (the spectrum) of $H_b(X)$ in the general case may be very complicated (see [3,5,6]) and due to the Aron–Berner extension [47,48] contains the second dual space X^{**}. If X has the approximation property, the spectrum of $H(X)$ consists of point-evaluation functionals, δ_x.*

Example 2. *Let $\mathcal{P}_0(X) = \mathcal{P}_f(X)$ be the subalgebra of polynomials of finite type. Recall that P is a polynomial of finite type if it is a finite algebraic combination of continuous linear functionals. Every semigroup of symmetry, in this case, is also trivial. The closure of $\mathcal{P}_f(X)$ in $H_b(X)$ is a so-called algebra of approximable analytic functions $H_{bA}(X)$. The spectrum of $H_{bA}(X)$ can be identified with X^{**} up to the Aron–Berner extension. For some special cases like if X is finite-dimensional, $X = c_0$, or X is the Tsirelson space, $H_{bA}(X) = H_b(X)$. Note that $\mathcal{P}_f(X)$ is not countably generated if X is infinite-dimensional because to obtain all linear functionals, the generating sequence must contain a*

Hamel basis of X^* but, as is well known, the Hamel basis of any infinite-dimensional Banach space is uncountable. However, if X^* has a Schauder basis (i.e., topological basis) $\{f_1, f_2, \ldots\}$, then the algebra of all algebraic combinations of functionals f_j is a dense countably generated subalgebra of $H_{bA}(X)$.

Example 3. *Let $X = \ell_p$, $1 \le p < \infty$ and $\mathcal{S}_0 = S$ be the group of operators of the form*

$$\nu_\sigma \colon (x_1, x_2, \ldots, x_n, \ldots) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}, \ldots),$$

where σ passes over all bijections of $\mathbb{N}$ to itself. Let $\mathcal{P}_s(\ell_p)$ be the algebra of all S-symmetric polynomials. In the literature, S-symmetric polynomials are referred as symmetric polynomials on ℓ_p. It is well known [49,50] that $\mathcal{P}_s(\ell_p)$ is countably generated and polynomials F_k, $k \ge \lceil p \rceil$

$$F_k(x) = \sum_{n=1}^{\infty} x_n^k$$

form an algebraic basis in $\mathcal{P}_s(\ell_p)$. By the definition of $\mathcal{P}_s(\ell_p)$, it is S-complete but the group S is not maximal. For example, the following linear operator

$$\eta \colon (x_1, x_2, \ldots, x_n, \ldots) \mapsto (x_1, 0, x_2, 0, \ldots, 0, x_n, 0, \ldots)$$

does not belong to S but $P(\eta(x)) = P(x)$ for every $P \in \mathcal{P}_s(\ell_p)$ and $x \in \ell_p$. According to [15], $x \sim y$ in ℓ_p if and only if for any $m \ge \lceil p \rceil$ and $k \ge m$ we have $F_k(x) = F_k(y)$. In other words, if we remove a finite number of polynomials F_k, then the algebra generated by the remaining polynomials will generate the same classes of equivalence on ℓ_p but it will obviously not be S-complete.

Spectra of algebras of symmetric analytic functions on ℓ_p were investigated in [15–18]. In particular, point evaluation functionals were constructed on ℓ_p for positive integers p, which are not point evaluation functionals.

Symmetric polynomials on ℓ_p and other Banach spaces are straightforward generalizations of classical symmetric polynomials of several variables. For classical symmetric polynomials and their applications, see [51]. There are a number of other generalizations of symmetric polynomials for infinite dimensions.

Example 4. *Let $\mathcal{S}_+$ be a semigroup of operators on ℓ_p generated by the following operators*

$$\sigma_j \colon (x_1, \ldots, x_j, x_{j+1}, \ldots) \mapsto (x_1, \ldots, x_j, 0, x_{j+1}, \ldots).$$

$\mathcal{S}_+$-symmetric polynomials on ℓ_p are called subsymmetric. The algebra of subsymmetric polynomials on ℓ_p is denoted by $\mathcal{P}_{sub}(\ell_p)$. It is known [52] that polynomials

$$P_{i_1,\ldots,i_s}(x) = \sum_{n_1 < \cdots < n_s} x_{n_1}^{i_1} \cdots x_{n_s}^{i_s},$$

$i_1 + \cdots + i_s = N$ form a linear basis in the space of N-homogeneous subsymmetric polynomials. From here it follows that $\mathcal{P}_{sub}(\ell_p)$ is countably generated. However, we do not know how $\mathcal{P}_{sub}(\ell_p)$ admits an algebraic basis (see [53,54] for some results in this direction). Also, nothing is known about the spectra of the corresponding algebras of analytic functions.

Example 5. *Let $X = \ell_p \oplus \ell_p$, $1 \le p < \infty$. For any element X, we represent this as*

$$(y|x) = (\ldots, y_2, y_1 | x_1, x_2, \ldots)$$

(in the literature, the notation (x/y) is also used), $x, y \in \ell_p$. We consider the semigroup of symmetry $\mathfrak{S}$ generated by the following affine operators

$$A_{\sigma,\omega,a} \colon (y|x) \mapsto (\ldots, y_{\omega(2)}, y_{\omega(1)}, a | a, x_{\sigma(1)}, x_{\sigma(2)}, \ldots),$$

where σ and ω are bijections of $\mathbb{N}$, and $a \in \mathbb{C}$. $\mathfrak{S}$-symmetric polynomials are called supersymmetric polynomials and it is known that the algebra of supersymmetric polynomials is $\mathfrak{S}$-complete and admits the following algebraic basis

$$T_k(y|x) = F_k(x) - F_k(y) = \sum_{n=1}^{\infty} x_n^k - \sum_{n=1}^{\infty} y_n^k, \quad k > \lceil p \rceil.$$

The properties of algebras generated by supersymmetric polynomials on Banach spaces, and structures of their spectra, were considered in [55,56]. Supersymmetric polynomials of several variables and related algebraic structures were studied in [57–59].

Example 6. *Let $X = L_p[0,1]$, $1 \le p \le \infty$, and we denote by Ξ the group of measurable automorphisms of $[0,1]$ which preserve the Lebesgue measure. Ξ-symmetric polynomials are called symmetric polynomials on $L_p[0,1]$ and the algebra of symmetric polynomials is denoted by $\mathcal{P}_s(L_p[0,1])$. If $p < \infty$, then $\mathcal{P}_s(L_p[0,1])$ is finitely generated and polynomials*

$$R_k(x) = \int_{[0,1]} (x(t))^k dt, \quad x = x(t) \in L_p[0,1] \tag{2}$$

form an algebraic basis in $\mathcal{P}_s(L_p[0,1])$ for $k = 1, \ldots, \lfloor p \rfloor$ [49]. If $p = \infty$, then polynomials (2) are well defined and form an algebraic basis in $\mathcal{P}_s(L_\infty[0,1])$ [19]. The algebra $H_{bs}(L_\infty[0,1])$) of symmetric analytic functions on $L_\infty[0,1]$, its spectrum, and its analytic structures on the spectrum were studied in [19,20]. In particular, it was proved that the spectrum consists of the point evaluation functionals, homeomorphic to the space of exponential-type functions of a complex variable. These results can be generalized for spaces of essentially bounded integrable functions in the union of Lebesgue–Rohlin spaces [60].

