axioms

Article

Fractional p-Laplacian Coupled Systems with Multi-Point Boundary Conditions

Ayub Samadi [1], Sotiris K. Ntouyas [2] and Jessada Tariboon [3,*]

[1] Department of Mathematics, Miyaneh Branch, Islamic Azad University, Miyaneh 5315836511, Iran; ayubtoraj1366@gmail.com
[2] Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece; sntouyas@uoi.gr
[3] Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
* Correspodence: jessada.t@sci.kmutnb.ac.th

Abstract: This article is allocated to the existence and uniqueness of solutions for a system of nonlinear differential equations consisting of the Caputo fractional-order derivatives. Our main results are proved via standard tools of fixed point theory. Finally, the presented results are clarified by constructing some examples.

Keywords: p-laplacian operator; fractional calculus; Caputo fractional derivative; Riemann–Liouville fractional integral

MSC: 26A33; 34A08; 34B15

check for updates

Citation: Samadi, A.; Ntouyas, S.K.; Tariboon, J. Fractional p-Laplacian Coupled Systems with Multi-Point Boundary Conditions. *Axioms* **2023**, *12*, 866. https://doi.org/10.3390/axioms12090866

Academic Editor: Chris Goodrich

Received: 28 July 2023
Revised: 3 September 2023
Accepted: 5 September 2023
Published: 7 September 2023

1. Introduction

Fractional differential equations (FDEs) have been applied to qualify multiplex problems in various fields of natural sciences, as physics, biology, chemistry, image processing and so on, see [1–5]. In the literature, different types of fractional derivatives and fractional integrals have been presented, where many of those definitions deal with the Riemann–Liouville and Caputo fractional derivatives. These definitions have been applied in many nonlocal boundary problems to present some physical phenomena (see [6–9]). Additionally, FDEs with a p-Laplacian operator appeared when Leibenson [10] was working to deduce an exact formula to construct a turbulent flow. Recently, many applications of FDEs with p-Laplacian operator have been obtained in the various areas such as glaciology and dynamics, see [11–15] and references therein. Hence, it is important to consider FDEs consisting of the p-Laplacian operator in different spaces. Recently, Srivastava et al. [16] have applied the p-Laplacian operator to consider the existence of solutions for a class of nonlinear differential equations of the form

$$ {}^{C}\mathcal{D}^{\rho}\left(\theta_{P}\left[{}^{C}\mathcal{D}^{\mu}\sigma(w)\right]\right)+\kappa(w,\sigma(w))=\theta, \quad w\in[c,d], \tag{1} $$

supplemented with coupled nonlocal boundary conditions

$$ \sigma(c)={}^{C}\mathcal{D}^{\mu}(c)=\theta, \quad \sigma(d)=\sum_{i=1}^{m}\lambda_i\sigma(\eta_i), \quad c<\eta_i<d, \tag{2} $$

in which $\kappa:[c,d]\times E\longrightarrow E$ is a given function, E is a Banach space with zero element θ, $0<\rho\le 1$, $1<\mu\le 2$, λ_i, $i=1,2,\ldots,m$ are real constants, ${}^{C}\mathcal{D}^{\rho}$, denote the Caputo fractional derivatives, θ_p indicates a p-Laplacian operator, that is

$$ \theta_p(s)=|s|^{p-2}s, \quad p>1, \quad \theta_p^{-1}=\theta_q, \quad \frac{1}{p}+\frac{1}{q}=1. $$

The aim of this work is to extend the problem (1) and (2) on coupled system of differential equations by introducing a new coupled system of FDEs. The significance of coupled systems of FDEs is that such systems have appeared in various fields of sciences, see [17–19]. In this paper, the existence of solutions for a system of nonlinear differential equations consisting of the Caputo fractional-order derivatives of the form

$$\begin{cases} {}^{C}\mathcal{D}^{\rho_1}\left(\theta_{p_1}\left[{}^{C}\mathcal{D}^{\mu_1}\sigma_1(w)\right]\right) = \kappa_1(w,\sigma_1(w),\sigma_2(w)), \quad w \in [c_1,d_1], \\ {}^{C}\mathcal{D}^{\rho_2}\left(\theta_{p_2}\left[{}^{C}\mathcal{D}^{\mu_2}\sigma_2(w)\right]\right) = \kappa_2(w,\sigma_1(w),\sigma_2(w)), \end{cases} \tag{3}$$

supplemented with coupled nonlocal boundary conditions

$$\begin{cases} \sigma_1(c_1) = {}^{C}\mathcal{D}^{\mu_1}\sigma_1(c_1) = 0, \quad \sigma_1(d_1) = \sum_{i=1}^{m} \lambda_i \sigma_2(\eta_i), \\ \sigma_2(c_1) = {}^{C}\mathcal{D}^{\mu_2}\sigma_2(c_1) = 0, \quad \sigma_2(d_1) = \sum_{j=1}^{n} \overline{\lambda}_j \sigma_1(\overline{\eta}_j) \end{cases} \tag{4}$$

is investigated, in which $0 < \rho_1, \rho_2 \le 1$, $1 < \mu_1, \mu_2 \le 2$, ${}^{C}\mathcal{D}^{\delta}$ denotes the Caputo fractional derivative of order $\delta \in \{\rho_1, \rho_2, \mu_1, \mu_2\}$, $\theta_p \ (p = p_1, p_2)$ indicates a p-Laplacian operator, $\kappa_1, \kappa_2 : [c_1, d_1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions with some conditions, $\lambda_i, \overline{\lambda}_j$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ are real constants and $c_1 < \eta_i, \overline{\eta}_j < d_1$.

The tools of fixed point theory will be applied to develop the existence theory for the problem (3) and (4). In fact, the Banach contraction mapping principle is applied to prove a uniqueness result, while two existence results are derived via Leray–Schauder alternative and Krasnosel'skiĭ fixed point theorem. The presented system in this article is not only new, but some other systems are special cases of this system, for example, by taking $\mu_1 = \mu_2 = p_1 = p_2 = 2$ and $\rho_1 = \rho_2 = 1$, we have the third order system of the form

$$\begin{cases} \sigma_1'''(w) = \kappa(w,\sigma_1(w),\sigma_2(w)), \quad w \in [c_1,d_1], \\ \sigma_2'''(w) = \kappa(w,\sigma_1(w),\sigma_2(w)), \\ \sigma_1(c_1) = \sigma_1''(c_1) = 0, \ \ \sigma_1(d_1) = \sum_{i=1}^{m} \lambda_i \sigma_2(\eta_i), \ \ c_1 < \eta_i < d_1, \\ \sigma_2(c_1) = \sigma_2''(c_1) = 0, \ \ \sigma_2(d_1) = \sum_{j=1}^{n} \overline{\lambda}_j \sigma_1(\overline{\eta}_j), \ \ c_1 < \overline{\eta}_j < d_1. \end{cases} \tag{5}$$

The structure of this article is as follows: In the next section, some definitions and basic Lemmas are collected, which will be required to prove the main results. In Section 3, an auxiliary lemma is proved to convert the problem (3) and (4) into a fixed point problem. The existence and uniqueness results are considered in Section 4. In Section 5, some examples are included to illustrate the obtained results.

2. Preliminaries

In this section, some results and concepts concerning the fractional calculus are presented.

Definition 1 ([2,20])**.** *The fractional integral of Riemann–Liouville type and order $\eta > 0$ of a function $\sigma \in L^1[c_1, d_1]$ is defined by*

$${}^{RL}\mathbb{I}^{\eta}\sigma(w) = \frac{1}{\Gamma(\eta)} \int_{c_1}^{w} (w-s)^{\eta-1}\sigma(s)ds, \ \ (w > c_1, \eta > 0),$$

where Γ is the Gamma function.

Definition 2 ([2]). *The fractional derivative of Caputo type and order η of a function $\sigma \in AC^n[c_1, d_1]$ is represented by*

$$ {}^{C}\mathcal{D}^{\eta}\sigma(w) = \begin{cases} \dfrac{1}{\Gamma(n-\eta)} \displaystyle\int_{c_1}^{w} (w-s)^{n-\eta-1}\sigma^{(n)}(s)ds, & \eta \notin \mathbb{N}, \\ \sigma^{(n)}(w), & \eta \in \mathbb{N}, \end{cases} \tag{6} $$

where $\sigma^{(n)}(w) = \frac{d^n\sigma(w)}{dw^n}$, $\eta > 0$ *and* $n = [\eta] + 1$.

Lemma 1 ([2]). *Let $\eta > \rho > 0$ and $\sigma \in L^1[c_1, d_1]$. Then,*

(i) ${}^{RL}\mathbb{I}^{\eta}\ {}^{C}\mathcal{D}^{\eta}\sigma(w) = \sigma(w) + \sum_{i=0}^{n-1} b_i(w-c_1)^i$, *for some* $b_i \in \mathbb{R}$ $(i = 0, 1, 2, \ldots, n-1)$, *where* $n = [\eta] + 1$,
(ii) ${}^{C}\mathcal{D}^{\eta}\ {}^{RL}\mathbb{I}^{\eta}\sigma(w) = \sigma(w)$,
(iii) ${}^{C}\mathcal{D}^{\rho}\ {}^{RL}\mathbb{I}^{\eta}\sigma(w) = {}^{RL}\mathbb{I}^{\eta-\rho}\sigma(w)$.

