

Article

Representations of Generalized Loop Planar Galilean Conformal Algebras $W(\Gamma)$

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Abstract: In this article, we study the representation of generalized loop planar Galilean conformal algebra $W(\Gamma)$; we will determine the Harish-Chandra modules, Verma modules, Whittaker modules, and $U(h)$ -free modules of $W(\Gamma)$.

Keywords: generalized loop planar Galilean conformal algebra; Harish-Chandra modules; Verma modules; Whittaker modules; free $U(h)$ -modules

MSC: 17B10; 17B35; 17B65; 17B68

1. Introduction

Galilean conformal algebra has been studied in the non-relativistic limit of the ADS/CFT conjecture [1]. The finite-dimensional Galilean conformal algebra is associated with certain non-semisimple Lie algebra, which is regarded as a non-relativistic analogue of conformal algebras. It was found that the finite Galilean conformal algebra could be given an infinite-dimensional lift for all space-time dimensions (see [1–3]). The planar Galilean conformal algebra was first introduced by Bahturin and Gopakumar in reference [1]. In Section 2 of reference [4], the Galilean conformal algebra of finite dimension in the d spatial dimension was introduced. For a given d , Galilean conformal algebra is labelled by a half-integer l . The infinite dimension $d = 2, l = 1$ is planar Galilean conformal algebra.

The generalized planar Galilean conformal algebra is an infinite-dimensional Lie algebra, which is generated by $\{L_m, H_m, I_m, J_m | m \in \Gamma\}$ over $\mathbb{F}$, where $\mathbb{F}$ is an algebraic closed field of characteristic 0, and Γ is a proper additive subgroup of $\mathbb{F}$, satisfying the following relations:

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n}, & [L_m, H_n] &= -nH_{m+n}, \\ [L_m, I_n] &= (m - n)I_{m+n}, & [L_m, J_n] &= (m - n)J_{m+n}, \\ [H_m, I_n] &= J_{m+n}, & [H_m, J_n] &= -I_{m+n}, \\ [H_m, H_n] &= [I_m, I_n] = [I_m, J_n] = [J_m, J_n] = 0, \end{aligned}$$

for $m, n \in \Gamma$. The center of W is $\{0\}$.

The generalized loop planar Galilean conformal algebra $W(\Gamma)$ is the tensor product $W \otimes \mathbb{F}[t^{\pm 1}]$ of the generalized planar Galilean conformal algebra W with the Laurent polynomial algebra $\mathbb{F}[t^{\pm 1}]$, which is generated by $\{L_{m,i}, H_{m,i}, I_{m,i}, J_{m,i} | m \in \Gamma, i \in \mathbb{Z}\}$ over $\mathbb{F}$, subject to the following bracket relations:

$$\begin{aligned} [L_{m,i}, L_{n,j}] &= (m - n)L_{m+n,i+j}, & [L_{m,i}, H_{n,j}] &= -nH_{m+n,i+j}, \\ [L_{m,i}, I_{n,j}] &= (m - n)I_{m+n,i+j}, & [L_{m,i}, J_{n,j}] &= (m - n)J_{m+n,i+j}, \\ [H_{m,i}, I_{n,j}] &= J_{m+n,i+j}, & [H_{m,i}, J_{n,j}] &= -I_{m+n,i+j}, \\ [H_{m,i}, H_{n,j}] &= [I_{m,i}, I_{n,j}] = [I_{m,i}, J_{n,j}] = [J_{m,i}, J_{n,j}] = 0, \end{aligned}$$



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where $X_{m,i} = X_m \otimes t^i$ for any $X \in \{L, H, I, J\}$, $m, n \in \Gamma$ and $i, j \in \mathbb{Z}$. We see that the center of W and $W(\Gamma)$ are all $\{0\}$. We have studied the structures of $W(\Gamma)$ in reference [5].

The representation theory of Lie algebra is important in many infinite dimensional algebras. The properties of modules in Lie algebras are the main topics of the representation theory of Lie algebras. The simple weight modules of the Virasoro algebra have been studied in reference [6]. Mathieu classified all simple Harish-Chandra modules over the Virasoro algebra. Recently, some researchers have investigated the Harish-Chandra modules of many important Lie algebras and Lie super algebras [7–9], such as twisted Heisenberg–Virasoro algebra [8], Schrödinger–Virasoro algebra [9] and deformative twisted Schrödinger–Virasoro Lie algebra [9]. In references [10–17], the Verma modules of some Lie algebras and super algebras have been determined. The Verma modules of generalized Virasoro algebras were determined in reference [18]. From reference [12], we know the classification of the Verma modules of the $W(2, 2)$. However, the Harish-Chandra modules and Verma modules of the generalized loop planar Galilean conformal algebra have not been studied; we will solve these problems in this article.

Recently, the non-weight modules of Lie (super)algebras have attracted much attention from mathematicians. In particular, the Whittaker modules and $U(h)$ -free modules have been widely studied for many Lie algebras. The Whittaker modules for many other Lie algebras have been investigated, such as in reference [19–27]. The Whittaker modules for the affine Lie algebra $A_1^{(1)}$ were determined in reference [24]. The Whittaker modules of super-Virasoro algebras were studied in reference [19]. The notation of $U(h)$ -free modules was first introduced by Nilsson [28]. Tan and Zhao showed that any free $U(h)$ -modules of rank 1 over the Witt algebra is isomorphism to $\Omega(\lambda, \alpha)$ for some $\lambda \in \mathbb{F}^*$, $\alpha \in \mathbb{F}$ [29]. Later, $U(h)$ -free modules for many infinite-dimensional Lie algebras were determined, for example, for Heisenberg–Virasoro algebras [30], the algebras $Vir(a, b)$ [31], Loop–Virasoro algebras and Block type algebras [32]. In this paper, we aim to study the free $U(h)$ -module over generalized loop planar Galilean conformal algebras.

The representations of the planar Galilean conformal algebra was determined in reference [33]; Gao studied the simple restricted modules over W , including the highest weight modules and Whittaker modules. However, the representations of $W(\Gamma)$ have not been studied up until now, hence we will study the representations of $W(\Gamma)$. The present paper is organized as follows. In Section 2, we will determine the Verma modules of $W(\Gamma)$. In Section 3, we will calculate the Harish–Chandra modules of $W(\Gamma)$. In Sections 4 and 5, we will give some conclusions for the Whittaker modules of $W(\Gamma)$. In Section 6, we will determine the rank 1 free $U(\mathbb{F}L_{0,0})$ -modules of $W(\Gamma)$. The main results of the paper are stated in Theorems 1, 3–5 and 7.

2. Verma Modules of $W(\Gamma)$

First, we fix a total order $<$ on Γ which is compatible with the addition, $x \preceq y$ implies $x + z \preceq y + z$ for any $z \in \Gamma$ [34]. We write $x < y$ if $x \preceq y$ and $x \neq y$. Assume that

$$\Gamma^+ = \{x \in \Gamma | x \succ 0\}, \quad \Gamma^- = \{x \in \Gamma | x < 0\}.$$

Then, we have $\Gamma = \Gamma^+ \cup \{0\} \cup \Gamma^-$, and $W(\Gamma)$ has a triangular decomposition

$$W(\Gamma) = W(\Gamma)^- W(\Gamma)_0 W(\Gamma)^+,$$

where

$$W(\Gamma)^- = \bigoplus_{m < 0} \mathbb{F}L_{m,i} \oplus \bigoplus_{m < 0} \mathbb{F}H_{m,i} \oplus \bigoplus_{m < 0} \mathbb{F}I_{m,i} \oplus \bigoplus_{m < 0} \mathbb{F}J_{m,i},$$

$$W(\Gamma)_0 = L_{0,i} \oplus H_{0,i} \oplus I_{0,i} \oplus J_{0,i},$$

$$W(\Gamma)^+ = \bigoplus_{m > 0} \mathbb{F}L_{m,i} \oplus \bigoplus_{m > 0} \mathbb{F}H_{m,i} \oplus \bigoplus_{m > 0} \mathbb{F}I_{m,i} \oplus \bigoplus_{m > 0} \mathbb{F}J_{m,i}.$$

The universal enveloping algebra of $W(\Gamma)$ is given by

$$U(W(\Gamma)) = U(W(\Gamma)^-)U(W(\Gamma)_0)U(W(\Gamma)^+).$$

Let $a, b, c, d \in \mathbb{F}$, V_h is a 1-dimensional vector space over $\mathbb{F}$ spanned by v_h and $V_h = \mathbb{F}v_h$. View V_h as a $W(\Gamma)_0$ -module, such that $L_{0,i}v_h = av_h$, $I_{0,i}v_h = bv_h$, $J_{0,i}v_h = cv_h$, $H_{0,i}v_h = dv_h$ for all $i \in \mathbb{Z}$. Then, V_h is a $\mathbb{B} = W(\Gamma)^+ \oplus W(\Gamma)_0$ -module by setting $W(\Gamma)^+v_h = 0$.

Definition 1. The induced module $V(a, b, c, d) = \text{Ind}_{\mathbb{B}}^{W(\Gamma)} V_h = U(W(\Gamma)) \otimes_{U(\mathbb{B})} V_h$ is called a Verma module of $W(\Gamma)$ with the highest weight (a, b, c, d) .

Let $U = U(W(\Gamma))$ and $I(a, b, c, d)$ be the left ideal of U , which is generated by the elements in $\{L_{m,i}, H_{m,i}, I_{m,i}, J_{m,i} | m \in \Gamma^+\} \cup \{L_{0,i} - a, I_{0,i} - b, J_{0,i} - c, H_{0,i} - d\}$ for any $a, b, c, d \in \mathbb{F}$. We define $V(a, b, c, d) = U/I(a, b, c, d)$ as the Verma module with the highest weight (a, b, c, d) for $W(\Gamma)$, and get a basis of $V(a, b, c, d)$ consisting of all vectors of the form

$$v_h, L_{-\alpha_1, i_1} \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} v_h,$$

with

$$\begin{aligned} 0 \preceq (\alpha_1, i_1) \preceq \cdots \preceq (\alpha_k, i_k), \quad 0 \preceq (\Gamma_1, p_1) \preceq \cdots \preceq (\Gamma_t, p_t), \\ 0 \preceq (l_1, q_1) \preceq \cdots \preceq (l_u, q_u), \quad 0 \preceq (\beta_1, j_1) \preceq \cdots \preceq (\beta_s, j_s), \end{aligned}$$

where $\alpha_1, \dots, \alpha_k, \Gamma_1, \dots, \Gamma_t, l_1, \dots, l_u, \beta_1, \dots, \beta_s \in \Gamma$ and $i_1, \dots, i_k, j_1, \dots, j_s, p_1, \dots, p_t, q_1, \dots, q_n \in \mathbb{Z}$. (For any $\alpha, \beta \in \Gamma, i, j \in \mathbb{Z}$, $(\alpha, i) \preceq (\beta, j) \iff \alpha \preceq \beta$ or $\alpha = \beta$ and $i \preceq j$). If $L_{0,i}v = \lambda v$ for all $i \in \mathbb{Z}$, then we call a vector $v \in V(a, b, c, d)$ a weight vector with weight λ .

