axioms

Article

An Extended Hilbert-Type Inequality with Two Internal Variables Involving One Partial Sums

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Abstract: By the use of the techniques of analysis and some useful formulas, we give a new extension of Hilbert-type inequality with two internal variables involving one partial sums, which is a refinement of a published inequality. We provide a few equivalent conditions of the best possible constant related to multi parameters. We obtain the equivalent inequalities, the operator expressions as well as a few inequalities with the particular parameters as applications.

Keywords: weight coefficient; parameter; partial sums; equivalent form; mid-value theorem

MSC: 26D15; 47A05

1. Introduction

Assuming that $p > 1 (q > 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m,\ b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, we have the well known Hardy-Hilbert's inequality as follows (ref. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}} \tag{1}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

In 2006, by putting $\lambda_i \in (0,2] (i = 1,2)$, $\lambda_1 + \lambda_2 = \lambda \in (0,4]$, Krnic et al. [2] gave a generalization of Inequality (1) as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\ \lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}, \tag{2}$$

with the best possible constant factor $B(\lambda_1,\ \lambda_2)$ ($B(u,v) := \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt\,(u,v > 0)$ is the Beta function). For $p = q = 2$, $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$ in Inequality (2), it deduces to the inequality in Yang's paper [3].

Inequalities (1) and (2) with their integral analogues played an important role in analysis and its applications (ref. [4–14]). By using the weight functions in 2016, Hong et al. [15] obtained a few equivalent conditions of the extension of Inequality (1) with the best constant factor related to multi parameters. Some further results were provided by [16–25].

In 2019, by using Inequality (2), Adiyasuren et. al. [26] gave an extension of Inequality (2) involving partial sums: if $\lambda_i \in (0,1] \cap (0,\lambda)$ $(\lambda \in (0,2];\, i = 1,2)$, $\lambda_1 + \lambda_2 = \lambda$, $a_m,\ b_n \geq 0$, then it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < \lambda_1\lambda_2 B(\lambda_1,\ \lambda_2)\left(\sum_{m=1}^{\infty} m^{-p\lambda_1-1} A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-q\lambda_2-1} B_n^q\right)^{\frac{1}{q}}, \tag{3}$$

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with the best constant $\lambda_1\lambda_2 B(\lambda_1,\ \lambda_2)$, involving two partial sums $A_m := \sum_{i=1}^{m} a_i$ and $B_n := \sum_{k=1}^{n} b_k$ $(m,n \in \{1,2,\ \cdots\})$, $A_m = o(e^{tm})$, $B_n = o(e^{tn})$ $(t > 0;\ m,n \to \infty)$, such that

$$0 < \sum_{m=1}^{\infty} m^{-p\lambda_1 - 1} A_m^p < \infty \text{ and } 0 < \sum_{n=1}^{\infty} n^{-q\lambda_2 - 1} B_n^q < \infty. \tag{4}$$

In 2021, Liao et al. [27] gave an extension of Inequality (3) with the kernel as $\frac{1}{(m^\alpha + n^\beta)^\lambda}(\alpha, \beta \in (0,1])$ involving one partial sums. But the constant factor in this inequality seems not to be the best possible unless at $\alpha = \beta = 1$. In 2023, by using the mid-value theorem, [28,29] gave a new inequality as well as the reverse with the same kernel as [27] involving two partial sums, and proved that the constant factor is the best possible in some conditions.

In this article, by applying the methods in [28], and using the techniques of introduced parameters, some useful formulas and the mid-value theorem, we give a new extension of Hardy-Hilbert's inequality with the internal variables involving one partial sums, which is a refinement of the inequality in [27]. We also provide a few equivalent statements of the best possible constant factor related to several parameters. As applications, we obtain the equivalent inequalities, the operator expressions as well as a few inequalities with the particular parameters. The lemmas and theorems provide an extensive account of this type of inequalities.

2. Some Lemmas

In what follows of this article, we assume that $p > 1 (q > 1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in (0,5]$, $\alpha, \beta \in (0,1]$, $\lambda_1 \in (-1, \frac{2}{\alpha} - 1] \cap (-1, \lambda)$, $\lambda_2 \in (0, \frac{2}{\beta}] \cap (0, \lambda + 1)$, $\hat{\lambda}_1 := \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}$, $\hat{\lambda}_2 := \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}$. We still suppose that a_m, $b_n \geq 0$, $A_m := \sum_{j=1}^{m} a_j$, $(m, n \in \mathrm{N} = \{1,2,\ \cdots\})$, such that $A_m = o(e^{tm^\alpha})$ $(t > 0; m \to \infty)$, and

$$0 < \sum_{m=1}^{\infty} m^{-p\alpha\hat{\lambda}_1 - 1} A_m^p < \infty,\ 0 < \sum_{n=1}^{\infty} n^{q(1-\beta\hat{\lambda}_2) - 1} b_n^q < \infty. \tag{5}$$

For showing the main results, we give the following key lemma by using the mid-value theorem:

Lemma 1. *If $t > 0$, then the following inequality holds:*

$$\sum_{m=1}^{\infty} e^{-tm^\alpha} a_m \leq t\alpha \sum_{m=1}^{\infty} e^{-tm^\alpha} m^{\alpha-1} A_m. \tag{6}$$

Proof. For $A_m e^{-tm^\alpha} = o(1)(m \to \infty)$, in view of Abel's summation formula, we obtain

$$\begin{aligned}\sum_{m=1}^{\infty} e^{-tm^\alpha} a_m &= \lim_{m\to\infty} A_m e^{-tm^\alpha} + \sum_{m=1}^{\infty} A_m \left[e^{-tm^\alpha} - e^{-t(m+1)^\alpha}\right] \\ &= \sum_{m=1}^{\infty} A_m \left[e^{-tm^\alpha} - e^{-t(m+1)^\alpha}\right].\end{aligned}$$

We set function $g(x) = e^{-tx^\alpha}$, $x \in [m, m+1]$. Then we find that $g\prime(x) = -t\alpha x^{\alpha-1} e^{-tx^\alpha}$, and for $x \in [m, m+1]$, $\alpha \in (0,1]$, the function $h(x) := x^{\alpha-1} e^{-tx^\alpha}$ is decreasing. In view of the mid-value theorem, it follows that

$$\begin{aligned}\sum_{m=1}^{\infty} e^{-tm^\alpha} a_m &= -\sum_{m=1}^{\infty} A_m (g(m+1) - g(m)) \\ &= -\sum_{m=1}^{\infty} A_m g\prime(m+\theta) = t\alpha \sum_{m=1}^{\infty} h(m+\theta) A_m \\ &\leq t\alpha \sum_{m=1}^{\infty} h(m) A_m = t\alpha \sum_{m=1}^{\infty} m^{\alpha-1} e^{-tm^\alpha} A_m (\theta \in (0,1)),\end{aligned}$$

namely, Equation (6) follows.