Example 7. *If a semigroup $\mathfrak{g}$ has representations as semigroups of operators $\mathcal{S}_1, \ldots, \mathcal{S}_s$ on Banach spaces $X_1, \ldots, X_s$, then it naturally acts on the Cartesian product $X_1 \times \cdots \times X_s$. More precisely, if $\mathbf{x} = (x^{(1)}, \ldots, x^{(s)})$, $x^{(j)} \in X_j$ and $\pi_j(\sigma)$ is a representation of an element $\sigma \in \mathfrak{g}$ in X_j, then we can define a semigroup $\mathcal{S}$ of operators on $X_1 \times \cdots \times X_s$ as a representation of $\mathfrak{g}$ in the following way:*

$$\pi(\sigma)(\mathbf{x}) = \big(\pi_1(\sigma)x^{(1)}, \ldots, \pi_s(\sigma)x^{(s)}\big).$$

Often, $\mathcal{S}$-symmetric polynomials on $X_1 \times \cdots \times X_s$ are called block-symmetric polynomials. Algebras of block-symmetric analytic functions were studied in [22,25,32,61–63] for the case when $X_j = \ell_{p_j}$ and $\mathcal{S}_j$ are groups of permutations of basis vectors in ℓ_{p_j}, and in [20,31] for the case when $X_j = L_\infty(\Omega_j)$ and $\mathcal{S}_j$ are groups of measurable, measure-preserving automorphisms of measure spaces Ω_j.

Example 8. *Let $X = \ell_p$, $1 \le p \le \infty$ or $X = c_0$. Denote by $\mathbf{I} = \{I_1^X, \ldots, I_n^X, \ldots\}$ the sequence of polynomials $I_n(x) = I_n^X(x) = x_n^n$, $x = (x_1, x_2, \ldots) \in X$ and by $\mathcal{P}_\mathbf{I}(X)$ the algebra of polynomials on X, generated by $\mathbf{I}$. The group of symmetry $\mathcal{C}$ of $\mathcal{P}_\mathbf{I}(X)$ is generated by linear operators*

$$(x_1, x_2, \ldots, x_n, \ldots) \mapsto \big(x_1, \alpha_{j_2}^{(2)} x_2, \ldots, \alpha_{j_n}^{(n)} x_n, \ldots\big),$$

where $\alpha_{j_n}^{(n)}$ is an n-power root of 1. If $X = \ell_\infty$, then $\mathcal{P}_\mathbf{I}(X)$ is not $\mathcal{C}$-complete. Indeed, every polynomial P on ℓ_∞ / c_0 can be naturally defined on ℓ_∞ and is $\mathcal{C}$-invariant but does not belong to $\mathcal{P}_\mathbf{I}(X)$. It is easy to check that for $X = \ell_p$ or c_0, $\mathcal{P}_\mathbf{I}(X)$ is $\mathcal{C}$-complete.

4. Algebras Generated by Sequences of Polynomials

Let $\mathbf{P} = \{P_n\}$ be a sequence of polynomials on a Banach space X. Throughout this section, we will assume that each P_n is either a norm one n-homogeneous polynomial or is equal to zero, and nonzero polynomials in this sequence are algebraically independent. We consider the minimal unital algebra $\mathcal{P}_{\mathbf{P}}(X)$ generated by polynomials in $\mathbf{P}$. We denote by $H_{b\mathbf{P}}(X)$ the closure of $\mathcal{P}_{\mathbf{P}}(X)$ in $H_b(X)$ and by $H_{\mathbf{P}}(X)$ the algebra of all entire functions

$$f = \sum_{n=0}^{\infty} f_n(x)$$

on X such that all Taylor polynomials f_n are in $\mathcal{P}_{\mathbf{P}}(X)$. The algebra $H_{\mathbf{P}}(X)$ is a topological algebra with respect to the topology of uniform convergence on compact subsets of X. Let $\mathbf{Q} = \{Q_n\}$ be another algebraically independent sequence of n-homogeneous norm one (or zero) polynomials on a Banach space Y. We consider the following question:

- Under which conditions can the algebraic isomorphism

$$J_0 = J_0^{PQ} \colon \mathcal{P}_{\mathbf{P}}(X) \to \mathcal{P}_{\mathbf{Q}}(Y)$$

$$J_0 \colon P_n \mapsto Q_n$$

  be extended to a topological isomorphism $J = J^{PQ} \colon H_{\mathbf{P}}(X) \to H_{\mathbf{Q}}(Y)$ or to a topological isomorphism $J_b = J_b^{PQ} \colon H_{b\mathbf{P}}(X) \to H_{b\mathbf{Q}}(Y)$?

Clearly, the sequence $\mathbf{P}$ is an algebraic basis in the algebra $\mathcal{P}_{\mathbf{P}}(X)$, and if $\mathbf{Q}$ is another algebraic basis in $\mathcal{P}_{\mathbf{P}}(X)$, then $H_{b\mathbf{P}}(X)$ is isomorphic to $H_{b\mathbf{Q}}(X)$. However, in this case, J_0 is not necessarily continuous. The following example shows that J_0 may be discontinuous even if $Q_n = a_n P_n$ for some numbers a_n with $|a_n| = 1$.

Example 9. *Let $X = \ell_1$ and*

$$P_n(x) = F_n(x) = \sum_{k=1}^{\infty} x_k^n.$$

Then, $H_{b\mathbf{P}}(X) = H_{bs}(\ell_1)$ is the algebra of symmetric analytic functions of bounded type on ℓ_1 and $\{F_n\}$ forms an algebraic basis in the algebra of all symmetric polynomials $\mathcal{P}_S(X) \subset H_{bs}(\ell_1)$ (see Example 3). In [45], it is proved that the mapping $F_n \mapsto aF_n$ can be extended to a continuous homomorphism of $H_{bs}(\ell_1)$ if and only if a is a positive integer.

However, there are interesting and surprising examples where the isomorphism J_0 is continuous.

Example 10. *Let $X = L_\infty[0,1]$,*

$$P_n(x) = R_n(x) := \int_{[0,1]} (x(t))^n dt$$

(see Example 6), $Y = \ell_\infty$, and $Q_n(x) = I_n(x) = x_n^n$ (see Example 8). In [34], it is proved that the mapping $R_n \mapsto I_n$ can be extended to a continuous isomorphism from $H_{bs}(L_\infty[0,1])$ to $H_{b\mathbf{I}}(\ell_\infty)$.

Note that the existence of an isomorphism between $H_{b\mathbf{P}}(X)$ and $H_{b\mathbf{Q}}(Y)$ does not imply the existence of an isomorphism between $H_{\mathbf{P}}(X)$ and $H_{\mathbf{Q}}(Y)$.

Example 11. *In [33], it is proved that J_0 can be extended to a continuous isomorphism J_b between $H_{b\mathbf{I}}(c_0)$ and $H_{b\mathbf{I}}(\ell_\infty)$, where I_n as in Example 8. But, $H_{\mathbf{I}}(c_0)$ is not isomorphic to $H_{\mathbf{I}}(\ell_\infty)$ because $H_{\mathbf{I}}(c_0)$ contains a nontrivial function of unbounded type while $H_{\mathbf{I}}(\ell_\infty) = H_{b\mathbf{I}}(\ell_\infty)$ [64].*

Definition 1. *We say that an algebraic basis $\mathbf{Q}$ in $\mathcal{P}_{\mathbf{P}}(X)$ is equivalent to $\mathbf{P}$ if J_b^{PQ} is an isomorphism of $H_{b\mathbf{P}}(X)$ to itself.*

Since the sequence of polynomials $\mathbf{P} = \{P_1, P_2, \ldots, P_n, \ldots\}$ forms an algebraic basis in $\mathcal{P}_{\mathbf{P}}(X)$ and $\mathcal{P}_{\mathbf{P}}(X)$ is a dense subalgebra in $H_{b\mathbf{P}}(X)$, any continuous complex homomorphism φ on $H_{b\mathbf{P}}(X)$ is completely defined by its values on the basic polynomials $P_n \in \mathbf{P}(X)$. Thus, the spectrum $M_{b\mathbf{P}}(X))$ of $H_{b\mathbf{P}}(X)$ can be identified with the set

$$\Gamma_{\mathbf{P}}(M_{b\mathbf{P}}(X)) = \{\Gamma_{\mathbf{P}}(\varphi) := (\varphi(P_1), \varphi(P_2), \ldots, \varphi(P_n), \ldots) \colon \varphi \in M_{b\mathbf{P}}(X)\}.$$