Lemma 1. If $V(a, b, c, d)$ is a weight module of $W(\Gamma)$ and $V(a, b, c, d) = \bigoplus_{\lambda \in a - W(\Gamma)^+} V_\lambda$, we obtain

$$\begin{aligned} V_\lambda = \text{span}_{\mathbb{F}} \{ L_{-\alpha_1, i_1} \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} \mid \sum_{i=1}^k \alpha_i \\ + \sum_{j=1}^s \beta_j + \sum_{p=1}^t \Gamma_p + \sum_{q=1}^u l_q = \lambda - a \} \end{aligned}$$

which is the weight vector with weight λ .

Proof. If v_h is the maximal weight vector with a , which satisfies $L_{0,0}v_h = av_h$. Then, we have

$$\begin{aligned} & L_{0,0} \cdot L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &= [L_{0,0}, L_{-\alpha_1,i_1}] \cdots [L_{0,0}, L_{-\alpha_k,i_k}] I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &+ \cdots L_{-\alpha_1,i_1} \cdots [L_{0,0}, L_{-\alpha_k,i_k}] I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &+ L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} [L_{0,0}, I_{-\Gamma_1,p_1}] \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &+ L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots [L_{0,0}, I_{-\Gamma_t,p_t}] J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &+ L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} [L_{0,0}, J_{-l_1,q_1}] \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &+ L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots [L_{0,0}, J_{-l_u,q_u}] H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &+ L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} [L_{0,0}, H_{-\beta_1,j_1}] \cdots H_{-\beta_s,j_s} v_h \\ &+ L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots [L_{0,0}, H_{-\beta_s,j_s}] v_h \\ &+ L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} L_{0,0} v_h \\ &= \left(\sum_{i=1}^k \alpha_i + \sum_{j=1}^s \beta_j + \sum_{p=1}^t \Gamma_i + \sum_{q=1}^u l_i + a \right) L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} \\ &J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &= \lambda L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h. \end{aligned}$$

So, we have

$$\sum_{i=1}^k \alpha_i + \sum_{j=1}^s \beta_j + \sum_{p=1}^t \Gamma_i + \sum_{q=1}^u l_i = \lambda - a.$$

□

For $x \in V(a, b, c, d)$, we set

$$x = \sum A_{\alpha,i,\Gamma,p,l,q,\beta,j} L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h,$$

where $A_{\alpha,i,\Gamma,p,l,q,\beta,j} \in \mathbb{F}$, $\alpha = (\alpha_1, \dots, \alpha_k)$, $i = (i_1, i_2, \dots, i_k)$, $\Gamma = (\Gamma_1, \dots, \Gamma_t)$, $p = (p_1, \dots, p_t)$, $l = (l_1, \dots, l_u)$, $q = (q_1, \dots, q_u)$, $\beta = (\beta_1, \dots, \beta_s)$, $j = (j_1, \dots, j_s)$ with only finite terms $A_{\alpha,i,\Gamma,p,l,q,\beta,j} \neq 0$.

Definition 2. We define

$$A_w = \{(\alpha, i) | A_{\alpha,i,\beta,j,r,p,l,q} \neq 0 \text{ for some } \beta, j, r, p, l, q\}.$$

$$l = \max\{k | \alpha = (\alpha_1, \dots, \alpha_k), i = (i_1, \dots, i_k) \in A_w\},$$

where $l = \text{len}(w)$, l is the length of w . We define

$$V_k = \text{span}_{\mathbb{F}}\{w | \text{len}(w) \leq k\}.$$

If $k \leq -1$, then $V_k = 0$.

The following theorem is our main result in this section.

Theorem 1.

(1) The Verma module $V(a, b, c, d)$ is an irreducible $W(\Gamma)$ module if $(b, c) \neq (0, 0)$.

(2) If $b = c = 0$, when $a = d = 0$, the Verma module $V(0, 0, 0, 0)$ contains a unique maximal submodule $N(0, 0, 0, 0)$, which is generated by $\{L_{m,i}v_h, H_{m,i}v_h, I_{m,i}v_h, J_{m,i}v_h | m \in \Gamma^+, i \in \mathbb{Z}\}$. If $a \neq 0$ and $d = 0$, $N(a, 0, 0, 0)$ is generated by $\{H_{m,i}v_h, I_{m,i}v_h, J_{m,i}v_h | m \in \Gamma^+, i \in \mathbb{Z}\}$. If $a = 0$ and $d \neq 0$, $N(0, 0, 0, d)$ is generated by $\{L_{m,i}v_h, I_{m,i}v_h, J_{m,i}v_h | m \in \Gamma^+, i \in \mathbb{Z}\}$. If $a \neq 0$ and $d \neq 0$, $N(a, 0, 0, d)$ is generated by $\{I_{m,i}v_h, J_{m,i}v_h | m \in \Gamma^+, i \in \mathbb{Z}\}$.

Proof. (1) Let us assume that w_0 is any nonzero weight vector in $V(a, b, c, d)$, the submodule of the weight module is the weight module. We will prove $v_h \in U(W(\Gamma))w_0$. Suppose

$$\begin{aligned} w_0 &= \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} v_h \\ &\equiv \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots L_{-\alpha_l, i_l} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} \\ &\quad H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} v_h \pmod{V_{l-1}}, \end{aligned}$$

where

$$\alpha = (\alpha_1, \dots, \alpha_k), i = (i_1, \dots, i_k), \beta = (\beta_1, \dots, \beta_s), j = (j_1, \dots, j_s),$$

$$r = (r_1, \dots, r_t), p = (p_1, \dots, p_t), l = (l_1, \dots, l_u), q = (q_1, \dots, q_u)$$

with only finite terms $A_{\alpha, i, \beta, j, \Gamma, p, l, q} \neq 0$.

Case 1: $b \neq 0, c \neq 0$.

We define

$$\alpha_{(0)} = \max\{\alpha_l | \alpha = (\alpha_1, \dots, \alpha_l), A_{\alpha, i, \beta, j, \Gamma, p, l, q} \neq 0\},$$

$$\beta_{(0)} = \max\{\beta_s | \beta = (\beta_1, \dots, \beta_s), A_{\alpha, i, \beta, j, \Gamma, p, l, q} \neq 0\}.$$

We only prove the case of $\alpha_{(0)} > \beta_{(0)}$, and the case of $\beta_{(0)} > \alpha_{(0)}$ is similar.

Let

$$\begin{aligned} w_1 &= I_{\alpha_{(0)}, k} w_0 \\ &= \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} I_{\alpha_{(0)}, k} L_{-\alpha_1, i_1} \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} v_h \\ &= \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots [I_{\alpha_{(0)}, k}, L_{-\alpha_m, i_m}] \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} \\ &\quad J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} v_h + \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} \\ &\quad J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots [I_{\alpha_{(0)}, k}, H_{-\beta_n, j_n}] \cdots H_{-\beta_s, j_s} v_h \\ &\quad + \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} I_{\alpha_{(0)}, k} v_h. \end{aligned}$$

Due to $\alpha_{(0)} > 0$, we have $I_{\alpha_{(0)}, k} v_h = 0$. Then, we have

$$\begin{aligned} &\sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} \\ &\quad H_{-\beta_1, j_1} \cdots [I_{\alpha_{(0)}, k}, H_{-\beta_n, j_n}] \cdots H_{-\beta_s, j_s} v_h \\ &= - \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} \\ &\quad H_{-\beta_1, j_1} \cdots \widehat{H_{-\beta_n, j_n}} \cdots H_{-\beta_n, j_n} J_{\alpha_{(0)} - \beta_n, k + j_n} v_h. \end{aligned}$$

By the definition of $\alpha_{(0)}$, it is easy to know that $J_{\alpha_{(0)} - \beta_n, k + j_n} v_h = 0$. Therefore, we obtain

$$\begin{aligned} w_1 &= \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots [I_{\alpha_{(0)}, k}, L_{-\alpha_m, i_m}] \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} \\ &\quad J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} v_h \\ &= (\alpha_{(0)} + \alpha_m) \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots \widehat{L_{-\alpha_m, i_m}} I_{\alpha_{(0)} - \alpha_m, k + i_m} \cdots L_{-\alpha_k, i_k} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} \\ &\quad J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} v_h \\ &\equiv (\alpha_{(0)} + \alpha_m) \sum A_{\alpha, i, \beta, j, \Gamma, p, l, q} L_{-\alpha_1, i_1} \cdots L_{-\alpha_{p-1}, i_{p-1}} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} \\ &\quad H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} v_h \pmod{V_{p-2}}. \end{aligned}$$

Repeating the process above, we have

$$w_2 = \sum A_{\beta, j, \Gamma, p, l, q} I_{-\Gamma_1, p_1} \cdots I_{-\Gamma_t, p_t} J_{-l_1, q_1} \cdots J_{-l_u, q_u} H_{-\beta_1, j_1} \cdots H_{-\beta_s, j_s} v_h.$$

Then, we apply $I_{\beta_{(0)},q}$ to w_2 , we have

$$\begin{aligned} w_3 &= I_{\beta_{(0)},q} w_2 \\ &= \sum A_{\beta,j,\Gamma,p,l,q} I_{\beta_{(0)},q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &\equiv \sum A_{\beta,j,\Gamma,p,l,q} I_{\beta_{(0)},q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta'_1,j'_1} \cdots H_{-\beta'_{s-1},j'_{s-1}} v_h \pmod{V_{s-2}}. \end{aligned}$$

Repeating the process, we have

$$w_4 = \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h.$$

Then, we define

$$\Gamma_{(0)} = \max\{\Gamma_t | \alpha = (\Gamma_1, \dots, \Gamma_t), a_{\beta,j,\Gamma,p,l,q} \neq 0\},$$

$$l_{(0)} = \max\{l_u | \Gamma = (l_1, \dots, l_u), a_{\beta,j,\Gamma,p,l,q} \neq 0\}.$$

We only prove the case of $\Gamma_{(0)} > l_{(0)}$, and the case of $l_{(0)} > \Gamma_{(0)}$ is similar.

We apply $H_{\Gamma_{(0)},\eta}$ (or $L_{\Gamma_{(0)},\eta}$) to w_4

$$\begin{aligned} w_5 &= H_{\Gamma_{(0)},\eta} w_4 = \sum A_{\Gamma,p,l,q} H_{\Gamma_{(0)},\eta} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \\ &= \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots [H_{\Gamma_{(0)},\eta}, I_{-\Gamma_m,p_m}] \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \\ &\quad + \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots [H_{\Gamma_{(0)},\eta}, J_{-l_n,q_n}] \cdots J_{-l_u,q_u} v_h \\ &\quad + \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{\Gamma_{(0)},\eta} v_h, \end{aligned}$$

where

$$\begin{aligned} &\sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots [H_{\Gamma_{(0)},\eta}, J_{-l_n,q_n}] \cdots J_{-l_u,q_u} v_h \\ &= - \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots \widehat{J_{-l_n,q_n}} \cdots J_{-l_u,q_u} I_{\Gamma_{(0)}-l_n,\eta+q_n} v_h. \end{aligned}$$

Due to $H_{\Gamma_{(0)},\eta} v_h = I_{\Gamma_{(0)}-l_n,\eta+q_n} v_h = 0$, we have

$$\begin{aligned} w_5 &= \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots [H_{\Gamma_{(0)},\eta}, I_{-\Gamma_m,p_m}] \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \\ &\equiv \sum A_{\Gamma',p',l,q} I_{-\Gamma'_1,p'_1} \cdots I_{-\Gamma'_{t-1},p'_{t-1}} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \pmod{V_{t-2}}. \end{aligned}$$

Since $c \neq 0$, we have $w_5 \neq 0$. Repeating the process above, we have $w_6 = \sum A_{l,q} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h$.

