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On the Study of Pseudo $\mathcal{S}$-Asymptotically Periodic Mild Solutions for a Class of Neutral Fractional Delayed Evolution Equations

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Abstract: The goal of this paper is to investigate the existence and uniqueness of pseudo $\mathcal{S}$-asymptotically periodic mild solutions for a class of neutral fractional evolution equations with finite delay. We essentially use the fractional powers of closed linear operators, the semigroup theory and some classical fixed point theorems. Furthermore, we provide an example to illustrate the applications of our abstract results.

Keywords: pseudo $\mathcal{S}$-asymptotically periodic function; neutral fractional evolution equation; mild solution; fractional power

MSC: 26D15; 26A33; 60E15

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1. Introduction

In this research article, we analyze the existence and uniqueness of pseudo-$\mathcal{S}$-asymptotically periodic mild solutions for the following abstract fractional Cauchy problem

$$\begin{cases} {}^cD_t^q\big(u(t) - G(t,u_t)\big) + Au(t) = F(t,u_t), & t \geq 0, \\ u(t) = \varphi(t), & -r \leq t \leq 0, \end{cases} \tag{1}$$

where the fractional derivative ${}^cD_t^q$, $q \in (0,1)$ is taken in the sense of Caputo approach and $(A, D(A))$ is a closed linear operator in a Banach space $(X, \|\cdot\|)$. We assume that $-A$ generates an exponentially stable analytic semigroup $(T(t))_{t\geq 0}$. Here,

$$F, G : \mathbb{R}^+ \times E \to X,\ r > 0,$$

are two continuous functions, where $\mathbb{R}^+ = [0, \infty)$ and $E = C([-r, 0], X)$. By u_t, we denote the classical history function defined by

$$u_t(s) := u(t+s) \quad \text{for } -r \leq s \leq 0,$$

while the data $u(\cdot)$ belongs to the space E. Let us recall that a bounded continuous function $f : \mathbb{R}^+ \to X$ is said to be pseudo $\mathcal{S}$-asymptotically periodic if there exists $\omega > 0$ such that

$$\lim_{h\to+\infty} \frac{1}{h} \int_0^h \|f(t+\omega) - f(t)\| \, dt = 0. \tag{2}$$

The class of pseudo $\mathcal{S}$-asymptotically periodic functions was introduced in [1]. In that paper, the authors have considered the classical version of (1) with $q = 1$ and established several interesting results concerning the existence and uniqueness of pseudo $\mathcal{S}$-asymptotically periodic mild solutions for such problems. The class of pseudo $\mathcal{S}$-asymptotically periodic functions is a natural generalization of the class of $\mathcal{S}$-asymptotically periodic functions; see [2]. Consequently, our study can be viewed as an extension and continuation of the research study in [3], where the solvability of problem (1) was discussed and optimal results about the existence and uniqueness of $\mathcal{S}$-asymptotically periodic mild solutions were successfully established. The investigation of existence and uniqueness of pseudo $\mathcal{S}$-asymptotically periodic mild solutions for various classes of the abstract fractional Cauchy problems is an attractive field and was the principal subject of many works. For example, in [4] the authors have examined the existence and uniqueness of pseudo $\mathcal{S}$-asymptotically periodic solutions of the second-order abstract Cauchy problems. Another interesting class of the abstract fractional equations was analyzed in [5], where the authors have considered the following fractional integro-differential neutral equations

$$\begin{cases} \dfrac{d}{dt}D(t,u_t) = \int\limits_0^t g_{\alpha-1}(t-s)AD(s,u_s)\,ds + f(t,u_t), \; t \geq 0, \\ u_0 = \varphi, \end{cases}$$

with $1 < \alpha < 2$, $(A, D(A))$ being a sectorial densely defined operator in the sense of [6] and

$$D(t,\varphi) = \varphi(0) + g(t,\varphi),$$

with f, g and φ being suitable vector-valued functions. Here,

$$g_\alpha(t) := t^{\alpha-1}/\Gamma(\alpha),$$

$\alpha > 0$, $t > 0$, where $\Gamma(\alpha)$ is the Euler Gamma function. For more details, we refer the reader to the recent research monographs [7,8] and the references cited therein. We would like to point out that this work is the first phase of another project, the aim of which is to study our main problem (1) set on non-regular domains. This study is motivated by the fact that these types of domains are often encountered in several concrete situations. Thus, the theory of such problems is interesting in itself. Moreover, the knowledge of the structure of solutions is useful in numerical analysis; see [9–12]. The organization of this paper can be briefly described as follows. In the Section 2, we collect some necessary definitions and results needed to justify our main results. In the Section 3, we furnish some sufficient conditions for the existence of a pseudo $\mathcal{S}$-asymptotically periodic mild solution for the problem (1). In the Section 5, we provide an example to demonstrate the abstract results obtained in this work.

2. Preliminaries

We use the standard notation throughout this paper. Unless stated otherwise, we will always assume that X denotes a complex Banach space equipped with the norm $\|\cdot\|$, and A is a closed linear operator with $0 \in \rho(A)$, where $\rho(A)$ denotes the resolvent set of A. In this work, we need to use the notion of fractional power of closed linear operators. Then, we know that, for every $\alpha > 0$, the operator $A^{-\alpha}$ is well defined and has the following explicit representation:

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\, dt,$$

for more details about fractional powers of closed linear operators, we refer the reader to the research monograph [6]. For $\alpha \in (0,1)$, we set

$$X_\alpha := D(A^\alpha).$$

In the particular situation $\alpha = 0$, we consider that $A^0 := I$ and $X_0 := X$. The fractional power space X_α is a Banach space when it is endowed with its natural norm

$$\|\cdot\|_\alpha = \|A^\alpha \cdot\|.$$

Furthermore, for $0 \le \alpha \le \beta \le 1$, one has

$$X_\beta \hookrightarrow X_\alpha,$$

and the embedding $X_\beta \hookrightarrow X_\alpha$ is compact whenever the resolvent operator of A is compact. In the sequel, we consider the Banach space

$$B_\alpha := C\big([-r,0], X_\alpha\big),$$

of all continuous vector-valued functions from $[-r,0]$ into X_α, equipped with the norm

$$\|\varphi\|_{B_\alpha} := \max_{s\in[-r,0]} \|\varphi(s)\|_\alpha.$$

Among the effective tools used in our work we cite the well known semigroup's theory. For the the reader's convenience, we refer him or her to the research monograph [13]. In this work, our basic assumption is that $-A$ generates an exponentially stable analytic semigroup in X. We need also to recall the following results (see, e.g., [13]):

Theorem 1. *Assume that $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge0}$ and let $\|T(t)\| \le Me^{-\delta t}$, $t \ge 0$ for some $\delta > 0$. If $0 \in \rho(A)$, then one has*

(i) *$A^{-\alpha}$ is a bounded linear operator for $0 \le \alpha \le 1$.*
(ii) *$T(t) : X \to D(A^\alpha)$ for every $t > 0$ and $\alpha \ge 0$.*
(iii) *For every $t \ge 0$ and $x \in D(A^\alpha)$, we have*

$$T(t)A^\alpha x = A^\alpha T(t)x.$$

(vi) *For every $t > 0$, the operator $A^\alpha T(t)$ is bounded and*

$$||A^\alpha T(t)|| \le M_\alpha t^{-\alpha} e^{-\delta t}. \tag{3}$$

(v) *If $0 < \alpha < 1$ and $x \in D(A^\alpha)$, then*

$$||T(t)x - x|| \le C_\alpha t^\alpha ||A^\alpha x||.$$

Now, let us define

$$U(t) := \int_0^\infty \xi_q(\tau) T\big(t^q \tau\big)\, d\tau \text{ and } V(t) := q \int_0^\infty \tau \xi_q(\tau) T\big(t^q \tau\big)\, d\tau,\ t \ge 0, \tag{4}$$

where

$$\xi_q(\tau) := \frac{1}{\pi q} \sum_{n\ge1} (-\tau)^{n-1} \frac{\Gamma(nq-1)}{n!} \sin(n\pi q), \quad \tau \in \mathbb{R}^+ - \{0\},$$

is a probability density function defined on $\mathbb{R}^+ - \{0\}$. Let us recall that (see, e.g., [14])

$$\xi_q(\tau) \geq 0,\ \tau > 0 \text{ and } \int_0^\infty \tau^\alpha \xi_q(\tau)\, d\tau = 1$$

and

$$\int_0^\infty \tau^v \xi_q(\tau)\, d\tau = \frac{\Gamma(1+v)}{\Gamma(1+qv)}, \quad 0 \leq v \leq 1. \tag{5}$$

The main properties of these families are summarized in the following lemma (see [14]):

Lemma 1. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup. Then, the operator families $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ defined by (4) have the following properties:*

(i) $(U(t))_{t\geq 0}$ *and* $(V(t))_{t\geq 0}$ *are strongly continuous.*
(ii) *If $(T(t))_{t\geq 0}$ is uniformly bounded, then $U(t)$ and $V(t)$ are linear bounded operators for any fixed $t \geq 0$.*
(iii) *If $(T(t))_{t\geq 0}$ is compact, then $U(t)$ and $V(t)$ are compact operators for any $t \geq 0$.*
(vi) *If $x \in X$, $\alpha,\ \beta \in (0,1)$ and $t > 0$, then*

$$AV(t)x = A^{1-\beta}V(t)A^\beta x$$

and

$$\|A^\alpha V(t)\| \leq \frac{M_\alpha}{t^{q\alpha}} \frac{q\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))}. \tag{6}$$