In particular, for point-evaluation functionals δ_x, for $x \in X$ we can write

$$\Gamma_{\mathbf{P}}(X) = \{\Gamma_{\mathbf{P}}(\delta_x) = \Gamma_{\mathbf{P}}(x) = (P_1(x), P_2(x), \ldots, P_n(x), \ldots) \colon x \in X\}.$$

Often, instead of $\Gamma_{\mathbf{P}}$, it is convenient to use the following multi-valued map from X to ℓ_∞ defined by

$$\mathfrak{P}_{\mathbf{P}}(x) = \Big(P_1(x), (P_2(x))^{1/2}, \ldots, (P_n(x))^{1/n}, \ldots\Big),$$

where $a^{1/n}$ is the multi-valued n-root function. In [64], it is observed that $\mathfrak{P}_{\mathbf{P}}$ maps the ball $r\mathcal{B}_X$ into the ball $r\mathcal{B}_{\ell_\infty}$, $r \geq 0$ and if z is in the range of $\mathfrak{P}_{\mathbf{P}}(x)$, then $P_n(x) = I_n(z)$.

Theorem 2. *Let us suppose that $\mathfrak{P}_{\mathbf{P}}(X) = \ell_\infty$, that is, $\mathfrak{P}_{\mathbf{P}}$ is a surjection from X to ℓ_∞. Then,*

1. *$M_{b\mathbf{P}}(X)$ consists of point-evaluation functionals.*
2. *$H_{b\mathbf{P}}(X) = H_{\mathbf{P}}(X)$.*

Proof. (1) Let

$$r = R(\varphi) = \limsup_{n\to\infty} \|\varphi_n\|^{1/n}$$

be the radius function of some $\varphi \in M_{b\mathbf{P}}(X)$. Then,

$$\left|(\varphi(P_n))^{1/n}\right| \leq \left(r^n \|P_n\|\right)^{1/n} = r.$$

Thus, $\mathfrak{P}_{\mathbf{P}}(M_{b\mathbf{P}}(X)) \subset \ell_\infty$. But $\mathfrak{P}_{\mathbf{P}}(X) = \ell_\infty$ and $\mathfrak{P}_{\mathbf{P}}(X) \subset \mathfrak{P}_{\mathbf{P}}(M_{b\mathbf{P}}(X))$. So $\mathfrak{P}_{\mathbf{P}}(X) \subset \mathfrak{P}_{\mathbf{P}}(M_{b\mathbf{P}}(X)) = \ell_\infty$.

Item (2) is proved in [64]. □

Proposition 3. *If $\Gamma_{\mathbf{P}}(X) = \Gamma_{\mathbf{Q}}(Y)$, then the mapping $J_0 \colon P_n \mapsto Q_n$ can be extended to an algebraic isomorphism J from $H_{\mathbf{P}}(X)$ to $H_{\mathbf{Q}}(Y)$.*

Proof. Since $\Gamma_{\mathbf{P}}(X) = \Gamma_{\mathbf{Q}}(Y)$, for every $y_0 \in Y$ there exists $x_0 \in X$ such that $\Gamma_{\mathbf{P}}(x_0) = \Gamma_{\mathbf{Q}}(y_0)$. If

$$f = \sum_{n=0}^{\infty} f_n \in H_{\mathbf{P}}(X)$$

and

$$g = J(f) = \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} J_0(f_n),$$

and then $g(y_0) = f(x_0)$. So, g is well defined on Y. Clearly, g is G-analytic (that is, the restriction of g to any finite-dimension subspace is analytic). If it is not continuous, then a homogeneous polynomial g_n is discontinuous. However, any g_n is an algebraic combination of $Q_1, \ldots, Q_n$ and is continuous. Thus, $g \in H_{\mathbf{Q}}(Y)$ and $J\big(H_{\mathbf{P}}(X)\big) \subset H_{\mathbf{Q}}(Y)$. By the same reason, $J^{-1}\big(H_{\mathbf{Q}}(Y)\big) \subset H_{\mathbf{P}}(X)$. Hence, J is a bijection. The linearity and multiplicativity of J can be easily checked. □

Corollary 2. *Let $\Gamma_{\mathbf{P}}(X) = \Gamma_{\mathbf{Q}}(Y)$. Suppose that $H_{b\mathbf{P}}(X) = H_{\mathbf{P}}(X)$, and $H_{b\mathbf{Q}}(Y) = H_{\mathbf{Q}}(Y)$. Then, $J_b = J$ is a topological isomorphism from $H_{b\mathbf{P}}(X)$ to $H_{b\mathbf{Q}}(Y)$.*

Proof. By Proposition 3, J_b is an algebraic isomorphism. It is known (see, e.g., [65,66]) that there is a unique Fréchet topology on a commutative semi-simple algebra. Thus, any algebraic isomorphism between commutative semi-simple Fréchet algebras is topological, because if we have a discontinuous algebraic isomorphism, then preimages of open sets with respect to this isomorphism give us a different Fréchet topology. □

By Theorem 2 (c.f. [64]), if $\mathfrak{P}_{\mathbf{P}}(X) = \ell_\infty$, then $H_{b\mathbf{P}}(X) = H_{\mathbf{P}}(X)$. So, we have the following corollary:

Corollary 3. *If $\mathfrak{P}_{\mathbf{P}}(X) = \mathfrak{P}_{\mathbf{Q}}(Y) = \ell_\infty$, then $H_{b\mathbf{P}}(X)$ is isomorphic to $H_{b\mathbf{Q}}(Y)$.*

A subset $\Omega \subset X$ is $H_{b\mathbf{P}}(X)$-*bounding* if every function in $H_{b\mathbf{P}}(X)$ is bounded on Ω.

Definition 2. *A Fréchet algebra $\mathcal{F}$ is functionally continuous if every complex homomorphism on $\mathcal{F}$ is continuous.*

The statement that *every Fréchet algebra must be functionally continuous* is the famous Michael Conjecture, open since 1952 [67]. There are many results (see, e.g., [66]) about classes of Fréchet algebra which are functionally continuous (e.g., algebras of analytic functions of several variables). On the other hand, it is known ([37], p. 240) that if X is an infinite dimensional Banach space with a topological basis, then $H_b(X)$ is a so-called *test algebra*. That is, if every complex homomorphism on $H_b(X)$ is continuous, then every Fréchet algebra is functionally continuous. In [20], it is proved that the countably generated algebra $H_{bs}(L_\infty[0,1])$ is a test algebra.