The lemma is proved. □

For showing the inequalities in Lemma 3, we need the following lemma:

Lemma 2. (ref. [4], (2.2.3)). *(i) Assuming that* $(-1)^k \frac{d^k}{dt^k} g(t) > 0$, $t \in [m, \infty)$, $g^{(k)}(\infty) = 0$ $(k = 0, 1, 2, 3)$, $P_j(t)$, $B_j (j \in \mathrm{N})$ *are Bernoulli functions and Bernoulli numbers of j-order, it follows that*

$$\int_m^\infty P_{2q-1}(t)g(t)dt = -\varepsilon_q \frac{B_{2q}}{2q} g(m) (0 < \varepsilon_q < 1; q = 1, 2, \cdots). \tag{7}$$

In particular, for $q = 1$, $B_2 = \frac{1}{6}$, *we have*

$$-\frac{1}{12}g(m) < \int_m^\infty P_1(t)g(t)dt < 0; \tag{8}$$

for $q = 2$, $B_4 = -\frac{1}{30}$, *we have*

$$0 < \int_m^\infty P_3(t)g(t)dt < \frac{1}{120}g(m). \tag{9}$$

(ii) (ref. [4], *(2.3.2)) If* $f(t)(> 0) \in C^3[m, \infty)$, $f^{(k)}(\infty) = 0 (k = 0, 1, 2, 3)$, *then the following Euler-Maclaurin summation formula holds*:

$$\sum_{k=m}^\infty f(k) = \int_m^\infty f(t)dt + \frac{1}{2}f(m) + \int_m^\infty P_1(t)f'(t)dt, \tag{10}$$

$$\int_m^\infty P_1(t)f'(t)dt = -\frac{1}{12}f'(m) + \frac{1}{6}\int_m^\infty P_3(t)f'''(t)dt. \tag{11}$$

Lemma 3. *For* $s \in (0, 6]$, $s_2 \in (0, \frac{2}{\beta}] \cap (0, s)$, $k_s(s_2) := B(s_2, s - s_2)$, *indicate the weight coefficient as follows*:

$$\varpi_s(s_2, m) := m^{\alpha(s-s_2)} \sum_{n=1}^\infty \frac{\beta n^{\beta s_2 - 1}}{(m^\alpha + n^\beta)^s} (m \in \mathrm{N}). \tag{12}$$

We have

$$0 < k_s(s_2)(1 - O(\frac{1}{m^{\alpha s_2}})) < \varpi_s(s_2, m) < k_s(s_2)(m \in \mathrm{N}). \tag{13}$$

where, we indicate $O(\frac{1}{m^{\alpha s_2}}) := \frac{1}{k_s(s_2)} \int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s} du > 0$.

Proof. For fixed $m \in \mathrm{N}$, we define $g(m, t)$ as follows:

$$g(m, t) := \frac{\beta t^{\beta s_2 - 1}}{(m^\alpha + t^\beta)^s} (t > 0).$$

In view of Inequality (10), it follows that

$$\begin{aligned} \sum_{n=1}^\infty g(m, n) &= \int_1^\infty g(m, t)dt + \tfrac{1}{2}g(m, 1) + \int_1^\infty P1(t)g'(m, t)dt \\ &= \int_0^\infty g(m, t)dt - h(m), \\ h(m) &:= \int_0^1 g(m, t)dt - \tfrac{1}{2}g(m, 1) - \int_1^\infty P1(t)g'(m, t)dt. \end{aligned}$$

We obtain that $-\frac{1}{2}g(m,1) = \frac{-\beta}{2(m^{\alpha}+1)^{s}}$ and find

$$\begin{aligned}
&\int_0^1 g(m,t)dt = \beta\int_0^1 \frac{t^{\beta s_2-1}}{(m^{\alpha}+t^{\beta})^{s}}dt \overset{u=t^{\beta}}{=} \int_0^1 \frac{u^{s_2-1}}{(m^{\alpha}+u)^{s}}du \\
&= \frac{1}{s_2}\int_0^1 \frac{du^{s_2}}{(m^{\alpha}+u)^{s}} = \frac{1}{s_2}\frac{u^{s_2}}{(m^{\alpha}+u)^{s}}|_0^1 + \frac{s}{s_2}\int_0^1 \frac{u^{s_2}}{(m^{\alpha}+u)^{s+1}}du \\
&= \frac{1}{s_2}\frac{1}{(m^{\alpha}+1)^{s}} + \frac{s}{s_2(s_2+1)}\int_0^1 \frac{du^{s_2+1}}{(m^{\alpha}+u)^{s+1}} \\
&> \frac{1}{s_2}\frac{1}{(m^{\alpha}+1)^{s}} + \frac{s}{s_2(s_2+1)}\Big[\frac{u^{s_2+1}}{(m^{\alpha}+u)^{s+1}}\Big]_0^1 + \frac{s(s+1)}{s_2(s_2+1)(m^{\alpha}+1)^{s+2}}\int_0^1 u^{s_2+1}du \\
&= \frac{1}{s_2}\frac{1}{(m^{\alpha}+1)^{s}} + \frac{\lambda}{s_2(s_2+1)}\frac{1}{(m^{\alpha}+1)^{s+1}} + \frac{s(s+1)}{s_2(s_2+1)(s_2+2)}\frac{1}{(m^{\alpha}+1)^{s+2}}, \\
&-g'(m,t) = -\frac{\beta(\beta s_2-1)t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s}} + \frac{\beta^2 s t^{\beta+\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s+1}} \\
&= -\frac{\beta(\beta s_2-1)t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s}} + \frac{\beta^2 s(m^{\alpha}+t^{\beta}-m^{\alpha})t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s+1}} = \frac{\beta(\beta s-\beta s_2+1)t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s}} - \frac{\beta^2 s m^{\alpha} t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s+1}}.
\end{aligned}$$

For $0 < s_2 \le \frac{2}{\beta}, 0 < \beta \le 1,\ s_2 < s \le 6$, it follows that

$$(-1)^i \frac{d^i}{dt^i}\Big[\frac{t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s}}\Big] > 0, (-1)^i \frac{d^i}{dt^i}\Big[\frac{t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s+1}}\Big] > 0 (i = 0,1,2,3).$$