3. An Auxiilliary Result

Lemma 2. *Let $h_1, h_2 \in C^2([c_1, d_1], \mathbb{R})$. Then, the unique solution of the system*

$$ \begin{cases} {}^{C}\mathcal{D}^{\rho_1}\left(\theta_{P_1}\left[{}^{C}\mathcal{D}^{\mu_1}\sigma_1(w)\right]\right) = h_1(w), & w \in [c_1, d_1], \\ {}^{C}\mathcal{D}^{\rho_2}\left(\theta_{P_2}\left[{}^{C}\mathcal{D}^{\mu_2}\sigma_2(w)\right]\right) = h_2(w), \end{cases} \tag{7} $$

supplemented with coupled nonlocal boundary conditions

$$ \begin{cases} \sigma_1(c_1) = {}^{C}\mathcal{D}^{\mu_1}\sigma_1(c_1) = 0, & \sigma_1(d_1) = \sum_{i=1}^{m} \lambda_i \sigma_2(\eta_i), \\ \sigma_2(c_1) = {}^{C}\mathcal{D}^{\mu_2}\sigma_2(c_1) = 0, & \sigma_2(d_1) = \sum_{j=1}^{n} \overline{\lambda}_j \sigma_2(\overline{\eta}_j), \end{cases} \tag{8} $$

is given by

$$ \begin{aligned} \sigma_1(w) &= {}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(w)\right)\right) + \frac{w-c_1}{\mathbb{A}}\Big[(d_1-c_1)\sum_{i=1}^{m}\lambda_i {}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(\eta_i)\right)\right) \\ &\quad -(d_1-c_1){}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(d_1)\right)\right) + \mathbb{A}_1\Big(\sum_{j=1}^{n}\overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(\overline{\eta}_j)\right)\right) \\ &\quad -{}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(d_1)\right)\right)\Big)\Big], \\ \sigma_2(w) &= {}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_2(w)\right)\right) + \frac{w-c_1}{\mathbb{A}}\Big[\mathbb{A}_2\Big(\sum_{i=1}^{m}\lambda_i {}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(\eta_i)\right)\right)\Big) \\ &\quad -(d_1-c_1){}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(d_1)\right)\right) + (d_1-c_1)\sum_{j=1}^{n}\overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(\overline{\eta}_j)\right)\right) \\ &\quad -(d_1-c_1){}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(d_1)\right)\right)\Big], \end{aligned} \tag{9} $$

where

$$ \mathbb{A}_1 = \sum_{i=1}^{m}\lambda_i(\eta_i - c_1), \quad \mathbb{A}_2 = \sum_{j=1}^{n}\overline{\lambda}_j(\overline{\eta}_j - c_1), \quad \frac{1}{p_1} + \frac{1}{q_1} = 1, \quad \frac{1}{p_2} + \frac{1}{q_2} = 1, \tag{10} $$

with

$$\mathbb{A} = (d_1 - c_1)^2 - \mathbb{A}_1\mathbb{A}_2. \tag{11}$$

Proof. Let $\phi_1(w) = \theta_{p_1}\left[{}^{C}\mathcal{D}^{\mu_1}\sigma_1(w)\right]$ and $\phi_2(w) = \theta_{p_2}\left[{}^{C}\mathcal{D}^{\mu_2}\sigma_2(w)\right]$. Then, the system (7) can be divided into two problems:

$$\begin{cases} {}^{C}\mathcal{D}^{\rho_1}\left[\phi_1(w)\right] = h_1(w), \quad w \in [c_1, d_1], \\ {}^{C}\mathcal{D}^{\rho_2}\left[\phi_2(w)\right] = h_2(w), \\ \phi_1(c_1) = 0, \quad \phi_2(c_1) = 0. \end{cases} \tag{12}$$

and

$$\begin{cases} {}^{C}\mathcal{D}^{\mu_1}\sigma_1(w) = \theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(w)\right), \quad w \in [c_1, d_1], \\ {}^{C}\mathcal{D}^{\mu_2}\sigma_2(w) = \theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(w)\right), \end{cases} \tag{13}$$

supplemented with coupled nonlocal boundary conditions

$$\begin{cases} \sigma_1(c_1) = 0, \quad \sigma_1(d_1) = \sum_{i=1}^{m} \lambda_i \sigma_2(\eta_i), \\ \sigma_2(c_1) = 0, \quad \sigma_2(d_1) = \sum_{j=1}^{n} \overline{\lambda}_j \sigma_1(\overline{\eta}_j). \end{cases} \tag{14}$$

The solution of the system (12) can be written as $\phi_1(w) = {}^{RL}\mathbb{I}^{\rho_1}h_1(w)$ and $\phi_2(w) = {}^{RL}\mathbb{I}^{\rho_2}h_2(w)$, respectively. On the other hand, by applying the Lemma 1 and the fractional integrals ${}^{RL}\mathbb{I}^{\mu_1}$ and ${}^{RL}\mathbb{I}^{\mu_2}$ on both sides of the equations in (13), we obtain

$$\begin{aligned} \sigma_1(w) &= {}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(w)\right) + e_0 + e_1(w - c_1),\right. \\ \sigma_2(w) &= {}^{RL}\mathbb{I}^{\mu_2}\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(w)\right) + f_0 + f_1(w - c_1). \end{aligned} \tag{15}$$

Now, by applying the boundary conditions $\sigma_1(c_1) = \sigma_2(c_1) = 0$ in (15), we obtain $e_0 = f_0 = 0$. Hence, we have

$$\begin{aligned} \sigma_1(w) &= {}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(w)\right)\right) + e_1(w - c_1), \\ \sigma_2(w) &= {}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(w)\right)\right) + f_1(w - c_1). \end{aligned} \tag{16}$$

By the boundary conditions $\sigma_1(d_1) = \sum_{i=1}^{m} \lambda_i \sigma_2(\eta_i)$, $\sigma_2(d_1) = \sum_{j=1}^{n} \overline{\lambda}_j \sigma_1(\overline{\eta}_j)$ and (16), we obtain

$$\begin{aligned} {}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(d_1)\right)\right) + e_1(d_1 - c_1) &= \sum_{i=1}^{m} \lambda_i {}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(\eta_i)\right)\right) + f_1 \sum_{i=1}^{m} \lambda_i(\eta_i - c_1), \\ {}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(d_1)\right)\right) + f_1(d_1 - c_1) &= \sum_{j=1}^{n} \overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(\overline{\eta}_j)\right)\right) + e_1 \sum_{j=1}^{n} \overline{\lambda}_j(\overline{\eta}_j - c_1). \end{aligned} \tag{17}$$

Consequently,

$$\begin{cases} e_1(d_1 - c_1) - f_1\mathbb{A}_1 = \sum_{i=1}^{m} \lambda_i {}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(\eta_i)\right)\right) - {}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(d_1)\right)\right), \\ -e_1\mathbb{A}_2 + f_1(d_1 - c_1) = \sum_{j=1}^{n} \overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\left(\theta_{q_1}\left({}^{RL}\mathbb{I}^{\rho_1}h_1(\overline{\eta}_j)\right)\right) - {}^{RL}\mathbb{I}^{\mu_2}\left(\theta_{q_2}\left({}^{RL}\mathbb{I}^{\rho_2}h_2(d_1)\right)\right). \end{cases} \tag{18}$$

By solving the above system, we obtain

$$\begin{aligned} e_1 &= \frac{1}{\mathbb{A}}\Big[(d_1-c_1)\sum_{i=1}^{m}\lambda_i {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}h_2(\eta_i)\big)\big) - (d_1-c_1){}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}h_1(d_1)\big)\big) \\ &\quad +\mathbb{A}_1\Big(\sum_{j=1}^{n}\overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}h_1(\overline{\eta}_j)\big)\big)\Big) - {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}h_2(d_1)\big)\big)\Big)\Big], \\ f_1 &= \frac{1}{\mathbb{A}}\Big[\mathbb{A}_2\Big(\sum_{i=1}^{m}\lambda_i {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}h_2(\eta_i)\big)\big)\Big) - (d_1-c_1){}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}h_1(d_1)\big)\big) \\ &\quad +(d_1-c_1)\sum_{j=1}^{n}\overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}h_1(\overline{\eta}_j)\big)\big) - (d_1-c_1){}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}h_2(d_1)\big)\big)\Big]. \end{aligned}$$

Replacing e_1 and f_1 in (16), we have the solution (9). The converse can be proved by direct computation. This completes the proof. □

4. Existence and Uniqueness Results

Let $\mathbb{X}^* = \{w(q) : w(q) \in C([c_1,d_1],\mathbb{R})\}$ be the Banach space of all continuous functions from $[c_1,d_1]$ into $\mathbb{R}$, which has been equipped with the norm

$$\|v\| = \sup_{q\in[c_1,d_1]} |v(q)|.$$

The space $(\mathbb{X}^*, \|.\|)$ is a Banach space. In addition, the product space $(\mathbb{X}^*\times\mathbb{X}^*, \|.\|)$ is also a Banach space with the norm $\|(v,w)\| = \|v\| + \|w\|$ for $(v,w) \in \mathbb{X}^*\times\mathbb{X}^*$. In view of Lemma 2, we define an operator $\mathbb{B} : \mathbb{X}^*\times\mathbb{X}^* \longrightarrow \mathbb{X}^*\times\mathbb{X}^*$ as $\mathbb{B}(v,w)(q) = (\mathbb{B}_1(v,w)(q), \mathbb{B}_2(v,w)(q))$ where

$$\begin{aligned} \mathbb{B}_1(v,w)(q) &= {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(q,v(q),w(q))\big)\big) \\ &\quad +\frac{q-c_1}{\mathbb{A}}\Big[(d_1-c_1)\sum_{i=1}^{m}\lambda_i {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(\eta_i,v(\eta_i),w(\eta_i))\big)\big) \\ &\quad -(d_1-c_1){}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(d_1,v(d_1),w(d_1))\big)\big) \\ &\quad +\mathbb{A}_1\Big(\sum_{j=1}^{n}\overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(\overline{\eta}_j,v(\overline{\eta}_j),w(\overline{\eta}_j))\big)\big)\Big) \\ &\quad -{}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(d_1,v(d_1),w(d_1))\big)\big)\Big)\Big], \quad q\in[c_1,d_1], \end{aligned} \tag{19}$$

and

$$\begin{aligned} \mathbb{B}_2(v,w)(q) &= {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(q,v(q),w(q))\big)\big) \\ &\quad +\frac{q-c_1}{\mathbb{A}}\Big[\mathbb{A}_2\Big(\sum_{i=1}^{m}\lambda_i {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(\eta_i,v(\eta_i),w(\eta_i))\big)\big)\Big) \\ &\quad -(d_1-c_1){}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(d_1,v(d_1),w(d_1))\big)\big) \\ &\quad +(d_1-c_1)\sum_{j=1}^{n}\overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(\overline{\eta}_j,v(\eta_j),w(\eta_j))\big)\big) \\ &\quad -(d_1-c_1){}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(d_1,v(d_1),w(d_1))\big)\big)\Big]. \end{aligned} \tag{20}$$