Theorem 3. *Suppose that $\Gamma_{\mathbf{P}}(X) = \Gamma_{\mathbf{Q}}(Y)$ and for every bounded subset $W \subset Y$ there is an $H_{b\mathbf{P}}(X)$-bounding subset $\Omega \subset X$ such that $\Gamma_{\mathbf{P}}(\Omega) \supset \Gamma_{\mathbf{Q}}(W)$. Then, the restriction J_b of J to $H_{b\mathbf{P}}(X)$ is an injective algebra homomorphism from $H_{b\mathbf{P}}(X)$ to $H_{b\mathbf{Q}}(Y)$ with a dense range. If $H_{b\mathbf{P}}(X)$ is functionally continuous, then J_b is continuous.*

Proof. Let $f \in H_{\mathbf{P}}(X)$ and $g = J(f)$. Let W be a bounded subset in Y. Then

$$\sup_{y\in W} |g(y)| \le \sup_{x\in\Omega} |f(x)| < \infty$$

because Ω is bounding. Thus, g is of the bounded type. Hence, J_b is an algebraic homomorphism from $H_{b\mathbf{P}}(X)$ to $H_{b\mathbf{Q}}(Y)$. The range of J_b is dense because it contains all polynomials in $H_{b\mathbf{Q}}(Y)$. It is known that every homomorphism from a functionally continuous Fréchet algebra to a semi-simple Fréchet algebra is continuous [66]. Since $H_{b\mathbf{Q}}(Y)$ is a semi-simple Fréchet algebra, the homomorphism J_b is continuous if $H_{b\mathbf{P}}(X)$ is functionally continuous. □

Corollary 4. *Let $\Gamma_{\mathbf{P}}(X) = \Gamma_{\mathbf{Q}}(Y)$. Suppose that for every bounded subset $W \subset Y$, there is an $H_{b\mathbf{P}}(X)$-bounding subset $\Omega \subset X$ such that $\Gamma_{\mathbf{P}}(\Omega) \supset \Gamma_{\mathbf{Q}}(W)$ and vice versa, for every bounded subset $W' \subset X$ there is an $H_{b\mathbf{Q}}(Y)$-bounding subset $\Omega' \subset Y$ such that $\Gamma_{\mathbf{Q}}(\Omega') \supset \Gamma_{\mathbf{P}}(W')$. Then, J_b is a topological isomorphism from $H_{b\mathbf{P}}(X)$ to $H_{b\mathbf{Q}}(Y)$.*

Proof. By Theorem 3, both J_b^{PQ} and J_b^{QP} are well defined and continuous, but $J_b^{QP} = (J_b^{PQ})^{-1}$. Thus, $J_b = J_b^{PQ}$ is the isomorphism. □

The following theorem gives us a simple condition that $H_{b\mathbf{P}}(X)$ is isomorphic to $H_{b\mathbf{Q}}(Y)$.

Theorem 4. *Let*

$$\Gamma_{\mathbf{P}}\big(M_{b\mathbf{P}}(X)\big) = \Gamma_{\mathbf{Q}}\big(M_{b\mathbf{Q}}(Y)\big).$$

Then, J_b is a topological isomorphism from $H_{b\mathbf{P}}(X)$ to $H_{b\mathbf{Q}}(Y)$.

Proof. As in Proposition 3, for every $\theta \in M_{b\mathbf{Q}}(Y)$, there exists $\varphi \in M_{b\mathbf{P}}(X)$ such that $\Gamma_{\mathbf{P}}(\varphi) = \Gamma_{\mathbf{Q}}(\theta)$. If

$$f = \sum_{n=0}^{\infty} f_n \in H_{b\mathbf{P}}(X)$$

and

$$g = J_b(f) = \sum_{n=0}^{\infty} g_n = \sum_{n=0}^{\infty} J_0(f_n),$$

and then $\widehat{g}(\theta) = \widehat{f}(\varphi)$. So, the formal series

$$\widehat{g} = \sum_{n=0}^{\infty} \widehat{g}_n$$

is well defined on $M_{b\mathbf{Q}}(Y)$. We write that $\varphi = J_b^*(\theta)$.

If $J_b = J_b^{PQ}$ is well defined on $H_{b\mathbf{P}}(X)$ and surjective, then it is bijective and so continuous. If J_b is discontinuous, then it is not defined (as a homomorphism from $H_{b\mathbf{P}}(X)$ to $H_{b\mathbf{Q}}(Y)$), and so there is $f \in H_{b\mathbf{P}}(X)$ such $g = J_b(f)$ is not of the bounded type.

Thus, the radius of boundedness of the formal series

$$g = \sum_{n=1}^{\infty} g_n$$

is equal to $r_0 < \infty$. Hence, there is a bounded sequence $y_k \in Y$ such that $r_0 < \|y_k\| < r_1 < \infty$ and the set

$$\left\{ \sum_{n=1}^{m} g_n(y_{k_i}) : m, i \in \mathbb{N} \right\}$$

is unbounded for every subsequence (y_{k_i}) of (y_k). On the other hand, $R(\delta_{y_k}) < r_1$, and for the reason of compactness, the set of characters $\{\delta_{y_k}\}$ contains a cluster point θ_0. Let $\varphi_0 = J_b^*(\theta_0)$. As mentioned above,

$$\theta_0(g) = \varphi_0(f) = \sum_{n=1}^{\infty} \varphi_0(f_n) = \sum_{n=1}^{\infty} \widehat{f}_n(\varphi_0)$$

is well defined. But,

$$\varphi_0(f) = \sum_{n=1}^{\infty} \varphi_0(f_n) = \sum_{n=1}^{\infty} \theta_0(g_n) = \sum_{n=1}^{\infty} \widehat{g}_n(\theta_0)$$

and the right-hand series diverges because by the definition of θ_0 there is a subsequence (y_{k_i}) of (y_k) such that

$$\theta_0(g_n) = \lim_{i \to \infty} g_n(y_{k_i})$$

for every polynomial g_n, which is a contradiction. □

Example 12. *Clearly,*

$$\Gamma_{\mathbf{I}}(c_0) \neq \Gamma_{\mathbf{I}}(\ell_\infty),$$

but

$$\Gamma_{\mathbf{I}}\big(M_{b\mathbf{I}}(c_0)\big) = \Gamma_{\mathbf{I}}\big(M_{b\mathbf{I}}(\ell_\infty)\big) = \{(I_1(x), \ldots, I_n(x), \ldots) : x \in \ell_\infty\}$$

(c.f. [33]). Thus, by Theorem 4, $H_{b\mathbf{I}}(c_0)$ is isomorphic to $H_{b\mathbf{I}}(\ell_\infty)$.

In [35], it is proved that for every $1 < p \leq \infty$, there exists a point $z \in \ell_p$ and $f \in H_{b\mathbf{I}}(\ell_1)$ such that $f(z)$ is not defined. In other words, $\delta_z : P \mapsto P(z)$ is an unbounded complex homomorphism of $\mathcal{P}_{\mathbf{I}}(\ell_1)$. From this fact, it follows that

$$\Gamma_{\mathbf{I}}\big(M_{b\mathbf{I}}(\ell_1)\big) \neq \Gamma_{\mathbf{I}}\big(M_{b\mathbf{I}}(\ell_p)\big)$$

if $p > 1$.

Example 13. *In [56], it is shown that the mapping* J_b^{TF} *from the algebra of supersymmetric analytic functions of the bounded type* $H_{bsup}(\ell_1 \oplus \ell_1)$ *on* $\ell_1 \oplus \ell_1$ *(see Example 5) to the algebra of symmetric analytic functions of the bounded type on* ℓ_1*,* $J_b^{TF}\colon T_k \mapsto F_k$*,* $f \in \mathbb{N}$ *is a continuous homomorphism with a dense range. The inverse map* J_b^{FT} *is defined on a dense subspace and is unbounded. Thus, the spectrum* $M_{bsup}(\ell_1 \oplus \ell_1)$ *of* $H_{bsup}(\ell_1 \oplus \ell_1)$ *is a subset of* $M_{bs}(\ell_1)$ *in the sense that the restriction of any character in* $M_{bs}(\ell_1)$ *to the range of* $H_{bsup}(\ell_1 \oplus \ell_1)$ *under* J_b *is given as an element in* $M_{bsup}(\ell_1 \oplus \ell_1)$*. On the other hand, from Theorem 4, we have that* $\Gamma_{\mathbf{T}}\big(M_{bsup}(\ell_1 \oplus \ell_1)\big) \neq \Gamma_{\mathbf{F}}\big(M_{bs}(\ell_1)\big)$*.*

