axioms

Review

The Magnificent Realm of Affine Quantization: Valid Results for Particles, Fields, and Gravity

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Abstract: Affine quantization is a relatively new procedure, and it can solve many new problems. This essay reviews this new, and novel, procedure for particle problems, as well as those of fields and gravity. New quantization tools, which are extremely close to, and even constructed from, the tools of canonical quantization, are able to fully solve selected problems that using the standard canonical quantization would fail. In particular, improvements can even be found with an affine quantization of fields, as well as gravity.

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1. A Preface

The scope of quantum physics has expanded remarkably, as will be clear in this presentation. Many problems have new and novel results.

The basic rules of quantization were largely set around mid-(1920), and have changed very little thereafter. There are many problems that using those rules can lead to acceptable results, but there are many more problems that those rules are inadequate. As an example, the traditional harmonic oscillator, which is set on the whole real line, can be fully solved with the original rules. However, if the harmonic oscillator is set only on the positive real line, it can not be solved with the old quantization rules despite the fact that it can be solved classically. Many problems that can be solved classically cannot be solved with old rules known as canonical quantization (CQ). Those procedures fail on non-renormalizable examples which include certain relativistic scalar fields and Einstein's gravity.

A new quantization procedure, called affine quantization (AQ), has now been added to the old rules. This procedure is now about 30 years old. AQ is not well known and it deserves to be as strongly known as CQ. While CQ chooses the momentum, e.g., p, and the coordinate, e.g., q, to promote to quantum operators, AQ chooses what we call the dilation, namely $d = pq$.

We start slowly with simple models to appreciate what AQ is able to accomplish. Already, using Monte Carlo methods, several non-renormalizable relativistic scalar models have confirmed what AQ can do for them. Einstein's gravity is more complicated, but the rules of AQ offer considerably positive results.

A brief example of the affine procedures, and a brief integration interval, $H = \int_{-1}^{1}[\pi(x)^2 + |\varphi(x)|^p]\,dx < \infty$, with $p = 1, 2, 3, \ldots$, then H can be finite if the integrand reaches infinity, e.g., $\pi(x)^2 = A/|x|^{1/4}$. This is proper mathematics, but the fields of nature should never reach infinity. Our solution introduces a new field, $\kappa(x) = \pi(x)\,\varphi(x)$. Now, $H = \int H(x)\ dx = \int[\kappa(x)^2/\varphi(x)^2 + |\varphi(x)|^p]\ dx < \infty$. To represent $\pi(x)$, then $0 < |\varphi(x)| < \infty$, $0 \leq |\kappa(x)| < \infty$, and $H(x) < \infty$. While $\varphi(x) \neq 0$, $\hat{\pi}(x)^\dagger \neq \hat{\pi}(x)$, then

$\hat{\kappa}(x) = [\hat{\pi}(x)^\dagger \hat{\varphi}(x) + \hat{\varphi}(x)\hat{\pi}(x)]/2$, and after scaling, then $\mathcal{H} = \int [\hat{\pi}(x)^2 + 2\hbar^2/\hat{\varphi}(x)^2 + |\hat{\varphi}(x)|^p]\ dx < \infty$. Note that $|\hat{\varphi}(x)|^p < \infty$ for all x, and all p, $0 < p < \infty$, due to $0 < |\hat{\varphi}(x)| < \infty$ as well as $0 \leq |\hat{\pi}(x)| < \infty$!

2. An Introduction to the Variables

Quantum operators are promoted from classical variables that can play an important role and need to be presented here because it is poorly covered. Our story involves three sets of classical variables that will, later, find their importance when they are promoted to basic quantum variables.

2.1. A Survey of Principal Topics

The common examination of quantum topics starts with a classical review, and we shall do the same. Our focus features three different classical versions. These three have some similar features as well as their differences, but they all play a role in the quantum story.

The three versions of quantum theory, which develop from the three classical versions, have important and distinct roles to play. After studying the procedures, we will apply them to specific problems. It follows that the various procedures fit specific sets of problems, and fail when the wrong procedures are applied to any wrong set of problems. In particular, problems that are non-renormalizable quantum problems, and which have been unsolved for decades, can, in fact, be properly solved by using the correct quantum procedures instead of the wrong procedures. While they may have been favored, they also may have been the incorrect procedure for decades! In later chapters, we will solve non-renormalizable covariant scalar fields as well as Einstein's gravity.

2.2. A Familiar Example of Classical Variables

The everyday behavior of most objects consists of its position, abbreviated by q, and its momentum, namely its mass multiplied by its velocity, like $p = mv$. These objects also change place and/or motion, which is represented by $q(t)$ and $p(t)$, with t serving as time. In an ideal universe, there would be no friction to slow motion down, while instead energy is typically considered to be a constant. The important Hamilton expression, $H(p,q)$, and the equations of motion are given by

$$\dot{q}(t) = \partial H(p,q)/\partial p(t)\ ,\ \dot{p}(t) = -\partial H(p,q)/\partial q(t)\ . \tag{1}$$

A common example is the harmonic oscillator, for which, like all systems, the energy is contained in the Hamiltonian,

$$H(p,q) = [p(t)^2/m + \omega^2\, m\, q(t)^2]/2\ . \tag{2}$$

This leads to the equations of motion given by $\dot{q}(t) = p(t)/m$, while $\dot{p}(t) = -\omega^2\, m\, q(t)$. These equations lead to $\ddot{q}(t) = -\omega^2 q(t)$ and $\ddot{p}(t) = -\omega^2 p(t)$, with solutions given by

$$q(t) = A\cos(\omega t) + B\sin(\omega t) \tag{3}$$

$$p(t) = Bm\omega\cos(\omega t) - Am\omega\sin(\omega t) \tag{4}$$

2.3. Selected Canonical Topics

The action functional is an important expression that also leads to the same equations that we dealt with in the section above, e.g.,

$$A = \int_0^T [p(t)\,\dot{q}(t) - H(p(t), q(t))]\ dt\ , \tag{5}$$

and which leads to tiny variations in the variables, $\delta q(t)$ and $\delta p(t)$, and now $\delta q(T) = \delta q(0) = 0$ as well as $\delta p(T) = \delta p(0) = 0$. The variations lead to

$$\begin{aligned}\delta A = 0 = \textstyle\int_0^T \{\, [\dot{q}(t) - \delta H(p(t), q(t))/\delta p(t)]\ \delta p(t) \\ +[-\dot{p}(t) - \delta H(p(t), q(t))/\delta q(t)]\ \delta q(t)\ \}\ dt\end{aligned} \tag{6}$$

which leads to the correct equations of motion being recovered when arbitrary variations are implied.

3. Phase Space, Poisson Brackets, and Constant Curvature Spaces

Phase space consists of a collection of general, continuous, functions $p(t)$ and $q(t)$. These functions can be turned into different functions, such as $\overline{f}(t) = \overline{F}(f(t))$; a simple example is $\overline{f}(t) = f(t)^3$. The family of functions is chosen to observe the integral

$$\int \overline{F}(\overline{p}(t), \overline{q}(t))\ d\overline{p}(t)\ d\overline{q}(t) = \int F(p(t), q(t))\ dp(t)\ dq(t)\,. \tag{7}$$

The Poisson brackets for these variables is given by

$$\{g(p,q), f(p,q)\} = \frac{\partial g(p,q)}{\partial q}\,\frac{\partial f(p,q)}{\partial p} - \frac{\partial g(p,q)}{\partial p}\,\frac{\partial f(p,q)}{\partial q}\ . \tag{8}$$

Poisson brackets play a reducing lever putting multiple expressions into fixed sets. For example, $\{q, p\} = 1$ and $\{q^3/3, p/q^2\} = 1$, and also as $\{q, pq\} = q$.

The pair of functions, $p(t)$ & $q(t)$, also has a geometric role to play. Let us assume we choose to create a flat, two-dimensional surface, by using the following expression,

$$d\sigma^2 = \omega^{-1}\, dp(t)^2 + \omega\, dq(t)^2\ , \tag{9}$$

where ω is a positive constant that does not depend on $p(t)$ or $q(t)$ in any way. A common name for this case is 'Cartesian variables'. It is noteworthy that this two-dimensional surface is completely identical if you move to any other location. That property may be called a 'constant zero curvature'.

Moreover, such a mathematical plane is infinitely big, meaning that $-\infty < p$ & $q < \infty$.