For computational convenience, we set

$$\Phi_1 = \frac{\Gamma(\rho_1(q_1-1)+1)}{\Gamma(\mu_1)\Gamma(\rho_1+1)^{q_1-1}\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1}$$

$$
\begin{aligned}
&+\frac{(d_1-c_1)^2\Gamma(\rho_1(q_1-1)+1)}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1}\\
&+\frac{|\mathbb{A}_1|\Gamma(\rho_1(q_1-1)+1)}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)\Gamma(\rho_1(q_1-1)+\mu_1)}\sum_{j=1}^{n}|\overline{\lambda}_j|(\overline{\eta}_j-c_1)^{\rho_1(q_1-1)+\mu_1},\\
\Phi_2 \;=\;& \frac{\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}\sum_{i=1}^{m}|\lambda_i|(\eta_i-c_1)^{\rho_2(q_2-1)+\mu_2}\\
&+\frac{|\mathbb{A}_1|(d_1-c_1)^2\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2},\\
\Phi_3 \;=\;& \frac{\Gamma(\rho_2(q_2-1)+1)}{\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2}\\
&+\frac{\Gamma(\rho_2(q_2-1)+1)|\mathbb{A}_2|}{|\mathbb{A}|\Gamma(\rho_2+1)^{q_2-1}\Gamma(\mu_2)\Gamma(\rho_2(q_2-1)+\mu_2)}\sum_{i=1}^{m}|\lambda_i|(\eta_i-c_1)^{\rho_2(q_2-1)+\mu_2}\\
&+\frac{(d_1-c_1)^2\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\rho_2+1)^{q_2-1}\Gamma(\mu_2)\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2},\\
\Phi_4 \;=\;& \frac{|\mathbb{A}_2|(d_1-c_1)^2\Gamma(\rho_1(q_1-1)+1)}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1}\\
&+\frac{\Gamma(\rho_1(q_1-1)+1)}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)\Gamma(\rho_1(q_1-1)+\mu_1)}\sum_{j=1}^{n}|\overline{\lambda}_j|(\overline{\eta}_j-c_1)^{\rho_1(q_1-1)+\mu_1},\\
\Phi_1^* \;=\;& \Phi_1-\frac{\Gamma(\rho_1(q_1-1)+1)}{\Gamma(\mu_1)\Gamma(\rho_1+1)^{q_1-1}\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1},\\
\Phi_3^* \;=\;& \Phi_3-\frac{\Gamma(\rho_2(q_2-1)+1)}{\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2}. \qquad (21)
\end{aligned}
$$

The following lemma is used in the sequel.

Lemma 3. *It holds:*

$$\frac{1}{\Gamma(\mu)}\int_a^t (t-s)^{\mu-1}(s-a)^{\rho}ds=\frac{\Gamma(\rho+1)}{\Gamma(\mu+\rho+1)}(t-a)^{\mu+\rho}.$$

4.1. Uniqueness Result

In the following theorem, the Banach contraction mapping principle is applied to establish the existence and uniqueness result for the system (3) and (4).

Theorem 1. *Assume that the functions $\kappa_1,\kappa_2:[c_1,d_1]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies the condition*

(H_1) *There exist real constants m_i, n_i $(i=1,2)$ such that for all $q\in[c_1,d_1]$ and $v_i,w_i\in\mathbb{R}, i=1,2$, we have*

$$
\begin{aligned}
|\kappa_1(q,v_1,w_1)-\kappa_1(q,v_2,w_2)| &\le m_1|v_1-v_2|+m_2|w_1-w_2|,\\
|\kappa_2(q,v_1,w_1)-\kappa_2(q,v_2,w_2)| &\le n_1|v_1-v_2|+n_2|w_1-w_2|.
\end{aligned}
$$

Then, the system (3) and (4) has a unique solution on $[c_1,d_1]$ if

$$(\Phi_1+\Phi_4)(m_1+m_2)+(\Phi_2+\Phi_3)(n_1+n_2)<1, \qquad (22)$$

where Φ_i $(i=1,2,3,4)$ are given in (21).

Proof. The hypotheses of Banach's contraction mapping principle will be considered in the following steps:

Step 1. $\mathbb{B}(\mathfrak{B}_x) \subseteq \mathfrak{B}_x$ where $\mathfrak{B}_x = \{(v,w) \in \mathbb{X}^* \times \mathbb{X}^* : \|(v,w)\| \leq x\}$ with

$$x \geq \frac{(\Phi_1+\Phi_4)M + (\Phi_2+\Phi_3)N}{1-[(\Phi_1+\Phi_4)(m_1+m_2)+(\Phi_2+\Phi_3)(n_1+n_2)]},$$

$M = \sup_{q\in[c_1,d_1]} |\kappa_1(q,0,0)|$, $N = \sup_{q\in[c_1,d_1]} |\kappa_2(q,0,0)|$.

Step 2. $\mathbb{B}$ is a contraction.

For Step 1, let $(v,w) \in \mathfrak{B}_x$. Then, we obtain

$$\begin{aligned}
&|\mathbb{B}_1(v,w)(q)| \\
\leq\; & {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}|\kappa_1(q,v(q),w(q)) - \kappa_1(q,0,0)| + |\kappa_1(q,0,0)|\big)\big) \\
&+\frac{q-c_1}{|\mathbb{A}|}\Big[(d_1-c_1)\sum_{i=1}^{m}|\lambda_i|{}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}|\kappa_2(\eta_i,v(\eta_i),w(\eta_i)) - \kappa_1(\eta_i,0,0)| \\
&+|\kappa_2(\eta_i,0,0)|\big)\big) + (d_1-c_1){}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}|\kappa_1(d_1,v(d_1),w(d_1)) - \kappa_1(d_1,0,0)| \\
&+|\kappa_1(d_1,0,0)|\big)\big) + \mathbb{A}_1\Big(\sum_{j=1}^{n}|\overline{\lambda}_j|{}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}|\kappa_1(\overline{\eta}_j,v(\overline{\eta}_j),w(\overline{\eta}_j)) - \kappa_1(\overline{\eta}_j,0,0)| \\
&+|\kappa_1(\overline{\eta}_j,0,0)|\big)\big) + {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}|\kappa_2(d_1,v(d_1),w(d_1)) - \kappa_2(d_1,0,0)| \\
&+|\kappa_2(d_1,0,0)|\big)\big)\Big)\Big] \\
\leq\; & \frac{1}{\Gamma(\mu_1)}\int_{c_1}^{q}(q-s)^{\mu_1-1}\theta_{q_1}\Big(\int_{c_1}^{s}(m_1\|v\|+m_2\|w\|+M)\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\Big)ds \\
&+\frac{|(d_1-c_1)^2|}{\Gamma(\mu_2)|\mathbb{A}|} \\
&\times\sum_{i=1}^{m}|\lambda_i|\int_{c_1}^{\eta_i}(\eta_i-s)^{\mu_2-1}\theta_{q_2}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}(n_1\|v\|+n_2\|w\|+N)d\tau\Big)ds \\
&+\frac{(d_1-c_1)^2}{\Gamma(\mu_1)|\mathbb{A}|}\int_{c_1}^{d_1}(d_1-s)^{\mu_1-1}\theta_{q_1}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}(m_1\|v\|+m_2\|w\|+M)d\tau\Big)ds \\
&+\frac{|\mathbb{A}_1|(d_1-c_1)}{\mathbb{A}}\left(\sum_{j=1}^{n}|\overline{\lambda}_j|\frac{1}{\Gamma(\mu_1)}\int_{c_1}^{\overline{\eta}_j}(\overline{\eta}_j-s)^{\mu_1-1}\theta_{q_2}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}(m_1\|v\|\right. \\
&+m_2\|w\|+M)d\tau\Big)ds \\
&\left.+\frac{1}{\Gamma(\mu_2)}\int_{c_1}^{d_1}(d_1-s)^{\mu_2-1}\theta_{q_2}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}(n_1\|v\|+n_2\|w\|+N)d\tau\Big)ds\right) \\
\leq\; & (m_1\|v\|+m_2\|w\|+M)\left\{\frac{1}{\Gamma(\mu_1)}\int_{c_1}^{q}(q-s)^{\mu_1-1}\theta_{q_1}\Big(\frac{(s-c_1)^{\rho_1}}{\Gamma(\rho_1+1)}\Big)ds\right. \\
&+\frac{|d_1-c_1|^2}{|\mathbb{A}|}\frac{1}{\Gamma(\mu_1)}\int_{c_1}^{d_1}(d_1-s)^{\mu_1-1}\theta_{q_1}\Big(\frac{(s-c_1)^{\rho_1}}{\Gamma(\rho_1+1)}\Big)ds \\
&\left.+\frac{|\mathbb{A}_1||d_1-c_1|}{|\mathbb{A}|}\sum_{j=1}^{n}\frac{1}{\Gamma(\mu_1)}\int_{c_1}^{\overline{\eta}_j}(\overline{\eta}_j-s)^{\mu_2-1}\theta_{q_2}\Big(\frac{(s-c_1)^{\rho_1}}{\Gamma(\rho_1+1)}\Big)ds\right\} \\
&+(n_1\|v\|+n_2\|w\|+N)\left\{\frac{(d_1-c_1)^2}{|\mathbb{A}|\Gamma(\mu_2)}\sum_{i=1}^{m}|\lambda_i|\int_{c_1}^{\eta_i}(\eta_i-s)^{\mu_2-1}\theta_{q_2}\Big(\frac{(s-c_1)^{\rho_2}}{\Gamma(\rho_2+1)}\Big)ds\right. \\
&\left.+\frac{|\mathbb{A}_1||d_1-c_1|}{|A|\Gamma(\mu_2)}\int_{c_1}^{d_1}(d_1-s)^{\mu_2-1}\theta_{q_2}\Big(\frac{(s-c_1)^{\rho_2}}{\Gamma(\rho_2+1)}\Big)ds\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad (m_1\|v\| + m_2\|w\| + M)\Bigg\{ \frac{1}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}} \int_{c_1}^{q} (q-s)^{\mu_1-1}(s-c_1)^{\rho_1(q_1-1)}ds \\
&\quad + \frac{(d_1-c_1)^2}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)} \int_a^b (b-s)^{\mu_1-1}(s-c_1)^{\rho_1(q_1-1)}ds \\
&\quad + \frac{|\mathbb{A}_1|}{\Gamma(\rho_1+1)^{q_2-1}\Gamma(\mu_2)} \sum_{j=1}^{n} |\overline{\lambda}_j| \int_{c_1}^{\overline{\eta}_j} (\overline{\eta}_j - s)^{\mu_2-1}(s-c_1)^{\rho_1(q_2-1)}ds \Bigg\} \\
&\quad + (n_1\|v\| + n_2\|w\| + N)\Bigg\{ \frac{(d_1-c_1)^2}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}} \\
&\quad \times \sum_{i=1}^{n} |\lambda_i| \int_{c_1}^{\eta_i} (\eta_i - s)^{\mu_2-1}(s-c_1)^{\rho_2(q_2-1)}ds \\
&\quad + \frac{|A_1|(d_1-c_1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}} \int_{c_1}^{d_1} (d_1-s)^{\mu_2-1}(s-c_1)^{\rho_2(q_2-1)}ds \Bigg\} \\
&\leq \quad (m_1\|v\| + m_2\|w\| + M)\Bigg\{ \frac{\Gamma(\rho_1(q_1-1)+1)}{\Gamma(\mu_1)\Gamma(\rho_1+1)^{q_1-1}\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1} \\
&\quad + \frac{(d_1-c_1)^2\Gamma(\rho_1(q_1-1)+1)}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1} \\
&\quad + \frac{|A_1|\Gamma(\rho_1(q_2-1)+1)}{\Gamma(\rho_1+1)^{q_2-1}\Gamma(\mu_2)\Gamma(\rho_1(q_2-1)+\mu_2)} \sum_{j=1}^{n} |\overline{\lambda}_j|(\overline{\eta}_j - c_1)^{\rho_1(q_2-1)+\mu_2} \Bigg\} \\
&\quad + (n_1\|v\| + n_2\|w\| + N) \\
&\quad \times \Bigg\{ \frac{(d_1-c_1)^2\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(\eta_i - c_1)^{\mu_2+\rho_2(q_2-1)} \\
&\quad + \frac{|\mathbb{A}_1|(d_1-c_1)\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2} \Bigg\} \\
&= \quad (m_1\|v\| + m_2\|w\| + M)\Phi_1 + (n_1\|v\| + n_2\|w\| + N)\Phi_2 \\
&= \quad (m_1\Phi_1 + n_1\Phi_2)\|v\| + (m_2\Phi_1 + n_2\Phi_2)\|w\| + \Phi_1 M + \Phi_2 N \\
&\leq \quad (m_1\Phi_1 + n_1\Phi_2 + m_2\Phi_1 + n_2\Phi_2)x + \Phi_1 M + \Phi_2 N.
\end{aligned}
$$