At last, we apply $H_{l_{(0)},s}$ to w_6 . Due to $l_{(0)} = \max\{l_u | \Gamma = (l_1, \dots, l_u), a_{\beta,j,\Gamma,p,l,q} \neq 0\}$, we assume that $l_{(0)} = l_p$, where $l_p \in (l_1, \dots, l_u)$. Then, we obtain that

$$\begin{aligned} w_7 &= H_{l_{(0)},s} w_6 = \sum a_{l,q} H_{l_{(0)},s} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \\ &= \sum A_{l,q} J_{-l_1,q_1} \cdots [H_{l_{(0)},s}, J_{-l_p,q_p}] \cdots J_{-l_u,q_u} v_h \\ &\equiv A'_{l',q'} J_{-l'_1,q'_1} \cdots J_{-l'_{u-1},q'_{u-1}} v_h \pmod{V_{u-2}}. \end{aligned}$$

Since $b \neq 0$, repeating the process, we deduce that

$$0 \neq w_8 = Abv_h \in U(W(\Gamma))w_0$$

Therefore, $v_h \in U(W(\Gamma))w_0$. In this case, $V(a, b, c, d)$ is irreducible.

Case 2: $b \neq 0, c = 0$ (the case of $b = 0, c \neq 0$ is similar)

$$w_9 = \sum A_{\alpha,i,\beta,j,\Gamma,p,l,q} L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h.$$

If $\alpha_{(0)} > \beta_{(0)}$, we apply $I_{\alpha_{(0)},k}$ to w_9 . Using the result in Case 1, we can obtain

$$w_{10} = \sum A_{\beta,j,\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h.$$

Next, applying $J_{\beta_{(0)},k}$ to w_{10} , we assume that $\beta_{(0)} = \beta_m$, where $\beta_m \in (\beta_1, \dots, \beta_u)$. Then, we have

$$\begin{aligned} w_{11} &= J_{\beta_{(0)},k} w_{10} \\ &= \sum A_{\Gamma,p,l,q,\beta,j} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots [J_{\beta_0,k}, H_{-\beta_m,j_m}] \cdots H_{-\beta_s,j_s} v_h \\ &= \sum A_{\Gamma,p,l,q,\beta',j'} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} I_{0,k+j_m} v_h \\ &\equiv \sum A_{\Gamma,p,l,q,\beta',j'} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta'_1,j'_1} \cdots H_{-\beta'_{s-1},j'_{s-1}} v_h \pmod{V_{s-2}}. \end{aligned}$$

Since $b \neq 0$, repeating the process, we have

$$\begin{aligned} w_{12} &= \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \\ &\equiv \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \pmod{V_{t-1}}. \end{aligned}$$

By the definition of $r_{(0)}$ and $l_{(0)}$ in case 1, applying $L_{r_{(0)},\mu}$ to w , we get

$$\begin{aligned} w_{13} &= L_{r_{(0)},\mu} w_{12} = \sum A_{\Gamma,p,l,q} L_{r_{(0)},\mu} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \\ &= \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots [L_{r_{(0)},\mu}, I_{-\Gamma_\xi,p_\xi}] \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \\ &\quad + \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots [L_{r_{(0)},\mu}, J_{-\Gamma_\tau,p_\tau}] J_{-l_u,q_u} v_h \\ &\quad + \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} L_{r_{(0)},\mu} v_h. \end{aligned}$$

Similar with case 1, we deduce that $I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots [L_{r_{(0)},\mu}, J_{-\Gamma_\tau,p_\tau}] \cdots J_{-l_u,q_u} v_h = 0$ and $L_{r_{(0)},\mu} v_h = 0$. Therefore, we have

$$w_{13} \equiv -(\Gamma_\xi + r_0) \sum A_{\Gamma,p,l,q} I_{-\Gamma_1,p_1} \cdots \widehat{I_{-\Gamma_\xi,p_\xi}} \cdots I_{-\Gamma_t,p_t} I_{r_0-\Gamma_\xi,\mu+p_\xi} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h \pmod{V_{t-2}}.$$

Repeating the process above, we have

$$w_{14} = \sum A_{l,q} J_{-l_1,q_1} \cdots J_{-l_u,q_u} v_h.$$

Finally, we apply $H_{l_{(0)},\sigma}$ to w_{14} , then it is easy to prove that $v_h \in U(W(\Gamma))u_0$. Therefore, if $b \neq 0$ and $c = 0$, $V_h \in U(W(\Gamma))w^0$ is also irreducible.

(2) Case 3: $b = c = 0$.

If $a = d = 0$, by the definition of $N(0,0,0,0)$, we know that all the basis elements of $V(0,0,0,0)$ except v_h are clearly in $N(0,0,0,0)$. We can show that $v_h \notin N(0,0,0,0)$. For any weight vector $v \in N(0,0,0,0)$, suppose that weight of v is λ . For any basis element

$$L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s}$$

of $U(W(\Gamma))$, such that $\sum_{i=1}^k \alpha_i + \sum_{i=1}^s \beta_i + \sum_{i=1}^t \Gamma_i + \sum_{i=1}^u l_i = -\lambda$, we have

$$\begin{aligned} &L_{-\alpha_1,i_1} \cdots L_{-\alpha_k,i_k} I_{-\Gamma_1,p_1} \cdots I_{-\Gamma_t,p_t} J_{-l_1,q_1} \cdots J_{-l_u,q_u} H_{-\beta_1,j_1} \cdots H_{-\beta_s,j_s} v_h \\ &= (aL_{0,0} + bI_{0,0} + cJ_{0,0} + dH_{0,0})v_h = 0 \end{aligned}$$

for $a, b, c, d \in \mathbb{F}$, which follows that $v_h \notin N(0,0,0,0)$.

If $a \neq 0$ and $d = 0$, then $U(W(\Gamma))^- v_h \notin N(a,0,0,0)$, $W(\Gamma)^- = \bigoplus_{\mu < 0} \mathbb{F} L_{\mu,i}$, which means that $N(a,0,0,0)$ is a proper submodule of $V(a,0,0,0)$. Suppose that V is any submodule of $V(a,0,0,0)$, such that $N(a,0,0,0) \subsetneq V$, then there exists $i_1 \cdots i_r \in W(\Gamma)^+$,

$r \in \mathbb{N}$, such that $L_{-i_1, k_1} \cdots L_{-i_r, k_r} v_h \in V$. If $r = 0$, $v_h \in V$ and $V = V(a, 0, 0, 0)$, suppose $r \geq 1$, we denote $i = i_1 + \cdots + i_r$, then

$$L_{i, k} L_{-i_1, k_1} \cdots L_{-i_r, k_r} v_h = (i + i_1)(i - i_1 + i_2) \cdots (i - i_1 - i_2 - \cdots - i_{r-1} + i_r) a v_h \in V.$$

Since $(i + i_1)(i - i_1 + i_2) \cdots (i - i_1 - i_2 - \cdots - i_{r-1} + i_r) a \neq 0$, we have $v_h \in V$ and $V = V(a, 0, 0, 0)$. Therefore, $N(a, 0, 0, 0)$ is the unique maximal submodule of $V = V(a, 0, 0, 0)$.

The proof of case of $a = 0$ and $d \neq 0$ is similar.

Finally, we can easily prove that if $a \neq 0$ and $d \neq 0$, $N(a, 0, 0, d)$ is generated by $\{I_{m, i} v_h, J_{m, i} v_h | m \in \Gamma^+, i \in \mathbb{Z}\}$. We have omitted some details. $\square$

3. Harish-Chandra Modules of $W(\Gamma)$

Definition 3. A $W(\Gamma)$ -module V is called a weight module if V is the sum of all its weight spaces

$$V_\lambda = \{v \in V | L_{0,0} v = \lambda v\},$$

where $\lambda \in \mathbb{F}$. For a weight module V , we define $\text{supp} V = \{\lambda \in \mathbb{F} | V_\lambda \neq 0\}$, which is called the weight set of V . An irreducible weight $W(\Gamma)$ -module V is called the intermediate series module if all its weight spaces are one dimensional. A weight module is called a uniformly bounded module if there exists $\mathbb{N} > 0$, such that $\dim V_\lambda \leq \mathbb{N}$ for all $\lambda \in \text{supp} V$. In addition, if all weight spaces V_λ of a weight $W(\Gamma)$ -module V are finitely dimensional, the module V is called the Harish-Chandra module.

Definition 4. A weight module $W(\Gamma)$ -module V is called a highest (resp. lowest) weight module with highest weight (resp. lowest weight) λ , if there exists a nonzero weight vector $v \in V_\lambda$, such that V is generated by v as $W(\Gamma)$ -module, and

$$W(\Gamma)_+ v = 0 \quad (\text{resp. } W(\Gamma)_- v = 0).$$

We define the evaluation module of $W(\Gamma)$ similarly to that in Section 2 of reference [35] and Section 5 of reference [36].

Definition 5. A $W(\Gamma)$ -module V is called a evaluation module, if for any $X \in \{L, H, I, J\}$, $m \in \Gamma$, $i \in \mathbb{Z}$, $0 \neq e \in \mathbb{F}$ and $v \in V$, we have

$$X_{m, i} v = e^i X_m v$$

Theorem 2. Theorem 5.7 of reference [36]. If V is an irreducible Harish-Chandra module over $W(\Gamma)$, then V is a highest weight or lowest weight module or evaluation module.

Next, we determine the intermediate series modules of $W(\Gamma)$. By Definition 3, we know that intermediate series modules are special Harish-Chandra modules, we assume that $V = \bigoplus_{k \in \Gamma} V_k$, where V_k is spanned by v_k . According to the definition of intermediate series modules, we have $\dim V_k \leq 1$. We also assume that

$$\begin{aligned} L_{m, i} v_k &= e^i L_m v_k = e^i a_{m, k} v_{m+k}, & H_{m, i} v_k &= e^i H_m v_k = e^i b_{m, k} v_{m+k}, \\ I_{m, i} v_k &= e^i I_m v_k = e^i c_{m, k} v_{m+k}, & J_{m, i} v_k &= e^i J_m v_k = e^i d_{m, k} v_{m+k}, \end{aligned} \quad (1)$$

for any $m, k \in \Gamma$, $i \in \mathbb{Z}$, $0 \neq e \in \mathbb{F}$.

Then, we obtain that

Lemma 2.

$$c_{m, k} = d_{m, k} = 0, \quad (2)$$

for any $m, k \in \Gamma$.