Observe that this property of p & q is *complete*, which means *every* point in $\mathbb{R}^2$ is included. There is no case where $q = 17$, for example, is excluded from the rest of $-\infty < q < \infty$.

3.1. A Brief Review of Spin Quantization

The operators in this story are S_i with $i = 1, 2, 3$, and which – here $i = \sqrt{-1}$, as well – satisfy $[S_i, S_j] = i\hbar\,\epsilon_{ijk}\,S_k$. These operators obey $\Sigma_{l=1}^3 S_l^2 = \hbar^2 s(s+1)\mathbb{1}_{2s+1}$, where $2s + 1 = 2, 3, 4, \ldots$ is the dimension of the spin matrices. The normalized eigenvectors of S_3 are $S_3\,|s, m\rangle = m\,\hbar\,|s, m\rangle$, where $m \in \{-s, \ldots, s-1, s\}$.

3.1.1. Spin Coherent States

The spin coherent states are defined by

$$|\theta, \varphi\rangle \equiv e^{-i\varphi S_3/\hbar}\, e^{-i\theta S_2/\hbar}\, |s, s\rangle\ , \tag{10}$$

where $-\pi < \varphi \le \pi$, and $-\pi/2 \le \theta \le \pi/2$. It follows that

$$\begin{aligned} d\sigma(\theta, \varphi)^2 &\equiv 2\hbar\,[\,\|\, d|\theta, \varphi\rangle\|^2 - |\langle\theta, \varphi|\, d|\theta, \varphi\rangle|^2\,] \\ &= (s\hbar)[d\theta^2 + \cos(\theta)^2\, d\varphi^2\,]\ . \end{aligned} \tag{11}$$

We can also introduce $q = (s\hbar)^{1/2}\,\varphi$ and $p = (s\hbar)^{1/2}\,\sin(\theta)$, with $|p,q\rangle = |p(\theta,\varphi), q(\theta,\varphi)\rangle$, which leads to

$$\begin{aligned} d\sigma(p,q)^2 &\equiv 2\hbar\,[\,\|\,d|p,q\rangle\|^2 - |\langle p,q|\,d|p,q\rangle|^2\,] \\ &= (1 - p^2/s\hbar)^{-1} dp^2 + (1 - p^2/s\hbar)\,dq^2\;. \end{aligned} \tag{12}$$

Equation (11) makes it clear that we are dealing with a spherical surface with a radius of $(s\hbar)^{1/2}$; this space is also known as a 'constant positive curvature' surface, and it has been created!

These classical variables can not lead to a physically correct canonical quantization. Instead, they offer a distinct quantization procedure that applies to different problems.

However, Equation (12) makes it clear that if $s \to \infty$, in which case both p and q span the real line, we are led to 'Cartesian coordinates', a basic property of canonical quantization.

3.1.2. A Brief Review of Affine Quantization

Consider a classical system for which $-\infty < p < \infty$, but $0 < q < \infty$, that does not lead to a self-adjoined quantum operator P, i.e., $P^\dagger \neq P$. Perhaps we can do better if we change classical variables. For example, $0 < q < \infty$ – or it may arise instead that $-\infty < q < 0$. To capture these possibilities for q —and thus also for Q $(=Q^\dagger)$—we are led to $d = pq \Rightarrow D = (P^\dagger Q + QP)/2\ (= D^\dagger)$, which leads to $[Q, D] = i\hbar Q$. This expression happens to be like the Lie algebra of the "affine group" [1], and, incidentally, that name has been adopted by "affine quantization". Again, it is useful to choose dimensions such that q & Q are dimensionless while d & D have the dimensions of $\hbar$.

3.1.3. Affine Coherent States

The affine coherent states involve the quantum operators D and Q, where now $Q > 0$. We use the classical variables p and $\ln(q)$, with $q > 0$. Specifically, we choose

$$|p;q\rangle \equiv e^{ipQ/\hbar}\,e^{-i\ln(q)\,D/\hbar}\,|\beta\rangle\;, \tag{13}$$

where the fiducial vector $|\beta\rangle$ fulfills the condition $[(Q - \mathbb{1}) + iD/\beta\hbar]|\beta\rangle = 0$, which implies that $\langle\beta|Q|\beta\rangle = 1$ and $\langle\beta|D|\beta\rangle = 0$ (the semicolon in $|p;q\rangle$ distinguishes the affine ket from the canonical ket $|p,q\rangle$. If $-\infty < q < 0$, change $\ln(q)$ to $\ln(|q|)$, but keep $q \to Q < 0$ so that $|q|Q = q|Q|$). This expression leads to

$$\begin{aligned} H'(pq,q) &= \langle p;q|\mathcal{H}'(D,Q)|p;q\rangle = \langle\beta|\mathcal{H}'(D + pqQ, qQ)|\beta\rangle \\ &= \mathcal{H}'(pq,q) + \mathcal{O}'(\hbar;pq,q)\;, \end{aligned} \tag{14}$$

and, as $\hbar \to 0$, and $\mathcal{O}'(\hbar;pq,q) = 0$, $H'(d,q) = \mathcal{H}'(d,q)$, a relation very similar to $H(p,q) = \mathcal{H}(p,q)$ when using CQ.

It follows that the Fubini–Study metric, for $q > 0$, becomes

$$\begin{aligned} d\sigma(p;q)^2 &\equiv 2\hbar[\|\,d|p;q\rangle\|^2 - |\langle p;q|\,d|p;q\rangle|^2] \\ &= (\beta\hbar)^{-1} q^2\,dp^2 + (\beta\hbar)\,q^{-2}\,dq^2\;. \end{aligned} \tag{15}$$

This expression leads to a surface that has a 'constant negative curvature' [2] of magnitude $-2/\beta\hbar$, which, like the other curvatures, has been 'created' (as noted, while constant zero and positive curvatures can be seen in our three spatial dimensions, a visualization of a complete constant negative curvature is not possible. A glance of one would be a single point on a saddle, namely the highest point from the rider's feet direction, and the lowest point from the horse's head direction). The set of classical variables can not lead to a physically correct canonical quantization. Instead, they offer a distinct quantization procedure that applies to different problems. Any use of classical variables that do not form a 'constant negative curvature' subject to an affine quantization is very likely not a physically correct quantization.

The inner product of two affine coherent state vectors is given by

$$\langle p';q'|p;q\rangle = \Big[[(q'/q)^{1/2} + (q/q')^{1/2}]/2 + i(q'q)^{1/2}\,(p'-p)/2\beta\hbar\Big]^{-2\beta}\,, \tag{16}$$

while $\int\int |p;q\rangle\langle p;q|(1-1/2\beta)\;dp\;dq/2\pi\hbar = \mathbb{1}$, provided that $\beta > 1/2$. While the variable change for all $p,q \to cp, q/c$ leaves a Cartesian metric still Cartesian, it can be seen that there is no change whatsoever in (15), illustrating the significance of the affine Fubini–Study metric.

The rule that $0 < q < \infty$ is limited and we can easily consider $0 < q+k < \infty$, where $k > 0$. This changes the coherent states from $\ln(q)$ to $\ln(q+k)$, which then changes the Fubini–Study metric to $(\beta\hbar)^{-1}(q+k)^2\,dp^2 + (\beta\hbar)\,(q+k)^{-2}\,dq^2$. If we choose to let $k \to \infty$, and at the same time let $\beta\hbar \to (\beta\hbar + \omega k^2)$, we are led to $\omega^{-1}dp^2 + \omega\,dq^2$, now with $q \in \mathbb{R}$, which, once again, applies to canonical quantization. Briefly stated, we can arrange that AQ $\to$ CQ!

3.2. Summarizing Constant Curvatures and Coherent States

These three stories complete our family of 'constant curvature' spaces, specifically, constant positive, zero, and negative curvatures. Additionally, the various coherent states can be used to build "bridges" in each case that enable one to pass from the classical realm to the quantum realm or pass in the other direction [3].

4. Learning to Quantize Selected Problems

We begin with two different quantization procedures, and two simple, but distinct, problems, one of which is successful and the other one is a failure in trying to use both of the quantization procedures on each example.

This exercise serves as a prelude to a valid and straightforward clarification of the fact that affine quantization and canonical quantization solve completely different sets of problems. This fact will help us when we turn to the quantization of field theories and of gravity in later chapters.