Similarly, we can show that

$$
|\mathbb{B}_2(v,w)(q)| \leq (m_1\Phi_4 + n_1\Phi_3 + m_2\Phi_4 + n_2\Phi_3)x + \Phi_4 M + \Phi_3 N.
$$

Consequently, we obtain

$$
\begin{aligned}
\|\mathbb{B}(v,w)\| &= \|\mathbb{B}_1(v,w)\| + \|\mathbb{B}_2(v,w)\| \\
&\leq \big[(\Phi_1+\Phi_4)(m_1+m_2) + (\Phi_2+\Phi_3)(n_1+n_2)\big]x \\
&\quad + (\Phi_1+\Phi_4)M + (\Phi_2+\Phi_3)N \\
&\leq x,
\end{aligned}
$$

which implies that $\mathbb{B}(\mathfrak{B}_x) \subseteq \mathfrak{B}_x$.

Now, we prove Step 2, that is, the operator $\mathbb{B}$ is a contraction. Let $(v_1,w_1),(v_2,w_2) \in \mathbb{X}^* \times \mathbb{X}^*$ and $q \in [c_1,d_1]$. Then, we have

$$
\begin{aligned}
&|\mathbb{B}_1(v_1,w_1)(q) - \mathbb{B}_1(v_2,w_2)(q)| \\
\leq\ & {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}|\kappa_1(q,v_1(q),w_1(q)) - \kappa_1(q,v_2(q),w_2(q))|\big)\big) \\
&+ \frac{q-c_1}{|\mathbb{A}|}\Big[(d_1-c_1)\sum_{i=1}^{m}|\lambda_i|{}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}|\kappa_2(\eta_i,v_1(\eta_i),w_1(\eta_i))
\end{aligned}
$$

$$
\begin{aligned}
& -\kappa_2(\eta_i, v_2(\eta_i), w_2(\eta_i))))\Big) \\
& +(d_1-c_1)^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}|\kappa_1(d_1, v_1(d_1), w_1(d_1)) - \kappa_1(d_1, v_2(d_1), w_2(d_1))|\big)\big) \\
& +\mathbb{A}_1\Big(\sum_{j=1}^{n}|\overline{\lambda}_j|^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}|\kappa_1(\overline{\eta}_j, v_1(\overline{\eta}_j), w_1(\overline{\eta}_j)) - \kappa_1(\overline{\eta}_j, v_2(\overline{\eta}_j), w_2(\overline{\eta}_j))\big)\big)\big) \\
& +{}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}|\kappa_2(d_1, v_1(d_1), w_1(d_1)) - \kappa_2(d_1, v_2(d_1), w_2(d_1))|\big)\big)\Big)\Big] \\
\leq\ & \frac{1}{\Gamma(\mu_1)}\int_{c_1}^{q}(q-s)^{\mu_1-1}\theta_{q_1}\Big(\int_{c_1}^{s}(m_1\|v_1-w_1\|+m_2\|w_1-w_2\|)\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\Big)ds \\
& +\frac{|(d_1-c_1)^2|}{\Gamma(\mu_2)|\mathbb{A}|} \\
& \times\sum_{i=1}^{m}|\lambda_i|\int_{c_1}^{\eta_i}(\eta_i-s)^{\mu_2-1}\theta_{q_2}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}(n_1\|v_1-v_2\|+n_2\|w_1-w_2\|)d\tau\Big)ds \\
& +\frac{(d_1-c_1)^2}{\Gamma(\mu_1)|\mathbb{A}|}\int_{c_1}^{d_1}(d_1-s)^{\mu_1-1}\theta_{q_1}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}(m_1\|v_1-v_2\|+m_2\|w_1-w_2\|)d\tau\Big)ds \\
& +\frac{|\mathbb{A}_1|(d_1-c_1)}{\mathbb{A}}\Big(\sum_{j=1}^{n}|\overline{\lambda}_j|\frac{1}{\Gamma(\mu_1)}\int_{c_1}^{\overline{\eta}_j}(\overline{\eta}_j-s)^{\mu_1-1}\theta_{q_2}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}(m_1\|v_1-v_2\| \\
& +m_2\|w_1-w_2\|)d\tau\Big)ds \\
& +\frac{1}{\Gamma(\mu_2)}\int_{c_1}^{d_1}(d_1-s)^{\mu_2-1}\theta_{q_2}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}(n_1\|v_1-v_2\|+n_2\|w_1-w_2\|)d\tau)ds\Big)\Big) \\
\leq\ & (m_1\|v_1-v_2\|+m_2\|w_1-w_2\|)\Bigg\{\frac{1}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}}\int_{c_1}^{q}(q-s)^{\mu_1-1}(s-a)^{\rho_1(q_1-1)}ds \\
& +\frac{(d_1-c_1)^2}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)}\int_{a}^{b}(b-s)^{\mu_1-1}(s-c_1)^{\rho_1(q_1-1)}ds \\
& +\frac{|\mathbb{A}_1|}{\Gamma(\rho_1+1)^{q_2-1}\Gamma(\mu_2)}\sum_{j=1}^{n}|\overline{\lambda}_j|\int_{c_1}^{\overline{\eta}_j}(\overline{\eta}_j-s)^{\mu_2-1}(s-c_1)^{\rho_1(q_2-1)}ds\Bigg\} \\
& +(n_1\|v_1-v_2\|+n_2\|w_1-w_2\|)\Bigg\{\frac{(d_1-c_1)^2}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}} \\
& \times\sum_{i=1}^{n}|\lambda_i|\int_{c_1}^{\eta_i}(\eta_i-s)^{\mu_2-1}(s-c_1)^{\rho_2(q_2-1)}ds \\
& +\frac{|A_1|(d_1-c_1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}}\int_{c_1}^{d_1}(d_1-s)^{\mu_2-1}(s-c_1)^{\rho_2(q_2-1)}ds\Bigg\} \\
\leq\ & (m_1\|v_1-v_2\|+m_2\|w_1-w_2\|) \\
& \times\Bigg\{\frac{\Gamma(\rho_1(q_1-1)+1)}{\Gamma(\mu_1)\Gamma(\rho_1+)^{q_1-1}\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1} \\
& +\frac{(d_1-c_1)^2\Gamma(\rho_1(q_1-1)+1)}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1} \\
& +\frac{|A_1|\Gamma(\rho_1(q_2-1)+1)}{\Gamma(\rho_1+1)^{q_2-1}\Gamma(\mu_2)\Gamma(\rho_1(q_2-1)+\mu_2)}\sum_{j=1}^{n}|\overline{\lambda}_j|(\overline{\eta}_j-c_1)^{\rho_1(q_2-1)+\mu_2}\Bigg\} \\
& +(n_1\|v_1-v_2\|+n_2\|w_1-w_2\|) \\
& \times\Bigg\{\frac{(d_1-c_1)^2\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(\eta_i-c_1)^{\mu_2+\rho_2(q_2-1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\mathbb{A}_1|(d_1-c_1)\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2}\bigg\} \\
=\ & (m_1\|v_1-v_2\|+m_2\|w_1-w_2\|)\Phi_1+(n_1\|v_1-v_2\|+n_2\|w_1-w_2\|)\Phi_2 \\
=\ & (m_1\Phi_1+n_1\Phi_2)\|v_1-v_2\|+(m_2\Phi_1+n_2\Phi_2)\|w_1-w_2\|.
\end{aligned}
$$

Hence,

$$\|\mathbb{B}_1(v_2,w_2)-\mathbb{B}_1(v_1,w_1)\| \le (m_1\Phi_1+n_1\Phi_2+m_2\Phi_1+n_2\Phi_2)(\|v_2-v_1\|+\|w_2-w_1\|). \tag{23}$$

Similarly, one can find that

$$\|\mathbb{B}_2(v_2,w_2)-\mathbb{B}_2(v_1,w_1)\| \le (m_1\Phi_4+n_1\Phi_3+m_2\Phi_4+n_2\Phi_3)(\|v_2-v_1\|+\|w_2-w_1\|). \tag{24}$$

By (23) and (24), we infer that

$$
\begin{aligned}
& \|\mathbb{B}(v_2,w_2)-\mathbb{B}(v_1,w_1)\| \\
\le\ & ((\Phi_1+\Phi_4)(m_1+m_2)+(\Phi_2+\Phi_3)(n_1+n_2))(\|v_2-v_1\|+\|w_2-w_1\|),
\end{aligned} \tag{25}
$$

which implies that $\mathbb{B}$ is a contraction. Thus, by applying the Banach contraction mapping principle, the system (3) and (4) has a unique solution on $[c_1,d_1]$. The proof is completed. □

4.2. Existence Results

Now, two existence results for the system (3) and (4) are proved via Leray–Schauder alternative [21] and Krasnosel'skiĭ fixed point theorem [22].