(v) *If $t \geq 0$ and $x \in X_\alpha$, then*

$$\|U(t)x\|_\alpha \leq M\|x\|_\alpha$$

and

$$\|V(t)x\|_\alpha \leq M\frac{q}{\Gamma(1+q)}\|x\|_\alpha.$$

For $I \subseteq \mathbb{R}$, we denote by $C_b(I, X)$ the Banach space consisting of all bounded and continuous functions from I into X, equipped with the norm

$$\|u\|_{C_b(I,X)} := \sup_{t\in I}\|u(t)\|.$$

In order to develop our results, we need to introduce the following spaces $PSAP_\omega(X)$ and $PSAP_{\omega,p}(X)$. The space $PSAP_\omega(X)$ consists of all bounded and continuous functions from $\mathbb{R}^+$ to X satisfying (2), while the space $PSAP_{\omega,p}(X)$ is defined as follows:

Definition 1. *Let $p > 0$ and $u \in PSAP_\omega(X)$. Then, we say that $u(\cdot)$ is pseudo-$\mathcal{S}$-asymptotically ω-periodic of class p ($u \in PSAP_{\omega,p}(X)$, equivalently) if*

$$\lim_{h\to+\infty} \frac{1}{h}\int_p^h \sup_{t\in[\xi-p,\xi]} \|f(t+\omega) - f(t)\|\, d\xi = 0.$$

More information about this class of functions can be founded in [1,5,15], from which we find the following:

Proposition 1. *Let $p \geq 0$. Then,*

(i) $PSAP_{\omega,p}(X) \subseteq PSAP_\omega(X)$.
(ii) $PSAP_{\omega,p}(X)$ *is a closed subspace of* $C_b(\mathbb{R}^+, X)$.
(iii) *Assume that $f \in C_b(\mathbb{R}^+, X)$. Then, $f \in PSAP_{\omega,p}(X)$ if and only if, for every $\varepsilon > 0$, we have*

$$\lim_{h\to+\infty}\frac{1}{h}\mu(M_{h,\varepsilon}(f))=0,$$

where $\mu(\cdot)$ denotes the classical Lebesgue measure and

$$M_{h,\varepsilon}(f)=\left\{t\in[p,h]:\sup_{t\in[\xi-p,\xi]}\|f(t+\omega)-f(t)\|\geq\varepsilon\right\}.$$

At this level, for Banach space $(Z,\|\cdot\|_Z)$ and $(W,\|\cdot\|_W)$ we define another functional framework which will be used henceforth.

Definition 2. *We say that a function $f\in C_b(\mathbb{R}^+\times Z,W)$ is uniformly (Z,W) pseudo-$\mathcal{S}$-asymptotically ω-periodic of class p if*

$$\lim_{h\to\infty}\frac{1}{h}\int_p^h\left(\sup_{t\in[\xi-p,\xi]}\left(\sup_{\|x\|_Z\leq L}\|f(t+\omega,x)-f(t,x)\|_W\right)\right)d\xi=0,$$

for any $L>0$.

The notation $PSAP_{\omega,p}(\mathbb{R}^+\times Z,W)$ is adopted to denote the set formed by functions of this type. From the previous cited references, we also know the following:

Lemma 2. *Let $u\in C_b([-r,\infty),X)$ and*

$$u|_{\mathbb{R}^+}\in PSAP_{\omega,p}(X).$$

Then, the function $t\mapsto u_t$ belongs to $PSAP_{\omega,p}(E)$.

Lemma 3. *Let $f\in PSAP_{\omega,p}(\mathbb{R}^+\times E,X)$ and*

(1) there exists $L_f\in C_b([0;\infty);\mathbb{R}^+)$ such that for all $(t,\phi_i)\in[0;\infty)\times E$:

$$||f(t,\phi_1)-f(t,\phi_2)||\leq L_f(t)||\phi_1-\phi_2||_{C_b([-r,0],X)},$$

(2) $u\in C_b([-r,\infty),X)$,

(3) $u|_{\mathbb{R}^+}\in PSAP_{\omega,p}(X)$.

Then, the function $t\mapsto f(t,u_t)$ belongs to $PSAP_{\omega,p}(X)$.

The following result can be regarded as a generalization of Lemma 3.

Lemma 4. *Let $\alpha,\beta\in[0,1]$ and $f\in PSAP_{\omega,p}(\mathbb{R}^+\times B_\alpha,X_\beta)$. We assume that*

(1) there exists $L_f\in C_b([0;\infty);\mathbb{R}^+)$ such that for all $(t,\phi_i)\in[0;\infty)\times B_\alpha$:

$$||f(t,\phi_1)-f(t,\phi_2)||_\beta\leq L_f(t)||\phi_1-\phi_2||_{B_\alpha},$$

(2) $u\in C_b([-r,\infty),X_\alpha)$,

(3) $u|_{\mathbb{R}^+}\in PSAP_{\omega,p}(X_\alpha)$.

Then, the function $t\mapsto f(t,u_t)$ belongs to $PSAP_{\omega,p}(X_\beta)$.

Proof. First, according to Lemma 2, we can see that $t\mapsto u_t\in PSAP_{\omega,p}(B_\alpha)$. Now, for $h>0$, one has

$$
\begin{aligned}
& \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t+\omega, u_{t+\omega}) - f(t,u_t)\|_\beta \right) d\xi \\
\leq \; & \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t+\omega, u_{t+\omega}) - f(t,u_{t+\omega})\|_\beta \right) d\xi + \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t, u_{t+\omega}) - f(t,u_t)\|_\beta \right) d\xi,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t+\omega, u_{t+\omega}) - f(t,u_t)\|_\beta \right) d\xi \\
\leq \; & \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \left(\sup_{\|\phi\|_{B_\alpha}\leq L} \|f(t+\omega,\phi) - f(t,\phi)\|_\beta \right) \right) d\xi \\
& + \left\|L_f\right\|_{C_b(\mathbb{R}^+,\mathbb{R}^+)} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|u_{t+\omega} - u_t\|_{B_\alpha} \right) d\xi.
\end{aligned}
$$

As a result, we obtain

$$
\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|f(t+\omega, u_{t+\omega}) - f(t,u_t)\|_\beta \right) d\xi = 0.
$$

The proof is complete. □

In our study, we will use the following types of solutions:

Definition 3. *A function $u \in C([-r,+\infty), X_\alpha)$ is said to be an α-mild solution for problem (1) if u satisfies problem (1) and*

$$
u(t) = \varphi(t),
$$

for $\varphi \in B_\alpha$ and $t \in [-r,0]$. In this case, u is defined explicitly as follows:

$$
\begin{aligned}
u(t) = \; & U(t)(\varphi(0) - G(0,\varphi) + G(t,u_t) \\
& - \int_0^t \Big((t-s)^{q-1} AV(t-s) G(s,u_s) \Big) \, ds \\
& + \int_0^t \Big((t-s)^{q-1} V(t-s) F(s,u_s) \Big) \, ds, \quad t \geq 0.
\end{aligned}
$$

Moreover, if $u|_{\mathbb{R}_+} \in PSAP_{\omega,p}(X_\alpha)$, then $u(\cdot)$ is called pseudo $\mathcal{S}$-asymptotically ω-periodic α-mild solution of class p for problem (1).

In this study, our strategy is based on the use of two versions of the well known fixed point theorems; that is,

Theorem 2. ***(Banach Contraction Principle).*** *Let (E,d) be a complete metric space and $f : E \to E$ be contractive. Then, f has a unique fixed point u, and $f^n(y) \to u$ for each $y \in E$.*

Theorem 3. ***(Krasnoselskii's Fixed Point Theorem).*** *Let Ω be a closed convex nonempty subset of a Banach space $(X, \|\cdot\|)$. Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ map Ω into X such that*

- *$\mathcal{A}_1 x + \mathcal{A}_2 y \in \Omega$ for every pair $x, y \in \Omega$;*
- *$\mathcal{A}_1$ is continuous and $\mathcal{A}_1(\Omega)$ is contained in a compact set;*
- *$\mathcal{A}_2$ is a contraction.*

  Then, there exists $y \in \Omega$ with $\mathcal{A}_1 y + \mathcal{A}_2 y = y$.

For more details, we refer the reader to [16,17].

3. Main Results

In this section, we discuss some questions related to the existence and uniqueness of pseudo $\mathcal{S}$-asymptotically ω-periodic α-mild solutions of class p to problem (1). Our standing hypotheses are

(A1) $F \in PSAP_{\omega,p}(\mathbb{R}^+ \times B_\alpha, X)$ and $G \in PSAP_{\omega,p}(\mathbb{R}^+ \times B_\alpha, X_1)$.

(A2) There exists a function $L_G(\cdot) \in C_b(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\|AG(t,\phi_1) - AG(t,\phi_2)\| \leq L_G(t)\|\phi_1 - \phi_2\|_{B_\alpha},$$

for all $(t, \phi_i) \in \mathbb{R}^+ \times B_\alpha$.

(A3) There exists a function $L_F(\cdot) \in C_b(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\|F(t,\phi_1) - F(t,\phi_1)\| \leq L_F(t)\|\phi_1 - \phi_2\|_{B_\alpha},$$

for all $(t, \phi_i) \in \mathbb{R}^+ \times B_\alpha$.