4.1. Choosing a Canonical Quantization

The classical variables, p & q, which are elements of a constant zero curvature, better known as Cartesian variables, such as those featured by Dirac [4], are promoted to self-adjoined quantum operators $P(=P^\dagger)$ and $Q\,(=Q^\dagger)$, ranged so that $-\infty < P$ & $Q < \infty$, and scaled so that $[Q,P] = i\hbar\mathbb{1}$ (in particular, in [4], the mid-page of 114, Dirac wrote "However, if the system does have a classical analogue, its connexion with classical mechanics is specially close and one can usually assume that the Hamiltonian is the same function of the canonical coordinates and momenta in the quantum theory as in the classical theory † Footnote †: This assumption is found in practice to be successful only when applied with the dynamical coordinates and momenta referring to a Cartesian system of axes and not to more general curvilinear coordinates").

4.1.1. First Canonical Example

Our example is just the familiar harmonic oscillator, for which $-\infty < p$ & $q < \infty$ and a Poisson bracket $\{q,p\} = 1$, also a classical Hamiltonian, with the common factors $m = \omega = 1$, given by $H(p,q) = (p^2+q^2)/2$. The quantum Hamiltonian is $\mathcal{H}(P,Q) = (P^2+Q^2)/2$, and Schrödinger's representation is given by $P = -i\hbar(\partial/\partial x)$ and $Q = x$, for $-\infty < x < \infty$. Finally, for our example, Schrödinger's equation is given by

$$i\hbar(\partial\psi(x,t)/\partial t) = (-\hbar^2\partial^2/\partial x^2 + x^2)/2\;\psi(x,t)\,. \tag{17}$$

Solutions to Equation (7) for our example are well known. In particular, for the harmonic oscillator, the eigenvalues are given by $E_n = \hbar(n+1/2)$ for $n = 0,1,2,,\ldots,$ and the eigenfunctions (with $\hbar = 1$) are given by $\psi_n(x) = N_n\, H_n(x)\, e^{-x^2/2}$ with $n = 0, 1, 2, \ldots$. Here, N_n serves to enforce normalization, and the remainder is

$$H_n(x)\, e^{-x^2/2} = e^{x^2/2}(-d/dx)^n e^{-x^2} \,. \tag{18}$$

This model is one of the most well understood of all examples.

4.1.2. Second Canonical Example

For our next example, we keep the same classical Hamiltonian, and we retain $-\infty < p < \infty$, but now we restrict $0 < q < \infty$. This new model is called the 'half-harmonic oscillator'. It follows that the operator $P^\dagger \neq P$, which leads to a different behavior to that when P is self adjoined, i.e., $P^\dagger = P$. In particular, this can lead to infinitely many different self-adjoined Hamiltonians each of which passes to the same classical Hamiltonian that would be $(p^2 + q^2)/2$ in this case. Just two of the different quantum Hamiltonians could be $\mathcal{H}_0(P, Q) = (PP^\dagger + Q^2)/2$, while the other is $\mathcal{H}_1(P, Q) = (P^\dagger P + Q^2)/2$. Clearly, both of these quantum Hamiltonians lead to the same classical Hamiltonian, namely $(p^2 + q^2)/2$, when $\hbar \to 0$ (here is one example of infinitely many quantum Hamiltonians for the half-harmonic oscillator, when $P^\dagger \neq P$, would be $[(P^{\dagger\, n+4}/P^{n+2} + P^{n+4}/P^{\dagger\, n+2})/2 + Q^2]/2$, for all $n = 0, 1, 2, \ldots$).

This judgement renders the canonical quantization of the half-harmonic oscillator to be an invalid quantization.

We interrupt our present story to bring the reader an important message.

A Simple Truth

Consider A×B = C, as well as A = B/C

If $B = C = 0$**, what is** A**?**
If $B = C = \infty$**, what is** A**?**

To ensure getting A **one must require** $0 < |B|$ & $|C| < \infty$**.**

This is good mathematics, but physics has an opinion as well.

Consider $mv = p$**. If the velocity** $v = 0$**, then the momentum** $p = 0$**, which makes good sense. However, if the mass** $m = 0$ **and the velocity** $v = 9$**, then the momentum,** $p = 0$**, makes bad physics. However, if any of them are infinite, that is certainly bad math as well as bad physics.**

We will especially use this topic for the dilation variable $d = pq$**, where** q **is the coordinate of a position and** p **denotes its time derivative (times its mass too). The position** $q(t)$ **is continuous, while** $p(t)$ **is traditionally continuous, but it can change sign, like bouncing a ball off a wall.**

We may point to an ABC-item to remind the reader of its relevance.

This important notification is finished.

4.1.3. First Affine Example

The traditional classical affine variables are $d \equiv pq$ and $q > 0$ (ABC), and they have a Poisson bracket given by $\{q, d\} = q$. In addition, we can choose a different dilation variable, $d = p(q + b)$, for which $-b < q < \infty$, generally, with $b > 0$. For very large b we can approximate a full-line harmonic oscillator and even see what happens if we choose $b \to \infty$ to mimic the full-line story.

The classical affine variables now are $-\infty < d \equiv p(q + b) < \infty$ and $0 < (q + b) < \infty$, while the classical harmonic oscillator Hamiltonian is given by $H'(d, q) = [d^2/(q + b)^2 + q^2]/2$, an expression that obeys $H(p, q) = (p^2 + q^2)/2$ albeit that $-b < q < \infty$.

Now, we consider basic quantum operators, namely $D = [P^\dagger(Q+b) + (Q+b)P)]/2$ and $Q+b$, which lead to $[Q+b, D] = i\hbar\,(Q+b)$, along with $Q+b>0$ The quantum 'partial-harmonic oscillator' is now given by

$$H'(D,Q) = [D(Q+b)^{-2}D + Q^2]/2 = [P^2 + (3/4)\hbar^2/(Q+b)^2 + Q^2]/2\,, \qquad (19)$$

while, in a proper limit, an affine quantization *becomes* a canonical quantization when the partial real line ($-b < q$ & $Q < \infty$) is stretched to its full length, ($-\infty < q$ & $Q < \infty$).

Evidently, an affine quantization fails to quantize a full harmonic oscillator.

4.1.4. Second Affine Example

The common canonical operator expression, $[Q,P] = i\hbar\mathbb{1}$, directly implies $[Q,(P^\dagger Q + QP)/2] = i\hbar\,Q$, which are the basic affine operators.

To confirm this affine expression, let us multiply $i\hbar\mathbb{1} = [Q,P]$ by Q, which gives $i\hbar\,Q = (Q^2P - QPQ + QPQ - PQ^2)/2$, i.e., $i\hbar\,Q = [Q,(QP+PQ)/2]$, which is the basic affine expression, $[Q,D] = i\hbar\,Q$, where $D \equiv (PQ+QP)/2 \equiv (P^\dagger Q + QP)/2$. This derivation assumes that $Q>0$ or $Q<0$. Canonical quantization implies affine quantization, but adds a limitation, for classical as well as quantum, on the coordinates.

Regarding our problem, now $b=0$, and so the classical affine variables are $d \equiv pq$ and $q>0$, which lead to the half-harmonic oscillator $H'(d,q) = (d^2/q^2 + q^2)/2$. The basic affine quantum operators are D and Q, where $D\,(=D^\dagger)$ and $Q>0\,(=Q^\dagger>0)$. These quantum variables lead to $[Q,D] = i\hbar\,Q$. The half-harmonic oscillator quantum Hamiltonian is given by $\mathcal{H}'(D,Q) = (DQ^{-2}D + Q^2)/2$, and Schrödinger's representation is given by $Q \to x > 0$ and

$$D = -i\hbar[x(\partial/\partial x) + (\partial/\partial x)x)]/2 = -i\hbar[x(\partial/\partial x) + 1/2]\,. \qquad (20)$$

Finally, Schrödinger's equation is given by

$$\begin{aligned} &i\hbar(\partial\psi(x,t)/\partial t) = \\ &\quad= [-\hbar^2(x(\partial/\partial x)+1/2)\,x^{-2}\,(x(\partial/\partial x)+1/2) + x^2]/2\;\psi(x,t) \\ &\quad= [-\hbar^2\,(\partial^2/\partial x^2) + (3/4)\hbar^2/x^2 + x^2]/2\;\psi(x,t)\,. \end{aligned} \qquad (21)$$

We note that kinetic factors, such as P and D, can annihilate separate features. Adopting Schrödinger's representation, it follows thar $P\,1 = 0$ while $Dx^{-1/2} = 0$. We will exploit this simple fact in later chapters.