Theorem 2. *Let $\kappa_1,\kappa_2 : [c_1,d_1]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be two continuous functions such that for all $q\in[c_1,d_1]$ and $v_i,w_i\in\mathbb{R}$, we have*

$$
\begin{aligned}
|\kappa_1(q,v_1,w_1)| &\le t_0+t_1|v_1|+t_2|w_1|, \\
|\kappa_2(q,v_2,w_2)| &\le u_0+u_1|v_2|+u_2|w_2|,
\end{aligned}
$$

where t_i,u_i are real constants with $t_0,u_0>0$. Then, the system (3) and (4) has at least one solution on $[c_1,d_1]$, provided that

$$(\Phi_1+\Phi_4)t_1+(\Phi_2+\Phi_3)u_1<1, \quad \text{and} \quad (\Phi_1+\Phi_4)t_2+(\Phi_2+\Phi_3)u_2<1, \tag{26}$$

where Φ_i, $i=1,2,3,4$ are defined in (21).

Proof. In view of the continuity property of the functions κ_1 and κ_2, we conclude that the operator $\mathbb{B}$ is continuous. Next, the completely continuous property of the operator $\mathbb{B}$ is showed. Let S be a bounded set of $\mathbb{X}^*\times\mathbb{X}^*$. Then, there exist positive constant $\mathbb{D}_1$ and $\mathbb{D}_2$ such that for all $(v,w)\in S$ we have $|\kappa_1(q,v(q),w(q))|\le\mathbb{D}_1$ and $|\kappa_2(q,v(q),w(q))|\le\mathbb{D}_2$. In consequence, for all $(u,v)\in S$, we have

$$
\begin{aligned}
& |\mathbb{B}_1(v,w)(q)| \\
\le\ & \mathbb{D}_1\bigg\{\frac{\Gamma(\rho_1(q_1-1)+1)}{\Gamma(\mu_1)\Gamma(\rho_1+)^{q_1-1}\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1} \\
&+\frac{(d_1-c_1)^2\Gamma(\rho_1(q_1-1)+1)}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1} \\
&+\frac{|A_1|\Gamma(\rho_1(q_2-1)+1)}{\Gamma(\rho_1+1)^{q_2-1}\Gamma(\mu_2)\Gamma(\rho_1(q_2-1)+\mu_2)}\sum_{j=1}^{n}|\overline{\lambda}_j|(\overline{\eta}_j-c_1)^{\rho_1(q_2-1)+\mu_2}\bigg\}
\end{aligned}
$$

$$+\mathbb{D}_2\bigg\{\frac{(d_1-c_1)^2\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(\eta_i-c_1)^{\mu_2+\rho_2(q_2-1)}$$
$$+\frac{|\mathbb{A}_1|(d_1-c_1)\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2}\bigg\},$$

which yields

$$\|\mathbb{B}_1(v,w)\|\leq\Phi_1\mathbb{D}_1+\Phi_2\mathbb{D}_2.$$

Similarly, we have

$$\|\mathbb{B}_2(v,w)\|\leq\Phi_3\mathbb{D}_1+\Phi_4\mathbb{D}_2.$$

Hence, we obtain

$$\|\mathbb{B}(v,w)\|=\|\mathbb{B}_1(v,w)\|+\|\mathbb{B}_2(v,w)\|\leq(\Phi_1+\Phi_3)\mathbb{D}_1+(\Phi_2+\Phi_4)\mathbb{D}_2.$$

Consequently, the uniformly boundedness property of the operator $\mathbb{B}$ is obtained. Now the equicontinuous property of the operator $\mathbb{B}$ is verified. Let $\bar{q}_1,\bar{q}_2\in[c_1,d_1]$ with $\bar{q}_1<\bar{q}_2$, Then, we obtain

$$\begin{aligned}
&|\mathbb{B}_1(v,w)(\bar{q}_2)-\mathbb{B}_1(v,w)(\bar{q}_1)|\\
\leq\;&\frac{\mathbb{D}_1}{\Gamma(\mu_1)}\int_{c_1}^{\bar{q}_1}\Big|(\bar{q}_2-s)^{\mu_1-1}-(\bar{q}_1-s)^{\mu_1-1}\Big|\theta_{q_1}\bigg(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\bigg)ds\\
&+\frac{\mathbb{D}_1}{\Gamma(\mu_1)}\int_{\bar{q}_1}^{\bar{q}_2}\Big|(\bar{q}_2-s)^{\mu_1-1}\Big|\theta_{q_1}\bigg(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\bigg)ds\\
&+\frac{(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_2)}(d_1-c_1)\mathbb{D}_1\sum_{i=1}^{m}|\lambda_i|\int_{c_1}^{\eta_i}(\eta_i-s)^{\mu_2-1}\theta_{q_2}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}d\tau\bigg)ds\\
&+\frac{(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_1)}\int_{c_1}^{d_1}(d_1-s)^{\mu_1-1}\theta_{q_1}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\bigg)ds\\
&+\frac{|\mathbb{A}_1|(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_1)}\sum_{j=1}^{n}|\lambda_j|\int_{c_1}^{\overline{\eta}_j}(\overline{\eta}_j-s)^{\mu_1-1}\theta_{q_1}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\bigg)ds\\
&+\frac{|\mathbb{A}_1|(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_2)}\int_{c_1}^{d_1}(d_1-s)^{\mu_2-1}\theta_{q_2}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}d\tau\bigg)ds\\
\leq\;&\frac{\mathbb{D}_1}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}}\int_{c_1}^{\bar{q}_1}\Big|(\bar{q}_2-s)^{\mu_1-1}-(\bar{q}_1-s)^{\mu_1-1}\Big|\theta_{q_1}\Big(\frac{(s-c_1)^{\rho_1}}{\Gamma(\rho_1+1)}\Big)ds\\
&+\frac{\mathbb{D}_1}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}}\int_{\bar{q}_1}^{\bar{q}_2}\Big|(\bar{q}_2-s)^{\mu_1-1}\Big|\theta_{q_1}\Big(\frac{(s-c_1)^{\rho_1}}{\Gamma(\rho_1+1)}\Big)ds\\
&+\frac{(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_2)}(d_1-c_1)\mathbb{D}_1\sum_{i=1}^{m}|\lambda_i|\int_{c_1}^{\eta_i}(\eta_i-s)^{\mu_2-1}\theta_{q_2}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}d\tau\bigg)ds\\
&+\frac{(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_1)}\int_{c_1}^{d_1}(d_1-s)^{\mu_1-1}\theta_{q_1}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\bigg)ds\\
&+\frac{|\mathbb{A}_1|(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_1)}\sum_{j=1}^{n}|\lambda_j|\int_{c_1}^{\overline{\eta}_j}(\overline{\eta}_j-s)^{\mu_1-1}\theta_{q_1}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\bigg)ds\\
&+\frac{|\mathbb{A}_1|(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_2)}\int_{c_1}^{d_1}(d_1-s)^{\mu_2-1}\theta_{q_2}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}d\tau\bigg)ds\\
\leq\;&\frac{\mathbb{D}_1}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}}\int_{c_1}^{\bar{q}_1}\Big|(\bar{q}_2-s)^{\mu_1-1}-(\bar{q}_1-s)^{\mu_1-1}\Big|(s-c_1)^{\rho_1}ds\\
&+\frac{\mathbb{D}_1}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}}\int_{\bar{q}_1}^{\bar{q}_2}\Big|(\bar{q}_2-s)^{\mu_1-1}\Big|(s-c_1)^{\rho_1}ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_2)}(d_1-c_1)\mathbb{D}_1\sum_{i=1}^{m}|\lambda_i|\int_{c_1}^{\eta_i}(\eta_i-s)^{\mu_2-1}\theta_{q_2}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}d\tau\bigg)ds\\
&+\frac{(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_1)}\int_{c_1}^{d_1}(d_1-s)^{\mu_1-1}\theta_{q_1}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\bigg)ds\\
&+\frac{|\mathbb{A}_1|(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_1)}\sum_{j=1}^{n}|\lambda_j|\int_{c_1}^{\overline{\eta}_j}(\overline{\eta}_j-s)^{\mu_1-1}\theta_{q_1}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\bigg)ds\\
&+\frac{|\mathbb{A}_1|(\bar{q}_2-\bar{q}_1)}{|\mathbb{A}|\Gamma(\mu_2)}\int_{c_1}^{d_1}(d_1-s)^{\mu_2-1}\theta_{q_2}\bigg(\int_{c_1}^{s}\mathbb{D}_1\frac{(s-\tau)^{\rho_2-1}}{\Gamma(\rho_2)}d\tau\bigg)ds\to 0,
\end{aligned}
$$

as $\bar{q}_1\to\bar{q}_2$, independently of $(v,w)\in S$. Hence, $\mathbb{B}_1(v,w)$ is equicontinuous. Similarly, we can show that $\mathbb{B}_2(v,w)$ is equicontinuous. Consequently, $\mathbb{B}(v,w)$ is equicontinuous.