By Inequalities (8)–(11), we obtain

$$\begin{aligned}
&\beta(\beta s - \beta s_2 + 1)\int_1^{\infty} P_1(t)\frac{t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s}}dt > -\frac{\beta(\beta s-\beta s_2+1)}{12(m^{\alpha}+1)^{s}}, \\
&-\beta^2 m^{\alpha} s\int_1^{\infty} P_1(t)\frac{t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s+1}}dt \\
&= \frac{\beta^2 m^{\alpha} s}{12(m^{\alpha}+1)^{s+1}} - \frac{\beta^2 m^{\alpha} s}{6}\int_1^{\infty} P_3(t)\Big[\frac{t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s+1}}\Big]'' dt \\
&> \frac{\beta^2 m^{\alpha} s}{12(m^{\alpha}+1)^{s+1}} - \frac{\beta^2 m^{\alpha} s}{720}\Big[\frac{t^{\beta s_2-2}}{(m^{\alpha}+t^{\beta})^{s+1}}\Big]''_{t=1} \\
&> \frac{\beta^2(m^{\alpha}+1-1)s}{12(m^{\alpha}+1)^{s+1}} - \frac{\beta^2(m^{\alpha}+1)s}{720}\Big[\frac{(s+1)(s+2)\beta^2}{(m^{\alpha}+1)^{s+3}} + \frac{\beta(s+1)(5-\beta-2\beta s_2)}{(m^{\alpha}+1)^{s+2}} + \frac{(2-\beta s_2)(3-\beta s_2)}{(m^{\alpha}+1)^{s+1}}\Big] \\
&= \frac{\beta^2 s}{12(m^{\alpha}+1)^{s}} - \frac{\beta^2 s}{12(m^{\alpha}+1)^{s+1}} \\
&\quad -\frac{\beta^2 s}{720}\Big[\frac{(s+1)(s+2)\beta^2}{(m^{\alpha}+1)^{s+2}} + \frac{\beta(s+1)(5-\beta-2\beta s_2)}{(m^{\alpha}+1)^{s+1}} + \frac{(2-\beta s_2)(3-\beta s_2)}{(m^{\alpha}+1)^{s}}\Big].
\end{aligned}$$

Then it follows that $h(m) > \frac{1}{(m^{\alpha}+1)^{s}}h_1 + \frac{\lambda}{(m^{\alpha}+1)^{s+1}}h_2 + \frac{s(s+1)}{(m^{\alpha}+1)^{s+2}}h_3$, where, we set

$$h_1 := \frac{1}{s_2} - \frac{\beta}{2} - \frac{\beta - \beta^2 s_2}{12} - \frac{\beta^2 s(2-\beta s_2)(3-\beta s_2)}{720},$$

$$h_2 := \frac{1}{s_2(s_2+1)} - \frac{\beta^2}{12} - \frac{\beta^3(s+1)(5-\beta-2\beta s_2)}{720},$$

and $h_3 := \frac{1}{s_2(s_2+1)(s_2+2)} - \frac{\beta^4(s+2)}{720}$. We find

$$h_1 \ge \frac{1}{s_2} - \frac{\beta}{2} - \frac{\beta-\beta^2 s_2}{12} - \frac{s\beta^2(2-\beta s_2)(3-\beta s_2)}{720} = \frac{g(s_2)}{720 s_2},$$

where, we indicate the function $g(\sigma)(\sigma \in (0, \frac{2}{\beta}])$ as follows:

$$g(\sigma) := 720 - (420\beta + 6s\beta^2)\sigma + (60\beta^2 + 5s\beta^3)\sigma^2 - s\beta^4\sigma^3.$$

We obtain that for $\beta \in (0,1], s \in (0,6]$,

$$\begin{aligned}
g'(\sigma) &= -(420\beta + 6s\beta^2) + 2(60\beta^2 + 5s\beta^3)\sigma - 3\beta^4\sigma^2 \\
&\le -420\beta - 6s\beta^2 + 2(60\beta^2 + 5s\beta^3)\frac{2}{\beta} = (14s\beta - 180)\beta < 0,
\end{aligned}$$

and then it follows that $h_1 \geq \frac{g(s_2)}{720s_2} \geq \frac{g(2/\beta)}{720s_2} = \frac{1}{6s_2} > 0$. We find that for $s_2 \in (0, \frac{2}{\beta}]$,

$$h_2 > \frac{\beta^2}{6} - \frac{\beta^2}{12} - \frac{5(s+1)\beta^2}{720} = (\frac{1}{12} - \frac{s+1}{140})\beta^2 > 0,$$

and $h_3 \geq (\frac{1}{24} - \frac{s+2}{720})\beta^3 > 0 (0 < s \leq 6)$.

Therefore, we have $h(m) > 0$, and then setting $u = m^{-\alpha}t^{\beta}$, it follows that

$$\begin{aligned} \varpi_s(s_2, m) &= m^{\alpha(s-s_2)} \sum_{n=1}^{\infty} g(m,n) < m^{\alpha(s-s_2)} \int_0^{\infty} g(m,t)dt \\ &= \beta m^{\alpha(s-s_2)} \int_0^{\infty} \frac{t^{\beta s_2 - 1}dt}{(m^\alpha + t^\beta)^s} = \int_0^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} = B(s_2, s-s_2) = k_s(s_2). \end{aligned}$$

By Inequality (10), it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)dt + \tfrac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt \\ &= \int_1^{\infty} g(m,t)dt + H(m), \\ H(m) &:= \tfrac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt. \end{aligned}$$

We have fond that $\frac{1}{2}g(m,1) = \frac{\beta}{2(m^\alpha+1)^s}$, and

$$g'(m,t) = -\frac{\beta(\beta s - \beta s_2 + 1)t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s} + \frac{\beta^2 s m^\alpha t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}}.$$

For $s_2 \in (0, \frac{2}{\beta}] \cap (0, s), 0 < s \leq 6$, in view of Inequality (7), it follows that

$$-\beta(\beta s - \beta s_2 + 1)\int_1^{\infty} P_1(t)\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^s}dt > 0,$$

and

$$\beta^2 m^\alpha s \int_1^{\infty} P_1(t)\frac{t^{\beta s_2 - 2}}{(m^\alpha + t^\beta)^{s+1}}dt > -\frac{\beta^2 m^\alpha s}{12(m^\alpha + 1)^{s+1}} > -\frac{\beta^2 s}{12(m^\alpha + 1)^s}.$$