(A4) Setting $L_G = \sup\limits_{t\in\mathbb{R}^+} L_G(t)$ and $L_F = \sup\limits_{t\in\mathbb{R}^+} L_F(t)$, we assume also that

$$\left(C_{\alpha-1} L_G + (L_G + L_F)\frac{M_\alpha \Gamma(1-\alpha)}{|v_0|^{1-\alpha}} \right) < 1,$$

with

$$C_{\alpha-1} = ||A^{\alpha-1}||.$$

Theorem 4. *Let* (A1)–(A4) *hold and let $-A$ be the generator of an exponentially stable analytic semigroup $T(t)_{(t\geq 0)}$. For $\alpha \in [0,1)$, we assume that $\varphi \in B_\alpha$, $F : \mathbb{R}^+ \times B_\alpha \to X$ and $G : \mathbb{R}^+ \times B_\alpha \to X_1$ are continuous functions. Then, problem (1) has a unique pseudo $\mathcal{S}$-asymptotic ω-periodic α-mild solution of class p.*

Proof. We consider the Banach space

$$C_{b,0}(X_\alpha) := \Big\{ x : [-r, +\infty) \to X_\alpha : x|_{[-r,0]} = 0 \text{ and } x|_{\mathbb{R}^+} \in C_b(\mathbb{R}^+, X_\alpha) \Big\},$$

endowed with the norm

$$\|x\|_{C_{b,0}} = \|x_0\|_{B_\alpha} + \sup_{t\geq 0}\|x(t)\|_\alpha = \sup_{t\geq 0}\|x(t)\|_\alpha.$$

According to Proposition 1, we define the closed subspace of $C_{b,0}(X_\alpha)$ as follows:

$$PSAP_{\omega,p,0}(X_\alpha) := \Big\{ x : [-r, +\infty) \to X_\alpha : x|_{[-r,0]} = 0 \text{ and } x|_{\mathbb{R}^+} \in PSAP_{\omega,p}(X_\alpha) \Big\}.$$

Throughout the proof, y denotes the function defined by

$$y(t) := \begin{cases} 0, & t \geq 0, \\ \varphi(t), & t \in [-r, 0]. \end{cases}$$

Let $x \in PSAP_{\omega,p,0}(X_\alpha)$. Due to the continuity of $F : \mathbb{R}^+ \times B_\alpha \to X$ and $G : \mathbb{R}^+ \times B_\alpha \to X_1$ and by taking into account assumptions (A1)–(A3) and Lemma 4, we can conclude that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|AG(t+\omega, x_{t+\omega}+y_{t+\omega}) - AG(t, x_t+y_t)\| \right) d\xi = 0 \tag{7}$$

and

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|F(t+\omega, x_{t+\omega}+y_{t+\omega}) - F(t, x_t+y_t)\| \right) d\xi = 0,$$

which means that $F \in PSAP_{\omega,p}(X)$ and $G \in PSAP_{\omega,p}(X_1)$. Further, there exist $M_F > 0$ and $M_G > 0$ such that

$$||F(t, x_t+y_t)|| \le M_F \text{ and } ||AG(t, x_t+y_t)|| \le M_G \text{ for all } t \ge 0. \tag{8}$$

Now, we need to introduce the following operator

$$\begin{array}{rccc} \mathcal{N}: & PSAP_{\omega,p,0}(X_\alpha) & \longrightarrow & PSAP_{\omega,p,0}(X_\alpha) \\ & x & \mapsto & \mathcal{N}x \end{array},$$

where

$$\begin{aligned} \mathcal{N}x(t) \;:=\; & U(t)(\varphi(0) - G(0,\varphi)) + G(t, x_t+y_t) \\ & - \int_0^t \Big((t-s)^{q-1} AV(t-s) G(s, x_s+y_s) \Big) ds \\ & + \int_0^t \Big((t-s)^{q-1} V(t-s) F(s, x_s+y_s) \Big) ds, \end{aligned}$$

with $t \in \mathbb{R}^+$. In what follows, we show that the operator $\mathcal{N}$ has a unique fixed point in $PSAP_{w,p,0}(X_\alpha)$. Firstly, we check that $\mathcal{N}$ is well defined. From Fubini's theorem and the definition of the operator V given by (4), it follows from (3), (5) and (8) that

$$\begin{aligned} & \int_0^t \Big((t-s)^{q-1} \|A^\alpha V(t-s)\| \|F(s, x_s+y_s)\| \Big) ds \\ \le\; & qM_F M_\alpha \int_0^t \left((t-s)^{q(1-\alpha)-1} \int_0^\infty \Big(\tau^{1-\alpha} \xi_q(\tau) e^{-|\nu_0|(t-s)^q \tau} \Big) d\tau \right) ds \\ \le\; & \frac{M_F M_\alpha \Gamma(1-\alpha)}{|\nu_0|^{1-\alpha}} \\ <\; & +\infty. \end{aligned}$$

Similarly,

$$\begin{aligned} & \int_0^t \Big((t-s)^{q-1} \|A^\alpha V(t-s)\| \|AG(s, x_s+y_s)\| \Big) ds \\ \le\; & qM_G M_\alpha \int_0^t \left((t-s)^{q(1-\alpha)-1} \int_0^\infty \Big(\tau^{1-\alpha} \xi_q(\tau) e^{-|\nu_0|(t-s)^q \tau} \Big) d\tau \right) ds \end{aligned}$$

$$\begin{aligned} &\leq \frac{M_G M_\alpha \Gamma(1-\alpha)}{|\nu_0|^{1-\alpha}} \\ &< +\infty, \end{aligned}$$

for every $x \in PSAP_{\omega,p,0}(X_\alpha)$. Consequently, we can see that $t \mapsto \mathcal{N}x(t)$ is a bounded function. Then, it remains to show that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|\mathcal{N}x(t+\omega) - \mathcal{N}x(t)\|_\alpha \right) d\xi = 0.$$

A direct computation allows us to obtain

$$\mathcal{N}x(t+\omega) - \mathcal{N}x(t) = \sum_{i=1}^{6} J_i(t),$$

where

$$J_1(t) = (U(t+\omega) - U(t))(\varphi(0) - G(0,\varphi)),$$

$$J_2(t) = G(t+\omega, x_{t+\omega} + y_{t+\omega}) - G(t, x_t + y_t),$$

$$J_3(t) = \int_0^\omega \Big((t+\omega-s)^{q-1} V(t+\omega-s) AG(s, x_s + y_s) \Big) ds,$$

$$J_4(t) = \int_0^t \Big((t-s)^{q-1} V(t-s) (AG(s+\omega, x_{s+\omega} + y_{s+\omega}) - AG(s, x_s + y_s)) \Big) ds,$$

$$J_5(t) = \int_0^\omega \Big((t+\omega-s)^{q-1} V(t+\omega-s) F(s, x_s + y_s) \Big) ds,$$

$$J_6(t) = \int_0^t \Big((t-s)^{q-1} V(t-s) (F(s+\omega, x_{s+\omega} + y_{s+\omega}) - F(s, x_s + y_s)) \Big) ds.$$

This implies that

$$\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|\mathcal{N}x(t+\omega) - \mathcal{N}x(t)\|_\alpha \right) d\xi \leq \sum_{i=1}^{6} \left(\int_p^h \sup_{t\in[\xi-p,\xi]} \|J_i(t)\|_\alpha \, d\xi \right).$$

Keeping in mind the exponential stability of semigroup $(T(t))_{t\geq 0}$ and the definition of the operator U given by (4), we deduce that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$, such that

$$\|U(t)\| \leq \frac{\varepsilon}{2}, \text{ for all } t \geq t_\varepsilon.$$

First, let us start with the estimation of the quantity J_1. We have

$$\|J_1(t)\|_\alpha \leq (\|U(t+\omega)\| + \|U(t)\|) \|\varphi(0) - G(0,\varphi)\|_\alpha\,,$$

so

$$\frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_1(t)\|_\alpha \right) d\xi$$

$$\leq \frac{1}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]} (\|U(t+\omega)\| + \|U(t)\|)\|\varphi(0) - G(0,\varphi)\|_\alpha \right) d\xi$$
$$\leq \|\varphi(0) - G(0,\varphi)\|_\alpha \left(\frac{2Mt_\varepsilon}{h} + \varepsilon\left(1 - \frac{(p+t_\varepsilon)}{h}\right)\right),$$

which implies that

$$\lim_{h\to+\infty} \frac{1}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_1(t)\|_\alpha \right) d\xi = 0.$$

From (7) and taking into account that $X_1 \hookrightarrow X_\alpha$, we obtain

$$\frac{1}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|G(t+\omega, x_{t+\omega} + y_{t+\omega}) - G(t, x_t + y_t)\|_\alpha \right) d\xi$$
$$\leq \frac{C_{\alpha-1}}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|AG(t+\omega, x_{t+\omega} + y_{t+\omega}) - AG(t, x_t + y_t)\| \right) d\xi;$$

hence,

$$\lim_{h\to+\infty} \frac{1}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|G(t+\omega, x_{t+\omega} + y_{t+\omega}) - G(t, x_t + y_t)\|_\alpha \right) d\xi = 0$$

and

$$t \mapsto G(t, x_t + y_t) \in PSAP_{\omega,p}(X_\alpha).$$

At this level, let us show that

$$\lim_{h\to+\infty} \frac{1}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_i(t)\|_\alpha \right) d\xi = 0,\ i = 3,4,5,6.$$