Finally, it will be indicated that the set $\mathcal{E}=\{(v,w)\in\mathbb{X}^*\times\mathbb{X}^*;\|(v,w)\|=\lambda\mathbb{B}(v,w)\},0\le\lambda\le 1\}$ is bounded. Let $(v,w)\in\mathcal{E}$, then $\|(v,w)\|=\lambda\mathbb{B}(v,w)$ and for all $q\in[c_1,d_1]$ we have $v(q)=\lambda\mathbb{B}_1(v,w)(q)$ and $w(q)=\lambda\mathbb{B}_2(v,w)(q)$. Thus, we have

$$
\begin{aligned}
&|v(q)|\\
\le\;&(t_0+t_1\|v\|+t_2\|w\|)\bigg\{\frac{\Gamma(\rho_1(q_1-1)+1)}{\Gamma(\mu_1)\Gamma(\rho_1+)^{q_1-1}\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1}\\
&+\frac{(d_1-c_1)^2\Gamma(\rho_1(q_1-1)+1)}{|\mathbb{A}|\Gamma(\rho_1+1)^{q_1-1}\Gamma(\mu_1)\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1}\\
&+\frac{|A_1|\Gamma(\rho_1(q_2-1)+1)}{\Gamma(\rho_1+1)^{q_2-1}\Gamma(\mu_2)\Gamma(\rho_1(q_2-1)+\mu_2)}\sum_{j=1}^{n}|\overline{\lambda}_j|(\overline{\eta}_j-c_1)^{\rho_1(q_2-1)+\mu_2}\bigg\}\\
&+(u_0+u_1\|v\|+u_2\|w\|)\\
&\times\bigg\{\frac{(d_1-c_1)^2\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(\eta_i-c_1)^{\mu_2+\rho_2(q_2-1)}\\
&+\frac{|\mathbb{A}_1|(d_1-c_1)\Gamma(\rho_2(q_2-1)+1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2}\bigg\}.
\end{aligned}
$$

and consequently

$$\|v\|\le(t_0+t_1\|v\|+t_2\|w\|)\Phi_1+(u_0+u_1\|v\|+u_2\|w\|)\Phi_2.$$

Similarly, we obtain

$$\|w\|\le(t_0+t_1\|v\|+t_2\|w\|)\Phi_4+(u_0+u_1\|v\|+u_2\|w\|)\Phi_3.$$

Hence, we have

$$
\begin{aligned}
\|v\|+\|w\|\;\le\;&(\Phi_1+\Phi_4)t_0+(\Phi_2+\Phi_3)u_0+[((\Phi_1+\Phi_4)t_1+(\Phi_2+\Phi_3)u_1]\|v\|\\
&+[((\Phi_1+\Phi_4)t_2+(\Phi_2+\Phi_3)u_2]\|w\|,
\end{aligned}
$$

which implies that

$$\|(v,w)\|\le\frac{(\Phi_1+\Phi_4)t_0+(\Phi_2+\Phi_3)u_0}{\mathrm{N}_0},$$

where

$$\mathrm{N}_0=\min\{1-[(\Phi_1+\Phi_4)t_1+(\Phi_2+\Phi_3)u_1],1-[(\Phi_1+\Phi_4)t_2+(\Phi_2+\Phi_3)u_2]\}.$$

Thus, the Leray–Schauder alternative implies that the operator $\mathbb{B}$ has at least one fixed point. Hence, the system (3) and (4) has at least one solution on $[c_1,d_1]$. The proof is completed. □

Now, Krasnosel'skiĭ's fixed point theorem [22] is applied to prove the second existence result.

Theorem 3. *Let $\kappa_1, \kappa_2 : [c_1, d_1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be two continuous functions satisfying the condition (H_1) of Theorem 1. Moreover, suppose that*
(H_2) There exist $\mathcal{R}, \mathcal{S} \in C([c_1, d_1], \mathbb{R}_+)$ such that

$$|\kappa_1(q, v, w)| \leq \mathcal{R}(q), \ \ |\kappa_2(q, v, w)| \leq \mathcal{S}(q), \ for\ each\ (q, v, w) \in [c_1, d_1] \times \mathbb{R} \times \mathbb{R}.$$

Then, the problem (3) and (4) has at least one solution on $[c_1, d_1]$, provided that

$$[\Phi_1^* + \Phi_4](m_1 + m_2) + [\Phi_3^* + \Phi_2](n_1 + n_2) < 1. \tag{27}$$

Proof. We split the operator $\mathbb{B}$ into four operator $\mathbb{B}_{1,1}, \mathbb{B}_{1,2}, \mathbb{B}_{2,1}, \mathbb{B}_{2,2}$ as

$$\begin{aligned}
\mathbb{B}_{1,1}(v, w)(q) &= {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(q, v(q), w(q))\big)\big), \quad q \in [c_1, d_1], \\
\mathbb{B}_{1,2}(v, w)(q) &= \frac{q - c_1}{\mathbb{A}}\Big[(d_1 - c_1)\sum_{i=1}^{m}\lambda_i {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(\eta_i, v(\eta_i), w(\eta_i))\big)\big) \\
&\quad -(d_1 - c_1){}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(d_1, v(d_1), w(d_1))\big)\big) \\
&\quad +\mathbb{A}_1\Big(\sum_{j=1}^{n}\overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(\overline{\eta}_j, v(\overline{\eta}_j), w(\overline{\eta}_j))\big)\big)\Big) \\
&\quad -{}^{RL}\mathbb{I}^{\mu_2}\Big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(d_1, v(d_1), w(d_1))\big)\Big)\Big], \\
\mathbb{B}_{2,1}(v, w)(q) &= {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(q, v(q), w(q))\big)\big), \\
\mathbb{B}_{2,2}(v, w)(q) &= \frac{q - c_1}{\mathbb{A}}\Big[\mathbb{A}_2\Big(\sum_{i=1}^{m}\lambda_i {}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(\eta_i, v(\eta_i), w(\eta_i))\big)\big)\Big) \\
&\quad -(d_1 - c_1){}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(d_1, v(d_1), w(d_1))\big)\big) \\
&\quad +(d_1 - c_1)\sum_{j=1}^{n}\overline{\lambda}_j {}^{RL}\mathbb{I}^{\mu_1}\big(\theta_{q_1}\big({}^{RL}\mathbb{I}^{\rho_1}\kappa_1(\overline{\eta}_j, v(\eta_j), w(\eta_j))\big)\big) \\
&\quad -(d_1 - c_1){}^{RL}\mathbb{I}^{\mu_2}\big(\theta_{q_2}\big({}^{RL}\mathbb{I}^{\rho_2}\kappa_2(d_1, v(d_1), w(d_1))\big)\big)\Big].
\end{aligned}$$

Obviously, $\mathbb{B}_1 = \mathbb{B}_{1,1} + \mathbb{B}_{1,2}$ and $\mathbb{B}_2 = \mathbb{B}_{2,1} + \mathbb{B}_{2,2}$. Let $\mathbb{B}_x = \{(v, w) \in \mathbb{X}^* \times \mathbb{X}^*; \|(v, w)\| \leq x\}$ with $x \geq (\Phi_1 + \Phi_3)\|\mathcal{R}\| + (\Phi_2 + \Phi_4)\|\mathcal{S}\|$. According to the proof of Theorem 2, we have

$$\|\mathbb{B}_{1,1}(v_1, v_2) + \mathbb{B}_{1,2}(w_1, w_2)\| \leq \Phi_1\|\mathcal{R}\| + \Phi_2\|\mathcal{S}\|,$$

and

$$\|\mathbb{B}_{2,1}(v_1, v_2) + \mathbb{B}_{2,2}(w_1, w_2)\| \leq \Phi_3\|\mathcal{R}\| + \Phi_4\|\mathcal{S}\|.$$

Hence, $\mathbb{B}_1(v_1, v_2) + \mathbb{B}_2(w_1, w_2) \in \mathbb{B}_x$. Next, it will be proved that the operator $(\mathbb{B}_{1,2}, \mathbb{B}_{2,2})$ is a contraction. As in the proof of Theorem 1, for $(v_1, w_1), (v_2, w_2) \in B_x$ we have

$$\begin{aligned}
&|\mathbb{B}_{1,2}(v_1, v_2)(q) - \mathbb{B}_{1,2}(w_1, w_2)(q)| \\
\leq\ & (m_1\|v_1 - w_1\| + m_2\|v_2 - w_2\|) \\
&\times\Bigg\{\frac{(d_1 - c_1)^2}{|\mathbb{A}|\Gamma(\rho_1 + 1)^{q_1 - 1}\Gamma(\mu_1)}\int_{c_1}^{d_1}(b - s)^{\mu_1 - 1}(s - c_1)^{\rho_1(q_1 - 1)}ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\mathbb{A}_1|}{\Gamma(\rho_1+1)^{q_2-1}\Gamma(\mu_2)}\sum_{j=1}^{n}|\overline{\lambda}_j|\int_{c_1}^{\overline{\eta}_j}(\overline{\eta}_j-s)^{\mu_2-1}(s-c_1)^{\rho_1(q_2-1)}ds\Big\}\\
&+(n_1\|v_1-v_2\|+n_2\|w_1-w_2\|)\Big\{\frac{(d_1-c_1)^2}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}}\\
&\times\sum_{i=1}^{n}|\lambda_i|\int_{c_1}^{\eta_i}(\eta_i-s)^{\mu_2-1}(s-c_1)^{\rho_2(q_2-1)}ds\\
&+\frac{|A_1|(d_1-c_1)}{|\mathbb{A}|\Gamma(\mu_2)\Gamma(\rho_2+1)^{q_2-1}}\int_{c_1}^{d_1}(d_1-s)^{\mu_2-1}(s-c_1)^{\rho_2(q_2-1)}ds\Big\}\\
\leq\ &(\Phi_1^*(m_1\|v_1-w_1\|+\|v_2-w_2\|)+\Phi_2(n_1\|v_1-w_1\|+\|v_2-w_2\|)\\
=\ &[\Phi_1^*m_1+\Phi_2n_1]\|v_1-w_1\|+[\Phi_1^*m_2+\Phi_2n_2]\|v_2-w_2\|, \qquad (28)
\end{aligned}
$$

and consequently

$$
\begin{aligned}
&\|\mathbb{B}_{1,2}(v_1,v_2)-\mathbb{B}_{1,2}(w_1,w_2)\|\\
\leq\ &[\Phi_1^*m_1+\Phi_2n_1]\|v_1-w_1\|+[\Phi_1^*m_2+\Phi_2n_2]\|v_2-w_2\|.
\end{aligned}
$$