Solutions of (21) have been provided by L. Gouba [5]. Her solutions for the half-harmonic oscillator contain eigenvalues that are equally spaced as are the eigenvalues of the full-harmonic oscillator, although the spacing itself differs in the two cases. The relevant differential equation in (21) is known as a 'spiked harmonic oscillator', and its solutions are based on confluent hypergeometric functions. It is noteworthy that every eigenfunction, $\psi_n(x) \propto x^{3/2}(polynomial_n)e^{-x^2/2\hbar}$, which applies for all $n = 0,1,2,\ldots$. The leading factor of the eigenfunctions, i.e., $x^{3/2}$, provides a continuous result after the first derivative, but the second derivative could lead to an $x^{-1/2}$ behavior, except that $[-d^2/dx^2 + (3/4)/x^2]\;x^{3/2} = 0$. This zero ensures that after two derivatives, the wave function is still finite, continuous, and belongs in a Hilbert space (there are examples in which a/x^2, with $a>0$, such potentials are studied, but some are negative, i.e., $-a/x^2$, with $a>0$, which has a completely different behavior).

It is interesting to consider an increase in the coordinate space by choosing $x+b>0$. This leads to a related Schrödinger's equation, given by

$$[-\hbar^2\,(\partial^2/\partial x^2) + (3/4)\hbar^2/(x+b)^2 + x^2]/2\;\psi(x,t) = E_b\,\psi(x)\,, \qquad (22)$$

which has been shown to also have equally spaced eigenvalues that become narrower as b becomes larger. Moreover, if $b\to\infty$, then the $\hbar$-term disappears and the full-harmonic oscillator, with its canonical quantization features, is fully recovered [6]. In this fashion, we observe that AQ can pass to CQ, but the reverse is, apparently, impossible.

Finally, we can assert that an affine quantization of the half-harmonic oscillator can be considered to be a correctly solved problem.

4.1.5. A Canonical Version of the Half-Harmonic Oscillator

We start again with the classical Hamiltonian for the half-harmonic oscillator which is still $H = (p^2 + q^2)/2$ and $q > 0$, but this time we will use different coordinates. To let our new coordinate variables span the whole real line, which makes them 'Ashtekar-like' [7], we choose $q = s^2$, where $-\infty < s < \infty$. Thus, s is the new coordinate. For the new momentum, r , we choose $p = r/2s$. We choose it because the Poisson bracket $\{s, r\} = \{\sqrt{q}, 2p\sqrt{q}\} = 1$ (it may be noticed that while $q > 0$, and now $q = s^2$, this would imply that $s \neq 0$. However, we will skip over this "unimportant point" until later). The classical Hamiltonian now becomes $H = (p^2 + q^2)/2 = (r^2/4s^2 + s^4)/2$.

4.1.6. A CQ Attempt to Solve the Half-Harmonic Oscillator

For quantization, the new variables use canonical quantum operators, $r \to R$ and $s \to S$, with $[S, R] = i\hbar\mathbb{1}$. Following the CQ rules, this leads to $\mathcal{H}_{CQ} = [R\, S^{-2}R/4 + S^4]/2 \leq \infty$. This quantum operator, using canonical operators where $[S, R] = i\hbar\mathbb{1}$, is quite different from the affine expression $\mathcal{H}_{AQ} = [DQ^{-2}D + Q^2]/2 < \infty$; rearranged into canonical operators with $[Q, P] = i\hbar\mathbb{1}$, that leads to $\mathcal{H}_{AQ} = [P^2 + (3/4)\hbar^2/Q^2 + Q^2]/2 < \infty$.

It is self-evident that these two canonical quantum Hamiltonian operators, $\mathcal{H}_{AQ}$ and $\mathcal{H}_{CQ}$, have different eigenfunctions and eigenvalues. Does it matter that $\mathcal{H}_{AQ} < \infty$ while $\mathcal{H}_{CQ} \leq \infty$, due to $S = 0$ while $R \neq 0$? It is clear that answer to this question is "No". Trying to quantize the half-harmonic oscillator, using CQ variables, has led to physically incorrect results.

Now, we examine a very different model using both CQ and AQ.

5. Using CQ and AQ to Examine 'The Particle in a Box'

5.1. An Example That Needs More Analysis

This model has often been used in teaching and it is introduced early in the process as an easy example to solve. The classical Hamiltonian for this model is simply $H = p^2$, allowing, for simplicity, that $2m = 1$. Now, the coordinate space is $-b < q < b$, where $0 < b < \infty$ (which also may be chosen as $0 < q < 2b \equiv L < \infty$). To accommodate the CQ operators, we assume that outside the box there are infinte potentials that force any wave functions to be zero in the entire outside region where $|x| \geq b$. Inside the box, we have the quantum equation

$$-\hbar^2(d^2\,\phi_n(x)/dx^2) = E_n\,\phi_n(x)\,. \tag{23}$$

Evidently, cos and sin are relevant. In particular, $\phi(-b) = \phi(b) = 0$ is necessary to continuously join the squashed wave functions, $\phi(|x| \geq b) = 0$. This leads to eigenfunctions which are $\cos(n\pi x/2b)$ for $n = 1, 3, 5, \ldots$ and $\sin(n\pi x/2b)$ for $n = 2, 4, 6\ldots$. That leads to the eigenvalues $n^2\pi^2/4b^2$, now for $n = 1, 2, 3, 4, \ldots$.

5.1.1. Failure of the Canonical Quantization of the Particle in a Box

While the statements in the last section seem to be correct, there is a problem. Let us focus on the ground state, $\cos(\pi x/2b)$. We need to consider two derivatives of this function, so let us start with $\cos'(\pi x/2b) = -(\pi/2b)\sin(\pi x/2b)$ which leads to $\cos'(\pm\pi/2) = \mp\pi/2b$, i.e., the first derivative is *not* a continuous function with the squashed portion. This forces the second derivative to contain two factors proportional to $\delta(|x| = b)$, the Dirac delta function $\delta(x)$, which vanishes everywhere but $x = 0$ where $\delta(0) = \infty$, such that $\int_{-a}^{a(>0)} \delta(x)\; dx = 1$. It now follows that $\int \delta(x)^2\; dx = \infty$, which then excludes such a function, which is supposed to be finite, e.g., $\int |\phi(x)|^2\, dx < \infty$, to join any Hilbert space.

It was remarked in Wikipedia's discussion of the particle in a box [8] that the first derivative was not continuous as it should have been, but it was effectively ignored afterwards.

In summary, we conclude that by using CQ, the standard treatment and results for the particle in a box are incorrect.

The reduced coordinate space now requires a newly named dilation variable, $d' = p(b+q)(b-q) = p(b^2 - q^2)$, along with accepting only $-b < q < b$. Using affine variables, the classical Hamiltonian now becomes $H' = d'^2/(b^2 - q^2)^2$. Following the affine quantization rules means that the $D' = [P^\dagger(b^2 - Q^2) + (b^2 - Q^2)P]/2$, and the quantum Hamiltonian is

$$\mathcal{H}' = D'(b^2 - Q^2)^{-2}D' = P^2 + \hbar^2[2Q^2 + b^2]/[b^2 - Q^2]^2 \,. \tag{24}$$

The new $\hbar$-expression is unravelled later in the Appendix A to Section 5.

When comparing the different $\hbar$-terms, we find, with using $Q \to x$, that if $x \simeq \pm b$, then $[2x^2 + b^2]/(b^2 - x^2)^2 \simeq 3b^2/(b \mp x)^2 4b^2$, which mimics the $(3/4)$-factor for the half-harmonic oscillator. This implies that the x term in eigenfunctions, extremely close to either $\pm b$, should be like $\psi(x) \simeq (b^2 - x^2)^{3/2}\,(remainder)$.

For a moment, we take an about face.

A very different use of (24) is to *accept* the outside space, $|x| > b$, and *reject* $|x| < b$, which then becomes an 'anti-box'.

Note that this system has a similarity to a toy 'black hole'. It could happen that particles would pile up close to an 'end of space', while having been attracted there by a simple, "gravity-like", pull of a potential, such as $V(x) = W^2x^4$. If you choose AQ, then the barracked, $\hbar$-like term, in (24), would prevent the particles from falling 'out of space', while the shores exhibit light from the fires of trapped trash.