Taking into account that

$$t + \omega - s \geq \frac{t+\omega}{\omega}(\omega - s)$$

and the estimates (6) and (8), we deduce that

$$\frac{1}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_3(t)\|_\alpha \right) d\xi$$
$$\leq M_G M_\alpha \frac{\omega\Gamma(1-\alpha)}{\Gamma(q(1-\alpha)+1)} \left(\frac{1}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]} (t+\omega)^{q(1-\alpha)-1} \right) d\xi \right)$$
$$\leq M_G M_\alpha \frac{\omega\Gamma(1-\alpha)}{\Gamma(q(1-\alpha)+1)} \left(\frac{1}{h}\int_p^h (\xi - p + \omega)^{q(1-\alpha)-1}\, d\xi \right),$$

which implies that

$$\lim_{h\to+\infty} \frac{1}{h}\int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_3(t)\|_\alpha \right) d\xi = 0,$$

and

$$\lim_{h\to+\infty}\frac{1}{h}\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_5(t)\|_\alpha\right)d\xi=0.$$

In fact, it suffices to see that

$$\begin{aligned}
&\frac{1}{h}\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_5(t)\|_\alpha\right)d\xi\\
\leq\ & M_F M_\alpha\frac{\omega\Gamma(1-\alpha)}{\Gamma(q(1-\alpha)+1)}\left(\frac{1}{h}\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}(t+\omega)^{q(1-\alpha)-1}\right)d\xi\right)\\
\leq\ & M_F M_\alpha\frac{\omega\Gamma(1-\alpha)}{\Gamma(q(1-\alpha)+1)}\left(\frac{1}{h}\int\limits_{p}^{h}(\xi-p+\omega)^{q(1-\alpha)-1}d\xi\right).
\end{aligned}$$

It remains to show that

$$\lim_{h\to+\infty}\frac{1}{h}\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_i(t)\|_\alpha\right)d\xi=0,\ i=4,6.$$

We examine the term I_4. To make the notation less cluttered, we define the function $\mathcal{Q}_G$ as follows

$$\mathcal{Q}_G(t)=AG(t+\omega,x_{t+\omega}+y_{t+\omega})-AG(t,x_t+y_t),\ t\geq 0.$$

Thanks to (7), we can deduce that, for every $\varepsilon>0$, there exists $h_\varepsilon>0$ such that

$$\frac{1}{h}\int\limits_{p}^{h}\sup_{t\in[\xi-p,\xi]}\|\mathcal{Q}_G(t)\|\,d\xi\leq\varepsilon,\ \text{for all } h\geq h_\varepsilon. \tag{9}$$

Consequently,

$$\begin{aligned}
&\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_4(t)\|_\alpha\right)d\xi\\
\leq\ & \int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\int\limits_{0}^{\xi-p}\left((t-s)^{q-1}\|A^\alpha V(t-s)\|\|\mathcal{Q}_G(s)\|\right)ds\right)d\xi\\
&+\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\int\limits_{\xi-p}^{t}\left((t-s)^{q-1}\|A^\alpha V(t-s)\|\|\mathcal{Q}_G(s)\|\right)ds\right)d\xi\\
&:=\sum_{i=1}^{2}J_4^i(t).
\end{aligned}$$

Taking into account the definition of the operator V given in (4) and the estimates (3), we obtain

$$\begin{aligned}
&J_4^1(t)\\
\leq\ & qM_\alpha\int\limits_{p}^{h}\sup_{t\in[\xi-p,\xi]}\int\limits_{0}^{\xi-p}(t-s)^{q(1-\alpha)-1}\left(\int_0^\infty\tau^{1-\alpha}\xi_q(\tau)e^{-|\nu_0|(t-s)^q\tau}\,d\tau\right)\|\mathcal{Q}_G(s)\|\,dsd\xi
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad qM_\alpha \int\limits_p^h \int\limits_0^{-p} (\xi - p - s)^{q(1-\alpha)-1} \left(\int_0^\infty \tau^{1-\alpha} \xi_q(\tau) e^{-|\nu_0|(\xi-p-s)^q \tau} d\tau \right) \| \mathcal{Q}_G(s) \| \, ds \, d\xi \\
&= \quad qM_\alpha \int\limits_p^h \int\limits_p^{\xi} (\xi - s)^{q(1-\alpha)-1} \left(\int_0^\infty \tau^{1-\alpha} \xi_q(\tau) e^{-|\nu_0|(\xi-s)^q \tau} d\tau \right) \| \mathcal{Q}_G(s-p) \| ds \, d\xi.
\end{aligned}
$$

Using the classical Fubini's theorem, we obtain

$$
\begin{aligned}
&J_4^1(t) \\
&\leq \quad qM_\alpha \int\limits_p^h \| \mathcal{Q}_G(s-p) \| \left(\int\limits_s^h (\xi - s)^{q(1-\alpha)-1} \left(\int_0^\infty \tau^{1-\alpha} \xi_q(\tau) e^{-|\nu_0|(\xi-s)^q \tau} d\tau \right) d\xi \right) ds \\
&\leq \quad qM_\alpha \int\limits_p^h \| \mathcal{Q}_G(s-p) \| \left(\int\limits_0^{h-s} (\xi)^{q(1-\alpha)-1} \left(\int_0^\infty \tau^{1-\alpha} \xi_q(\tau) e^{-|\nu_0|(\xi)^q \tau} d\tau \right) d\xi \right) ds.
\end{aligned}
$$

Similarly, as before, we obtain

$$
J_4^1(t) \leq M_\alpha \frac{\Gamma(1-\alpha)}{|\nu_0|^{1-\alpha}} \int\limits_p^h \| \mathcal{Q}_G(s-p) \| \, ds;
$$

it follows from (9) and Proposition 1 that

$$
\lim_{h \to +\infty} \frac{1}{h} J_4^1(t) = 0.
$$

Assume that $h \geq 2p$; then, we obtain

$$
\begin{aligned}
J_4^2(t) \quad &= \quad \int\limits_p^{2p} \left(\sup_{t \in [\xi-p,\xi]} \int\limits_{\xi-p}^{t} \left((t-s)^{q-1} \| A^\alpha V(t-s) \| \| \mathcal{Q}_G(s) \| \right) ds \right) d\xi \\
&\quad + \int\limits_{2p}^{h} \left(\sup_{t \in [\xi-p,\xi]} \int\limits_{\xi-p}^{t} \left((t-s)^{q-1} \| A^\alpha V(t-s) \| \| \mathcal{Q}_G(s) \| \right) ds \right) d\xi \\
&:= \quad \sum_{i=1}^{2} J_4^{2,i}(t).
\end{aligned}
$$

Thanks to (8) and the estimate (6), we obtain

$$
\begin{aligned}
J_4^{2,1}(t) \quad &\leq \quad 2M_G M_\alpha \frac{\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \int\limits_p^{2p} \left(\sup_{t \in [\xi-p,\xi]} \int\limits_{\xi-p}^{t} \left((t-s)^{q(1-\alpha)-1} \right) ds \right) d\xi \\
&= \quad 2M_G M_\alpha \frac{\Gamma(1-\alpha)}{\Gamma(1+q(1-\alpha))} \int\limits_p^{2p} \left(\sup_{t \in [\xi-p,\xi]} (t - \xi + p)^{q(1-\alpha)} \right) d\xi \\
&\leq \quad 2M_G M_\alpha \frac{\Gamma(1-\alpha)}{\Gamma(1+q(1-\alpha))} \int\limits_p^{2p} p^{q(1-\alpha)} \, d\xi,
\end{aligned}
$$

and hence,

$$
\lim_{h \to +\infty} \frac{1}{h} J_4^{2,1}(t) = 0.
$$

Concerning the term $J_4^{2,2}(t)$, we have

$$\begin{aligned} J_4^{2,2}(t) &\leq M_\alpha \frac{q\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \int_{2p}^{h} \left(\sup_{t\in[\xi-p,\xi]} \int_{\xi-p}^{t} (t-s)^{q(1-\alpha)-1} \|\mathcal{Q}_G(s)\|\, ds \right) d\xi \\ &\leq M_\alpha \frac{q\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \int_{2p}^{h} \left(\int_0^p (s)^{q(1-\alpha)-1} \sup_{t\in[\xi-p,\xi]} \|\mathcal{Q}_G(t-s)\|\, ds \right) d\xi, \end{aligned}$$

for $h \geq h_\varepsilon$, and the use of Fubini theorem implies

$$\begin{aligned} &\frac{1}{h} J_4^{2,2}(t) \\ \leq\ & M_\alpha \frac{\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \int_0^p (s)^{q(1-\alpha)-1} \left(\frac{1}{h} \int_{2p}^{h} \sup_{t\in[\xi-p,\xi]} \|\mathcal{Q}_G(t-s)\|\, d\xi \right) ds \\ \leq\ & M_\alpha \frac{\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \int_0^p (s)^{q(1-\alpha)-1} \left(\frac{1}{h} \int_{p}^{h} \sup_{t\in[\xi-p,\xi]} \|\mathcal{Q}_G(t)\|\, d\xi \right) ds \\ \leq\ & \varepsilon M_\alpha \frac{\Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \int_0^p s^{q(1-\alpha)-1}\, ds \\ \to\ & 0, \text{ as } \varepsilon \to 0. \end{aligned}$$

Combining all the previous estimates, we conclude that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_4(t)\|_\alpha \right) d\xi = 0.$$

Similarly,

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_6(t)\|_\alpha \right) d\xi = 0.$$

Summing up, one can deduce that

$$\mathcal{N}(PSAP_{\omega,p,0}(X_\alpha)) \subseteq PSAP_{\omega,p,0}(X_\alpha). \tag{10}$$