Proof. According to the definition of $W(\Gamma)$, we obtain several equations as follows:

$$\begin{aligned} [L_{m,i}, I_{n,j}]v_k &= (m-n)e^{i+j}c_{m+n,k}v_{m+n+k}, & [L_{m,i}, J_{n,j}]v_k &= (m-n)e^{i+j}d_{m+n,k}v_{m+n+k}, \\ [I_{m,i}, I_{n,j}]v_k &= [J_{m,i}, J_{n,j}]v_k = 0. \end{aligned}$$

Using (1), we have

$$c_{n,k}a_{m,n+k} - a_{m,k}c_{n,m+k} = (m-n)c_{m+n,k}, \quad (3)$$

$$d_{n,k}a_{m,n+k} - a_{m,k}d_{n,m+k} = (m-n)d_{m+n,k}, \quad (4)$$

$$c_{n,k}c_{m,n+k} = c_{m,k}c_{n,m+k}, \quad (5)$$

$$d_{n,k}d_{m,n+k} = d_{m,k}d_{n,m+k}. \quad (6)$$

Using (6) with $n = 0$, we have

$$d_{m,k}(d_{0,m+k} - d_{0,k}) = 0. \quad (7)$$

Using (4) with $n = 0$, we have

$$-a_{m,k}(d_{0,m+k} - d_{0,k}) = md_{m,k}$$

$$-a_{m,k}d_{m,k}(d_{0,m+k} - d_{0,k}) = m(d_{m,k})^2 \quad (8)$$

Using (7) and (8), we get $m(d_{m,k})^2 = 0$. So, we have

$$d_{m,k} = 0, \quad \forall k \in \Gamma, \quad m \neq 0. \quad (9)$$

On the other hand, we can deduce that $d_{0,k} = 0$ for any $k \in \Gamma$ using Lemma 1.2 of reference [37]. Therefore, we have $d_{m,k} = 0$ for any $m, k \in \Gamma$. Similarly, using (3) and (5), we get $c_{m,k} = 0$ for any $m, k \in \Gamma$. $\square$

From what we have discussed above, we obtain the intermediate series modules of $W(\Gamma)$ as follows:

Theorem 3. Intermediate series module V over $W(\Gamma)$ must be one of $A_{a,b,c}(e)$, $A_{a,d}(e)$, $B_{a,d}(e)$, $U_d(e)$, $V_d(e)$, $\widetilde{U}_d(e)$ and $\widetilde{V}_c(e)$, or their subquotient module:

- (1) $A_{a,b,c}(e) : L_{m,i}v_k = e^i(a+k+bm)v_{m+k}, \quad H_{m,i}v_k = ce^i v_{m+k},$
 $I_{m,i}v_k = J_{m,i}v_k = 0.$
- (2) $A_{a,d}(e) : L_{m,i}v_k = e^i(k+m)v_{m+k}(k \neq 0),$
 $L_{m,i}v_0 = e^i m(m+a)v_m, H_{m,i}v_k = 0(k \neq 0),$
 $H_{m,i}v_0 = e^i dm v_m, I_{m,i}v_k = J_{m,i}v_k = 0.$
- (3) $B_{a,d}(e) : L_{m,i}v_k = e^i k v_{m+k}(k+m \neq 0),$
 $L_{m,i}v_{-m} = e^i m(m+a)v_0, H_{m,i}v_k = 0(k+m \neq 0),$
 $H_{m,i}v_{-m} = e^i dm v_0, I_{m,i}v_k = J_{m,i}v_k = 0.$
- (4) $U_d(e) : L_{m,i}v_k = e^i k v_{m+k}, H_{m,i}v_k = 0(k+n \neq 0),$
 $H_{m,i}v_{-m} = e^i dm v_0, I_{m,i}v_k = J_{m,i}v_k = 0.$
- (5) $V_d(e) : L_{m,i}v_k = e^i(m+k)v_{m+k}, H_{m,i}v_k = 0(k \neq 0),$
 $H_{m,i}v_0 = e^i dm v_m, I_{m,i}v_k = J_{m,i}v_k = 0.$
- (6) $\widetilde{U}_d(e) : L_{m,i}v_k = e^i k v_{m+k}(k+n \neq 0), L_{m,i}v_{-m} = 0, H_{m,i}v_k = 0(m+k \neq 0),$
 $H_{m,i}v_{-m} = e^i dm v_0, I_{m,i}v_k = J_{m,i}v_k = 0.$
- (7) $\widetilde{V}_c(e) : L_{m,i}v_k = e^i k v_{m+k}(m+k \neq 0), L_{m,i}v_{-m} = 0, H_{m,i}v_k = 0,$
 $H_{m,i}v_0 = e^i c v_m(m \neq 0), I_{m,i}v_k = J_{m,i}v_k = 0(k \neq 0 \text{ or } n = k = 0).$

For any $a, b, c, d, m, k \in \mathbb{F}$. $i \in \mathbb{Z}$.

Proof. Using Lemma 2, Theorem 3.2 of reference [8] and Theorem 5.1 of reference [7], the result can be checked. $\square$

4. Whittaker Modules of $W(\Gamma)$

First, we fix a total order $\prec$ on Γ which is compatible with an addition. So, $x \preceq y$ implies $x+z \preceq y+z$ for any $z \in \Gamma$ [34]. We write $x \prec y$ if $x \preceq y$ and $x \neq y$. Assume that

$$\Gamma^+ = \{x \in \Gamma | x \succ 0\}, \Gamma^- = \{x \in \Gamma | x \prec 0\}.$$

Then, we have $\Gamma = \Gamma^+ \cup \{0\} \cup \Gamma^-$, we get the triangular decomposition of $W(\Gamma)$,

$$W(\Gamma) = W(\Gamma)^- \oplus W(\Gamma)^0 \oplus W(\Gamma)^+,$$

where

$$W(\Gamma)^- = \bigoplus_{m \prec 0} \mathbb{F} L_{m,i} \oplus \bigoplus_{m \prec 0} \mathbb{F} H_{m,i} \oplus \bigoplus_{m \prec 0} \mathbb{F} I_{m,i} \oplus \bigoplus_{m \prec 0} \mathbb{F} J_{m,i},$$

$$W(\Gamma)^0 = L_{0,i} \oplus H_{0,i} \oplus I_{0,i} \oplus J_{0,i},$$

$$W(\Gamma)^+ = \bigoplus_{m \succ 0} \mathbb{F} L_{m,i} \oplus \bigoplus_{m \succ 0} \mathbb{F} H_{m,i} \oplus \bigoplus_{m \succ 0} \mathbb{F} I_{m,i} \oplus \bigoplus_{m \succ 0} \mathbb{F} J_{m,i}.$$

According to the definition of $W(\Gamma)$, we deduce that $W(\Gamma)^+$ is generated by $L_{1,i}, L_{2,j}, H_{1,k}, I_{1,l}, J_{1,q}$, where $i, j, k, l, q \in \mathbb{Z}$.

Definition 6. We define a partition u to be non-decreasing to the sequence of positive integers $u = (u_1, \dots, u_r)$, where $0 < u_1 \leq \dots \leq u_r$, and we also define a pseudopartition λ to be non-

decreasing to the sequence of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_s)$ where $0 \leq \lambda_1 \leq \dots \leq \lambda_s$. Let P represent the set of partitions, and $\tilde{P}$ denote the set of pseudopartition. Then, it is obvious that $P \subseteq \tilde{P}$. For $\lambda \in \tilde{P}$, we also write that $\lambda = (0^{\lambda(0)}, 1^{\lambda(1)}, \dots)$, where $\lambda(k)$ is the number of times of k appears in the pseudopartition and $\lambda(k) = 0$ for k sufficiently large. Then, a pseudopartition λ is a partition when $\lambda(0) = 0$. For $u = (u_1, u_2, \dots, u_r) \in \tilde{P}$, $\lambda = (\lambda_1, \dots, \lambda_s) \in \tilde{P}$, $v = (v_1, \dots, v_t) \in \tilde{P}$, $l = (l_1, \dots, l_k) \in \tilde{P}$, $i = (i_1, i_2, \dots, i_r) \in \mathbb{Z}^r$, $j = (j_1, \dots, j_s) \in \mathbb{Z}^s$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_t) \in \mathbb{Z}^t$, $q = (q_1, \dots, q_k) \in \mathbb{Z}^k$, we define

$$\begin{aligned} |u| &= u_1 + \dots + u_r, |\lambda| = \lambda_1 + \dots + \lambda_s, \\ |v| &= v_1 + \dots + v_t, |l| = l_1 + \dots + l_k, \\ L_{-u,i} &= L_{-u_r,i_r} \cdots L_{-u_1,i_1}, H_{-\lambda,j} = H_{-\lambda_s,j_s} \cdots H_{-\lambda_1,j_1}, \\ I_{-v,\varepsilon} &= I_{-v_t,\varepsilon_t} \cdots I_{-v_1,\varepsilon_1}, J_{-l,q} = J_{-l_k,q_k} \cdots J_{-l_1,q_1}. \end{aligned}$$

Definition 7. Let V be a $W(\Gamma)$ -module and $\varphi : W(\Gamma)^+ \rightarrow \mathbb{F}$ be a Lie algebra homomorphism, a vector $v \in V$ is called a Whittaker vector if $xv = \varphi(x)v$ for any $x \in W(\Gamma)^+$. The module V is said to be a Whittaker module of type φ if it is generated by a Whittaker vector of type φ . The Lie algebra homomorphism φ is called nonsingular, which means that it takes nonzero values on the $W(\Gamma)^+$. The Lie brackets in the definition of $W(\Gamma)$ force $\varphi(L_{i,p}) = \varphi(H_{j,w}) = \varphi(I_{m,s}) = \varphi(J_{l,t}) = 0$ for any $i \geq 3, j, m, l \geq 2$ and $p, w, s, t \in \mathbb{Z}$. We say that $W(\Gamma)^+$ acts on V locally and nilpotently if for any $v \in V$ there is $s \in \mathbb{N}$ depending on v , such that $x_1 x_2 \cdots x_s v = 0$ for any $x_1, x_2, \dots, x_s \in W(\Gamma)^+$.

Let $W(\Gamma)_\varphi^+ = \{x - \varphi(x) | x \in W(\Gamma)^+\}$, then we have the following result.

Lemma 3. Lemma 3.1 of reference [38] and Lemma 2.1 of reference [39]. Let V be a Whittaker $W(\Gamma)$ -module of type φ . Suppose that $W(\Gamma)^+$ acts locally and nilpotently on $W(\Gamma)/W(\Gamma)^+$. Then, the following statements hold.

- (i) $W(\Gamma)_\varphi^+$ acts locally and nilpotently on V . In particular, $x - \varphi(x)$ acts locally and nilpotently on V for any $x \in W(\Gamma)^+$.
- (ii) All Whittaker vectors in V are of type φ .
- (iii) Any nonzero submodule of V contains a Whittaker vector of type φ .
- (iv) If the vector space of Whittaker vectors of V is one-dimensional, then V is simple.

Definition 8. The Lie algebra homomorphism $\varphi : W(\Gamma) \rightarrow \mathbb{F}$ is called nonsingular if $\varphi(I_{1,s})\varphi(J_{1,t}) \neq 0$ for any $s, t \in \mathbb{Z}$.

Definition 9. For a Lie algebra homomorphism $\varphi : W(\Gamma)^+ \rightarrow \mathbb{F}$, we define C_φ to be a one-dimensional $W(\Gamma)^+$ -module given by $xa = \varphi(x)a$ for $x \in W(\Gamma)^+$ and $a \in \mathbb{F}$. Then, we have an induced $W(\Gamma)$ -module M_φ , which satisfies

$$M_\varphi = U(W(\Gamma)) \otimes_{U(W(\Gamma)^+)} C_\varphi.$$

Let $w = 1 \otimes 1$, using the PBW theorem, M_φ has a basis $\{L_{-u,i}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}w | (u, \lambda, v, l) \in \tilde{P} \times \tilde{P} \times \tilde{P} \times \tilde{P}\}$. Then, M_φ has the universal property in the sense that for any Whittaker module V of type ψ generated by w' , there is a surjective homomorphism $\varphi : M_\varphi \rightarrow V$ such that $uw \mapsto uw' (\forall u \in U(W(\Gamma)^-))$. Hence, we call M_φ the universal Whittaker module of type ψ .