5.1.2. Removing a Single Point

Assuming that we still have chosen the outside, $|x| > b$, coordinates, it is noteworthy that if we focus on the region where $b \to 0$, while insisting that $|x| > 0$. In this case, the $\hbar$-term becomes $2\hbar^2/x^2$. However, the previous eigenfunction behavior of $(x^2 - b^2)^{3/2}$, now with $x^2 > b^2$, implies that any eigenstates (again, having potentials, like $V(x) = |x|^r$, for $r \geq 2$, that reach infinity) must start like $\psi_n(x) \simeq x^3(remainder_n)$. This offers effective continuity for the eigenfunction and its first two derivatives, even though $x \neq 0$ can permit a more different behavior on either side of $x = 0$. This, then, is the 'cost' to remove a single point in the usual coordinate space, e.g., in this case, removing just the single point at $q = 0$.

This result has been made possible using AQ and not using CQ, which requires including all x, i.e., $-\infty < x < \infty$.

A Vector Version: The point we now wish to remove is $\overrightarrow{q} = 0$; stated, we want to retain all the variables that obey $\overrightarrow{q}^{\,2} > 0$ and all those of $\overrightarrow{p}^{\,2} \geq 0$. In addition, we introduce $\overrightarrow{P}^{\,2} = \Sigma_{j=1}^s P_j^2$ and $\overrightarrow{Q}^{\,2} = \Sigma_{j=1}^s Q_j^2 > 0$.

Using these variables, we are led to $d^* = |\overrightarrow{p}|(\overrightarrow{q}^{\,2} - b^2)$, which leads to $\overrightarrow{p}^{\,2} = d^{*2}/(\overrightarrow{q}^{\,2} - b^2)^2$. Quantizing, we have $D^* = [\,|\overrightarrow{P}|(\overrightarrow{Q}^{\,2} - b^2) + (\overrightarrow{Q}^{\,2} - b^2)|\overrightarrow{P}|\,]/2$ $(= D^{*\dagger})$. Adopting the kinetic factor, $D^*(\overrightarrow{Q}^{\,2} - b^2)^{-2}D^*$, that equation also unfolds, in a fashion similar to that shown in the Appendix A to Section 5, below, and leads to the quantum Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}\,[\,\overrightarrow{P}^{\,2} + \hbar^2(2\overrightarrow{Q}^{\,2} + b^2)/(\overrightarrow{Q}^{\,2} - b^2)^2] + V(\overrightarrow{Q}^{\,2}). \tag{25}$$

Just by sending $b^2 \to 0$, we achieve the situation where only the single point, i.e., $\overrightarrow{q}^{\,2} = 0 \to \overrightarrow{Q}^{\,2} = 0$ is removed from our s-dimensional space. The quantum Hamiltonian in this case is

$$\mathcal{H}_s = \tfrac{1}{2}\,[\,\overrightarrow{P}^{\,2} + 2\hbar^2/\overrightarrow{Q}^{\,2}\,] + V(\overrightarrow{Q}^{\,2})\,. \tag{26}$$

To offer a justification that this relation holds for all $\overrightarrow{Q}$ including just the case where $s = 1$, i.e., just Q^2. To do so, let us introduce the wave function $\psi(x) = U(x)\, W_B(Bx_j)$, by introducing a partial expectation of the Hamiltonian given by

$$\begin{aligned}&\int W_B(Bx_j)^*[\overrightarrow{P}^2 - 2V(x^2)]W_B(Bx_j)\; d^{s-1}x \\ &\quad = \int W_B(Bx_j)^*\, 2\hbar^2[2/(x^2 + \Sigma_{j=2}^s x_j^2)]\; W_B(Bx_j)\; d^{s-1}x\end{aligned} \tag{27}$$

in which we have integrated all x_j except $x = x_1$. Now, for all, but x_1, we let $x_j \to x_j/B$, which changes the previous equation to become

$$\begin{aligned}&\int W_C(x_j)^*[\overrightarrow{P}^2 - 2V(x^2 + B^{-2}\Sigma_j\, x_j^2)]W_C(x_j)\; d^{s-1}x \\ &\quad = \int W_C(x_j)^*\, 2\hbar^2[2/(x^2 + B^{-2}\Sigma_j\, x_j^2)]W_C(x_j)\; d^{s-1}x\,.\end{aligned} \tag{28}$$

The purpose of this exercise is to show that the original quantum Hamiltonian (26) for s many dimensions holds the equation for a final quantum Hamiltonian (28) as $B \to \infty$ for a single dimension.

Briefly stated, an $(s-1)$-dimensional reduction may be arranged that can force all of those coordinates to become zero. This leaves behind just one of the coordinates, which is part of a proper equation, and is already waiting to fulfill its duty (additional x_j factors may be made active by simply removing their B factor from the beginning).

5.2. Lessons from Canonical and Affine Quantization Procedures

An important lesson from the foregoing set of examples is that canonical quantization requires special classical variables, i.e., $-\infty < p \;\&\; q < \infty$, that create a flat surface, to be promoted to valid quantum operators that satisfy $-\infty < P \;\&\; Q < \infty$. However, an affine quantization requires different classical variables, e.g.,$-\infty < d_b = p(b+q) < \infty$ and $-b < q < \infty$, chosen so that $0 < b < \infty$, to be promoted to valid affine quantum operators, which satisfy $-\infty < D < \infty$ and $-b < Q < \infty$, provided that the classical variables arise from a constant negative curvature.

The essential information from this exercise is that affine quantization variables are created from canonical quantization variables and they permit classical and quantum alike in having a limited behavior where $-b < q \;\&\; Q < \infty$, b is finite, and $-\infty < d \;\&\; D < \infty$.

6. Ultralocal Field Models

6.1. Introduction

In some ways, our first example of a field theory is the hardest to deal with its quantization. An ultralocal form of any classical field theory eliminates all spatial (but *not* temporal) derivatives in its action functional, and specifically, in its classical Hamiltonian such as $H = \int\{\,\frac{1}{2}[\pi(x)^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p\}\; d^s x$, where $p = 4, 6, 8, \ldots$ and $s = 1, 2, 3, \ldots$.

If we can handle this model, we should be able to handle more relevant relativistic field models that restore spatial derivatives.

6.2. What Is the Meaning of Ultralocal

The phrase 'ultralocal' implies there are no spatial derivatives of the fields only a separate time derivative. Previously, one of the authors has quantized ultralocal scalar fields by affine quantization to show that these non-renormalizable theories can be correctly quantized by affine quantizations; the story of such scalar models is introduced in this chapter. The present chapter will also show that ultralocal gravity can be successfully quantized by affine quantization.

The purpose of this study is to show that a successful affine quantization of any ultralocal field problem would imply that, with properly restored spacial derivatives, the classical theory can, in principle, be guaranteed a successful quantization result using either a canonical quantization in some cases or an affine quantization in different cases.

In particular, Einstein's gravity requires an affine quantization, and it will be successful, as we will find out in a following chapter.

6.3. Classical and Quantum Scalar Field Theories

The purpose of this section is to review a modest summary of the results of canonical quantization when it has been used to study a variety of covariant scalar field models.

We interrupt our present story to upgrade 'A Simple Truth' to prepare the reader for its use with fields:

Another Simple Truth

Consider $A(x) \times B(x) = C(x)$ **as well as** $A(x) = C(x)/B(x)$

If $B(x) = C(x) = 0$ **what is** $A(x)$**?**
If $B(x) = C(x) = \infty$ **what is** $A(x)$**?**

To ensure getting $A(x)$ **one must require** $0 < |B(x)|$ & $|C(x)| < \infty$**.**

This is good mathematics, but physics has an opinion as well.