Next, we will show that $\mathcal{N}$ is a contraction mapping. Let $x, z \in PSAP_{\omega,p,0}(X_\alpha)$; taking into account the assumptions (A2) and (A3), we obtain

$$\begin{aligned} &\|\mathcal{N}x(t) - \mathcal{N}z(t)\|_\alpha \\ \leq\ & C_{\alpha-1} \|AG(t, x_t + y_t) - AG(t, z_t + y_t)\|_\alpha \\ & + \int_0^t \Big((t-s)^{q-1} \|A^\alpha V(t-s)\| \|AG(s, x_s + y_s) - AG(s, z_s + y_s)\| \Big) ds, \\ & + \int_0^t \Big((t-s)^{q-1} \|A^\alpha V(t-s)\| \|F(s, x_s + y_s) - F(s, z_s + y_s)\| \Big) ds \\ \leq\ & C_{\alpha-1} L_G \|x_t - z_t\|_{B_\alpha} \\ & + (L_G + L_F) \int_0^t \Big((t-s)^{q-1} \|A^\alpha V(t-s)\| \|x_s - z_s\|_{B_\alpha} \Big) ds \end{aligned}$$

$$\leq \left(C_{\alpha-1} L_G + (L_G + L_F) \frac{M_\alpha \Gamma(1-\alpha)}{|v_0|^{1-\alpha}} \right) ||x - z||_{C_{b,0}}.$$

As a result, we confirm that

$$||\mathcal{N}x - \mathcal{N}z||_{C_{b,0}} \leq \left(C_{\alpha-1} L_G + (L_G + L_F) \frac{M_\alpha \Gamma(1-\alpha)}{|v_0|^{1-\alpha}} \right) ||x - z||_{C_{b,0}}.$$

Hence, taking into account assumption (A4), we conclude that the mapping

$$\mathcal{N} : PSAP_{w,p,0}(X_\alpha) \to PSAP_{w,p,0}(X_\alpha)$$

is a contraction. Then, it follows from the Banach contraction principle that $\mathcal{N}$ has a unique fixed point $x \in PSAP_{w,p,0}(X_\alpha)$. Set $u(t) = x(t) + y(t)$ for $t \in [-r, +\infty)$; we can confirm that u is a unique pseudo $\mathcal{S}$-asymptotic ω-periodic α-mild solution of class p of the problem (1). $\square$

In the remainder of this section, we prove the existence of the pseudo $\mathcal{S}$-asymptotic ω-periodic α-mild solution of class p for problem (1) without assuming the Lipschitz property of the function F. Our strategy is based on the use of the Krasnoselskii's fixed point theorem. In order to accomplish that, we will need the following conditions:

(A5) Let $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$ be non-negative functions that satisfy the following estimate:

$$\lim_{h \to +\infty} \frac{1}{h} \int_p^h \left(\sup_{t \in [\xi - p, \xi]} \psi_i(t) \right) d\xi = 0, \quad i = 1, 2.$$

and assume that, for every $t \in \mathbb{R}^+$ and $\phi \in B_\alpha$, there exists $\omega > 0$ such that

$$\|F(t+\omega, \phi) - F(t, \phi)\| \leq \psi_1(t), \text{ and } \|AG(t+\omega, \phi) - AG(t, \phi)\| \leq \psi_2(t).$$

(A6) There exists a function $k : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $\delta > 0$ such that for all $\phi \in B_\alpha$,

$$||F(t, \phi)|| \leq k(t), \text{ for all } t \geq 0, \tag{11}$$

and k satisfies the following estimate:

$$\lim_{h \to +\infty} \frac{1}{h} \int_\delta^h \left((\xi - p)^{q(1-\alpha)} \sup_{s \in [0, \xi]} k(s) \right) d\xi = 0. \tag{12}$$

(A7) There exists a positive constant L_G such that

$$\|AG(t, \phi_1) - AG(t, \phi_2)\| \leq L_G \|\phi_1 - \phi_2\|_{B_\alpha},$$

for all $(t, \phi_i) \in \mathbb{R}^+ \times B_\alpha$.

(A8) Assume that

$$\left(C_{\alpha-1} L_G + L_G \frac{M_\alpha \Gamma(1-\alpha)}{|v_0|^{1-\alpha}} \right) < 1.$$

and one has the following interesting result:

Theorem 5. *Assume that* (A5)–(A8) *hold and $-A$ generates a compact, exponentially stable analytic semigroup $(T(t))_{t \geq 0}$ on X. For $\alpha \in [0, 1)$, we assume that $\varphi \in B_\alpha$, $F : \mathbb{R}^+ \times B_\alpha \to X_\alpha$ is a bounded continuous function and $G : \mathbb{R}^+ \times B_\alpha \to X_1$ is a continuous function that satisfies $G(t, 0) = 0$ for $t \geq 0$. Then, the problem* (1) *has at least one pseudo $\mathcal{S}$-asymptotic ω-periodic α-mild solution of class p.*

Proof. For the sake of convenience, we will conserve the notation adopted in the proof of the Theorem 4. In the sequel, our aim is to show that

$$\mathcal{N}(PSAP_{w,p,0}(X_\alpha)) \subseteq PSAP_{w,p,0}(X_\alpha),$$

which means that for any $x \in PSAP_{w,p,0}(X_\alpha)$,

$$\begin{aligned}\mathcal{N}x \quad : \quad & t \mapsto U(t)(\varphi(0) - G(0,\varphi) + G(t, x_t + y_t) \\ & - \int_0^t \Big((t-s)^{q-1} AV(t-s)G(s, x_s + y_s)\Big) ds \\ & + \int_0^t \Big((t-s)^{q-1} V(t-s)F(s, x_s + y_s)\Big) ds, \quad t \in \mathbb{R}^+,\end{aligned}$$

belongs to the space $PSAP_{\omega,p}(X_\alpha)$. Since the function F is bounded and G is continuous and satisfies the conditions (A5) and (A7), then by the embedding $X_\alpha \hookrightarrow X$ and Lemma (4), there exist two positive constants M'_F and M'_G such that

$$||F(t, x_t + y_t)|| \leq M'_F \text{ and } ||AG(t, x_t + y_t)|| \leq M'_G \text{ for all } t > 0. \tag{13}$$

for any $x \in PSAP_{\omega,p,0}(X_\alpha)$. Furthermore, one has

$$t \mapsto G(t, x_t + y_t) \in PSAP_{w,p}(X_1). \tag{14}$$

Note that (13) guarantees the boundedness of the function $\mathcal{N}x$. Now, we look to prove that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|\mathcal{N}x(t+\omega) - \mathcal{N}x(t)\|_\alpha \right) d\xi = 0,$$

i.e.,

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_i(t)\|_\alpha \right) d\xi = 0, \ i = \{1, 2, ..., 6\}.$$

From Theorem 4, it is immediate that

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_1(t)\|_\alpha \right) d\xi = 0.$$

Exploiting (14) and the fact that $X_1 \hookrightarrow X_\alpha$, we obtain

$$\lim_{h\to+\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_2(t)\|_\alpha \right) d\xi = 0. \tag{15}$$

Taking into account (13) and Theorem 4, we confirm that

$$\lim_{h\to\infty} \frac{1}{h} \int_p^h \left(\sup_{t\in[\xi-p,\xi]} \|J_i(t)\|_\alpha \right) d\xi = 0, \quad i = 3, 5.$$

Our objective now is to show that

$$\lim_{h\to\infty}\frac{1}{h}\int_p^h\left(\sup_{t\in[\xi-p,\xi]}\|J_i(t)\|_\alpha\right)d\xi=0,\quad i=4,6.$$

First of all, for $t\geq 0$, we set

$$\mathcal{Q}_F(t)=F(t+\omega,x_{t+\omega}+y_{t+\omega})-F(t,x_{t+\omega}+y_{t+\omega}).$$

According to (A5), it is evident to say that the function $\mathcal{Q}_F$ satisfies the following estimate:

$$\lim_{h\to+\infty}\frac{1}{h}\int_p^h\sup_{t\in[\xi-p,\xi]}\|\mathcal{Q}_F(t)\|\,d\xi=0. \tag{16}$$

On other side, we observe that

$$\int_p^h\left(\sup_{t\in[\xi-p,\xi]}\|J_6(t)\|_\alpha\right)d\xi\leq\sum_{i=1}^{2}\int_p^h\left(\sup_{t\in[\xi-p,\xi]}\left\|J_6^i(t)\right\|_\alpha\right)d\xi,$$

where

$$J_6^1(t)=\int_0^t\Big((t-s)^{q-1}V(t-s)\mathcal{Q}_F(t)\Big)ds,$$

and

$$J_6^2(t)=\int_0^t\Big((t-s)^{q-1}V(t-s)(F(s,x_{s+\omega}+y_{s+\omega})-F(s,x_s+y_s))\Big)ds.$$

Due to (13), (15) and (16) with Theorem 4,

$$\lim_{h\to+\infty}\frac{1}{h}\int_p^h\left(\sup_{t\in[\xi-p,\xi]}\|J_4(t)\|_\alpha\right)d\xi=0,$$

$$\lim_{h\to+\infty}\frac{1}{h}\int_p^h\left(\sup_{t\in[\xi-p,\xi]}\left\|J_6^1(t)\right\|_\alpha\right)d\xi=0.$$

It remains to show that

$$\lim_{h\to+\infty}\frac{1}{h}\int_p^h\left(\sup_{t\in[\xi-p,\xi]}\left\|J_6^2(t)\right\|_\alpha\right)d\xi=0.$$