For any $v = \sum P_{u,i,\lambda,j,v,p,l,q} L_{-u,i}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}w \in M_\varphi$ ($P_{u,i,\lambda,j,v,p,l,q} \in \mathbb{F}$), we define

$$\maxdeg(v) = \max\{|u| + |\lambda| + |v| + |l| | P_{u,i,\lambda,j,v,p,l,q} \neq 0\}$$

and $\maxdeg(0) = -\infty$.

For any $x \in U(W(\Gamma))^+$, $w' = uw$, $u \in U(W(\Gamma))^-$, we have

$$(x - \psi(x))w' = [x, u]w.$$

In particular, we have

$$(E_n - \psi(E_n))w' = [E_n, u]w,$$

where $E_n \in \{L_{n,p}, H_{n,p}, I_{n,p}, J_{n,p} | n \in \Gamma, p \in \mathbb{Z}\}$.

Lemma 4. For $m \in \Gamma^+ \cup \{0\}$, $s, p \in \mathbb{Z}$, $k \in \mathbb{N}$, $u, \lambda \in \tilde{P}$, $i \in \mathbb{Z}^r$, $j \in \mathbb{Z}^s$ we have the following results:

- (1) $\maxdeg([I_{m,p}, L_{-u,i}H_{-\lambda,j}]w) \leq |\lambda + u| - m + 1$.
- (2) Suppose that $\lambda = (0^{\lambda(0)}, 1^{\lambda(1)}, 2^{\lambda(2)}, \dots)$ and $k \in \mathbb{N}$ is the minimal, such that $\lambda(k) \neq 0$ or $u(k) \neq 0$, then we have

$$[I_{k+1,s}, L_{-u,i}H_{-\lambda,j}]w = v + u(k)(2k+1)\psi(I_{1,s+p})L_{-u',i'}H_{-\lambda,j'}w - \lambda(k)\psi(J_{1,s+p})L_{-u,i}H_{-\lambda',j'}w$$

where $\lambda'(k) = \lambda(k) - 1$, $u'(k) = u(k) - 1$, $\lambda'(i) = \lambda(i)$, $u'(i) = u(i)$ for all $i \neq k$. If $k > 0$, we have $\maxdeg(v) < |u + \lambda| - k$. If $k = 0$, we have $v = v' + v''$, $\maxdeg(v') < |\lambda + u| - k$, $\max_{L_0}(v'') < \lambda(k) + u(k) - 1$.

Proof. (1)

$$\begin{aligned} & [I_{m,p}, L_{-u,i}H_{-\lambda,j}]w \\ &= \sum_{t=1}^r L_{-u_r,i_r} \cdots [I_{m,p}, L_{-u_t,i_t}] \cdots L_{-u_1,i_1} H_{-\lambda,j} w + \sum_{q=1}^s L_{-u,i} H_{-\lambda_{s+j_s}} \cdots [I_{m,p}, H_{-\lambda_{q+j_q}}] \cdots H_{-\lambda_1,j_1} w \\ &= \sum P_{u',i',v',p',\lambda,j} L_{-u',i'} I_{-v',\varepsilon'} H_{-\lambda,j} w + \sum P_{u,i,\lambda',j',n,k} L_{-u,i} H_{-\lambda',j'} J_{n,k} w, \end{aligned}$$

where

$$|u + \lambda| - m = |u' + v' + \lambda| = |u + \lambda'| - n.$$

Since $J_{n,k}w = \psi(J_{n,k})w = 0$ when $n \geq 2$, we can get

$$\maxdeg([I_{m,p}, L_{-u,i}H_{-\lambda,j}]w) = \maxdeg(\sum P_{u',i',v',\varepsilon',n,k} L_{-u,i} H_{-\lambda,j} J_{n,k} w) \leq |u + \lambda| - m + 1.$$

(2) We denote

$$L_{-u,i} = L_{-u',i'} L_{-k,p_1} \cdots L_{-k,p_{u(k)'}}$$

where $k \in \Gamma$, $p_1 \cdots p_{u(k)} \in \mathbb{Z}$.

We have

$$\begin{aligned} & [I_{k+1,s}, L_{-k,p_1} \cdots L_{-k,p_{u(k)}}]w \\ &= \sum_{t=1}^{u(k)} L_{-k,p_1} \cdots L_{-k,p_{t-1}} [I_{k+1,s}, L_{-k,p_t}] L_{-k,p_{t+1}} \cdots L_{-k,p_{u(k)}} w \\ &= (2k+1) \sum_{t=1}^{u(k)} L_{-k,p_1} \cdots \widehat{L_{-k,p_t}} \cdots L_{-k,p_{u(k)}} \psi(I_{1,s+p_t})w. \end{aligned}$$

We see that $p_1 \cdots p_{u(k)}$ all have no relations with coefficients. On the other hand, if $i \geq 3$, $j, m, l \geq 2$, we have $\varphi(L_{i,p}) = \varphi(H_{j,w}) = \varphi(I_{m,s}) = \varphi(J_{l,t}) = 0$ for any $p, w, s, t \in \mathbb{Z}$.

Therefore, we use p replace $p_1 \cdots p_{u(k)}$ and assume that $L_{-k,p_1} \cdots L_{-k,p_t} \cdots L_{-k,p_{u(k)}} = L_{-k,p}^{u(k)} \psi(I_{1,s+p_t}) = \psi(I_{1,s+p_t})$.

Then, we have $L_{-u,i} = L_{-u',i'} L_{-k,p}^{u(k)}$ and $H_{-\lambda,j} = H_{-\lambda',j'} H_{-k,p}^{\lambda(k)}$. By calculating this, we have

$$[I_{k+1,s}, L_{-k,p}^a] = \sum_{i=0}^{a-1} L_{-k,p}^i [I_{k+1,s}, L_{-k,p}] L_{-k,p}^{a-1-i} = a(2k+1) L_{-k,p}^{a-1} I_{1,s+p}.$$

Similarly, we obtain

$$[I_{k+1,s}, H_{-k,p}^b] = -b H_{-k,p}^{b-1} J_{1,s+p}.$$

So, we deduce that

$$\begin{aligned} & [I_{k+1,s}, L_{-u,i} H_{-\lambda,j}] w \\ &= [I_{k+1,s}, L_{-u',i'}] L_{-k,p}^{u(k)} H_{-\lambda,j} w + L_{-u',i'} [I_{k+1,s}, L_{-k,p}^{u(k)}] H_{-\lambda,j} w + L_{-u,i} [I_{k+1,s}, H_{-\lambda',j'}] H_{-k,p}^{\lambda(k)} w \\ &+ L_{-u,i} H_{-\lambda',j'} [I_{k+1,s}, H_{-k,p}^{\lambda(k)}] w. \end{aligned}$$

According to the assumption of k , we know that $[I_{k+1,s}, L_{-u',i'}] \in U(W(\Gamma))^-$ and $[I_{k+1,s}, H_{-\lambda',j'}] \in U(W(\Gamma))^-$, and we can deduce that

$$\maxdeg([I_{k+1,s}, L_{-u',i'}] L_{-k,p}^{u(k)} H_{-\lambda,j}) \leq |u + \lambda| - k - 1,$$

$$\maxdeg(L_{-u,i} [I_{k+1,s}, H_{-\lambda',j'}] H_{-k,p}^{\lambda(k)}) \leq |u + \lambda| - k - 1.$$

and $|u + \lambda| - k - 1 = |u' + \lambda| - k - 1 = |u + \lambda'| - k - 1$, which means that $|u| = |u'| + ku(k)$ and $|\lambda| = |\lambda'| + k\lambda(k)$,

$$L_{-u',i'} [I_{k+1,s}, L_{-k,p}^{u(k)}] H_{-\lambda,j} w = u(k)(2k+1) \psi(I_{1,s+p}) L_{-u',i'} H_{-\lambda,j} w,$$

$$L_{-u,i} H_{-\lambda',j'} [I_{k+1,s}, H_{-k,p}^{\lambda(k)}] w = \lambda(k)(2k+1) \psi(J_{1,s+p}) L_{-u,i} H_{-\lambda',j'} w,$$

where $u'(k) = u(k) - 1$, $\lambda'(k) = \lambda(k) - 1$. $\square$

Lemma 5. For $m \in \Gamma^+ \cup \{0\}$, $k \in \mathbb{N}$, $u, \lambda \in \tilde{P}$, $s, p \in \mathbb{Z}$, $i \in \mathbb{Z}^r$, $j \in \mathbb{Z}^s$, we have the following results:

- (1) $\maxdeg([J_{m,p}, L_{-u,i} H_{-\lambda,j}] w) \leq |\lambda + u| - m + 1$.
- (2) We can get

$$[J_{k+1,s}, L_{-u,i} H_{-\lambda,j}] w = v + u(k)(2k+1) \psi(J_{1,s+p}) L_{-u',i'} H_{-\lambda,j} w + \lambda(k) \psi(I_{1,s+p}) L_{-u,i} H_{-\lambda',j'} w$$

where $\lambda'(k) = \lambda(k) - 1$, $u'(k) = u(k) - 1$, $\lambda'(i) = \lambda(i)$, $u'(i) = u(i)$ for all $i \neq k$. If $k > 0$, we have $\maxdeg(v) < |u + \lambda| - k$. If $k = 0$, we have $v = v' + v''$, $\maxdeg(v') < |\lambda + u| - k$, $\max_{L_0}(v'') < \lambda(k) + u(k) - 1$.

Proof. The proof is similar to that of Lemma (4). $\square$

Lemma 6. For $m, \lambda \in \Gamma^+ \cup \{0\}$, $k \in \mathbb{N}$, $u, v, l \in \tilde{P}$, $s, p \in \mathbb{Z}$, $i \in \mathbb{Z}^r$, $\varepsilon \in \mathbb{Z}^t$, $q \in \mathbb{Z}^k$, we have the following results:

- (1) $\maxdeg([H_{m,p}, L_{-u,i} I_{-v,\varepsilon} J_{-l,q}] w) \leq |u + v + l| - m + 1$.
- (2) If $u(i) = v(i) = l(i) = 0$ for all $0 \leq i \leq k$, then we get

$$\maxdeg([H_{k+1,s}, L_{-u,i} I_{-v,\varepsilon} J_{-l,q}]) \leq |u + v + l| - k - 1.$$

- (3) If $u(i) = 0$ for all $0 \leq i \leq k$, $v(j) = l(j) = 0$ for all $0 \leq j < k$ and $v(k) \neq 0$ or $l(k) \neq 0$, then we have

$$[H_{k+1,s}, L_{-u,i} I_{-v,\varepsilon} J_{-l,q}]w = v + v(k)\psi(I_{1,s+p})L_{-u,i} I_{-v,\varepsilon'} J_{-l,q}w + \lambda(k)\psi(I_{1,s+p})L_{-u,i} I_{-v,\varepsilon} J_{-l',q'}w$$

where $\maxdeg(v) < |u + v + l| - k$, $v'(k) = v(k) - 1$, $l'(k) = l(k) - 1$, $v'(i) = v(i)$, $l'(i) = l(i)$ for all $i \neq k$.

Proof.