Consider $k(x) = \pi(x)\varphi(x)$**, where** $\varphi(x)$ **is a chosen physical field,** $\pi(x)$ **is its momentum field, and their product is** $\kappa(x)$**, which we will call the dilation field. Since** $\pi(x)$ **serves as the time derivative of** $\varphi(x)$**, it can vanish along with** $\kappa(x)$**. However, requiring that both plus and minus sides of** $\varphi(x) \neq 0$ **are acceptable, since the derivative term ensures it will still seem to come from a continuous function. Moreover, if** $\varphi(x) = 0$ **it could be confused with any other field, e.g.,** $\alpha(x) = 0$ **(if you think dimensions can distinguish two such fields, we can eliminate dimensional features by first introducing** $\varphi(y) \neq 0$ **and** $\alpha(z) \neq 0$**. Now, dimensionless factors lead to** $\varphi(x)/\varphi(y) = 0 = \alpha(x)/\alpha(z)$**. Thus, omitting points, or streams of them, where** $\varphi(x) = 0$**, do not violate any physics. In fact, it may seem logical to say that** $\varphi(x) = 0$ **never even belonged in physics. It fact, since numbers were used to count physical things, in very early times, zero** $= 0$**, was** *banned for 1500 years*; **see [9]).**

It is good math for finite integrations if there are examples where the fields may reach infinity, e.g., $\int_{-1}^{1} \varphi^{-2/3}\, d\varphi < \infty$**. However, such cases are very likely to be bad physics because no item of nature reaches infinity. Accepting** $\kappa(x)$ $(= \pi(x)\,\varphi(x))$ **and** $\varphi(x) \neq 0$**, instead of** $\pi(x)$ **and** $\varphi(x)$**, as the basic variables, will have profound consequences.**

For example, the classical Hamiltonian expressed as

$$H = \int \{ \tfrac{1}{2}[\kappa(x)^2/\varphi(x)^2 + m^2\varphi(x)^2] + g\ \varphi(x)^p \}\ d^s x\ , \tag{29}$$

in which $0 \leq |\kappa(x)| < \infty$ **and** $0 < |\varphi(x)| < \infty$**, to well represent** $\pi(x)$**, fulfills the remarkable property that** $H(x) < \infty$**, where** $H = \int H(x)\ d^s x$**,** *as nature requires!* **This fact shows that** $\kappa(x)$ **and** $\varphi(x) \neq 0$ **should be the new variables!**

We now point to our new ABC-items to remind the reader of their relevance.

This important notification is finished.

6.4. Canonical Ultralocal Scalar Fields

These models have a classical (labelled by c) Hamiltonian given by

$$H_c = \int \{ \tfrac{1}{2}[\pi(x)^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p \}\ d^s x\ , \tag{30}$$

with $p = 4, 6, 8, \ldots$ and $s = 1, 2, 3, \ldots$. With $n = s + 1$ spacetime dimensions, and first using canonical quantization, we examine these models. In preparation for a possible path integration, the domain of H_c consists of all, momentum functions $\pi(x)$ and scalar fields $\varphi(x)$, for which $0 \leq H_c < \infty$.

Since all derivatives have now been removed, even stronger issues can be expected by path integrations being swamped by integrable infinities of the field, or by vast numbers of almost integrable infinities. However, effectively, that strong behavior fails to contribute to the path integration results, e.g., for $p \geq 4$, while the middle range contributions have the most influence on the final result.

To confirm that view, Monte Carlo computations have shown an effectively free-like behavior for analogous CQ models.

6.5. An Affine Ultralocal Scalar Field

Affine classical variables are given by $\kappa(x) \equiv \pi(x)\,\varphi(x)$ and $\varphi(x)$, with the restriction that $\varphi(x) \neq 0$ (ABC), and the Poisson bracket is given by $\{\varphi(x), \kappa(x')\} = \delta(x-x')^s \varphi(x)$. The ultralocal classical Hamiltonian, expressed in affine variables, is given by

$$H_u = \int \{ \tfrac{1}{2}[\kappa(x)^2/\varphi(x)^2 + m^2\,\varphi(x)^2] + g\,\varphi(x)^p \}\; d^s x\,. \tag{31}$$

The term $\kappa(x)^2/\varphi(x)^2$ requires that $0 < |\varphi(x)| < \infty$ to be fair to $\kappa(x)$, while $\kappa(x)$ is limited only by $|\kappa(x)| < \infty$ to be fair to $\varphi(x)$ (ABC). Observe that $0 < |\varphi(x)| < \infty$ now implies that $0 < |\varphi(x)|^p < \infty$ *for all* $0 < p < \infty$ and *for all* $s = 1, 2, 3, \ldots$.

The basic quantum operators are $\hat{\varphi}(x) \neq 0$ and $\hat{\kappa}(x)$, and their commutator is given by $[\hat{\varphi}(x), \hat{\kappa}(x')] = i\hbar\delta^s(x-x')\hat{\varphi}(x)$. The quantum, ultralocal, affine Hamiltonian, is now given by

$$\mathcal{H} = \int \{ \tfrac{1}{2}[\hat{\kappa}(x)\hat{\varphi}(x)^{-2}\hat{\kappa}(x) + m^2\hat{\varphi}(x)^2] + g\,\hat{\varphi}(x)^p \}\; d^s x\,, \tag{32}$$

with $\hat{\kappa}(x) = -\frac{1}{2}i\hbar[\varphi(x)(\delta/\delta\varphi(x)) + (\delta/\delta\varphi(x))\,\varphi(x)]$.

Clearly, this is a formal equation for the Hamiltonian operator, etc. Such expressions deserve a regularization and rescaling of these equations.

The kinetic term in (32) is $\mathcal{K}(\hat{\pi}, \hat{\varphi}) = \hat{\kappa}(x)\hat{\varphi}(x)^{-2}\hat{\kappa}(x) = \hat{\pi}(x)^2 + 2\hbar^2\,W/\hat{\varphi}(x)^2$, where $W = \delta(0)^{2s} = \infty$, and δ(x) is a special function of Dirac. Now, a kinetic scaling can be taken simply by first letting that $Z = (a^2 W)^{1/4}$ and then $\mathcal{K}(\hat{\pi}, \hat{\varphi})_{new} = Z^{-2}\,\mathcal{K}(Z\hat{\pi}, Z\hat{\varphi}) = \hat{\pi}(x)^2 + 2\hbar^2/a^2\hat{\varphi}(x)^2$, and since $0 < a < \infty$, any factor is allowed.

It is noteworthy that Monte Carlo computations have shown a reasonable active behavior for analogous AQ models [10].

7. An Ultralocal Gravity Model

An affine formulation would use the classical metric $g_{ab}(x)$, which, as before, has a positivity requirement, while the momentum field will be replaced by the dilation field, $\pi^a_b(x)$ [$\equiv \pi^{ac}(x)\,g_{bc}(x)$], summed by c. These basic affine variables are promoted to quantum operators, both of which can be self-adjoined, while the metric operator is also positive as required.

The principle of using ultralocal rules, as before, is that spacial derivatives must be eliminated. To satisfy that rule, we drop the factor ${}^{(3)}R(x)$, the Ricci scalar field composed of the metric field and its spacial derivatives, and replace it with a new function, $\Lambda(x)$, which will be called a 'Cosmological Function' to imitate the standard constant factor, Λ, known as the 'Cosmological Constant'. This new function is independent of the dilation and metric functions, and is simply used as a continuous function that obeys $0 < \Lambda(x) < \infty$, or otherwise.

With this substitution, the ultralocal classical Hamiltonian is now given by

$$H_u = \int \{ g(x)^{-1/2}[\pi^a_b(x)\pi^b_a(x) - \tfrac{1}{2}\,\pi^a_a(x)\pi^b_b(x)] + g(x)^{1/2}\,\Lambda(x) \}\; d^3x\,. \tag{33}$$

Since there are no spatial derivatives, we are given another example that every spatial point x labels a pair of distinct variables, namely $\pi^a_b(x)$ and $g_{cd}(x)$. Once again, we find a quantum wave function, using the Schrödinger representation for the metric field $g_{ab}(x)$, that is a product of independent spacial values of the form $\Psi(\{g\}) = \Pi_x W(x)$, where $\{g\}$ denotes $g_{ab}(\cdot)$ for all x.

When this Hamitonian is quantized, the only variables that are promoted to quantum operators are the metric field, $g_{ab}(x)$, and the dilation (sometimes known as 'momentric' to include *momen*tum and me*tric*) field, $\pi^a_b(x) = \pi^{ac}(x)\, g_{bc}(x)$, and the field $\Lambda(x)$ is **fixed** and **not** made into any operator.

7.1. An Affine Quantization of Ultralocal Gravity

The quantum operators are $\hat{g}_{ab}(x)$ and $\hat{\pi}^c_d(x)$, and their Schrödinger representations are given by $\hat{g}_{ab}(x) = g_{ab}(x)$ and $\hat{\pi}^a_b(x) = -i\frac{1}{2}\hbar[g_{bc}(x)\ (\delta/\delta g_{ac}(x)) + (\delta/\delta g_{ac}(x))g_{bc}(x)]$.