Taking into account the exponential stability of the semigroup $(T(t))_{t\geq 0}$ and the definition of the operator V as presented in (4), we can deduce that for every $\varepsilon>0$, there exists

$$t'_\varepsilon=\left(qC_{-\beta}\frac{M_\beta\Gamma(1-\beta)}{\Gamma(q(1-\beta))\varepsilon}\right)^{\frac{1}{\beta q}}>0\,,$$

such that $\|V(t)\|\leq\varepsilon$, for all $t\geq t'_\varepsilon$ and $\beta\in(0,1)$. In actual fact, it suffices to take into account the estimate (6) which allows us to write

$$\|V(t)x\|\leq\left\|A^{-\beta}\right\|\left\|A^\beta V(t)x\right\|\leq qC_{-\beta}\frac{M_\beta\Gamma(1-\beta)}{t^{q\beta}\Gamma(q(1-\beta))}\|x\|,$$

for any $\beta \in (0,1)$ and $x \in X$. Then, we can write

$$\begin{aligned}
&\int_p^h \sup_{t\in[\xi-p,\xi]} J_6^2(t)\, d\xi \\
= &\int_p^{t'_\varepsilon+p} \sup_{t\in[\xi-p,\xi]} \int_0^t \Big((t-s)^{q-1} V(t-s)(F(s,x_{s+\omega}+y_{s+\omega}) - F(s,x_s+y_s))\Big) ds d\xi \\
&+ \int_{t'_\varepsilon+p}^h \sup_{t\in[\xi-p,\xi]} \int_0^t \Big((t-s)^{q-1} V(t-s)(F(s,x_{s+\omega}+y_{s+\omega}) - F(s,x_s+y_s))\Big) ds d\xi.
\end{aligned}$$

Thanks to (13) and the estimate (6), we conclude that

$$\begin{aligned}
&\frac{1}{h}\int_p^{t'_\varepsilon+p} \left(\sup_{t\in[\xi-p,\xi]} \int_0^t \Big((t-s)^{q-1} \|A^\alpha V(t-s)\| \|(F(s,x_{s+\omega}+y_{s+\omega}) - F(s,x_s+y_s))\|\Big) ds \right) d\xi \\
\leq\ &\frac{2M'_F M_\alpha \Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \left(\frac{1}{h} \int_p^{t'_\varepsilon+p} \xi^{q(1-\alpha)} d\xi \right) \to 0, \quad \text{as } h \to +\infty.
\end{aligned}$$

Keeping in mind that the function

$$s \mapsto g(s) = (s + (\xi - p))^q - s^q,$$

is decreasing for $s \geq 0$, we obtain $g(0) \geq g(p)$ *i. e.*, $\xi^q - (\xi-p)^q \leq p^q$. Hence,

$$\begin{aligned}
&\frac{1}{h}\int_{t'_\varepsilon+p}^h \left(\sup_{t\in[\xi-p,\xi]} \int_0^{t-(\xi-p)} \Big((t-s)^{q-1} \|V(t-s)\| \|A^\alpha F(s,x_{s+\omega}+y_{s+\omega}) - A^\alpha F(s,x_s+y_s)\|\Big) ds \right) d\xi \\
\leq\ &\frac{2M'_F \varepsilon}{qC_{-\alpha}} \left(\frac{1}{h} \int_{t'_\varepsilon+p}^h (\xi^q - (\xi-p)^q) d\xi \right) \leq \frac{2M'_F \varepsilon}{qC_{-\alpha}} p^q \to 0, \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

At this level, the use of (11) justify the fact that

$$||F(t, x_s + y_s)|| \leq k(t).$$

Therefore, by (12), one can find

$$\begin{aligned}
&\frac{1}{h}\int_{t'_\varepsilon+p}^h \sup_{t\in[\xi-p,\xi]} \int_{t-(\xi-p)}^t \Big((t-s)^{q-1} \|A^\alpha V(t-s)\| \|F(s,x_{s+\omega}+y_{s+\omega}) - F(s,x_s+y_s)\|\Big) ds d\xi \\
\leq\ &\frac{2M_\alpha \Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \left(\frac{1}{h}\int_{t'_\varepsilon+p}^h \left(\sup_{t\in[\xi-p,\xi]} \int_{t-(\xi-p)}^t \Big((t-s)^{q(1-\alpha)-1} k(s)\Big) ds \right) d\xi \right) \\
\leq\ &\frac{2M_\alpha \Gamma(1-\alpha)}{\Gamma(q(1-\alpha))} \left(\frac{1}{h}\int_{t'_\varepsilon+p}^h \left((\xi-p)^{q(1-\alpha)} \sup_{s\in[0,\xi]} k(s) \right) d\xi \right),
\end{aligned}$$

then

$$\lim_{h\to+\infty}\left(\frac{1}{h}\int\limits_{t'_\varepsilon+p}^{h}\sup_{t\in[\xi-p,\xi]}\int\limits_{t-(\xi-p)}^{t}\Big((t-s)^{q-1}\|A^\alpha V(t-s)\|\|F(s,x_{s+\omega}+y_{s+\omega})-F(s,x_s+y_s)\|\Big)dsd\xi\right)$$
$$=0.$$

Thus,

$$\lim_{h\to\infty}\frac{1}{h}\int\limits_{p}^{h}\left(\sup_{t\in[\xi-p,\xi]}\|J_6(t)\|_\alpha\right)d\xi=0.$$

Summing up, the above results for J_i, $i\in\{1,2,\dots,6\}$, we conclude that

$$\mathcal{N}x\in PSAP_{\omega,p}(X_\alpha);$$

which justifies the following inclusion, that is,

$$\mathcal{N}(PSAP_{\omega,p,0}(X_\alpha))\subseteq PSAP_{\omega,p,0}(X_\alpha).$$

We are now in a position to show that the operator $\mathcal{N}$ has at least one fixed point $x\in PSAP_{\omega,p,0}(X_\alpha)$. For $R>0$, we define the closed ball of $PSAP_{\omega,p,0}(X_\alpha)$ with center 0 and radius R by

$$\Omega_R=\Big\{x\in PSAP_{\omega,p,0}(X_\alpha):||x||_{C_{b,0}}\le R\Big\}.$$

Set $\mathcal{N}=\mathcal{N}_1+\mathcal{N}_2$ with

$$\mathcal{N}_1x(t):=U(t)\varphi(0)+\int\limits_0^t\big((t-s)^{q-1}V(t-s)F(s,x_s+y_s)\big)ds,$$
$$\mathcal{N}_2x(t):=U(t)(G(0,\varphi))+G(t,x_t+y_t)-\int\limits_0^t\big((t-s)^{q-1}AV(t-s)G(s,x_s+y_s)\big)ds.$$

We first prove that there exists a positive constant R_0 such that $\mathcal{N}_1x+\mathcal{N}_2z\in\Omega_{R_0}$, for every pair $x,z\in\Omega_{R_0}$. For this purpose, we assume that for any $R>0$, there exist $x,z\in\Omega_R$ and $t\ge 0$ such that

$$\begin{aligned}R&\le\|\mathcal{N}_1x(t)+\mathcal{N}_2z(t)\|_\alpha\\&\le\|U(t)\|\|\varphi(0)\|_\alpha+\int\limits_0^t\Big((t-s)^{q-1}\|A^\alpha V(t-s)\|\|F(s,x_s+y_s)\|\Big)ds\\&\quad+\|U(t)\|\|G(0,\varphi)\|_\alpha+\|G(t,z_s+y_s)\|_\alpha+\int\limits_0^t\Big((t-s)^{q-1}\|A^\alpha V(t-s)\|\|AG(s,z_s+y_s)\|\Big)ds.\end{aligned}$$

Hence, we have

$$\begin{aligned}R&\le M\|\varphi\|_{B_\alpha}+M'_F\int\limits_0^t\Big((t-s)^{q-1}\|A^\alpha V(t-s)\|\Big)ds\\&\quad+MC_{-\alpha}L_G\|\varphi\|_{B_\alpha}+C_{\alpha-1}L_G\|z_s+y_s\|_{B_\alpha}+L_G\|z_s+y_s\|_{B_\alpha}\int\limits_0^t\Big((t-s)^{q-1}\|A^\alpha V(t-s)\|\Big)ds,\end{aligned}$$

which implies

$$\begin{aligned} R \quad \leq \quad & M\|\varphi\|_{B_\alpha} + M'_F M_\alpha \frac{\Gamma(1-\alpha)}{|\nu_0|^{1-\alpha}} \\ & + MC_{-\alpha} L_G \|\varphi\|_{B_\alpha} + C_{\alpha-1} L_G \left(R + \|\varphi\|_{B_\alpha}\right) + L_G \left(R + \|\varphi\|_{B_\alpha}\right) \frac{M_\alpha \Gamma(1-\alpha)}{|\nu_0|^{1-\alpha}}. \end{aligned}$$

Dividing both sides by R and taking the limit as R approaches infinity, we obtain

$$1 \leq C_{\alpha-1} L_G + L_G \frac{M_\alpha \Gamma(1-\alpha)}{|\nu_0|^{1-\alpha}}.$$

Combining all the above arguments, we can deduce that there exists a positive constant R_0, such that for any pair of $x, z \in \Omega_{R_0}$, one has $\mathcal{N}_1 x + \mathcal{N}_2 z \in \Omega_{R_0}$.