Hence, we have

$$H(m) > \frac{\beta}{2(m^\alpha + 1)^s} - \frac{\beta^2 s}{12(m^\alpha + 1)^s} \geq \frac{\beta}{2(m^\alpha + 1)^s} - \frac{6\beta}{12(m^\alpha + 1)^s} = 0,$$

and then it follows that

$$\begin{aligned} \varpi_s(s_2, m) &= m^{\alpha(s-s_2)} \sum_{n=1}^{\infty} g(m,n) > m^{\alpha(s-s_2)} \int_1^{\infty} g(m,t)dt \\ &= m^{\alpha(s-s_2)} \int_0^{\infty} g(m,t)dt - m^{\alpha(s-s_2)} \int_0^{1} g(m,t)dt \\ &= k_s(s_2)\left[1 - \frac{1}{k_s(s_2)} \int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s}du\right] > 0, \end{aligned}$$

where, $O(\frac{1}{m^{\alpha s_2}}) = \frac{1}{k_s(s_2)} \int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s}du$, satisfying

$$0 < \int_0^{\frac{1}{m^\alpha}} \frac{u^{s_2-1}}{(1+u)^s}du < \int_0^{\frac{1}{m^\alpha}} u^{s_2-1}du = \frac{1}{s_2 m^{\alpha s_2}}.$$

Hence, Inequalities (13) follow.

This proves the lemma. □

By applying the above lemma, we obtain an extended Hardy-Hilbert's inequality as follows.

Lemma 4. *The following inequality is valid:*

$$I_{\lambda+1} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m^{\alpha-1}}{(m^{\alpha}+n^{\beta})^{\lambda+1}}A_m b_n < (\tfrac{1}{\beta}k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(\tfrac{1}{\alpha}k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}} \times\left(\sum_{m=1}^{\infty}m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{q(1-\beta\hat{\lambda}_2)-1}b_n^q\right)^{\frac{1}{q}}. \tag{14}$$

Proof. By the symmetry, for $s_1 \in (0, \frac{2}{\alpha}] \cap (0, s), s \in (0, 6]$, the inequalities for the next weight coefficient is obtained as follows

$$0 < k_s(s_1)(1 - O(\tfrac{1}{n^{\beta s_1}})) < \omega_s(s_1, n) := n^{\beta(s-s_1)}\sum_{m=1}^{\infty}\frac{\alpha m^{\alpha s_1-1}}{(m^{\alpha}+n^{\beta})^{s}} < k_s(s_1) := B(s_1, s-s_1)(n \in \mathrm{N}), \tag{15}$$

where, we set $O(\frac{1}{n^{\beta s_1}}) := \frac{1}{k_s(s_1)}\int_0^{\frac{1}{n^{\beta}}}\frac{u^{s_1-1}}{(1+u)^{s}}du > 0(n \in \mathrm{N})$.

By using Hö lder's inequality (ref. [30]), we find

$$\begin{aligned} I_{\lambda+1} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m^{\alpha}+n^{\beta})^{\lambda+2}}\left[\frac{m^{\alpha(-\lambda_1)/q}(\beta n^{\beta-1})^{1/p}m^{\alpha-1}}{n^{\beta(1-\lambda_2)/p}(\alpha m^{\alpha-1})^{1/q}}A_m\right]\left[\frac{n^{\beta(1-\lambda_2)/p}(\alpha m^{\alpha-1})^{1/q}}{m^{\alpha(-\lambda_1)/q}(\beta n^{\beta-1})^{1/p}}b_n\right] \\ &\le \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\beta m^{p(\alpha-1)}}{(m^{\alpha}+n^{\beta})^{\lambda+1}}\frac{m^{\alpha(-\lambda_1)(p-1)}n^{\beta-1}A_m^p}{n^{\beta(1-\lambda_2)}(\alpha m^{\alpha-1})^{p-1}}\right]^{\frac{1}{p}} \\ &\quad\times\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\alpha}{(m^{\alpha}+n^{\beta})1}\frac{n^{\beta(1-\lambda_2)(q-1)}m^{\alpha-1}b_n^q}{m^{\alpha(-\lambda_1)}(\beta n^{\beta-1})^{q-1}}\right]^{\frac{1}{q}} \\ &= \left(\frac{1}{\beta}\sum_{m=1}^{\infty}\varpi_{\lambda+1}(\lambda_2, m)m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}} \\ &\quad\times\left(\frac{1}{\alpha}\sum_{n=1}^{\infty}\omega_{\lambda+1}(\lambda_1+1, n)n^{q(1-\beta\hat{\lambda}_2)-1}b_n^q\right)^{\frac{1}{q}}. \end{aligned}$$

Then by Inequalities (13) and (15) (for $s = \lambda + 1,\ s_1 = \lambda_1 + 1,\ s_2 = \lambda_2$), in view of Inequality (6), we have Inequality (14).

This proves the lemma. □

3. Main Results

In view of Lemma 1 and Lemma 3, we obtain the following main results:

Theorem 1. *We have an inequality as follows:*

$$I := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\lambda}} < (\tfrac{\alpha}{\beta})^{\frac{1}{p}}\lambda(k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}} \times\left(\sum_{m=1}^{\infty}m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\hat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{16}$$

In particular, for $\lambda_1 + \lambda_2 = \lambda$, we have

$$0 < \sum_{m=1}^{\infty}m^{-p\alpha\lambda_1-1}A_m^p < \infty,\ 0 < \sum_{n=1}^{\infty}n^{q(1-\beta\lambda_2)-1}b_n^q < \infty, \tag{17}$$

and the following inequality:

$$\begin{aligned} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\beta)^\lambda} &< (\tfrac{\alpha}{\beta})^{\frac{1}{p}}\lambda B(\lambda_1+1,\lambda_2) \\ &\times\left(\sum_{m=1}^{\infty} m^{-p\alpha\lambda_1-1}A_m^p\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\beta\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \quad (18)$$