The Schrödinger equation for the ultralocal Hamiltonian is then given by

$$\begin{aligned} i\hbar\,\partial\,\psi(\{g\},t)/\partial t = \textstyle\int \{\, \hat{\pi}^a_b(x)\, g(x)^{-1/2}\, \hat{\pi}^b_a(x) - \frac{1}{2}\hat{\pi}^a_a(x)\, g(x)^{-1/2}\, \hat{\pi}^b_b(x) \\ + g(x)^{1/2}\, \Lambda(x)\}\; d^3x\ \psi(\{g\},t)\ , \end{aligned} \tag{34}$$

where, as noted, the symbol $\{g\}$ denotes the full metric matrix. Solutions of (34) are governed by the Central Limit Theorem.

7.2. A Regularized Affine Ultralocal Quantum Gravity

Much like the regularization of the ultralocal scalar fields, we introduce a discrete version of the underlying space such as $x \to \mathbf{k}a$, where $\mathbf{k} \in \{\dots,-1,0,1,2,3,\dots\}^3$ and $a > 0$ is the spacing between rungs in which, for the Schrödinger representation, $g_{ab}(x) \to g_{ab\,\mathbf{k}}$ and $\hat{\pi}^c_d(x) \to \hat{\pi}^c_{d\,\mathbf{k}}$. It can be helpful by assuming that the metric has been diagonalized so that $g_{ab\,\mathbf{k}} \to \{g_{11\,\mathbf{k}}, g_{22\,\mathbf{k}}, g_{33\,\mathbf{k}}\}$, as it becomes

$$\begin{aligned} \hat{\pi}^c_{d\,\mathbf{k}} &= -i\tfrac{1}{2}\hbar[g_{de\,\mathbf{k}}(\partial/\partial g_{ce\,\mathbf{k}}) + (\partial/\partial g_{ce\,\mathbf{k}})g_{de\,\mathbf{k}}]\ a^{-s} \\ &= -i\hbar[g_{de\,\mathbf{k}}(\partial/\partial g_{ce\,\mathbf{k}}) + \delta^c_d/2]\ a^{-s}\,. \end{aligned} \tag{35}$$

Take note that $\hat{\pi}^a_{b\,\mathbf{k}}\, g_{\mathbf{k}}^{-1/2} = 0$, where $g_{\mathbf{k}} = \det(g_{ab\,\mathbf{k}})$. We will exploit such an expression one more time.

The regularized Schrödinger equation is now given by

$$\begin{aligned} &i\hbar\,\partial\psi(g,t)/\partial t \\ &\quad = \textstyle\sum_{\mathbf{k}}\{\hat{\pi}^a_{b\,\mathbf{k}}\, g_{\mathbf{k}}^{-1/2}\hat{\pi}^b_{a\,\mathbf{k}} - \frac{1}{2}\hat{\pi}^a_{a\,\mathbf{k}}\, g_{\mathbf{k}}^{-1/2}\hat{\pi}^b_{b\,\mathbf{k}}\ + g_{\mathbf{k}}^{1/2}\Lambda_{\mathbf{k}}\,\}\ a^s\ \psi(g,t)\ . \end{aligned} \tag{36}$$

Observe that $g_{\mathbf{k}} = \det(g_{ab\ \mathbf{k}})$ is now the only representative of the metric $g_{ab_{\mathbf{k}}}$.

A normalized, stationary solution to this equation may be given by some $Y(g_{\mathbf{k}})$, which obeys $\Pi_{\mathbf{k}}\int|Y(g_{\mathbf{k}})|^2(ba^3)/g_{\mathbf{k}}^{(1-ba^3)}\ dg_{\mathbf{k}} = 1$, which offers a unit normalization for

$$\Psi_Y(g) = \Pi_{\mathbf{k}} Y(g_{\mathbf{k}})\,(ba^3)^{1/2}\, g_{\mathbf{k}}^{-(1-ba^3)/2}\,. \tag{37}$$

The Characteristic Function for such expressions is given by

$$\begin{aligned} C_Y(f) &= \lim_{a\to 0}\Pi_{\mathbf{k}}\textstyle\int e^{if_{\mathbf{k}}g_{\mathbf{k}}}\,|Y(g_{\mathbf{k}})|^2(ba^3)g_{\mathbf{k}}^{-(1-ba^3)}\ dg_{\mathbf{k}} \\ &= \lim_{a\to 0}\Pi_{\mathbf{k}}\{1-(ba^3)\textstyle\int[1-e^{if_{\mathbf{k}}g_{\mathbf{k}}}]\}|Y(g_{\mathbf{k}})|^2\ g_{\mathbf{k}}^{-(1-ba^3)}\ dg_{\mathbf{k}} \\ &= \exp\{-b\textstyle\int d^3x\int[1-e^{if(x)\,g(x)}]\,|Y(g(x))|^2\ \ dg(x)/g(x)\}\ , \end{aligned} \tag{38}$$

where the scalar $g_{\mathbf{k}} \to g(x) > 0$ and Y accommodates any change due to $a \to 0$. The final result is a (generalized) Poisson distribution, which obeys the Central Limit Theorem.

The formulation of Characteristic Functions for gravity establishes the suitability of an affine quantization as claimed. Although this analysis was only for an ultralocal model, it nevertheless points to the existence of proper quantum solutions for Einstein's general relativity.

7.3. The Main Lesson from Ultralocal Gravity

Just like the success of quantizing ultralocal scalar models, we have also showed that ultralocal gravity can be quantized using affine quantization. The purpose of solving ultralocal scalar models was to ensure that non-renormalizable covariant fields can be solved using affine quantization. Likewise, the purpose of quantizing an ultralocal version of Einstein's gravity shows that we should, in principle, and using affine quantization, be able to quantize the genuine version of Einstein's gravity using affine quantization.

The analysis of certain gravity models with significant symmetry may provide examples that can be completely solved using the tools of affine quantization.

8. How to Quantize Relativistic Fields

If the reader thinks that canonical quantization is the best way to quantize relativistic field theories, the reader should read this chapter carefully.

8.1. Reexamining the Classical Territory

We now turn from ultralocal models to those that are relativistic. These are models that really can represent nature, and they are clearly the most important examples. The principal example of a covariant scalar field theory is the usual one that we focus on, namely

$$H = \int H(x)\, d^s x = \int \{ \tfrac{1}{2} [\pi(x)^2 + (\overrightarrow{\nabla} \varphi(x))^2 + m^2 \varphi(x)^2] + g\, \varphi(x)^p \}\, d^s x\,. \tag{39}$$

This example is meant to deal with fields that obey the rule that $|\pi(x)| + |\varphi(x)| < \infty$ to ensure that $H(x) < \infty$. That is a very reasonable restriction; however, a path integration can violate that rule. We have in mind integrable infinities, such as $\pi(x)^2 = 1/|x|^{2s/3}$, where s is the number of spatial coordinates, i.e., $|x|^2 = x_1^2 + x_2^2 + \cdots + x_s^2$, which from a classical viewpoint seem unlikely, but from a path integration point of view seem very likely. Such integrable infinities encountered here in the classical analysis lead to non-renormalizable behavior in which the domain of the variables for a free model, i.e., $g = 0$, becomes reduced then, when $g > 0$, and $p \geq 2n/(n-2)$, with $n = s + 1$. Since the domain of the classical variables becomes reduced, it remains that way when the coupling constant is reduced to zero using $g \to 0$. With such behavior for the classical analysis, there is every reason to expect considerable difficulties in using canonical quantization.

To make that statement clear, it is a fact that Monte Carlo calculations for the scalar fields φ_3^{12} and φ_4^4 apparently led to *free results*, using CQ, as if the coupling constant $g = 0$ when that was not the case, but offered reasonable results using AQ. Clearly, integrable infinities are *not* welcome!

This section will draw on Section 5 to a large extent, although it has been somewhat changed by the introduction of the gradient term. That may lead to some repeats of certain topics.

8.1.1. A Simple Way to Avoid Integrable-Infinities

Let us, again, introduce a new field, $\kappa(x) \equiv \pi(x)\, \varphi(x)$, as a featured variable rather than $\pi(x)$, to accompany $\varphi(x) \neq 0$ (ABC). We do not really 'change any variable', but just give the usual ones 'a new role'.