Now, let us show that the function $\mathcal{N}_1$ is compact and the function $\mathcal{N}_2$ is contraction. To accomplish that, we should perform several steps as follows:

Step 1. We show that the function $\mathcal{N}_1$ is continuous on Ω_{R_0}. In fact, due the continuity of the function F, for any sequence $(x^n) \in \Omega_{R_0}$ such that $x^n \to x$ on Ω_{R_0}, one can see

$$\|F(s, x^n_s + y_s) - F(s, x_s + y_s)\| \to 0, \text{ as } n \to +\infty.$$

Then, by the dominate convergence theorem, we can conclude that

$$\begin{aligned} \|\mathcal{N}_1 x^n(t) - \mathcal{N}_1 x(t)\|_\alpha \quad \leq \quad & \int_0^t \Big((t-s)^{q-1} \|A^\alpha V(t-s)\| \|F(s, x^n_s + y_s) - F(s, x_s + y_s)\| \Big) ds \\ \to \quad & 0, \text{ as } n \to +\infty. \end{aligned}$$

Step 2. Following [3], for $t \geq 0$, we define

$$\mathcal{N}_1^{\varepsilon,\delta} x(t) := U(t)\varphi(0) + q \int_0^t \left((t-s)^{q-1} \int_\delta^\infty (\tau \xi_q(\tau) T(t^q \tau) F(s, x_s + y_s)) d\tau \right) ds.$$

The compactness of the operator $T(t)$ and Lemma 1 implies that the set $\mathcal{N}_1^{\varepsilon,\delta}(\Omega_{R_0})(t)$ is relatively compact in X_α. Moreover, it follows from (3) and (13) that

$$\begin{aligned} & \left\| \mathcal{N}_1 x(t) - \mathcal{N}_1^{\varepsilon,\delta} x(t) \right\|_\alpha \\ \leq \quad & q \int_0^t \left((t-s)^{q-1} \int_0^\delta \big(\tau \xi_q(\tau) \|A^\alpha T(t^q \tau)\| \|F(s, x_s + y_s)\|\big) d\tau \right) ds \\ & + q \int_{t-\varepsilon}^t \left((t-s)^{q-1} \int_\delta^\infty \big(\tau \xi_q(\tau) \|A^\alpha T(t^q \tau)\| \|F(s, x_s + y_s)\|\big) d\tau \right) ds \\ \leq \quad & q M_\alpha M'_F \left[\int_0^\delta \left(\tau^{1-\alpha} \xi_q(\tau) \int_0^t \Big((t-s)^{q(1-\alpha)-1} e^{-|\nu_0|(t-s)^q \tau} \Big) ds \right) d\tau \right. \\ & \left. + \int_{t-\varepsilon}^t \Big((t-s)^{q(1-\alpha)-1} \Big) ds \int_0^\infty \Big(\tau^{1-\alpha} \xi_q(\tau) \Big) d\tau \right] \\ \leq \quad & M_\alpha M'_F \frac{\Gamma(1-\alpha)}{|\nu_0|^{1-\alpha}} \left[\int_0^\delta \xi_q(\tau) d\tau + \frac{1}{q(1-\alpha)} \varepsilon^{q(1-\alpha)} \right], \end{aligned}$$

in other words,

$$\lim_{\varepsilon,\delta\to 0}\left\|\mathcal{N}_1 x(t)-\mathcal{N}_1^{\varepsilon,\delta}x(t)\right\|_\alpha=0.$$

Consequently, the set $\mathcal{N}_1(\Omega_{R_0})(t)$ is relatively compact in X_α.

Step 3. Let $t_1>t_2\geq 0$ and $x\in\Omega_{R_0}$. Observe that, from Lemma 2.9 in [14], we deduce that

$$\|A^\alpha V(t_1-s)-A^\alpha V(t_2-s)\|\to 0,\ \text{as}\ t_1\to t_2$$

and

$$\|U(t_1)-U(t_2)\|\to 0,\ \text{as}\ t_1\to t_2.$$

On other side, one has

$$\begin{aligned}
&\int_0^{t_2}\left(\frac{(t_2-s)^{q-1}-(t_1-s)^{q-1}}{(t_2-s)^{\alpha q}}\right)ds\\
=&\int_0^{t_2}\left((t_2-s)^{q(1-\alpha)-1}-(t_1-s)^{q(1-\alpha)-1}\left(\frac{t_1-s}{t_2-s}\right)^{\alpha q}\right)ds\\
\leq&\int_0^{t_2}\left((t_2-s)^{q(1-\alpha)-1}-(t_1-s)^{q(1-\alpha)-1}\right)ds\\
\to&\ 0,\ \text{as}\ t_1\to t_2.
\end{aligned}$$

This gives

$$\begin{aligned}
&\|\mathcal{N}_1x(t_1)-\mathcal{N}_1x(t_2)\|_\alpha\\
\leq&\ \|U(t_1)-U(t_2)\|\|A^\alpha\varphi(0)-A^\alpha G(0,\varphi)\|\\
&+\int_0^{t_2}\left((t_1-s)^{q-1}\|A^\alpha V(t_1-s)-A^\alpha V(t_2-s)\|\|F(s,x_s+y_s)\|\right)ds\\
&+\int_0^{t_2}\left(\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]\|A^\alpha V(t_2-s)\|\|F(s,x_s+y_s)\|\right)ds\\
&+\int_{t_2}^{t_1}\left((t_1-s)^{q-1}\|A^\alpha V(t_1-s)\|\|F(s,x_s+y_s)\|\right)ds\\
\leq&\ \|U(t_1)-U(t_2)\|(\|A^\alpha\varphi(0)\|+C_{\alpha-1}\|AG(0,\varphi)\|)\\
&+M_F'\left[\int_0^{t_2}\left((t_1-s)^{q-1}\|A^\alpha V(t_1-s)-A^\alpha V(t_2-s)\|\right)ds\right.\\
&\left.+M_\alpha\int_0^{t_2}\left(\frac{(t_2-s)^{q-1}-(t_1-s)^{q-1}}{(t_2-s)^{\alpha q}}\right)ds+M_\alpha\int_{t_2}^{t_1}\left((t_1-s)^{q(1-\alpha)-1}\right)ds\right]
\end{aligned}$$

Then,

$$\lim_{t_1\to t_2}\|\mathcal{N}_1x(t_1)-\mathcal{N}_1x(t_2)\|_\alpha=0,$$

which means that $\mathcal{N}_1(\Omega_{R_0})$ is equicontinuous. Combining the above steps, the Arzela-Ascoli theorem guarantees that $\mathcal{N}_1$ is a compact operator on Ω_{R_0}.

Step 4. What is left is to show that $\mathcal{N}_2$ is a contraction. Let $x,z\in\Omega_{R_0}$, for $t\geq 0$; one has

$$\|\mathcal{N}_2x(t)-\mathcal{N}_2z(t)\|_\alpha$$

$$\leq \quad \|G(t, x_t + y_t) - G(t, z_t + y_t)\|_\alpha$$
$$+ \int_0^t \Big((t-s)^{q-1} \|A^\alpha V(t-s)\| \|AG(s, x_s + y_s) - AG(s, z_s + y_s)\| \Big) ds,$$

then

$$\|\mathcal{N}_2 x(t) - \mathcal{N}_2 z(t)\|_\alpha \leq \left(C_{\alpha-1} L_G + L_G \frac{M_\alpha \Gamma(1-\alpha)}{|\nu_0|^{1-\alpha}} \right) \|x - z\|_{C_{b,0}},$$

it follows from (A8) that $\mathcal{N}_2$ is contraction. Finally, by applying Theorem 3, we conclude that the operator $\mathcal{N}$ has at least one fixed point $x \in \Omega_{R_0} \subset PSAP_{\omega,p,0}(X_\alpha)$. Hence, we can affirm that $u = x + y$ is the pseudo $\mathcal{S}$-asymptotically ω-periodic α-mild solution of class p for problem (1). $\square$

4. Applications

In this section, we present a concrete example to apply of our abstract theoretical results. Of concern is the following delayed partial differential: equation

$$\begin{cases} \dfrac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}} \left(u(t,\xi) - k_2(t) \int\limits_{t-r}^{t} \left(\int_a^\xi b_2(s-t) u(s,\eta)\, d\eta \right) ds \right) - \dfrac{\partial^2}{\partial \xi^2} u(t,\xi) \\ \qquad = k_1(t) \int_{t-r}^t b_1(s-t) u(s,\xi)\, ds, \ \xi \in [0,\pi], \ t \in \mathbb{R}^+, \\ u(t,0) = u(t,\pi) = 0, \ \ t \in \mathbb{R}^+, \\ u(\tau,\xi) = \varphi(\tau)(\xi), \ \ \tau \in [-r,0], \xi \in [0,\pi], \end{cases} \tag{17}$$

where $\partial^{\frac{1}{2}} / \partial t^{\frac{1}{2}}$ is a Caputo fractional partial derivative of order $1/2$, $r > 0$ is a constant, $\varphi \in C\big([-r,0], L^2([0,\pi])\big)$ and $b_1(\cdot), b_2(\cdot) \in C([-r,0], \mathbb{R})$ and $k_1(.), k_2(.)$ are suitable functions. Let $X := L^2([0,\pi])$ and $A : D(A) \subseteq X \to X$ is the operator defined by

$$\begin{cases} Au := -u'', \\ D(A) = \{u \in X | -u'' \in X, \ u(0) = u(\pi) = 0\}. \end{cases}$$

Remark 1. *Most of the useful spectral properties of this operator can be founded in Section 5 in [18] and Example 5.1 in [19]. For the reader's convenience, we recall that*

- *A has a discrete spectrum with eigenvalues n^2, $n \in \mathbb{N}$. Furthermore,*
- *A generates an exponentially stable analytic semigroup $(T(t))_{(t \geq 0)}$ defined by*

$$T(t)u := \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, e_n \rangle e_n \ \text{ and } \ \|T(t)\| \leq e^{-t},$$

where $\{e_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of X and $e_n(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(n\xi)$ are the associated normalized eigenvectors .