- (1) We have

$$\begin{aligned} & [H_{m,p}, L_{-u,i} I_{-v,\varepsilon} J_{-l,q}]w \\ &= [H_{m,p}, L_{-u,i}]I_{-v,\varepsilon} J_{-l,q}w + L_{-u,i}[H_{m,p}, I_{-v,\varepsilon}]J_{-l,q}w + L_{-u,i}I_{-v,\varepsilon}[H_{m,p}, J_{-l,q}]w \\ &= \sum P_{u',i',v,\varepsilon,l,q} H_{-u',i'} I_{-v,\varepsilon} J_{-l,q}w + \sum P_{u,i,l'',q''} L_{-u,i} I_{-l'',q''}w + \sum P_{u,i,v''',\varepsilon'''} L_{-u,i} I_{-v''',\varepsilon'''}w \\ &= \sum P_{u',i',v,\varepsilon,l',q'} H_{-u',i'} I_{-v,\varepsilon} J_{-l',q'} J_{n,k}w + \sum P_{u,i,l'',q''} L_{-u,i} J_{-l'',q''}w \\ &+ \sum P_{u,i,v''',\varepsilon'''} L_{-u,i} I_{-v''',\varepsilon'''}w, \end{aligned}$$

where $|u + v + l| - q = |u' + v + l'| - n = |u + l''| + |u + v'''|$. Since $J_{n,k}w = \psi(J_{n,k})w = 0$ for $n > 1$, we can get

$$\maxdeg([H_{m,p}, L_{-u,i} I_{-v,\varepsilon} J_{-l,q}]w) \leq |u + v + l| - m + 1.$$

- (2) If $(u, v, l) = (0, 0, 0)$, then $\maxdeg([H_{k+1,s}, L_{-u,i} I_{-v,\varepsilon} J_{-l,q}]w) = -\infty$.
If $(u, v, l) \neq (0, 0, 0)$, then $\maxdeg([H_{k+1,s}, L_{-u,i} I_{-v,\varepsilon} J_{-l,q}]w) = |u + v + l| - k - 1$.
(3) We denote $I_{-v,\varepsilon} = I_{-v',\varepsilon'} I_{-k,p}^{v(k)}$ and $J_{-l,q} = J_{-l',q'} J_{-k,p}^{l(k)}$. Then, we have

$$\begin{aligned} & [H_{k+1,s}, L_{-u,i} I_{-v,\varepsilon} J_{-l,q}]w \\ &= [H_{k+1,s}, L_{-u,i}]I_{-v,\varepsilon} J_{-l,q}w + L_{-u,i}[H_{k+1,s}, I_{-v,\varepsilon}]J_{-l,q}w + L_{-u,i}I_{-v,\varepsilon}[H_{k+1,s}, J_{-l,q}]w \\ &= [H_{k+1,s}, L_{-u,i}]I_{-v,\varepsilon} J_{-l,q}w + L_{-u,i}[H_{k+1,s}, I_{-v',\varepsilon'}]I_{-k,p}^{v(k)} J_{-l,q}w + L_{-u,i}I_{-v',\varepsilon'} [H_{k+1,s}, I_{-k,p}^{v(k)}]J_{-l,q}w \\ &+ L_{-u,i}I_{-v,\varepsilon} [H_{k+1,s}, J_{-l',q'}]J_{-k,p}^{l(k)}w + L_{-u,i}I_{-v,\varepsilon} J_{-l',q'} [H_{k+1,s}, J_{-k,p}^{l(k)}]w \\ &= [H_{k+1,s}, L_{-u,i}]I_{-v,\varepsilon} J_{-l,q}w + L_{-u,i}[H_{k+1,s}, I_{-v',\varepsilon'}]I_{-k,p}^{v(k)} J_{-l,q}w \\ &+ v(k)\psi(I_{1,s+p})L_{-u,i} I_{-v',\varepsilon'} I_{-k,p}^{v(k-1)} J_{-l,q}w + L_{-u,i} I_{-v,\varepsilon} [H_{k+1,s}, J_{-l',q'}]J_{-k,p}^{l(k)}w \\ &- l(k)\psi(I_{1,s+p})L_{-u,i} I_{-v,\varepsilon} J_{-l',q'} J_{-k,p}^{l(k)-1}w. \end{aligned}$$

According to the assumption of k , we know that $[H_{k+1,s}, L_{-u,i}] \in U(W(\Gamma))^-$, $[H_{k+1,s}, I_{-v',\varepsilon'}] \in U(W(\Gamma))^-$, $[H_{k+1,s}, J_{-l',q'}] \in U(W(\Gamma))^-$, and we obtain the following conclusions:

$$\maxdeg([H_{k+1,s}, L_{-u,i}]I_{-v,\varepsilon} J_{-l,q}w) \leq |u + v + l| - k - 1$$

$$\maxdeg(L_{-u,i}[H_{k+1,s}, I_{-v',\varepsilon'}]I_{-k,p}^{v(k)} J_{-l,q}w) \leq |u + v + l| - k - 1$$

$$\maxdeg(L_{-u,i}I_{-v,\varepsilon} [H_{k+1,s}, J_{-l',q'}]J_{-k,p}^{l(k)}w) \leq |u + v + l| - k - 1$$

□

Lemma 7. For $m \in \Gamma^+ \cup \{0\}$, $k \in \mathbb{N}$, $u, \lambda, v, l \in \tilde{P}$, $s, p \in \mathbb{Z}$, $i \in \mathbb{Z}^r$, $\varepsilon \in \mathbb{Z}^t$, $q \in \mathbb{Z}^k$, we have the following results:

- (1) $\maxdeg([L_{m,p}, L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w) \leq |u + \lambda + v + l| - m + 2$.
(2) If $u(i) = \lambda(i) = v(i) = l(i) = 0$ for all $0 \leq i \leq k$, then we can get

$$\maxdeg([H_{k+1,s}, L_{-u,i}]) \leq |u + v + l| - k - 1,$$

(3) If $u(i) = \lambda(i) = 0$ for all $0 \leq i \leq k$, $v(j) = l(j) = 0$ for all $0 \leq j < k$ and $v(k) \neq 0$ or $l(k) \neq 0$, then we have

$$\begin{aligned} & [L_{k+1,s}, L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}] w \\ &= v + v(k)(2k+1)\psi(I_{1,s+p})L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w \\ &+ l(k)(2k+1)\psi(I_{1,s+p})L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w. \end{aligned}$$

where $\maxdeg(v) < |u + v + l| - k$, $v''(k) = v(k) - 1$, $l''(k) = u(k) - 1$, $v''(i) = v(i)$, $l''(i) = l(i)$ for all $i \neq k$.

Proof. The proof is similar to that of Lemma (6). $\square$

Theorem 4. Suppose $\psi(I_{1,t}) \neq 0$ and $\psi(J_{1,u}) \neq 0$ ($t, u \in \mathbb{Z}$), $w = 1 \otimes 1 \in M_\varphi$. Then, $v \in M_\varphi$ is a Whittaker vector if and only if $v = uw$ for some $u \in \mathbb{F}$ ($u \neq 0$).

Proof.

$$N = \max\{|u| + |\lambda| + |v| + |l| \mid P_{u,i,\lambda,j,v,p,l,q} \neq 0\}$$

and define that

$$\Lambda_N = \{(u, \lambda, v, l) \in \tilde{P} \times \tilde{P} \times \tilde{P} \times \tilde{P} \mid P_{u,i,\lambda,j,v,p,l,q} \neq 0, |u| + |\lambda| + |v| + |l| = N\}$$

Assume that $k = \min\{n \in N \mid u(n) \neq 0 \text{ or } \lambda(n) \neq 0 \text{ or } v(n) \neq 0 \text{ or } l(n) \neq 0 \text{ for some } (u, \lambda, v, l) \in \Lambda_N\}$.

Case 1: k satisfies $u(k) \neq 0$ or $\lambda(k) \neq 0$ for some $(u, \lambda, v, l) \in \Lambda_N$.

We have

$$\begin{aligned} & (I_{k+1,s} - \psi(I_{k+1,s}))w' \\ &= \sum_{(u,\lambda,v,l) \notin \Lambda_N} [I_{k+1,s}, L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}] w \\ &+ \sum_{(u,\lambda,v,l) \in \Lambda_N, \lambda(k)=u(k)=0} [I_{k+1,s}, L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}] w \\ &+ \sum_{(u,\lambda,v,l) \in \Lambda_N, \lambda(k) \neq 0 \text{ or } u(k) \neq 0} [I_{k+1,s}, L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}] w. \end{aligned} \quad (10)$$

For the first term on the right hand side of (10), using Lemma 4(1), we know that its degree is strictly smaller than $N - k$. For the second term on the right hand side of (10), note that $\lambda(i) = u(i) = 0$ for $0 \leq i \leq k$, we have

$$\begin{aligned} & [I_{k+1,s}, L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}] \\ &= [I_{k+1,s}, L_{-u,i}] H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} + L_{-u,i} [I_{k+1,s}, H_{-\lambda,j}] I_{-v,\varepsilon} J_{-l,q} \in U(W(\Gamma))^- \end{aligned}$$

Thus, its degree is also strictly smaller than $N - k$. Now, using Lemma 4(2) to the third term on the right hand side of (10), we know that it is of the following form:

$$v + u(k)(2k+1)\psi(I_{1,s+p})L_{-u',i'} H_{-\lambda,j} w - \lambda(k)\psi(J_{1,j+s})L_{-u,i} H_{-\lambda',j'} w.$$

If $k = 0$, then we have $\maxdeg(v) < N - k$, $\lambda'(k) = \lambda(k) - 1$, $u'(k) = u(k) - 1$, $\lambda'(i) = \lambda(i)$, $u'(i) = u(i)$. Thus, the degree of the third term is equal to $N - k$, this proves $(I_{k+1,s} - \psi(I_{k+1,s}))w' \neq 0$. Similarly, we deduce that $(J_{k+1,s} - \psi(J_{k+1,s}))w' \neq 0$ by Lemma 5.

Case 2: k satisfies $u(k) = \lambda(k) = 0$ for any $(u, \lambda, v, l) \in \Lambda_N$ and $v(k) \neq 0$ or $l(k) \neq 0$ for some $(u, \lambda, v, l) \in \Lambda_N$.

We have

$$\begin{aligned}
 & (H_{k+1,s} - \psi(H_{k+1,s}))w' \\
 &= \sum_{(u,\lambda,v,l) \notin \Lambda_{\mathbb{N}}} [H_{k+1,s}, L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w \\
 &+ \sum_{(u,\lambda,v,l) \in \Lambda_{\mathbb{N}}, \lambda(k)=u(k)=0} [H_{k+1,s}, L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w \\
 &+ \sum_{(u,\lambda,v,l) \in \Lambda_{\mathbb{N}}, \lambda(k) \neq 0 \text{ or } u(k) \neq 0} [H_{k+1,s}, L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w.
 \end{aligned} \tag{11}$$

By using Lemma 6(1) to the first term on the right hand side of (11), and using Lemma 6(2) and (3) to the second and third term, respectively, we have

$$(H_{k+1,s} - \psi(H_{k+1,s}))w' = v + v(k)\psi(I_{1,s+p})L_{-u,i}I_{-v'',\varepsilon''}J_{-l,q}w - l(k)L_{-u,i}I_{-v,\varepsilon}J_{-l'',q''}\psi(I_{1,s+p})w,$$

where $\maxdeg(v) < N - k$, $v''(i) = v(i)$, $l''(i) = l(i)$ for all $i \neq k$ and $v''(k) = v(k) - 1$, $l''(k) = l(k) - 1$. Thus, we can deduce that $(H_{k+1,s} - \psi(H_{k+1,s}))w' \neq 0$. Similarly, we can deduce that $(L_{k+1,s} - \psi(L_{k+1,s}))w' \neq 0$ by Lemma 7.