In a similar way, we obtain

$$
\begin{aligned}
&\|\mathbb{B}_{2,2}(v_1,v_2)-\mathbb{B}_{2,2}(w_1,w_2)\|\\
\leq\ &[\Phi_4m_1+\Phi_3^*n_1]\|v_1-w_1\|+[\Phi_4m_2+\Phi_3^*n_2]\|v_2-w_2\|. \qquad (29)
\end{aligned}
$$

By (28) and (29), we obtain

$$
\begin{aligned}
&\|(\mathbb{B}_{1,2},\mathbb{B}_{2,2})(v_1,v_2)-(\mathbb{B}_{1,2},\mathbb{B}_{2,2})(w_1,w_2)\|\\
\leq\ &\Big\{[\Phi_1^*+\Phi_4](m_1+m_2)+[\Phi_2+\Phi_3^*](n_1+n_2)\Big\}(\|v_1-w_1\|+\|v_2-w_2\|), \qquad (30)
\end{aligned}
$$

which, by applying the condition (27), the contraction property of the operator $(\mathbb{B}_{1,2},\mathbb{B}_{2,2})$ is obtained. On the other hand, in view of the continuity property of κ_1 and κ_2 , the operator $(\mathbb{B}_{1,1},\mathbb{B}_{2,1})$ is continuous. In addition,

$$
\begin{aligned}
&\|(\mathbb{B}_{1,2},\mathbb{B}_{2,1})(v,w)\|\\
\leq\ &\frac{\Gamma(\rho_1(q_1-1)+1)}{\Gamma(\mu_1)\Gamma(\rho_1+)^{q_1-1}\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1}\|\mathcal{R}\|\\
&+\frac{\Gamma(\rho_2(q_2-1)+1)}{\Gamma(\mu_2)\Gamma(\rho_2+)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2}\|\mathcal{S}\|,
\end{aligned}
$$

as

$$
\|\mathbb{B}_{1,2}(v,w)\|\leq\frac{\Gamma(\rho_1(q_1-1)+1)}{\Gamma(\mu_1)\Gamma(\rho_1+)^{q_1-1}\Gamma(\rho_1(q_1-1)+\mu_1)}(d_1-c_1)^{\rho_1(q_1-1)+\mu_1}\|\mathcal{R}\|,
$$

and

$$
\|\mathbb{B}_{2,1}(v,w)\|\leq\frac{\Gamma(\rho_2(q_2-1)+1)}{\Gamma(\mu_2)\Gamma(\rho_2+)^{q_2-1}\Gamma(\rho_2(q_2-1)+\mu_2)}(d_1-c_1)^{\rho_2(q_2-1)+\mu_2}\|\mathcal{S}\|.
$$

Thus, $(\mathbb{B}_{1,1},\mathbb{B}_{2,1})$ is uniformly bounded. In the final step, we prove that the operator $(\mathbb{B}_{1,1},\mathbb{B}_{2,1})\mathbb{B}_x$ is equicontinuous. For $\bar{q}_1,\bar{q}_2\in[c_1,d_1]$ with $\bar{q}_1<\bar{q}_2$ and for all $(v,w)\in\mathbb{B}_x$, we have

$$
\begin{aligned}
&|\mathbb{B}_{1,1}(v,w)(\bar{q}_2)-\mathbb{B}_{1,1}(v,w)(\bar{q}_1)|\\
\leq\ &\frac{\|\mathcal{R}\|}{\Gamma(\mu_1)}\int_{c_1}^{\bar{q}_1}\Big|(\bar{q}_2-s)^{\mu_1-1}-(\bar{q}_1-s)^{\mu_1-1}\Big|\theta_{q_1}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\|\mathcal{R}\|}{\Gamma(\mu_1)}\int_{\bar{q}_1}^{\bar{q}_2}\Big|(\bar{q}_2-s)^{\mu_1-1}\Big|\theta_{q_1}\Big(\int_{c_1}^{s}\frac{(s-\tau)^{\rho_1-1}}{\Gamma(\rho_1)}d\tau\Big)ds\\
\le\ &\frac{\|\mathcal{R}\|}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}}\int_{c_1}^{\bar{q}_1}\Big|(\bar{q}_2-s)^{\mu_1-1}-(\bar{q}_1-s)^{\mu_1-1}\Big|\theta_{q_1}\Big(\frac{(s-c_1)^{\rho_1}}{\Gamma(\rho_1+1)}\Big)ds\\
&+\frac{\|\mathcal{R}\|}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}}\int_{\bar{q}_1}^{\bar{q}_2}\Big|(\bar{q}_2-s)^{\mu_1-1}\Big|\theta_{q_1}\Big(\frac{(s-c_1)^{\rho_1}}{\Gamma(\rho_1+1)}\Big)ds\\
\le\ &\frac{\|\mathcal{R}\|}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}}\int_{c_1}^{\bar{q}_1}\Big|(\bar{q}_2-s)^{\mu_1-1}-(\bar{q}_1-s)^{\mu_1-1}\Big|(s-c_1)^{\rho_1}ds\\
&+\frac{\|\mathcal{R}\|}{\Gamma(\mu_1)(\Gamma(\rho_1+1))^{q_1-1}}\int_{\bar{q}_1}^{\bar{q}_2}\Big|(\bar{q}_2-s)^{\mu_1-1}\Big|(s-c_1)^{\rho_1}ds\to 0,
\end{aligned}
$$

independently of $(v,w)\in\mathbb{B}_x$. Similarly, one can find that

$$|\mathbb{B}_{2,1}(v,w)(\bar{q}_2)-\mathbb{B}_{2,1}(v,w)(\bar{q}_1)|\longrightarrow 0\ \ as\ \ \bar{q}_1\longrightarrow\bar{q}_2.$$

Thus, $|(\mathbb{B}_{1,1},\mathbb{B}_{2,1})(\bar{q}_2)-(\mathbb{B}_{1,1},\mathbb{B}_{2,1})(\bar{q}_1)|\longrightarrow 0$ as $\bar{q}_1\longrightarrow\bar{q}_2$. Hence, the operator $(\mathbb{B}_{1,1},\mathbb{B}_{1,2})$ is equicontinuous. Arzelá–Ascoli theorem implies that the operator $(\mathbb{B}_{1,1},\mathbb{B}_{2,1})$ is compact on $\mathbb{B}_x$ and hence by applying Krasnosel'skiǐ's fixed point theorem, we conclude that the operator $\mathbb{B}$ has at least one fixed point, which is a solution of the system (3) and (4). The proof is finished. □

5. Examples

In this section, we illustrate our main results by considering the following system of nonlinear differential equations consisting of the Caputo fractional-order derivatives of the form

$$
\begin{cases}
{}^{C}\mathcal{D}^{\frac{1}{2}}\big(\theta_2\big[{}^{C}\mathcal{D}^{\frac{4}{3}}\sigma_1(w)\big]\big)=\kappa_1(w,\sigma_1(w),\sigma_2(w)), & w\in[1/4,11/4],\\
{}^{C}\mathcal{D}^{\frac{1}{3}}\big(\theta_3\big[{}^{C}\mathcal{D}^{\frac{5}{3}}\sigma_2(w)\big]\big)=\kappa_2(w,\sigma_1(w),\sigma_2(w)),
\end{cases}
\tag{31}
$$

with coupled boundary conditions as

$$
\begin{cases}
\sigma_1\left(\frac{1}{4}\right)={}^{C}\mathcal{D}^{\frac{4}{3}}\sigma_1\left(\frac{1}{4}\right)=0,\\
\sigma_1\left(\frac{11}{4}\right)=\frac{1}{11}\sigma_2\left(\frac{1}{2}\right)+\frac{2}{13}\sigma_2\left(\frac{3}{4}\right)+\frac{3}{17}\sigma_2\left(\frac{5}{4}\right),\\
\sigma_2\left(\frac{1}{4}\right)={}^{C}\mathcal{D}^{\frac{5}{3}}\sigma_2\left(\frac{1}{4}\right)=0,\\
\sigma_2\left(\frac{11}{4}\right)=\frac{4}{19}\sigma_1\left(\frac{3}{2}\right)+\frac{5}{23}\sigma_1\left(\frac{7}{4}\right)+\frac{6}{29}\sigma_1\left(\frac{9}{4}\right)+\frac{7}{31}\sigma_1\left(\frac{5}{2}\right).
\end{cases}
\tag{32}
$$

Comparing problem (31) and (32) with (3) and (4), we have $\rho_1=1/2$, $\rho_2=1/3$, $\mu_1=4/3$, $\mu_2=5/3$, $p_1=2$, $p_2=3$, $q_1=2$, $q_2=3/2$, $c_1=1/4$, $d_1=11/4$, $m=3$, $\lambda_1=1/11$, $\lambda_2=2/13$, $\lambda_3=3/17$, $\eta_1=1/2$, $\eta_2=3/4$, $\eta_3=5/4$, $n=4$, $\overline{\lambda}_1=4/19$, $\overline{\lambda}_2=5/23$, $\overline{\lambda}_3=6/29$, $\overline{\lambda}_4=7/31$,$\overline{\eta}_1=3/2$, $\overline{\eta}_2=7/4$, $\overline{\eta}_3=9/4$, $\overline{\eta}_4=5/2$. By using the Maple program, we obtain $\mathbb{A}_1\approx 0.2761209379$, $\mathbb{A}_2\approx 1.511102471$, $\mathbb{A}\approx 5.832752968$, $\Phi_1\approx 13.37197426$, $\Phi_2\approx 1.880065187$, $\Phi_3\approx 12.91632487$, $\Phi_4\approx 10.85384001$, $\Phi_1^*\approx 6.985157994$ and $\Phi_3^*\approx 6.713970944$.