Proof. In view of the following equality relate to the Gamma function:

$$\frac{1}{(m^\alpha+n^\beta)^\lambda}=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(m^\alpha+n^\beta)t}dt,$$

and Inequality (6), we obtain

$$\begin{aligned} I &= \tfrac{1}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n\int_0^\infty t^{\lambda-1}e^{-(m^\alpha+n^\beta)t}dt \\ &= \tfrac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\left(\sum_{m=1}^{\infty} e^{-m^\alpha t}a_m\right)\left(\sum_{n=1}^{\infty} e^{-n^\beta t}b_n\right)dt \\ &\le \tfrac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}\left(t\alpha\sum_{m=1}^{\infty} e^{-m^\alpha t}m^{\alpha-1}A_m\right)\left(\sum_{n=1}^{\infty} e^{-n^\beta t}b_n\right)dt \\ &= \tfrac{\alpha}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\alpha-1}A_m b_n\int_0^\infty t^{\lambda}e^{-(m^\alpha+n^\beta)t}dt \\ &= \alpha\tfrac{\Gamma(\lambda+1)}{\Gamma(\lambda)}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^{\alpha-1}}{(m^\alpha+n^\beta)^{\lambda+1}}A_m b_n. \end{aligned}$$

Hence by Inequality (14), it follows that Inequality (16) is valid.
This proves the theorem. □

In the following two theorems, we give a few equivalent conditions of the best value related to multi parameters in Inequality (16).

Theorem 2. *Suppose that* $\lambda\in(0,5]$, $\lambda_1\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda)$, $\lambda_2\in(0,\frac{2}{\beta}]\cap(0,\lambda)$. *If* $\lambda_1+\lambda_2=\lambda$, *then the constant factor* $(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda\,(k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}$ *is the best possible in Inequality (16).*

Proof. Now, we prove that

$$(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda B(\lambda_1+1,\lambda_2)(=(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda_1 B(\lambda_1,\lambda_2))$$

is the best possible constant in Inequality (18).

For any $\varepsilon\in(0,\min\{p\lambda_1,q\lambda_2\})$, we put

$$\widetilde{a}_m:=m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1},\ \widetilde{b}_n:=n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}(m,n\in\mathrm{N}).$$

Since $0<\lambda_1-\frac{\varepsilon}{p}\le\frac{2}{\alpha}-1, 0<\alpha(\lambda_1-\frac{\varepsilon}{p})\le 2-\alpha<2$, in view of (2.2.24) (ref. [5]), we find

$$\begin{aligned} \widetilde{A}_m &:= \sum_{i=1}^{m}\widetilde{a}_i=\sum_{i=1}^{m} i^{\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)-1}=\int_1^m t^{\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)-1}dt \\ &\quad+\tfrac{1}{2}\left[m^{\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)-1}+1\right]+\tfrac{\varepsilon_0}{12}\left[\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)-1\right]\left[m^{\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)-2}-1\right] \\ &= \frac{1}{\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)}\left(m^{\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)}+c_1+O_1\left(m^{\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)-1}\right)\right) \\ &\le \frac{m^{\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)}}{\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)}\left(1+|c_1|m^{-\alpha\left(\lambda_1-\frac{\varepsilon}{p}\right)}+\left|O_1\left(m^{-1}\right)\right|\right)(\varepsilon_0\in(0,1);m\in\mathrm{N},\mathrm{m}\to\infty) \end{aligned}$$

(c_1 is a constant). We observe that $\widetilde{A}_m = o(e^{tm^\alpha})$ ($t > 0$; m $\to \infty$).

If there exists a positive constant $M(\leq (\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda B(\lambda_1+1,\ \lambda_2))$, such that Inequality (18) is value as we replace $(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda B(\lambda_1+1,\ \lambda_2)$ by M. Then, substitution of $a_m = \widetilde{a}_m$, $b_n = \widetilde{b}_n$ and $A_m = \widetilde{A}_m$ in Inequality (18), we find

$$\widetilde{I} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{(m^\alpha+n^\beta)^\lambda} < M\left(\sum_{m=1}^{\infty} m^{-p\alpha\lambda_1-1}\widetilde{A}_m^p\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\beta\lambda_2)-1}\widetilde{b}_n^q\right]^{\frac{1}{q}}. \tag{19}$$

Setting $a(x) \to 0(x\to\infty)$, we find

$$\lim_{x\to\infty}\frac{(1+a(x))^p-1}{a(x)} = \lim_{x\to\infty}\frac{p(1+a(x))^{p-1}a'(x)}{a'(x)} = \lim_{x\to\infty}p(1+a(x))^{p-1} = p,$$

namely, $(1+a(x))^p = 1+O(a(x))(x\to\infty)$. Then we obtain

$$\begin{aligned}&(1+|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})}+|O_1(m^{-1})|)p = 1+O(|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})}+|O^1(m^{-1})|)\\ &= 1+O(m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})})+\widetilde{O}(m^{-1})(m\in \mathrm{N}; m\to\infty).\end{aligned}$$

Hence, we have

$$\begin{aligned}\sum_{m=1}^{\infty} m^{-p\alpha\lambda_1-1}\widetilde{A}_m^p &\leq \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1}(1+|c_1|m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})}+|O_1(m^{-1})|)^p\\ &= \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1}(1+O(m^{-\alpha(\lambda_1-\frac{\varepsilon}{p})})+\widetilde{O}(m^{-1}))\\ &= \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\left[\sum_{m=1}^{\infty} m^{-\alpha\varepsilon-1}+\sum_{m=1}^{\infty}(O(m^{-\alpha(\lambda_1+\frac{\varepsilon}{p})-1}+\widetilde{O}(m^{-\alpha\varepsilon-2}))\right]\\ &= \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\left(\sum_{m=2}^{\infty} m^{-\alpha\varepsilon-1}+O_1(1)\right)\\ &< \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p\left(\int_1^{\infty} x^{-\alpha\varepsilon-1}dx+O_1(1)\right) = \left[\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\right]^p(\frac{1}{\alpha\varepsilon}+O_1(1)).\end{aligned}$$

We still find that

$$\sum_{n=1}^{\infty} n^{q(1-\beta\lambda_2)-1}\widetilde{b}_n^q = 1+\sum_{n=2}^{\infty} n^{-\beta\varepsilon-1} < 1+\int_1^{\infty} y^{-\beta\varepsilon-1}dy = 1+\frac{1}{\beta\varepsilon}.$$

Then, we obtain $\widetilde{I} < \frac{M}{\varepsilon}\frac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\left(\frac{1}{\alpha}+\varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\frac{1}{\beta}+\varepsilon\right)^{\frac{1}{q}}$.