Case 3: $u = 0$.

Let $w' = \sum P_{\lambda,j,v,\varepsilon,l,q} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}$, where $P_{\lambda,j,v,\varepsilon,l,q} \neq 0$. We have

$$\begin{aligned}
 & (L_{k+1,s} - \psi(L_{k+1,s}))w' = \sum_{(\lambda,v,l) \notin \Lambda_{\mathbb{N}}} [L_{k+1,s}, H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w \\
 &+ \sum_{(\lambda,v,l) \in \Lambda_{\mathbb{N}}, v(k)=l(k)=0} [L_{k+1,s}, H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w + \sum_{(\lambda,v,l) \in \Lambda_{\mathbb{N}}, v(k) \neq 0 \text{ or } l(k) \neq 0} [L_{k+1,s}, H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w \\
 &= \sum_{(\lambda,v,l) \notin \Lambda_{\mathbb{N}}} [L_{k+1,s}, H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w + \sum_{(\lambda,v,l) \in \Lambda_{\mathbb{N}}, v(k)=l(k)=0} [L_{k+1,s}, H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w \\
 &+ \sum_{(\lambda,v,l) \in \Lambda_{\mathbb{N}}, v(k) \neq 0 \text{ or } l(k) \neq 0} [L_{k+1,s}, H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}]w \\
 &+ \sum_{(\lambda,v,l) \in \Lambda_{\mathbb{N}}, v(k) \neq 0 \text{ or } l(k) \neq 0} H_{-\lambda,j} [L_{k+1,s}, I_{-v,\varepsilon} J_{-l,q}]w \\
 &+ \sum_{(\lambda,v,l) \in \Lambda_{\mathbb{N}}, v(k) \neq 0 \text{ or } l(k) \neq 0} H_{-\lambda,j} I_{-v,\varepsilon} [L_{k+1,s}, J_{-l,q}]w.
 \end{aligned}$$

We denote the five terms on the last equation by v_1, v_2, v_3, v_4, v_5 , respectively. We denote $I_{-v,\varepsilon} = I_{-v',\varepsilon'} I_{-k,p}^{v(k)}$, $J_{-l,q} = J_{-l',q'} J_{-k,p}^{l(k)}$. Note that $[L_{k+1,s}, H_{-\lambda,j}] \in U(W(\Gamma))^-$, $[L_{k+1,s}, H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}] \in U(W(\Gamma))^-$. We have $\maxdeg(v_i) < N - k$ when $i = 1, 2, 3$,

$$[L_{k+1,s}, I_{-k,p}^{v(k)}] = v(k)(2k+1)\psi(I_{1,s+p})I_{-k,p}^{v(k)-1}, [L_{k+1,s}, J_{-k,p}^{l(k)}] = v(k)(2k+1)\psi(I_{1,s+p})I_{-k,p}^{l(k)-1},$$

So, we have

$$v_4 = v_4' + \sum_{(\lambda,v,l) \in \Lambda_{\mathbb{N}}} v(k)(2k+1)\psi(I_{1,s+p})p_{\lambda,v,l} H_{-\lambda,j} I_{-k,p}^{v(k)-1} J_{-l,q} w,$$

and

$$v_5 = v_5' + \sum_{(\lambda,v,l) \in \Lambda_{\mathbb{N}}} v(k)(2k+1)\psi(I_{1,s+p})p_{\lambda,v,l} H_{-\lambda,j} I_{-v,\varepsilon} J_{-k,p}^{l(k)-1} w,$$

where $\maxdeg(v_t) < N - k$, $t = 4, 5$. Thus, we have $(L_{k+1} - \psi(L_{k+1}))w' \neq 0$ by Lemma 7.

Case 4: There exists some $u \neq 0$ for which $P_{u,i,\lambda,j,v,p,l,q} \neq 0$. Denote

$$N' = \max\{|u| + |\lambda| + |v| + |l| \mid u \neq 0, P_{u,i,\lambda,j,v,p,l,q} \neq 0\},$$

$$\Lambda_{\mathbb{N}'} = \{(u, \lambda, v, l) \mid u \neq 0, P_{u,i,\lambda,j,v,p,l,q} \neq 0, |u| + |\lambda| + |v| + |l| = N'\},$$

$l = \min\{n|\lambda = n^{\lambda(n)}, (n+1)^{\lambda(n+1)}, \dots \text{ such that } |u| + |\lambda| + |v| + |l| = N', P_{u,\lambda,j,v,p,l,q} \neq 0\}$.

Note that $u = i = 0$ for those (u, λ, v, l) satisfy $N' < |u| + |\lambda| + |v| + |l| < N$ and $P_{u,i,\lambda,j,v,p,l,q} \neq 0$. Thus, we have

$$\begin{aligned} w' &= \sum_{(u,\lambda,v,l) \in \Lambda_{N'}, u(l) \neq 0} P_{u,i,\lambda,j,v,\varepsilon,l,q} L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w \\ &+ \sum_{(u,\lambda,v,l) \in \Lambda_{N'}, u(l)=0} P_{u,i,\lambda,j,v,\varepsilon,l,q} L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w \\ &+ \sum_{|u|+|\lambda|+|v|+|l| < N'} P_{u,i,\lambda,j,v,\varepsilon,l,q} L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w \\ &+ \sum_{N' < |u|+|\lambda|+|v|+|l| < N} P_{\lambda,j,v,\varepsilon,l,q} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w. \end{aligned}$$

We apply $H_{k+1} - \psi(H_{k+1})$ to both sides of the last equation, the remaining proof is similar to that of Case 3.

Finally, note that the set

$$\{L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w | (u, \lambda, v, l) \in \tilde{P} \times \tilde{P} \times \tilde{P} \times \tilde{P}\}$$

forms a basis of M_φ . $\square$

5. Some Properties for the Whittaker Modules of $W(\Gamma)$

In this section, the Lie algebra homomorphism φ is assumed to be non-singular, which $\varphi(I_{1,t}) \neq 0$ and $\varphi(J_{1,u}) \neq 0 (t, u \in \mathbb{Z})$, $\varphi : W(\Gamma)^+ \rightarrow \mathbb{F}$ is an algebra homomorphism and a Whittaker module V of type φ . We may regard V as a $W(\Gamma)^+$ module by restriction. Define a modification of $W(\Gamma)^+$ by setting $x \cdot v = xv - \varphi(x)v$ for $x \in W(\Gamma)^+$ and $v \in V$. Then, it is clear that V is a $W(\Gamma)^+$ module under the dot action; we also have $E_{n,i} \cdot v = E_{n,i}v - \varphi(E_{n,i})v = [E_{n,i}, u]w$ for any $n \in \Gamma^+$, $i \in \mathbb{Z}_+$ and $v = uw \in V$.

Lemma 8. Let $(u, \lambda, v, l) \in \tilde{P} \times \tilde{P} \times \tilde{P} \times \tilde{P}$.

(1) For $n \in \Gamma^+$, $k \in \mathbb{Z}$, $i \in \mathbb{Z}^r$, $j \in \mathbb{Z}^s$, $\varepsilon \in \mathbb{Z}^t$, $q \in \mathbb{Z}^k$, we have

$$\begin{aligned} &E_{n,k} \cdot L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w \\ &\in \text{span}_{\mathbb{F}} \{L_{-u',i'} H_{-\lambda',j'} I_{-v',\varepsilon'} J_{-l',q'} w | |u'| + |\lambda'| + |v'| + |l'| + \lambda'(0) \leq |u| + |\lambda| + |v| + |l| + \lambda(0)\}. \end{aligned}$$

(2) Suppose V is a Whittaker module for $W(\Gamma)$, let $v \in V$. Regarding V as a $W(\Gamma)^+$ -module under the dot action, then $U(W(\Gamma))^+ \cdot v$ is a finite dimensional $W(\Gamma)^+$ submodule of V .

Proof. (1) The case of $(u, \lambda, v, l) = 0$ is obvious. Now, we prove the case of $(u, \lambda, v, l) > 0$ by induction. We only prove that $u \neq 0$, the case of $u = 0$ is similar.

Assume that $l = \max\{i | l(i) > 0\}$. Then

$$L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} = L_{-m,j} L_{-u',i'} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q},$$

where $u'(m) = u(m) - 1$, $u'(i) = u(i)$ for all $i \neq k$. Therefore, we have

$$\begin{aligned} E_{n,i} \cdot L_{-u,i} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w &= E_{n,i} \cdot L_{-m,j} L_{-u',i'} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w \\ &= [E_{n,i}, L_{-m,j}] L_{-u',i'} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q} w + L_{-m,j} [E_{n,i}, L_{-u',i'} H_{-\lambda,j} I_{-v,\varepsilon} J_{-l,q}] w. \end{aligned} \quad (12)$$

For the first term on the right hand side of (12), we only take $E_{n,i} = L_{n,i}$, and the case of $E_{n,i} = H_{n,i}$, $E_{n,i} = I_{n,i}$, $E_{n,i} = J_{n,i}$ is similar.

If $n - m \leq 0$, which is obvious that

$$[L_{n,i}, L_{m,j}]L_{-u',i'}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}w = (m+n)L_{n-m,i+j}L_{-u',i'}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}w,$$

has the desired form.

If $n - m > 0$, then we have

$$\begin{aligned} L_{n-m,i+j} \cdot L_{-u',i'}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}w &= [L_{n-m,i+j}, L_{-u',i'}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}]w \\ &+ \varphi(L_{n-m,i+j})L_{-u',i'}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}w. \end{aligned}$$

By assumption, we deduce that $[L_{n-m,i+j}, L_{-u',i'}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}]w$ also has the desired form.

For the second term on the right hand side of (12), by the induction hypothesis, we have

$$\begin{aligned} [E_{n,i}, L_{-u',i'}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}] &\in \text{span}_{\mathbb{F}}\{L_{-u'',i''}H_{-\lambda',j'}I_{-v',\varepsilon'}J_{-l',q'}w \mid |u'| + |\lambda'| + |v'| \\ &+ |l'| + \lambda'(0) \leq |u| + |\lambda| + |v| + |l| + \lambda(0)\}. \end{aligned}$$

Thus $L_{-m,j}[E_{n,i}, L_{-u',i'}H_{-\lambda,j}I_{-v,\varepsilon}J_{-l,q}]w$ has the desired form since $-m \leq 0$ and

$$m + |u'| + |\lambda| + |v| + |l| = |u| + |\lambda| + |v| + |l|.$$

The proof of (2) is a direct result of (1). $\square$

Theorem 5. Let $\varphi : W(\Gamma)^+ \rightarrow \mathbb{F}$ be a Lie algebra homomorphism of nonsingular type, then the Whittaker $W(\Gamma)$ -module is simple.

Proof. This follows from Lemma 3(iv) and Theorem 4. $\square$

6. Rank 1 Free $U(\mathbb{F}L_{0,0})$ -Modules of $W(\Gamma)$

In this section, we will determine free $U(h)$ -modules over the generalized loop planar Galilean conformal algebra $W(\Gamma)$, where $h = \mathbb{F}L_{0,0}$ is the canonical Cartan subalgebra of $W(\Gamma)$. Similarly to Theorem 2.4 of reference [32], we have the following result.