(i) Assume that two nonlinear functions are given by

$$\kappa_1(w,\sigma_1,\sigma_2)=\frac{2}{5(4w+63)}\left(\frac{\sigma_1^2+2|\sigma_1|}{1+|\sigma_1|}\right)+\frac{256}{(4w+11)^4}\left(\frac{|\sigma_2|}{1+|\sigma_2|}\right)+\frac{1}{4},\tag{33}$$

and

$$\kappa_2(w,\sigma_1,\sigma_2) = \frac{1}{2(8w+35)}|\sin\sigma_1| + \frac{1}{38(4w+1)^2}\left(\frac{\sigma_2^2+2|\sigma_2|}{1+|\sigma_2|}\right) + \frac{1}{5}. \tag{34}$$

Observe that both of them are unbounded and also satisfy the Lipschitz condition as

$$|\kappa_1(w,\sigma_1,\sigma_2) - \kappa_1(w,\delta_1,\delta_2)| \leq \frac{1}{80}|\sigma_1-\delta_1| + \frac{1}{81}|\sigma_2-\delta_2|,$$

and

$$|\kappa_2(w,\sigma_1,\sigma_2) - \kappa_2(w,\delta_1,\delta_2)| \leq \frac{1}{74}|\sigma_1-\delta_1| + \frac{1}{76}|\sigma_2-\delta_2|,$$

with Lipschitz constants $m_1 = 1/80$, $m_2 = 1/81$, $n_1 = 1/74$ and $n_2 = 1/76$. Since

$$(\Phi_1+\Phi_4)(m_1+m_2) + (\Phi_2+\Phi_3)(n_1+n_2) \approx 0.9965473651 < 1.$$

by Theorem 1, the problem (31) and (32), with functions κ_1, κ_1 given in (33) and (34), has a unique solution on the interval $[1/4, 11/4]$.

(ii) If two nonlinear functions are defined by

$$\kappa_1(w,\sigma_1,\sigma_2) = \frac{1}{2} + \frac{1}{2(4w+20)}\left(\frac{|\sigma_1|^{2023}}{1+\sigma_1^{2022}}\right) + \frac{1}{5(4w+7)}|\sigma_2|e^{-\sigma_1^2}, \tag{35}$$

and

$$\kappa_2(w,\sigma_1,\sigma_2) = \frac{1}{3} + \frac{1}{28(w+1)}|\sigma_1|\sin^{26}(\pi\sigma_2^4) + \frac{1}{2\pi(2w+9)}|\sigma_2|\tan^{-1}\sigma_1^8, \tag{36}$$

then they are bounded, since

$$|\kappa_1(w,\sigma_1,\sigma_2)| \leq \frac{1}{2} + \frac{1}{42}|\sigma_1| + \frac{1}{40}|\sigma_2|,$$

and

$$|\kappa_2(w,\sigma_1,\sigma_2)| \leq \frac{1}{3} + \frac{1}{35}|\sigma_1| + \frac{1}{38}|\sigma_2|.$$

By setting constants $t_0 = 1/2$, $t_1 = 1/42$, $t_2 = 1/40$, $u_0 = 1/3$, $u_1 = 1/35$, $u_2 = 1/38$, the inequalities (26) are satisfied as

$$(\Phi_1+\Phi_4)t_1 + (\Phi_2+\Phi_3)u_1 \approx 0.9995591034 < 1$$

and

$$(\Phi_1+\Phi_4)t_2 + (\Phi_2+\Phi_3)u_2 \approx 0.9950240426 < 1.$$

By Theorem 2, the fractional p-Laplacian coupled system with multi-point boundary conditions (31) and (32), with κ_1, κ_2 given in (35) and (36), has at least one solution on the interval $[1/4, 11/4]$.

(iii) Consider the nonlinear functions

$$\kappa_1(w,\sigma_1,\sigma_2) = \frac{1}{6} + \frac{1}{8(w+6)}\left(\frac{|\sigma_1|}{1+|\sigma_1|}\right) + \frac{1}{13(4w+1)^2}\sin|\sigma_2|, \tag{37}$$

and

$$\kappa_2(w,\sigma_1,\sigma_2) = \frac{1}{8} + \frac{1}{48(w+1)}\tan^{-1}|\sigma_1| + \frac{1}{29(4w+1)}\left(\frac{|\sigma_2|}{1+|\sigma_2|}\right). \tag{38}$$

It is clear that both of them are bounded by

$$|\kappa_1(w,\sigma_1,\sigma_2)| \leq \frac{1}{6} + \frac{1}{8(w+6)} + \frac{1}{13(4w+1)^2} := \mathcal{R}(w),$$

and

$$|\kappa_2(w,\sigma_1,\sigma_2)| \leq \frac{1}{8} + \frac{\pi}{96(w+1)} + \frac{1}{29(4w+1)} := \mathcal{S}(w).$$

In addition, they satisfy the Lipschitz condition as

$$|\kappa_1(w,\sigma_1,\sigma_2) - \kappa_1(w,\delta_1,\delta_2)| \leq \frac{1}{50}|\sigma_1 - \delta_1| + \frac{1}{52}|\sigma_2 - \delta_2|,$$

and

$$|\kappa_2(w,\sigma_1,\sigma_2) - \kappa_2(w,\delta_1,\delta_2)| \leq \frac{1}{60}|\sigma_1 - \delta_1| + \frac{1}{58}|\sigma_2 - \delta_2|,$$

with Lipschitz constants $m_1 = 1/50$, $m_2 = 1/52$, $n_1 = 1/60$, $n_2 = 1/58$. Unfortunately, the inequality

$$(\Phi_1 + \Phi_4)(m_1 + m_2) + (\Phi_2 + \Phi_3)(n_1 + n_2) \approx 1.452114004 > 1,$$

is failed and cannot be used to guarantee the uniqueness result. However, we have

$$(\Phi_1^* + \Phi_4)(m_1 + m_2) + (\Phi_2 + \Phi_3^*)(n_1 + n_2) \approx 0.9912445862 < 1,$$

which means that the inequality in (27) holds. Consequently, by Theorem 3, the fractional p-Laplacian system (31) and (32), with two nonlinear bounded Lipschitz functions (37) and (38), has at least one solution on the interval $[1/4, 11/4]$.

6. Conclusions

In this paper, the existence and uniqueness results have been established for a fractional p-Laplacian coupled system of nonlinear differential equations involving the Caputo fractional-order derivatives, subjected to multi-point boundary conditions. The fixed point technique has been applied to the given system by transforming it into a fixed point problem. Our problem is novel and some other problems are special cases of our problem. For example, by taking $\mu_1 = \mu_2 = p_1 = p_2 = 2$ and $\rho_1 = \rho_2 = 1$, our problem corresponds to the problem (5). For future work, we plan to extend the results of this paper to other kinds of boundary value problems.

Author Contributions: Conceptualization, S.K.N.; methodology, A.S., S.K.N. and J.T.; validation, A.S., S.K.N. and J.T.; formal analysis, A.S., S.K.N. and J.T.; writing—original draft preparation, A.S., S.K.N. and J.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Science, Research and Innovation Fund (NSRF), and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-66-11.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Hilfer R. (Ed.) *Applications of Fractional Calculus in Physics*; World Scientific Publishing Company: Singapore, 2000.
2. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematics Studies: Amsterdam, The Netherlands, 2006.
3. Miller, K.S.; Ross, B. *An Introduction to Fractional Calculus and Fractional Differential Equations*; A Wiley-Interscience Publication: New York, NY, USA, 1993.
4. Oldham, K.B. Fractional differential equations in electrochemistry. *Adv. Eng. Softw.* **2010**, *41*, 9–12 [CrossRef]
5. Sabatier, J.; Agrawal, O.P.; Machado, J.A.T. *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*; Springer: Dordrecht, The Netherlands, 2007.
6. Diethelm, K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer Science, Business Media: Berlin/Heidelberg, Germany, 2010.

7. Ma, R. A survey on nonlocal boundary value problems. *Appl. Math. E-Notes* **2007**, *7*, 257–279.
8. Ahmad, B.; Alsaedi, A.; Ntouyas, S.K.; Tariboon, J. *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*; Springer: Cham, Switzerland, 2017.
9. Ahmad, B.; Alghanmi, M.; Ntouyas, S.K.; Alsaedi, A. Fractional differential equations involving generalized derivativewith Stieltjes and fractional integral boundary conditions. *Appl. Math. Lett.* **2018**, *84*, 111–117. [CrossRef]
10. Leibenson, L.S. General problem of the movement of a compressible fluid in a porous medium. *Bull. Acad. Sci. URSS. Ser. Geograph. Geophys.* **1945**, *9*, 7–10.
11. Liu, X.; Jia, M.; Xiang, X. On the solvability of a fractional differential equation model involving the p-Laplacian operator. *Comput. Math. Appl.* **2012**, *64*, 3267–3275. [CrossRef]
12. Merzoug, I.; Guezane-Lakoud, A.; Khaldi, R. Existence of solutions for a nonlinear fractional p-Laplacian boundary value problem. *Rend. Circ. Mat. Palermo Ser. II* **2020**, *69*, 1099–1106. [CrossRef]
13. Tan, J.; Cheng, C. Existence of solutions of boundary value problems for fractional differential equations with p-Laplacian operator in Banach spaces. *Numer. Funct. Anal. Optim.* **2017**, *38*, 738–753. [CrossRef]
14. Tan, J.; Li, M. Solutions of fractional differential equations with p-Laplacian operator in Banach spaces. *Bound. Value Probl.* **2018**, *15*, 1–13. [CrossRef]
15. Liu, X.; Jia, M.; Ge, W. The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator. *Appl. Math. Lett.* **2017**, *65*, 56–62. [CrossRef]
16. Srivastava, H.M.; Abbas,M.I.; Boutiara, A.; Hazarika, B. Fractional p-Laplacian differential equations with multi-point boundary conditions in Banach spaces. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **2023**, *117*, 1–16. [CrossRef]
17. Faieghi, M.; Kuntanapreeda, S.; Delavari, H.; Baleanu, D. LMI-based stabilization of a class of fractional-order chaotic systems. *Nonlinear Dyn.* **2013**, *72*, 301–309. [CrossRef]
18. Zhang, F.; Chen, G.; Li, C.; Kurths, J. Chaos synchronization in fractional differential systems. *Philos. T. R. Soc.* **2013**, *371*, 20120155. [CrossRef]
19. Xu,Y.; Li, W. Finite-time synchronization of fractional-order complex-valued coupled systems. *Phys. A Stat. Mech. Appl.* **2020**, *549*, 123903. [CrossRef]
20. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering*; Academic Press: New York, NY, USA, 1999; Volume 198.
21. Granas,A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2005.
22. Krasnosel'skiĭ, M.A. Two remarks on the method of successive approximations. *Uspekhi Mat. Nauk.* **1955**, *10*, 123–127.