In view of Inequality (15) (for $s=\lambda\in(0,5]$, $s_1=\lambda_1-\frac{\varepsilon}{p}\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda)$), we find

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}}{(m^\alpha+n^\beta)^\lambda}n^{\beta(\lambda_2-\frac{\varepsilon}{q})-1} = \frac{1}{\alpha}\sum_{n=1}^{\infty} n^{-\beta\varepsilon-1}\left[n^{\beta(\lambda_2+\frac{\varepsilon}{q})}\sum_{m=1}^{\infty}\frac{\alpha m^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}}{(m^\alpha+n^\beta)^\lambda}\right]\\ &\geq \frac{1}{\alpha}k_\lambda(\lambda_1-\frac{\varepsilon}{p})\sum_{n=1}^{\infty} n^{-\beta\varepsilon-1}\left(1-O(\frac{1}{n^{\beta(\lambda_1-\frac{\varepsilon}{p})}})\right)\\ &= \frac{1}{\alpha}k_\lambda(\lambda_1-\frac{\varepsilon}{p})\left(\sum_{n=1}^{\infty} n^{-\beta\varepsilon-1}-\sum_{n=1}^{\infty}O(\frac{1}{n^{\beta(\lambda_1+\frac{\varepsilon}{q})+1}})\right)\\ &> \frac{1}{\alpha}k_\lambda(\lambda_1-\frac{\varepsilon}{p})(\int_1^{\infty} y^{-\beta\varepsilon-1}dy-O_2(1))\\ &= \frac{1}{\alpha\varepsilon}B(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p})\left(\frac{1}{\beta}-\varepsilon O_2(1)\right).\end{aligned}$$

Based on the above results, we have

$$\begin{aligned}&\tfrac{1}{\alpha}B(\lambda_1-\tfrac{\varepsilon}{p},\lambda_2+\tfrac{\varepsilon}{p})\left(\tfrac{1}{\beta}-\varepsilon O_2(1)\right)\\&\quad<\varepsilon\widetilde{I}<M\tfrac{1}{\alpha(\lambda_1-\frac{\varepsilon}{p})}\left(\tfrac{1}{\alpha}+\varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\tfrac{1}{\beta}+\varepsilon O_2(1)\right)^{\frac{1}{q}}.\end{aligned}$$

Setting $\varepsilon\to 0^+$, based on the continuity of the Beta function, we find

$$\frac{1}{\alpha\beta}B(\lambda_1,\lambda_2)\leq M\frac{1}{\alpha\lambda_1}\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\beta}\right)^{\frac{1}{q}},$$

namely,

$$(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda B(\lambda_1+1,\ \lambda_2)=(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda_1 B(\lambda_1,\ \lambda_2)\leq M.$$

Hence, the constant factor $M=(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda B(\lambda_1+1,\ \lambda_2)$ in Inequality (18) is the best possible. The theorem is proved. □

Theorem 3. *Assume that* $\lambda\in(0,5]$, $\lambda_1\in(0,\frac{2}{\alpha}-1]\cap(0,\lambda)$, $\lambda_2\in(0,\frac{2}{\beta}]\cap(0,\lambda)$. *If the constant factor* $(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda\ (k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}$ *in Inequality (16) is the best value, then for*

$$\lambda-\lambda_1-\lambda_2\leq\min\{p(\frac{2}{\alpha}-1-\lambda_1),q(\frac{2}{\beta}-\lambda_2)\},$$

we have $\lambda_1+\lambda_2=\lambda$.

Proof. For $\hat{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}=\frac{\lambda-\lambda_1-\lambda_2}{p}+\lambda_1$, $\hat{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\frac{\lambda-\lambda_1-\lambda_2}{q}+\lambda_2$, we find $\hat{\lambda}_1+\hat{\lambda}_2=\lambda$, and $0<\hat{\lambda}_1,\ \hat{\lambda}_2<\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda$. For $\lambda-\lambda_1-\lambda_2\leq p(\frac{2}{\alpha}-1-\lambda_1)$, we have $\hat{\lambda}_1\leq\frac{2}{\alpha}-1$; for $\lambda-\lambda_1-\lambda_2\leq q(\frac{2}{\beta}-\lambda_2)$, we find $\hat{\lambda}_2\leq\frac{2}{\beta}$. For $\lambda_i=\hat{\lambda}_i(i=1,2)$ in Inequality (18), we obtain

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\beta)^\lambda}<(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda B(\hat{\lambda}_1+1,\hat{\lambda}_2)\\&\times\left(\sum_{m=1}^{\infty}m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\hat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned}\tag{20}$$

In view of Hö lder's inequality (ref. [30]), we obtain

$$\begin{aligned}&B(\hat{\lambda}_1+1,\hat{\lambda}_2)=k_{\lambda+1}(\tfrac{\lambda-\lambda_2}{p}+\tfrac{\lambda_1}{q}+1)\\&=\int_0^\infty\tfrac{1}{(1+u)^{\lambda+1}}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}}du=\int_0^\infty\tfrac{1}{(1+u)^{\lambda+1}}\left(u^{\frac{\lambda-\lambda_2}{p}}\right)\left(u^{\frac{\lambda_1}{q}}\right)du\\&\geq\left[\int_0^\infty\tfrac{1}{(1+u)^{\lambda+1}}u^{\lambda-\lambda_2}du\right]^{\frac{1}{p}}\left[\int_0^\infty\tfrac{1}{(1+u)^{\lambda+1}}u^{\lambda_1}du\right]^{\frac{1}{q}}\\&=\left[\int_0^\infty\tfrac{1}{(1+v)^{\lambda+1}}v^{\lambda_2-1}dv\right]^{\frac{1}{p}}\left[\int_0^\infty\tfrac{1}{(1+u)^{\lambda+1}}u^{\lambda_1}du\right]^{\frac{1}{q}}\\&=(k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}.\end{aligned}\tag{21}$$

If constant factor $(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda\ (k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}$ in Inequality (16) is the best possible, then by Inequalities (16) and (20), it follows that

$$(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda(k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}\leq(\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda B(\hat{\lambda}_1+1,\hat{\lambda}_2)(\in\mathrm{R}_+),$$

which means that

$$B(\hat{\lambda}_1+1,\hat{\lambda}_2) \geq (k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}. \tag{22}$$

Then by Inequality (21), we have

$$B(\hat{\lambda}_1+1,\hat{\lambda}_2) = (k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}};$$

Hence, Inequality (21) keeps the form of equality.
This proves the theorem. □

Remark 1. *Since the constant in Inequality (18) is the best value, inequality Inequality (18) is a refinement of the inequalities in [16].*