Example 14. *Let* $\mathfrak{B}\colon (x_1, x_2, x_3, \ldots) \to (x_2, x_3, \ldots)$ *be the backward shift operator on* $X = \ell_p$ *or* c_0*. Then, for a given algebra* $H_{b\mathbf{P}}(X)$ *the operator* J_b^{PQ} *is an isomorphism, where the sequence of polynomials* $\mathbf{Q}$ *is defined by* $Q_k = P_k \circ \mathfrak{B}$*. Indeed, it is easy to check that* $\Gamma_{\mathbf{P}}(X) = \Gamma_{\mathbf{Q}}(X)$ *and since* $\mathfrak{B}$ *is bounded, we can apply Corollary 4.*

Example 15. *Let* $H_{b\mathbf{P}}(X) = H_{b\mathbf{F}}(\ell_1) = H_{bs}(\ell_1)$ *be the algebra of symmetric analytic functions of the bounded type on* ℓ_1*, and*

$$\widetilde{F}_k(x) = F_k\big(\mathfrak{B}^k(x)\big) = \sum_{n=k+1}^{\infty} x_n^k.$$

For every $x \in \ell_1$*, there is a character* $\varphi_x \in M_{b\widetilde{\mathbf{F}}}(\ell_1)$ *such that* $\varphi_x\big(\widetilde{F}_k\big) = F_k(x)$ *in the following way. Let* $u_m = (\underbrace{0, \ldots, 0}_{m}, x_1, x_2, \ldots)$*. We set*

$$\varphi_x(g) = \lim_{m\to\infty} g(u_m), \quad g \in H_{b\widetilde{\mathbf{F}}}(\ell_1).$$

Since the sequence u_m *is bounded and* g *is of the bounded type, the limit formally exists via a free ultrafilter. But, evidently,* $\lim_{m\to\infty} \widetilde{F}_k(u_m) = F_k(x)$*,* $k \in \mathbb{N}$*. Thus, the limit does not depend on an ultrafilter and* $\varphi_x(g) = f(x)$ *for every* $g = J_b^{F\widetilde{F}}(f)$*. From here, it follows that* $J_b^{\widetilde{F}F}$ *maps* $H_{b\widetilde{\mathbf{F}}}(\ell_1)$ *onto* $H_{bs}(\ell_1)$*. Note that* φ_x *are not point evaluation functionals, in general. On the other hand, due to [17,18], we know that there is a family* ψ_λ *of "exceptional" characters on* $H_{bs}(\ell_1)$*, which are not point-evaluation functionals, such that* $\psi_\lambda(F_1) = \lambda \in \mathbb{C}$ *and* $\psi_\lambda(F_k) = 0$ *for* $k > 1$*. For the exceptional character* $\psi_\lambda \in M_{bs}(\ell_1)$*, the point* $y_\lambda = (0, \lambda, 0, \ldots)$ *is such that* $\psi_\lambda(f) = g(y_0)$*,* $g = J_b^{F\widetilde{F}}(f)$*. Indeed,*

$$\widetilde{F}_k(y_\lambda) = \begin{cases} \lambda & k = 1 \\ 0 & k > 1 \end{cases} = \psi_\lambda(F_k).$$

Proposition 4. *If* $y = (y_1, \ldots, y_n, \ldots)$ *is a vector in* ℓ_1 *such that only a finite number of coordinates* $y_{k_1}, \ldots y_{k_m}$*,* $m \neq 0$*, and* $k_j > 1$ *is not equal to zero. Then, there is no character* $\theta \in M_{bs}(\ell_1)$ *such that* $\theta(f) = g(y)$ *for all* $g = J_b^{F\widetilde{F}}(f)$ *(as in Example 15). The map* $J_b^{\widetilde{F}F}$ *is not projected.*

Proof. If such a vector y and a character θ exist, then $\theta(F_k) \neq 0$ only for a finite number of $k, k_1, \ldots k_m$, $m \neq 0$, $k_j > 1$. But, this is impossible according to [18]. If $J_b^{\widetilde{F}F}$ is projected, then it is bijective, and $J_b^{F\widetilde{F}} = \big(J_b^{\widetilde{F}F}\big)^{-1}$. Thus,

$$f \mapsto J_b^{F\widetilde{F}}(f)(y)$$

is a character for every $y \in \ell_1$. However, this is is not so, as we show above. □

5. The Equivalence of Algebraic Bases

We need the following technical result, which probably is known.

Lemma 1. *Let $p\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial of n complex variables:*

$$p(z_1, \ldots, z_n) = \sum_{m_1=0}^{M_1} \ldots \sum_{m_n=0}^{M_n} \beta_{m_1,\ldots,m_n} z_1^{m_1} \ldots z_n^{m_n}.$$

Then, for every $l_1, \ldots, l_n \in \mathbb{N} \cup \{0\}$ such that $l_j \in \{0, \ldots, M_j\}$ for every $j \in \{1, \ldots, n\}$, and for every $r > 0$

$$|\beta_{l_1,\ldots,l_n}| \le \frac{1}{r^{l_1+\ldots+l_n}} \sup_{|z_1|\le r,\ldots,|z_n|\le r} |p(z_1, \ldots, z_n)|.$$

Proof. We proceed by induction on n. In the case of $n = 1$, the polynomial p has the form

$$p(z_1) = \sum_{m_1=0}^{M_1} \beta_{m_1} z_1^{m_1}.$$

By the Cauchy estimate, for every $l_1 \in \{0, \ldots, M_1\}$ and $r > 0$

$$\sup_{|z_1|\le r} |\beta_{l_1} z_1^{l_1}| \le \sup_{|z_1|\le r} |p(z_1)|.$$

Since $\sup_{|z_1|\le r} |\beta_{l_1} z_1^{l_1}| = r^{l_1} |\beta_{l_1}|$, it follows that

$$|\beta_{l_1}| \le \frac{1}{r^{l_1}} \sup_{|z_1|\le r} |p(z_1)|.$$

This completes the proof for the $n = 1$ case.

Assume that the statement of the lemma holds for every $k \in \{1, \ldots, n-1\}$. Let $l_1, \ldots, l_n \in \mathbb{N} \cup \{0\}$ be such that $l_j \in \{0, \ldots, M_j\}$ for every $j \in \{1, \ldots, n\}$. For fixed $z_2, \ldots, z_n \in \mathbb{C}$, the function $p_1(z_1) = p(z_1, z_2, \ldots, z_n)$ is a polynomial of one complex variable. Note that

$$p_1(z_1) = \sum_{m_1=0}^{M_1} \gamma_{m_1} z_1^{m_1},$$

where

$$\gamma_{m_1} = \sum_{m_2=0}^{M_2} \ldots \sum_{m_n=0}^{M_n} \beta_{m_1,m_2,\ldots,m_n} z_2^{m_2} \ldots z_n^{m_n}.$$

Since the statement of the lemma holds for polynomials of one complex variable, it follows that for every $r > 0$,

$$|\gamma_{l_1}| \le \frac{1}{r^{l_1}} \sup_{|z_1|\le r} |p_1(z_1)|,$$

i.e.,

$$\left| \sum_{m_2=0}^{M_2} \ldots \sum_{m_n=0}^{M_n} \beta_{l_1,m_2,\ldots,m_n} z_2^{m_2} \ldots z_n^{m_n} \right| \le \frac{1}{r^{l_1}} \sup_{|z_1|\le r} |p(z_1, z_2, \ldots, z_n)|. \quad (3)$$

Let $q_{l_1} : \mathbb{C}^{n-1} \to \mathbb{C}$ be defined by

$$q_{l_1}(z_2, \ldots, z_n) = \sum_{m_2=0}^{M_2} \ldots \sum_{m_n=0}^{M_n} \beta_{l_1,m_2,\ldots,m_n} z_2^{m_2} \ldots z_n^{m_n}.$$

By the induction hypothesis, for every $r > 0$

$$|\beta_{l_1,l_2,\ldots,l_n}| \le \frac{1}{r^{l_2+\ldots+l_n}} \sup_{|z_2|\le r,\ldots,|z_n|\le r} |q_{l_1}(z_2, \ldots, z_n)|. \quad (4)$$