Theorem 6. We assume that $\mathfrak{N}$ is generalized loop-Virasoro algebra. Let M be a $U(\mathfrak{N})$ -module, such that the restriction of $U(\mathfrak{N})$ to $U(\mathbb{F}L_{0,0})$ is free of rank 1. Then, $M \cong \Omega(\lambda, \mu, \alpha)$ for some $\lambda, \mu \in \mathbb{F}^*$, $\alpha \in \mathbb{F}$. Moreover, M is simple if and only if $\lambda, \mu, \alpha \in \mathbb{F}^*$.

Definition 10. For any $m \in \Gamma$, $i \in \mathbb{Z}$, $\lambda, \mu \in \mathbb{F}^*$, $\alpha \in \mathbb{F}$, we denote by $\Omega_{W(\Gamma)}(\lambda, \mu, \alpha) = \mathbb{F}[t]$, the polynomial algebra. We define the $W(\Gamma)$ -module structure on $\Omega_{W(\Gamma)}(\lambda, \mu, \alpha)$ as follows

$$L_{m,i}(f(t)) = \lambda^{m-i}\mu^i(t - m\alpha)f(t - m), \quad H_{m,i}(f(t)) = I_{m,i}(f(t)) = J_{m,i}(f(t)) = 0. \quad (13)$$

where $f(t) \in \mathbb{F}[t]$.

Lemma 9. For any $\lambda, \mu \in \mathbb{F}^*$ and $\alpha \in \mathbb{F}$, $\Omega_{W(\Gamma)}(\lambda, \mu, \alpha)$ is an $W(\Gamma)$ -module.

Proof. It is easy to check this conclusion using (13). $\square$

Lemma 10. For any $m \in \Gamma$ and $j \in \mathbb{Z}$, the following formulas hold:

$$\begin{aligned} L_{m,i}L_{0,0}^j &= (L_{0,0} + m)^jL_{m,i}, & H_{m,i}L_{0,0}^j &= (L_{0,0} + m)^jH_{m,i}, \\ I_{m,i}L_{0,0}^j &= (L_{0,0} + m)^jI_{m,i}, & J_{m,i}L_{0,0}^j &= (L_{0,0} + m)^jJ_{m,i}. \end{aligned}$$

Proof. According to Definition 13, it is easy to check that

$$\begin{aligned} L_{m,i}L_{0,0} &= (L_{0,0} + m)L_{m,i}, & H_{m,i}L_{0,0} &= (L_{0,0} + m)H_{m,i}, \\ I_{m,i}L_{0,0} &= (L_{0,0} + m)I_{m,i}, & J_{m,i}L_{0,0} &= (L_{0,0} + m)J_{m,i}. \end{aligned}$$

Then, the lemma can be proven by an induction on j . $\square$

Lemma 11. Let Q is a free $U(\mathbb{F}L_{0,0})$ -module of rank 1 over the algebra $W(\Gamma)$. For any $m \in \Gamma$ and $i \in \mathbb{Z}$, we assume that

$$L_{m,i} \cdot \mathbf{1} = a_{m,i}(L_{0,0}), \quad H_{m,i} \cdot \mathbf{1} = b_{m,i}(L_{0,0}), \quad I_{m,i} \cdot \mathbf{1} = c_{m,i}(L_{0,0}), \quad J_{m,i} \cdot \mathbf{1} = d_{m,i}(L_{0,0}),$$

which $a_{m,i}(L_{0,0}), b_{m,i}(L_{0,0}), c_{m,i}(L_{0,0}), d_{m,i}(L_{0,0}) \in Q$, then $a_{m,i}(L_{0,0}), b_{m,i}(L_{0,0}), c_{m,i}(L_{0,0}), d_{m,i}(L_{0,0})$ completely determine the action of $L_{m,i}, H_{m,i}, I_{m,i}, J_{m,i}$ on Q .

Proof. Now, we take any $u(L_{0,0}) = \sum_{j \geq 0} a_j L_{0,0}^j \in U(\mathbb{F}L_{0,0})$. Then, using Lemma 4, we have

$$\begin{aligned} L_{m,i} \cdot u(L_{0,0}) &= L_{m,i} \cdot \sum_{j \geq 0} a_j L_{0,0}^j = \sum_{j \geq 0} a_j L_{m,i} L_{0,0}^j \\ &= \sum_{j \geq 0} a_j (L_{0,0} + m)^j L_{m,i} = \sum_{j \geq 0} a_j (L_{0,0} + m)^j a_{m,i}(L_{0,0}) \\ &= u(L_{0,0} + m) a_{m,i}(L_{0,0}). \end{aligned} \quad (14)$$

Similarly, we prove that

$$H_{m,i} \cdot u(L_{0,0}) = u(L_{0,0} + m) b_{m,i}(L_{0,0}), \quad (15)$$

$$I_{m,i} \cdot u(L_{0,0}) = u(L_{0,0} + m) c_{m,i}(L_{0,0}), \quad (16)$$

$$J_{m,i} \cdot u(L_{0,0}) = u(L_{0,0} + m) d_{m,i}(L_{0,0}), \quad (17)$$

$\square$

Lemma 12. For all $m \in \Gamma$ and $i \in \mathbb{Z}$, we have $b_{m,i}(L_{0,0}) \in \mathbb{F}$, $c_{m,i}(L_{0,0}) \in \mathbb{F}$ and $d_{m,i}(L_{0,0}) \in \mathbb{F}$.

Proof. We only prove $c_{m,i}(L_{0,0}) \in \mathbb{F}$, and the case of $d_{m,i}(L_{0,0}) \in \mathbb{F}$ is similar. Now, we assume that $c_{m,i}(L_{0,0}) = \sum_{l=0}^{k_m} c_{m,i}^l L_{0,0}^l$, which $c_{m,i} \in \mathbb{F}$, $k_m \in \mathbb{N}$ and $c_{m,k_m} \neq 0$. Now, we calculate $[I_{m,i}, I_{n,j}] \cdot \mathbf{1}$ as follows

$$\begin{aligned} 0 &= [I_{m,i}, I_{n,j}] \cdot \mathbf{1} = I_{m,i} \cdot I_{n,j} \cdot \mathbf{1} - I_{n,j} \cdot I_{m,i} \cdot \mathbf{1}, \\ &= \sum_{l=0}^{k_n} c_{n,i}(L_{0,0} + m)^l \sum_{l=0}^{k_m} c_{m,i}^l L_{0,0}^l - \sum_{l=0}^{k_m} c_{m,i}(L_{0,0} + n)^l \sum_{l=0}^{k_n} c_{n,i}^l L_{0,0}^l, \\ &\equiv c_{m,k_m} c_{n,k_n} (mk_n - nk_m) L_{0,0}^{k_m+k_n-1} \pmod{\bigoplus_{l=0}^{k_m+k_n-2} \mathbb{F} L_{0,0}^l}. \end{aligned}$$

Therefore, we have $mk_n = nk_m$, which means that $k_m = mk_1$. If $k_1 > 0$, it is nonsense for k_m if m is negative, which forces $k_1 = 0$. Hence, $k_m = 0$ for any $m \in \Gamma$ and $i \in \mathbb{Z}$. We obtain $c_{m,i}(L_{0,0}) = c_{m,i} \in \mathbb{F}$. Similarly, by $[H_{m,i}, H_{n,j}] = 0$ and $[J_{m,i}, J_{n,j}] = 0$, we obtain $b_{m,i}(L_{0,0}) = b_{m,i} \in \mathbb{F}$ and $d_{m,i}(L_{0,0}) = d_{m,i} \in \mathbb{F}$ for all $m \in \Gamma$ and $i \in \mathbb{Z}$. $\square$

Lemma 13. We have $b_{m,i} = c_{m,i} = d_{m,i} = 0$ for all $m \in \Gamma$ and $i \in \mathbb{Z}$.

Proof. By the definition of the loop Virasoro module $\Omega(\lambda, \mu, \alpha)$, we have

$$L_{m,i}(f(t)) = \lambda^{m-i} \mu^i (t - m\alpha) f(t - m), \quad m \in \Gamma, \quad i \in \mathbb{Z}, \quad f(t) \in \mathbb{F}[t].$$

Now, we consider the action of $H_{m,i}$'s, $I_{m,i}$'s, $J_{m,i}$'s. Similar to Proposition 2.3 of reference [32], we assume that

$$\begin{aligned} H_{m,i}(f(t)) &= H_{m,i}(f(L_{0,0})\mathbf{1}) = f(t - m)H_{m,i} \cdot \mathbf{1}, \\ I_{m,i}(f(t)) &= I_{m,i}(f(L_{0,0})\mathbf{1}) = f(t - m)I_{m,i} \cdot \mathbf{1}, \\ J_{m,i}(f(t)) &= J_{m,i}(f(L_{0,0})\mathbf{1}) = f(t - m)J_{m,i} \cdot \mathbf{1}. \end{aligned}$$

Using Lemma 7, we calculate $[L_{m,i}, I_{n,j}] \cdot \mathbf{1}$ as follows

$$\begin{aligned} (m - n)I_{m+n,i+j} \cdot \mathbf{1} &= [L_{m,i}, I_{n,j}] \cdot \mathbf{1} = L_{m,i} \cdot I_{n,j} \cdot \mathbf{1} - I_{n,j} \cdot L_{m,i} \cdot \mathbf{1} \\ &= c_{n,j}L_{m,i} \cdot \mathbf{1} - \lambda^{m-i} \mu^i (t - n - m\alpha) I_{n,j} \cdot \mathbf{1} \\ &= c_{n,j} \lambda^{m-i} \mu^i (t - m\alpha) - c_{n,j} \lambda^{m-i} \mu^i (t - n - m\alpha) \\ &= nc_{n,j} \lambda^{m-i} \mu^i. \end{aligned} \quad (18)$$

Taking $m = \pm 1$ in (18), we have

$$-(n + 1)c_{n-1,i+j} = nc_{n,j} \lambda^{-1-i} \mu^i \quad (\lambda, \mu \in \mathbb{F}^*), \quad (19)$$

and

$$(1 - n)c_{n+1,i+j} = nc_{n,j} \lambda^{1-i} \mu^i \quad (\lambda, \mu \in \mathbb{F}^*). \quad (20)$$

Using (19) and (20), we deduce that $c_{m,i} = 0$ for all $m \in \Gamma$ and $i \in \mathbb{Z}$. Similarly, using $[L_{m,i}, H_{n,j}] \cdot \mathbf{1} = -nH_{m+n,i+j} \cdot \mathbf{1}$ and $[L_{m,i}, J_{n,j}] \cdot \mathbf{1} = J_{m+n,i+j} \cdot \mathbf{1}$, we obtain $b_{m,i} = d_{m,i} = 0$. $\square$

From what we have discussed above, we have obtained the main result of this section:

Theorem 7. Let Q be a $U(W(\Gamma))$ -module, such that the restriction of $U(W(\Gamma))$ to $U(\mathbb{F}L_{0,0})$ is free of rank 1. Then, $Q \cong \Omega_{W(\Gamma)}(\lambda, \mu, \alpha)$ for some $\lambda, \mu \in \mathbb{F}^*$, $\alpha \in \mathbb{F}$. Moreover, Q is simple if and only if $\lambda, \mu, \alpha \in \mathbb{F}^*$.

Proof. We prove this theorem using Lemmas 9–13. $\square$

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