4. Equivalent Inequalities and Operator Expressions

Theorem 4. *The following inequality is valid equivalent to Inequality (16):*

$$\begin{aligned} J &:= \left\{\sum_{n=1}^{\infty} n^{p\beta\hat{\lambda}_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\beta)^\lambda}\right]^p\right\}^{\frac{1}{p}} \\ &< (\tfrac{\alpha}{\beta})^{\frac{1}{p}}\lambda(k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}\left(\sum_{m=1}^{\infty} m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}}. \end{aligned} \tag{23}$$

For $\lambda_1+\lambda_2=\lambda$, the following inequality is valid, which is equivalent to Inequality (18):

$$\begin{aligned} &\left\{\sum_{n=1}^{\infty} n^{p\beta\lambda_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\beta)^\lambda}\right]^p\right\}^{\frac{1}{p}} \\ &< (\tfrac{\alpha}{\beta})^{\frac{1}{p}}\lambda B(\lambda_1+1,\lambda_2)\left(\sum_{m=1}^{\infty} m^{-p\alpha\lambda_1-1}A_m^p\right)^{\frac{1}{p}}. \end{aligned} \tag{24}$$

Proof. Assuming that Inequality (23) is value, by Hö lder's inequality, we find

$$I = \sum_{n=1}^{\infty}\left[n^{-\frac{1}{p}+\beta\hat{\lambda}_2}\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\beta)^\lambda}\right](n^{\frac{1}{p}-\beta\hat{\lambda}_2}b_n) \leq J\left[\sum_{n=1}^{\infty} n^{q(1-\beta\hat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{25}$$

By Inequalities (16) and (23) follows. Assuming that Inequality (16) is valid, setting

$$b_n := n^{p\beta\hat{\lambda}_2-1}\left[\sum_{m=1}^{\infty}\frac{a_m}{(m^\alpha+n^\beta)^\lambda}\right]^{p-1}, n \in \mathrm{N},$$

it follows that

$$\sum_{n=1}^{\infty} n^{q(1-\beta\hat{\lambda}_2)-1}b_n^q = J^p = I. \tag{26}$$

If $J=\infty$, then it is impossible that makes Inequality (23) value, namely, $J<\infty$; if $J=0$, then Inequality (23) is naturally valid. Assume that $0<J<\infty$. In view of Inequality (16), we find

$$J^p = I < (\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda(k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}\left(\sum_{m=1}^{\infty} m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}}J^{p-1},$$

$$J < (\frac{\alpha}{\beta})^{\frac{1}{p}}\lambda(k_{\lambda+1}(\lambda_2))^{\frac{1}{p}}(k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}\left(\sum_{m=1}^{\infty} m^{-p\alpha\hat{\lambda}_1-1}A_m^p\right)^{\frac{1}{p}}.$$

Hence, we have Inequality (23), which is equivalent to Inequality (16).
This proves the theorem. □

By the equivalency of Inequalities (16) and (23), we have

Theorem 5. *Assume that*

$$\lambda \in (0,5],\ \lambda_1 \in (0, \frac{2}{\alpha} - 1] \cap (0, \lambda),\ \lambda_2 \in (0, \frac{2}{\beta}] \cap (0, \lambda).$$

If $\lambda_1 + \lambda_2 = \lambda$, then $(\frac{\alpha}{\beta})^{\frac{1}{p}} \lambda\, (k_{\lambda+1}(\lambda_2))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1+1))^{\frac{1}{q}}$ is the best possible constant factor in Inequality (23). If the same constant factor in Inequality (23) is the best possible, then for $\lambda - \lambda_1 - \lambda_2 \leq \min\Big\{ p(\frac{2}{\alpha} - 1 - \lambda_1), q(\frac{2}{\beta} - \lambda_2) \Big\}$, we have $\lambda_1 + \lambda_2 = \lambda$.

Proof. We prove that the following constant factor

$$(\frac{\alpha}{\beta})^{\frac{1}{p}} \lambda B(\lambda_1 + 1, \lambda_{21}) (= (\frac{\alpha}{\beta})^{\frac{1}{p}} \lambda_1 B(\lambda_1, \lambda_{21}))$$

in Inequality (24) is the best possible. Otherwise, we would reach a contradiction that the same constant in Inequality (18) is not the best value by using Inequality (25) (for $\lambda_1 + \lambda_2 = \lambda$).

If the constant in Inequality (23) is the best value, then the same constant in Inequality (16) is still the best value. Otherwise, by Inequality (26) (for $\lambda_1 + \lambda_2 = \lambda$), we would reach a contradiction that the same constant factor in Inequality (24) is not the best value.

This proves theorem. □

We indicate the functions as follows: $\phi(m) := m^{-p\alpha\hat{\lambda}_1 - 1}$, $\psi(n) := n^{q(1-\beta\hat{\lambda}_2)-1}$, wherefrom,

$$\psi^{1-p}(n) := n^{p\beta\hat{\lambda}_2 - 1} (m, n \in \mathrm{N}).$$

We still indicate some linear spaces as follows:

$$l_{q,\psi} := \left\{ b = \{b_n\}_{n=1}^{\infty}; \|b\|_{q,\psi} = \left(\sum_{n=1}^{\infty} \psi(n) |b_n|^q \right)^{\frac{1}{q}} < \infty \right\},$$
$$l_{p,\psi^{1-p}} := \left\{ c = \{c_n\}_{n=1}^{\infty}; \|b\|_{p,\psi^{1-p}} = \left(\sum_{n=1}^{\infty} \psi^{1-p}(n) |c_n|^p \right)^{\frac{1}{p}} < \infty \right\},$$
$$l_{p,\varphi} := \left\{ d = \{d_m\}_{m=1}^{\infty}; \|d\|_{p,\varphi} = \left(\sum_{m=1}^{\infty} \varphi(m) |d_m|^p \right)^{\frac{1}{p}} < \infty \right\},$$
$$B := \big\{ a = \{a_m\}_{m=1}^{\infty}; A = \{A_m\}_{m=1}^{\infty} \in l_{p,\varphi} \big\} (A_m = \sum_{i=1}^{m} a_i).$$

For $a = \{a_m\}_{m=1}^{\infty} (> 0) \in B$, setting $c = \{c_n\}_{n=1}^{\infty} : c_n := \sum_{m=1}^{\infty} \frac{a_m}{(m^\alpha + n^\beta)^\lambda}$, Inequality (23) can be rewritten as:

$$||c||_{p,\psi^{1-p}} < (\frac{\alpha}{\beta})^{\frac{1}{p}} \lambda (k_{\lambda+1}(\lambda_2))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1 + 1))^{\frac{1}{q}} ||A||_{p,\phi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