By (3) and (4),

$$|\beta_{l_1,l_2,\ldots,l_n}| \le \frac{1}{r^{l_2+\ldots+l_n}} \sup_{|z_2|\le r,\ldots,|z_n|\le r} \frac{1}{r^{l_1}} \sup_{|z_1|\le r} |p(z_1,z_2,\ldots,z_n)|$$
$$= \frac{1}{r^{l_1+l_2+\ldots+l_n}} \sup_{|z_1|\le r,|z_2|\le r,\ldots,|z_n|\le r} |p(z_1,z_2,\ldots,z_n)|.$$

□

Theorem 5. *Let* $\mathbf{P}$ *be a sequence of n-homogeneous algebraically independent polynomials such that* $P_n\colon \ell_p \to \mathbb{C}$ *for every* $n \in \mathbb{N}$ *and* $P_n(x)$ *does not depend on the coordinates* $x_{n+1}, x_{n+2}, \ldots$ *for every sequence* $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) \in \ell_p$, $1 \le p < \infty$*. Then, for every* $f \in H_{b\mathbf{P}}(\ell_p)$*,*

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\cdots+nk_n=n} a_{k_1,k_2,\ldots,k_n} P_1^{k_1}(x)P_2^{k_2}(x)\cdots P_n^{k_n}(x)$$

we have

$$|a_{k_1,k_2,\ldots,k_n}| \le \frac{1}{r^n}\|f_n\|_s, \tag{5}$$

where $\|f_n\|_s = \sup_{\{x\in\ell_p\colon \|x\|\le s\}} |f_n(x)|$ *and* $r > 0$ *is such that* $r \le \frac{s}{\sqrt[p]{n}}$*.*

Proof. Note that

$$\sup_{\{x=(x_1,\ldots,x_n,0,\ldots)\in\ell_p:|x_1|\le r,\ldots,|x_n|\le r\}} |f_n(x)| \le \sup_{\{x=(x_1,\ldots,x_n,0,\ldots)\in\ell_p:\|x\|\le s\}} |f_n(x)|. \tag{6}$$

Indeed, for a given $x = (x_1, \ldots, x_n, 0, \ldots) \in \ell_p$ with $|x_k| \le r$, $k = 1, \ldots, n$,

$$\|x\| = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p} \le \sqrt[p]{nr^p} = r\sqrt[p]{n} \le \frac{s}{\sqrt[p]{n}}\sqrt[p]{n} = s.$$

Therefore, $\{x = (x_1, \ldots, x_n, 0, \ldots) \in \ell_p\colon |x_1| \le r, \ldots, |x_n| \le r\} \subset \{x = (x_1, \ldots, x_n, 0, \ldots) \in \ell_p\colon \|x\| \le s\}$. Next, it is easy to see that

$$\sup_{\{x=(x_1,\ldots,x_n,0,\ldots)\in\ell_p\colon \|x\|\le s\}} |f_n(x)| = \|f_n\|_s$$

because polynomial f_n does not depend on the coordinates $x_{n+1}, x_{n+2}, \ldots$. Taking into account inequality (6), we have

$$\sup_{\{x=(x_1,\ldots,x_n,0,\ldots)\in\ell_p\colon |x_1|\le r,\ldots,|x_n|\le r\}} |f_n(x)| \le \|f_n\|_s. \tag{7}$$

Let $\widehat{K_n} : \mathbb{C}^n \to \ell_p$ be the linear embedding operator $\widehat{K_n}(z_1, \ldots, z_n) = (z_1, \ldots, z_n, 0, \ldots)$. Let us consider a polynomial $g\colon \mathbb{C}^n \to \mathbb{C}$ such that $g = f_n \circ \widehat{K_n}$, that is,

$$g(z_1, \ldots, z_n) = f_n(z) = \sum_{k_1+2k_2+\cdots+nk_n=n} a_{k_1,k_2,\ldots,k_n} z_1^{k_1} z_2^{2k_2} \cdots z_n^{nk_n}$$

for every sequence $z = (z_1, \ldots, z_n, 0, \ldots) \in \ell_p$. From Lemma 1, it follows that for all non-negative integers $k_1, \ldots, k_n$ such that $k_1 + 2k_2 + \cdots + nk_n = n$ and for every $r > 0$,

$$|a_{k_1,k_2,\ldots,k_n}| \le \frac{1}{r^{k_1+\cdots+nk_n}} \sup_{|z_1|\le r,\ldots,|z_n|\le r} |g(z_1, \ldots, z_n)|.$$

Therefore,

$$|a_{k_1,k_2,\ldots,k_n}| \le \frac{1}{r^n} \sup_{|z_1|\le r,\ldots,|z_n|\le r} |f_n(z)|.$$

From inequality (7) we have the estimation (5). □

The following theorem shows that for a special case of $\mathcal{P}_{\mathbf{P}}(X)$, all normalized bases of homogeneous polynomials are equivalent.

Theorem 6. *Let* $\mathbf{Q}$ *be an algebraic basis in* $\mathcal{P}_{\mathbf{P}}(X)$ *and* $\mathfrak{P}_{\mathbf{P}}(X) = \ell_\infty$, *and* $\|Q_n\| = 1$ *(i.e.,* $\mathbf{Q}$ *is normalized). Then,* $\mathbf{Q} \sim \mathbf{P}$.

Proof. Since $\mathbf{P}$ is an algebraic basis in $\mathcal{P}_{\mathbf{P}}(X)$, the basis $\mathbf{Q}$ can be represented via $\mathbf{P}$ in the following way

$$\begin{array}{rl} Q_1(x) = & c_1 P_1(x), \\ Q_2(x) = & c_{20}(P_1(x))^2 + c_{01}P_2, \\ \cdots & \\ Q_n(x) = & c_{n0\ldots0}(P_1(x))^n + \cdots + c_{k_1\ldots k_n}(P_1(x))^{k_1}\cdots(P_n(x))^{k_n} + \cdots + c_{0\ldots01}P_n(x) \\ \cdots & , \end{array} \tag{8}$$

where $c_{\underbrace{0\ldots01}_{n}} \neq 0$ for every n, and $k_1 + 2k_2 + \cdots + nk_n = n$. Let us first consider the case when $X = \ell_\infty$ and $\mathbf{P} = \mathbf{I}$. Then, from (8), it follows that each Q_n may depend only on $x_1, x_2, \ldots, x_n$. Taking into account that $\|Q_n\| = 1$ and Lemma 1, we have that $|c_{k_1\ldots k_n}| \le 1$. Thus, for every $x \in \ell_\infty$,

$$|Q_n(x)| \le \sum_{k_1+\cdots+nk_n=n} |c_{k_1\ldots k_n}| \|x\|^n \le \operatorname{part}(n)\|x\|^n,$$

where $\operatorname{part}(n)$ is the number of partitions of the integer number n. It is well known that

$$\operatorname{part}(n) \sim \frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}}$$

and so

$$\limsup_{n\to\infty} \sqrt[n]{\operatorname{part}(n)} = 1.$$

Hence,

$$\sqrt[n]{|Q_n(x)|} \le \|x\|,$$

that is, $((Q_1(x))^{1/n}, \ldots, (Q_n(x))^{1/n}, \ldots) \in \ell_\infty$.