- *The operator $A^{1/2}$ is well defined and can be characterized as follows*

$$\begin{cases} (A)^{\frac{1}{2}} u := \sum_{n=1}^\infty n \langle u, e_n \rangle e_n, \\ (A)^{-\frac{1}{2}} u := \sum_{n=1}^\infty \dfrac{1}{n} \langle u, e_n \rangle e_n, \\ D(A^{1/2}) := \{u \in X : \sum_{n=1}^\infty n \langle u, e_n \rangle e_n \in X\}. \end{cases}$$

- *For $u \in D(A^{1/2})$: $\|u\|_{\frac{1}{2}} = \|u'\|$.*

Let us introduce the following functions $F : \mathbb{R}^+ \times B_{\frac{1}{2}} \to X$ and $G : \mathbb{R}^+ \times B_{\frac{1}{2}} \to X_1$ as follows

$$\begin{cases} F(t,\phi)(\xi) = k_1(t) \int_{-r}^{0} b_1(s)\phi(s,\xi)\, ds, \\ \text{and} \\ G(t,\phi)(\xi) = k_2(t) \int_{-r}^{0} \int_{a}^{\xi} b_2(s)\phi(s,\eta)\, d\eta\, ds. \end{cases}$$

According to Theorem 4, we have the following result

Proposition 2. *Suppose that the functions $k_1,\ k_2$ belong to $PSAP_{w,p}(\mathbb{R})$ and*

$$(1+\pi)\left(\int_{-r}^{0}|b_1(s)|^2 ds\right)^{\frac{1}{2}}\|k_1\|_{C_b(\mathbb{R})} + \pi\left(\int_{-r}^{0}|b_2(s)|^2 ds\right)^{\frac{1}{2}}\|k_2\|_{C_b(\mathbb{R})} < r^{-\frac{1}{2}}. \tag{18}$$

Then, the problem (17) has a unique pseudo $\mathcal{S}$-asymptotic ω-periodic α-mild solution of class p.

Proof. Note that, for $t \geq 0$ and $\phi \in B_{\frac{1}{2}}$, one has

$$\begin{aligned} |F(t,\phi)(\xi)|^2 &\leq |k_1(t)|^2\left(\int_{-r}^{0}|b_1(s)||\phi(s,\xi)|ds\right)^2 \\ &\leq |k_1(t)|^2\int_{-r}^{0}|b_1(s)|^2 ds\int_{-r}^{0}|\phi(s,\xi)|^2 ds. \end{aligned}$$

Using the Fubini theorem, we have

$$\begin{aligned} \|F(t,\phi)\|^2 &\leq |k_1(t)|^2\int_{-r}^{0}|b_1(s)|^2 ds\int_{-r}^{0}\|\phi(s,\cdot)\|^2_{L^2([0,\pi])} ds \\ &\leq r|k_1(t)|^2\int_{-r}^{0}|b_1(s)|^2 ds \sup_{s\in[-r,0]}\|\phi(s,\cdot)\|^2_{L^2([0,\pi])}. \end{aligned}$$

Furthermore,

$$\|F(t,\phi)\| \leq r^{\frac{1}{2}}|k_1(t)|\left(\int_{-r}^{0}|b_1(s)|^2 ds\right)^{\frac{1}{2}}\|\phi\|_{B_{\frac{1}{2}}}$$

and

$$\begin{aligned} &\frac{1}{h}\int_{p}^{h}\sup_{t\in[\xi-p,\xi]}\ \sup_{\|\phi\|_{B_{\frac{1}{2}}}\leq L}\|F(t+\omega,\phi)-F(t,\phi)\|d\xi \\ \leq\ & Lr^{\frac{1}{2}}\left(\int_{-r}^{0}|b_1(s)|^2 ds\right)^{\frac{1}{2}}\left(\frac{1}{h}\int_{p}^{h}\sup_{t\in[\xi-p,\xi]}|k_1(t+\omega)-k_1(t)|d\xi\right), \end{aligned}$$

which implies that

$$\lim_{h\to+\infty}\frac{1}{h}\int_{p}^{h}\sup_{t\in[\xi-p,\xi]}\ \sup_{\|\phi\|_{B_{\frac{1}{2}}}\leq L}\|F(t+\omega,\phi)-F(t,\phi)\|d\xi = 0$$

and

$$F \in PSAP_{\omega,p}(\mathbb{R}^+ \times B_{\frac{1}{2}}, X). \tag{19}$$

Moreover, we can easily see that

$$\|F(t,\phi_1)-F(t,\phi_2)\| \leq r^{\frac{1}{2}}\|k_1\|_{C_b(\mathbb{R}^+,\mathbb{R}^+)}\left(\int_{-r}^{0}|b_1(s)|^2 ds\right)^{\frac{1}{2}}\|\phi_1-\phi_2\|_{B_{\frac{1}{2}}}, \tag{20}$$

for any $\phi_1,\ \phi_2 \in B_{\frac{1}{2}}$. Similarly, one has

$$\begin{aligned}\left|\frac{\partial^2}{\partial\xi^2}G(t,\phi)(\xi)\right|^2 &= |k_2(t)|^2\left|\int_{-r}^{0} b_2(s)\frac{\partial^2}{\partial\xi^2}\int_a^{\xi}\phi(s,\eta)\,d\eta\,ds\right|^2 \\ &\leq |k_2(t)|^2\left|\int_{-r}^{0} b_2(s)\frac{\partial}{\partial\xi}\phi(s,\xi)ds\right|^2 \\ &\leq |k_2(t)|^2\int_{-r}^{0}|b_2(s)|^2\,ds\int_{-r}^{0}\left|\frac{\partial}{\partial\xi}\phi(s,\xi)\right|^2 ds.\end{aligned}$$

This yields

$$\begin{aligned}\left\|\frac{\partial^2}{\partial\xi^2}G(t,\phi)\right\| &\leq r^{\frac{1}{2}}|k_2(t)|\left(\int_{-r}^{0}|b_2(s)|^2 ds\right)^{\frac{1}{2}}\sup_{s\in[-r,0]}\|\phi'(s,\cdot)\|_{L^2([0,\pi])} \\ &= r^{\frac{1}{2}}|k_2(t)|\left(\int_{-r}^{0}|b_2(s)|^2 ds\right)^{\frac{1}{2}}\|\phi\|_{B_{\frac{1}{2}}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{h}\int_p^h \sup_{t\in[\xi-p,\xi]}\ \sup_{\|\phi\|_{B_{\frac{1}{2}}}\leq L}\left\|\frac{\partial^2}{\partial\xi^2}G(t+\omega,\phi)-\frac{\partial^2}{\partial\xi^2}G(t,\phi)\right\|d\xi \\ \leq\ & Lr^{\frac{1}{2}}\left(\int_{-r}^{0}|b_2(s)|^2 ds\right)^{\frac{1}{2}}\left(\frac{1}{h}\int_p^h \sup_{t\in[\xi-p,\xi]}|k_2(t+\omega)-k_2(t)|d\xi\right),\end{aligned}$$

which means that

$$\lim_{h\to+\infty}\frac{1}{h}\int_p^h \sup_{t\in[\xi-p,\xi]}\ \sup_{\|\phi\|_{B_{\frac{1}{2}}}\leq L}\left\|\frac{\partial^2}{\partial\xi^2}G(t+\omega,\phi)-\frac{\partial^2}{\partial\xi^2}G(t,\phi)\right\|d\xi = 0$$

and consequently,

$$G \in PSAP_{\omega,p}(\mathbb{R}^+\times B_{\frac{1}{2}}, X_1). \tag{21}$$

On other side, we have

$$\begin{aligned}&\left\|\tfrac{\partial^2}{\partial\xi^2}G(t,\phi_1)-\tfrac{\partial^2}{\partial\xi^2}G(t,\phi_2)\right\| \\ \leq\ & r^{\frac{1}{2}}\|k_2\|_{C_b(\mathbb{R}^+,\mathbb{R}^+)}\left(\textstyle\int_{-r}^{0}|b_2(s)|^2 ds\right)^{\frac{1}{2}}\|\phi_1-\phi_2\|_{B_{\frac{1}{2}}},\end{aligned} \tag{22}$$

for any $\phi_1,\ \phi_2 \in B_{\frac{1}{2}}$. Observe that, from (19)–(22), we can deduce that the condition (A1), (A2) and (A3) from Section 3 hold. It is immediate that (18) implies that the condition (A4) holds with $\|A^{-1/2}\| = 1$, $M_{\frac{1}{2}} = \Gamma(1/2) = \sqrt{\pi}$ and $\nu_0 = -1$. By Theorem 4, we conclude that the problem (17) has a unique pseudo $\mathcal{S}$-asymptotical ω-periodic α-mild solution of class p. □

5. Conclusions

In this paper, we furnish some sufficient conditions for the existence of pseudo $\mathcal{S}$-asymptotically periodic mild solutions for the following abstract fractional Cauchy problem:

$$\begin{cases} {}^{c}D_t^q\big(u(t) - G(t, u_t)\big) + Au(t) = F\big(t, u_t\big), & t \geq 0, \\ u(t) = \varphi(t), & -r \leq t \leq 0, \end{cases}$$

where the fractional derivative ${}^{c}D_t^q$, $q \in (0,1)$ is taken in the sense of Caputo approach and $(A, D(A))$ is a closed linear operator in a Banach space $(X, \|\cdot\|)$. An important example is given to demonstrate the abstract results obtained in our work. We hope that our new results will have important applications in nonlinear analysis, mathematical physics, mechanics, biology and related fields in the future.

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