Definition 1. *Define an operator* $T : B \to l_{p,\psi^{1-p}}$ *as follows: For any* $a \in B$, *there exists a unique representation* $c = Ta \in l_{p,\psi^{1-p}}$, *such that for any* $n \in \mathrm{N}$, *we have* $Ta(n) = c_n$. *Define the formal inner product of* Ta *and* $b \in l_{q,\psi}$ *and the norm of* T *as follows*:

$$(Ta, b) := \sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{a_{mn}}{(m^\alpha + n^\beta)^\lambda},$$

$$||T|| := \sup_{a(\neq 0)\in B} \frac{||Ta||_{p,\psi^{1-p}}}{||A||_{p,\phi}}.$$

By using Theorems 2–4, we have

Theorem 6. *Assume that* $a(> 0) \in B, b(> 0) \in l_{q,\psi}, ||b||_{q,\psi} > 0$. *We have the following equivalent inequalities:*

$$(Ta, b) < (\frac{\alpha}{\beta})^{\frac{1}{p}} \lambda (k_{\lambda+1}(\lambda_2))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1 + 1))^{\frac{1}{q}} ||A||_{p,\phi} ||b||_{q,\psi}, \tag{27}$$

$$||Ta||_{p,\psi^{1-p}} < (\frac{\alpha}{\beta})^{\frac{1}{p}} \lambda (k_{\lambda+1}(\lambda_2))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1 + 1))^{\frac{1}{q}} ||A||_{p,\phi}. \tag{28}$$

Moreover, assume that $\lambda \in (0, 5]$, $\lambda_1 \in (0, \frac{2}{\alpha} - 1] \cap (0, \lambda)$, $\lambda_2 \in (0, \frac{2}{\beta}] \cap (0, \lambda)$. If $\lambda_1 + \lambda_2 = \lambda$, then the constant factor $(\frac{\alpha}{\beta})^{\frac{1}{p}} \lambda (k_{\lambda+1}(\lambda_2 + 1))^{\frac{1}{p}} (k_{\lambda+1}(\lambda_1))^{\frac{1}{q}}$ in Inequalities (27) and (28) is the best possible, namely, $||T|| = (\frac{\alpha}{\beta})^{\frac{1}{p}} \lambda_1 B(\lambda_1, \lambda_{21})$. On the other hand, if the constant factor in Inequality (26) (or Inequality (27)) is the best possible, then for

$$\lambda - \lambda_1 - \lambda_2 \leq \min\left\{ p(\frac{2}{\alpha} - 1 - \lambda_1), q(\frac{2}{\beta} - \lambda_2) \right\},$$

we have $\lambda_1 + \lambda_2 = \lambda$.

Remark 2. *(i) For* $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$ *in Inequalities (18) and (24), we have the equivalent inequalities as follows:*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m^\alpha + n^\beta} < (\frac{\alpha}{\beta})^{\frac{1}{p}} \frac{\pi}{q \sin(\pi/p)} \times \left[\sum_{m=1}^{\infty} m^{(1-p)\alpha - 1} A_m^p \right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{(q-1)(1-\beta)} b_n^q \right]^{\frac{1}{q}}, \tag{29}$$

$$\left[\sum_{n=1}^{\infty} n^{\beta - 1} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^\alpha + n^\beta} \right)^p \right]^{\frac{1}{p}} < (\frac{\alpha}{\beta})^{\frac{1}{p}} \frac{\pi}{q \sin(\pi/p)} \left[\sum_{m=1}^{\infty} m^{(1-p)\alpha - 1} A_m^p \right]^{\frac{1}{p}}. \tag{30}$$

(ii) For $\lambda = 1, \lambda_1 = \frac{1}{p}, \lambda_2 = \frac{1}{q}$ *in Inequalities (18) and (24), the equivalent inequalities are valid as follows:*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m^\alpha + n^\beta} < (\frac{\alpha}{\beta})^{\frac{1}{p}} \frac{\pi}{p \sin(\pi/p)} \times \left(\sum_{m=1}^{\infty} m^{-\alpha - 1} A_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{q - \beta - 1} b_n^q \right)^{\frac{1}{q}}, \tag{31}$$

$$\left[\sum_{n=1}^{\infty} n^{(p-1)\beta - 1} \left(\sum_{m=1}^{\infty} \frac{a_m}{m^\alpha + n^\beta} \right)^p \right]^{\frac{1}{p}} < (\frac{\alpha}{\beta})^{\frac{1}{p}} \frac{\pi}{p \sin(\pi/p)} \left(\sum_{m=1}^{\infty} m^{-\alpha - 1} A_m^p \right)^{\frac{1}{p}}. \tag{32}$$

(iii) when $p = q = 2$, both Inequalities (29) and (31) deduce to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^{\alpha}+n^{\beta}}< (\frac{\alpha}{\beta})^{\frac{1}{2}}\frac{\pi}{2}\left(\sum_{m=1}^{\infty} m^{-\alpha-1}A_m^2\sum_{n=1}^{\infty} n^{1-\beta}b_n^2\right)^{\frac{1}{2}}, \tag{33}$$

and both Inequalities (30) and (32) deduce to the following inequality equivalent to Inequality (33):

$$\left[\sum_{n=1}^{\infty} n^{\beta-1}\left(\sum_{m=1}^{\infty}\frac{a_m}{m^{\alpha}+n^{\beta}}\right)^2\right]^{\frac{1}{2}} < (\frac{\alpha}{\beta})^{\frac{1}{2}}\frac{\pi}{2}\left(\sum_{m=1}^{\infty} m^{-\alpha-1}A_m^2\right)^{\frac{1}{2}} \tag{34}$$

We observe that the above constants are all the best value.

5. Conclusions

In this article, by means of the idea of introduced parameters and the techniques of real analysis, using Euler-Maclaurin summation formula as well as the mid-value theorem, we give a new extended Hardy-Hilbert's inequality with the power functions as the internal variables involving one partial sums in Theorem 1, which is a refinement of a published inequality. We provide a few equivalent conditions of the best possible constant related to multi parameters in Theorems 2 and 3. As applications, the equivalent inequalities are fond in Theorems 4 and 5, and the operator expressions as well as a few inequalities with the particular parameters are considered in Theorem 5 and Remark 2.

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