Conversely, suppose that $z = (z_1, \ldots z_n, \ldots) \in \ell_\infty$. Let us show that there exists $x \in \ell_\infty$ such that $Q_n(x) = z_n^n$. Taking the inverse map of (8) for $P_n = I_n$ and $z_n^n = Q_n(x)$, we have that

$$x_m^m = I_m(x) = \sum_{k_1+\cdots+mk_m=m} a_{k_1\ldots k_m} z_1^{k_1}\cdots z_m^{k_m}, \quad m = 1, \ldots, n$$

for some constants $a_{k_1\ldots k_m}$. Using Lemma 1 and the same arguments as above, we obtain that $x \in \ell_\infty$. Thus, $\mathfrak{P}_{\mathbf{Q}}(\ell_\infty) = \ell_\infty$, and so any algebraic basis $\mathbf{Q}$ in $H_{b\mathbf{I}}(\ell_\infty)$ is equivalent to $\mathbf{I}$. The general case can be obtained using Corollary 3. □

6. Derivations in Countable Generated Algebras

A linear operator D from algebra A to itself is a *derivation* if it satisfies the Leibniz rule

$$D(ab) = D(a)b + aD(b) \quad \text{for all} \quad a, b \in A.$$

It is well known that any derivation on a semi-simple Fréchet commutative algebra with identity is continuous [68]. To define a derivation D on the algebra of polynomials $\mathcal{P}_{\mathbf{P}}(X)$, it is enough to define operator D on the algebraic basis $\mathbf{P}$ and extend it by the Leibniz rule and the linearity to the whole space $\mathcal{P}_{\mathbf{P}}(X)$. If the operator D is continuous on $\mathcal{P}_{\mathbf{P}}(X)$, then it can be extended to a derivation on $H_{b\mathbf{P}}(X)$, which we will denote by the same symbol D.

Let Z be a locally convex space endowed with topology generated on the basis of semi-norms cv (Z). A linear operator $D\colon Z \to Z$ is continuous if and only if for every $q \in \mathrm{cv}\,(Z)$ there exists $p \in \mathrm{cv}\,(Z)$ such that $q \circ D(z) \leq Cp(z)$ for some constant C and every $z \in Z$ (see, e.g., [69] (pp. 126–128)). In other words, if D is a derivation on $\mathcal{P}_{\mathbf{P}}(X)$, then it is continuous if and only if for every number $s > 0$ there is a number $r > 0$ such that D is bounded as a linear operator between normed spaces $(\mathcal{P}_{\mathbf{P}}(X), \|\cdot\|_r)$ and $(\mathcal{P}_{\mathbf{P}}(X), \|\cdot\|_s)$. That is, there is a constant $C > 0$ such that

$$\|D(P)\|_s = \sup_{\|x\| \leq s} \big|D(P(x))\big| \leq C\|P\|_r = \sup_{\|x\| \leq r} \big|P(x)\big|.$$

Let $\gamma(z) = \sum_{n=0}^{\infty} c_n z^n$ be a function of exponential type on $\mathbb{C}$. For a given derivation D on $\mathcal{P}_{\mathbf{P}}(X)$, we consider the following operator $\gamma(D)$:

$$\gamma(D)(P) = \sum_{n=0}^{\infty} c_n D^n, \quad P \in \mathcal{P}_{\mathbf{P}}(X).$$

Clearly, if $\gamma(z) = e^z$, then $\gamma(D)(P)$ is an algebraic homomorphism. If D is continuous, then $\gamma(D)(P)$ is so. Indeed,

$$\Big\| \sum_{m=1}^{\infty} c_m D^m(P) \Big\|_s \leq C \sum_{n=0}^{\infty} \frac{\|D(P)\|_r^n}{n!} = \exp\big(C\|P\|_r\big), \quad P \in \mathcal{P}_{\mathbf{P}}(X)$$

for some $C > 0$, and so this operator is continuous.

We denote by $H_{b\mathbf{P}}^n(X)$ the closed subalgebra in $H_{b\mathbf{P}}(X)$, generated by polynomials $P_1, \ldots, P_n$.

Theorem 7. *Suppose that $H_{b\mathbf{P}}(X)$ is such that $H_{b\mathbf{P}}^n(X)$ is isomorphic to $H(\mathbb{C}^n)$ for every $n \in \mathbb{N}$. If D is a derivation on $H_{b\mathbf{P}}(X)$ such that $D(P_k) = b_k \in \mathbb{C}$, then the operator*

$$\gamma(D)(P) = \sum_{m=1}^{\infty} c_m D^m,$$

is hypercyclic, where γ is a nonconstant function of exponential type.

Proof. Let J_n be the isomorphism from $H(\mathbb{C}^n)$ to $H_{b\mathbf{P}}^n(X)$ given by $J_n(t_k) = P_k$, $(t_1, \ldots, t_n) \in \mathbb{C}^n$. Then, $H_{b\mathbf{P}}^n(X)$ is an invariant subspace of D, and $d = J_n^{-1} D J_n$ is a derivation on $H(\mathbb{C}^n)$. It is known that any derivation on $H(\mathbb{C}^n)$ is of the form

$$d(f) = \sum_{k=1}^{n} h_k \frac{\partial}{\partial t_k} f$$

for some $h_1, \ldots, h_n \in H(\mathbb{C}^n)$ [70–72]. Taking into account that $d(t_k) = b_k$, we have

$$d(f)(t) = \sum_{k=1}^{n} b_k \frac{\partial}{\partial t_k} f(t).$$

It is well known that this operator satisfies the hypercyclicity criterion for the entire sequence on $H(\mathbb{C}^n)$. Moreover, for every nonconstant function of exponential type γ, the operator $\gamma(d)$ satisfies the hypercyclicity criterion for the entire sequence on $H(\mathbb{C}^n)$ [43]. Thus, the restriction of $\sum_{m=1}^{\infty} c_m D^m$ to any $H_{b\mathbf{P}}^n(X)$ satisfies the hypercyclicity criterion for the entire sequence. By [46], Lemma 3.2, the operator $\sum_{m=1}^{\infty} c_m D^m$ is hypercyclic on $H_{b\mathbf{P}}(X)$. □

In [28], the following theorem is proved.

Theorem 8. *Let X be a complex Banach space. Let $n \in \mathbb{N}$ and $\{Q_1, Q_2, \ldots, Q_n\}$ be a set of polynomials on X, which have the following properties:*

1. *For every $j \in \{1, \ldots, n\}$ the mapping $Q_j : X \to \mathbb{C}$ is a continuous d_j-homogeneous polynomial, where $d_j \in N$;*
2. *The set of polynomials $\{Q_1, Q_2, \ldots, Q_n\}$ is algebraically independent;*
3. *There exists a constant $C > 0$ such that for every vector $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, there exists an element $x_z \in X$ such that $\|x_z\| \leq C\|z\|_\infty$ and $Q_j(x_z) = z_j$ for every $j \in \{1, \ldots, n\}$, where $\|z\|_\infty = \max\{|z_1|, \ldots, |z_n|\}$.*

Let A be a closed subalgebra of the Fréchet algebra $H_b(X)$ such that for every function $f \in A$, each term of the Taylor series of this function is an algebraic combination of elements of the set $\{Q_1, Q_2, \ldots, Q_n\}$. Then, A and $H(\mathbb{C}^n)$ are isomorphic.

In particular, if each finite subsequence $\{P_1, P_2, \ldots, P_n\} \subset \mathbf{P}$ satisfies the conditions of Theorem 8 for every n, then J_n is an isomorphism from $H(\mathbb{C}^n)$ to $H^n_{b\mathbf{P}}(X)$, $n \in \mathbb{N}$.

Definition 3. *A polynomial mapping*

$$\mathcal{Q}_n : \mathbb{C}^n \to \mathbb{C}^n,$$

$$\mathcal{Q}_n : (x_1, \ldots, x_n) \mapsto \big(Q_1(x_1, \ldots, x_n), \ldots, Q_n(x_1, \ldots, x_n)\big)$$

is said to be proper if for every compact set $K \subset \mathbb{C}^n$, $\mathcal{Q}_n^{-1}(K)$ is a compact set in $\mathbb{C}^n$.

It is known (see, e.g., [73], Chapter 15) that a proper map is surjective and open. In [74], it is proved that if polynomials $Q_1, \ldots, Q_n$ are homogeneous, then $\mathcal{Q}_n$ is proper if and only if $\mathcal{Q}_n^{-1}(0) = 0$.

The following proposition gives the conditions for which Theorem 7 is applicable.

Proposition 5. *Suppose that X contains an n-dimensional complex subspace V_n such that the restriction $\mathcal{Q}_n$ of the polynomial map $x \mapsto \big(P_1(x), \ldots, P_n(x)\big)$, $P_k \in \mathbf{P}$ to V_n is proper, then J_n is an isomorphism from $H(\mathbb{C}^n)$ to $H^n_{b\mathbf{P}}(X)$.*

Proof. In the definition of the sequence $\mathbf{P}$, we already assumed that the homogeneous polynomials $P_1, \ldots, P_n$ are algebraically independent. So, it is enough to check item (3) in Theorem 8. Since $\mathcal{Q}_n$ is proper, it is surjective and the preimage of the closed unit ball in $(\mathbb{C}^n, \|\cdot\|_\infty)$ is a bounded set in a ball of radius $R > 0$ in $\mathbb{C}^n$. Let $z \in \mathbb{C}^n$ and $x_z \in \mathcal{Q}_n^{-1}(z)$. First, we consider the case when $\|z\|_\infty \leq 1$. Then, x_z is in the ball of radius R centered at the origin, and so $\|x_z\| \leq C\|z\|_\infty$ if $C \geq R$. If $\|z\| \geq 1$, then $\|x_z\| > 1$, because $\|\mathcal{Q}\| \leq 1$. Thus, to check item (3) it is enough to show that there is a constant $C > 0$ such that for every $x \in V_n$, $\|x\| > 1$ we have $\|x\| \leq C\mathcal{Q}(x)\|$. Let us suppose, on the contrary, that for every $C > 0$ there exists x, $\|x\| > 1$ such that $\|x\| \geq C\|\mathcal{Q}(x)\|$. We write $x = \lambda y$, where $\|y\| = 1$ and $\lambda > 1$. Then,

$$\|x\| = \lambda > C\big\|(\lambda Q_1(x), \ldots, \lambda^n Q_n(x))\big\|_\infty = C \max\big(\lambda|Q_1(x)|, \ldots, \lambda^n|Q_n(x)|\big).$$

Since we can find such $x = \lambda y$ for every C, there exists a sequence $y_1, y_2, \ldots$ with $\|y_k\| = 1$ such that $\|\mathcal{Q}(y_k)\| \to 0$ as $k \to 0$. Let y_0 be a limit point of the sequence. Then, $\|y_0\| = 1$ and $\mathcal{Q}(y_0) = 0$. But, this contradicts the property of proper maps in [74], mentioned above, suggesting that $\mathcal{Q}_n^{-1}(0) = 0$. □

Example 16.

1. *From Proposition 5, it follows that the algebras $H_{bs}(\ell_p)$, as in Example 3, $H_{bs}(L_\infty[0,1])$, as in Example 10, and $H_{bI}(X)$, as in Example 8, and algebras of supersymmetric analytic functions of bounded type on ℓ_p, $1 \le p < \infty$ satisfy the conditions of Theorem 8.*
2. *Let $X = \ell_p$, $1 \le p \le \infty$ or $X = c_0$, and $P_n(x) = x_1 x_2 \cdots x_n$. Then, for the algebra $H_{b\mathbf{P}}(X)$, we cannot apply Proposition 5 because the polynomial mapping*

$$(x_1, x_2 \ldots, x_n) \mapsto (x_1, x_1x_2, \ldots, x_1x_2 \cdots x_n)$$

is not proper.

Example 17. *For the algebra $H_{bs}(\ell_1) = H_{b\mathbf{F}}(\ell_1)$, the derivation defined on the algebraic basis $\mathbf{F}$ by $D_{k,\lambda}(F_m) = \lambda\delta_{km}$ is well defined and continuous only if $k = 1$ (c.f. [75]). Here, $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$, and δ_{km} is the Kronecker symbol. The operator $D_{1,\lambda}$ can be computed by*

$$D_{1,\lambda}(f)(x) = \lim_{n\to\infty} f\Big(\underbrace{\frac{\lambda}{n}, \ldots, \frac{\lambda}{n}}_{n}, x_1, x_2, \ldots\Big)$$

and if

$$f(x) = \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\cdots+nk_n=n} a_{k_1,k_2,\ldots,k_n} F_1^{k_1}(x) F_2^{k_2}(x) \cdots F_n^{k_n}(x),$$

then

$$\exp(D_{1,\lambda})(f)(x) = \sum_{n=0}^{\infty} \sum_{k_1+2k_2+\cdots+nk_n=n} a_{k_1,k_2,\ldots,k_n} (F_1+\lambda)^{k_1}(x) F_2^{k_2}(x) \cdots F_n^{k_n}(x).$$

In other words, the homomorphism $\exp(D_{1,\lambda})$ is defined on the algebraic basis $\mathbf{F}$ by

$$\exp(D_{1,\lambda})(F_1) = \lambda, \quad \text{and} \quad \exp(D_{1,\lambda})(F_m) = 0, \quad m > 1.$$

For the same reason, if $\exp(D_{k,\lambda})$ is continuous, then the homomorphism $\exp(D_{k,\lambda})$ defined by

$$\exp(D_{k,\lambda})(F_k) = \lambda, \quad \text{and} \quad \exp(D_{k,\lambda})(F_m) = 0, \quad m \ne k$$

is continuous. This is the case only for $k = 1$ [18,45].

Note that in [46], it is shown that for every $y \in \ell_1$, the operator

$$T_y(f)(x) = f(x_1, y_1, \ldots, x_n, y_n, \ldots)$$

is hypercyclic on $H_{bs}(\ell_1)$. We do not know how can we represent T_y as e^D for some continuous derivation.

Example 18. *In [35], it is proved that the operator $\mathcal{T}_y$, $y \in \ell_p$,*

$$\mathcal{T}_y\colon f(x) \mapsto f(x_1 + y_1, \sqrt{x_2 + y_2}, \ldots, \sqrt[n]{x_n + y_n}, \ldots)$$

is a continuous homomorphism of $H_{b\mathbf{I}}(\ell_p)$, $1 \le p \le \infty$ such that $\mathcal{T}_y(I_n)(x) = I_n(x) + I_n(y)$. From [45], it follows that $\mathcal{T}_y$ is hypercyclic if $p < \infty$.

7. Conclusions

Considering various examples of countably generated algebras of analytic functions on Banach spaces, we can see that these algebras may have quite different structures. For example, all algebraic bases of homogeneous polynomials on the algebra of symmetric analytic functions on $L_\infty[0,1]$ are equivalent, while the set of equivalent bases of the algebra of symmetric analytic functions of the bounded type on ℓ_p is very restricted.

However, we find some general conditions when the algebras are isomorphic, and the corresponding algebraic bases are equivalent. In addition, we considered derivations in countably generated algebras of analytic functions of the bounded type, and find the conditions of the hypercyclicity of operators, related to the derivations.

The obtained results pave the way for further studies of algebraic and analytic structures on spectra of countably generated algebras of analytic functions on Banach spaces. In particular, for many reasons, it is interesting to define an analytic manifold structure on the spectrum of a given algebra of analytic functions so that the set of Gelfand transforms of the characters coincides with a natural algebra of analytic functions on this manifold. Also, it is interesting to consider derivatives of the countably generated algebras from the point of view of Lie Algebras Theory. These and other problems will be considered in further investigations.

Author Contributions: Conceptualization, A.Z.; investigation, Z.N. and S.V.; writing—original draft preparation, Z.N. and S.V.; writing—review and editing, A.Z.; project administration, A.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the National Research Foundation of Ukraine, 2020.02/0025.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest.

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