Some care is needed in choosing $\kappa(x)$ and $\varphi(x)$ as the new pair of variables, and physics can be a good guide.

Let us recall the simple analog, namely $p = mv$. If the velocity $v = 0$, then physics agrees that the momentum $p = 0$. However, if the mass $m = 0$ and $v = 6$, then having $p = 0$, along with any term being infinity, is very bad physics. Instead, physics requires that $0 \leq |v|$ & $|p| < \infty$ and $0 < m < \infty$ makes good physics. This story can apply to other variables, and as has often been noted, we point to such items as (ABC).

In our case, we assume $\varphi(x)$ is a physical field, $\pi(x)$ is its time derivative, and $\kappa(x) \equiv \pi(x)\,\varphi(x)$, their product, which will be called the 'dilation field', serves as a kind of momentum. Now, using a similar argument as above, we accept the assertion that $0 \leq |\kappa(x)|$ & $|\pi(x)| < \infty$ and $0 < |\varphi(x)| < \infty$, which makes good physics.

The reader may still worry about requiring $\varphi(x) \neq 0$. Surely, integrations like $\int \varphi(x)^r\, d^sx$, $r > 0$, are not affected. However, there is a good reason to accept it. Such an equation lends itself to $\varphi(x) = 0 = \zeta(x)$ if two different fields might find this fact. If you worry about dimensions, or different charges (denoted here by *), you can use $\varphi(y) \neq 0$ and $\zeta(z) \neq 0$ or $\zeta^*(z) \neq 0$, which then leads to $\varphi(x)/\varphi(y) = 0 = \zeta(x)/\zeta(z)$ or $\varphi(x)/\varphi(y) = 0 = \zeta(x)^*/\zeta(z)^*$ which equates two fully dimensionless terms. Adopting $\varphi(x) \neq 0$ still leads to continuity thanks to the presence of the gradient term, $(\overrightarrow{\nabla}\varphi(x))^2$, which enforces a necessarily continuous field behavior.

8.1.2. The Absence of Infinities by Using Affine Field Variables

Now, let us use $\kappa(x)$ and $\varphi(x) \neq 0$ as the new variables to be used in the classical Hamiltonian (39), which then becomes

$$H = \int \{\, \tfrac{1}{2}\,[\,\kappa(x)^2/\varphi(x)^2 + (\overrightarrow{\nabla}\varphi(x))^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p\}\; d^sx\;. \tag{40}$$

Now, things are different. To represent $\pi(x)$, then $\kappa(x)$ and $\varphi(x)$, must serve their role. Hence we require that $0 < |\varphi(x)| < \infty$, which implies that $0 < |\varphi(x)|^p < \infty$ for all $0 < p < \infty$ and all s. In addition, we require that $|\kappa(x)| < \infty$ for a similar reason. The gradient term, which arises in the spacial derivative $(\overrightarrow{\nabla}\kappa(x)) = (\overrightarrow{\nabla}\pi(x))\varphi(x) + \pi(x)(\overrightarrow{\nabla}\varphi(x))$, creates another kind of (ABC) issue that leads to $|(\overrightarrow{\nabla}\varphi(x))| < \infty$. The Hamiltonian density, $H(x)$, is now *finite everywhere!* It follows that the Hamiltonian, $H = \int H(x)\; d^sx$, will be finite if it is confined to any finite spacial region, or if the field values taper off sufficiently, as is customary.

Although we have pointed out some difficulties that might arise in a canonical quantization, we follow a careful road to see how far we can get.

The usual continuum limit of the canonical quantum Hamiltonian leads to

$$\mathcal{H} = \int \{\, \tfrac{1}{2}[\hat\pi(x)^2 + (\overrightarrow{\nabla}\hat\varphi(x))^2 + m^2\hat\varphi^2(x)^2] + g\,\hat\varphi(x)^p\}\; d^sx\;, \tag{41}$$

but now there is some confusion.

The confusion arises in comparing $[Q_k, P_l] = i\hbar\delta_{kl}\mathbb{1}$ with $[\hat\varphi(x), \hat\pi(y)] = i\hbar\delta(x-y)\mathbb{1}$. As with the ultralocal case, it seems that we have a big difference in scale when $p \geq 2n/(n-2)$ and the domain reduction appears when the interaction term is active compered with if it is not active. The same issue applied to the ultralocal case, which the p-value happened even earlier due to the absence of the gradient term, which, then is $p > 2$. From a path integration viewpoint, fields like $|\varphi(x)| \gg 1$ are less likely to help their contribution. That can also apply to $|\varphi(x)| \ll 1$ about the fields. Indeed, having both $\pi(x)$ and $\varphi(x)$ fields in 'the middle' tends to make them more prominent features in a path integration.

8.2. Affine Quantization of Relativistic Field Models

8.2.1. Affine Classical Variables for Selected Field Theories

We first reexamine the features of a classical Hamiltonian once again, now with the affine variables $\kappa(x)$ and $\varphi(x) \neq 0$, which becomes

$$H = \int \{\, \tfrac{1}{2}[\kappa(x)^2/\varphi(x)^2 + (\overrightarrow{\nabla}\varphi(x))^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p\}\; d^sx\;. \tag{42}$$

In this case, we need $0 < |\varphi(x)| < \infty$, and $|\overrightarrow{\nabla}\varphi(x)|$ & $|\kappa(x)| < \infty$ (ABC). This requirement leads to the Hamiltonian *density*, $H(x)$, which will entirely be $0 \leq H(x) < \infty$, for all x, signaling that integrable infinities may be excluded. That is true, and it must be obeyed, also in a path integration. This rule, regarding quantization, already distinguishes AQ from CQ.

If new variables can calm down the classical Hamiltonian, is it possible that they might also calm down the quantum Hamiltonian? Let us see how we can do just that!

8.2.2. An Affine Quantization of Relativistic Fields

We first focus on the several field factors, which obey $\pi(x)^2 = \kappa(x)^2/\varphi(x)^2$. Using Schrödinger's representation, the quantization of these fields leads to $\pi(x)^2 \Rightarrow \hat{\kappa}(x)(\varphi(x))^{-2}\hat{\kappa}(x)$, where $\hat{\kappa}(x) = [\hat{\pi}(x)^\dagger\varphi(x) + \varphi(x)\hat{\pi}(x)]/2$. In a similar case, in an earlier chapter, we found that $p^2 = d^2/q^2 \to DQ^{-2}D = P^2 + 2\hbar^2/Q^2$, with $D = (P^\dagger Q + QP)/2$. Now, we follow the same path, more or less.

Still using Schrödinger's representation, then

$$\begin{aligned} \pi(x)^2 = \kappa(x)^2/\varphi(x)^2 &\Rightarrow \hat{\kappa}(x)(\varphi(x)^{-2})\hat{\kappa}(x) \\ &= \hat{\pi}(x)^2 + 2\hbar^2\delta(0)^{2s}/\varphi(x)^2\,, \end{aligned} \tag{43}$$

which involves the Dirac delta function much like $[\hat{\varphi}(x), \hat{\pi}(x)] = i\hbar\delta(0)^s\mathbb{1}$ does as well.

Now is the time to introduce some scaling. Such a feature can adopt $\pi_\kappa \to a^{-s/2}P_\kappa$ and $\varphi_\kappa \to a^{-s/2}Q_\kappa$, with $\hat{\kappa}_\kappa = (\hat{\pi}_\kappa^\dagger\hat{\varphi}_\kappa + \hat{\varphi}_\kappa\hat{\pi}_k)/2 = a^{-s}[P_\kappa^\dagger Q_\kappa + Q_\kappa P_\kappa]/2$. Now, we re-examine the kinetic factor for which $\hat{\kappa}_\kappa(\hat{\varphi}_\kappa^{-2})\hat{\kappa}_\kappa = a^{-s}P_\kappa^2 + 2a^{-2s}\hbar^2/a^{-s}Q_\kappa^2$. This regularization now leads to a regularized quantum Hamiltonian

$$\mathcal{H} = \Sigma_\kappa a^{-s}\{\tfrac{1}{2}[P_\kappa^2 + 2\hbar^2/Q_\kappa^2 + m^2Q_\kappa^2] + gQ_\kappa^p\}\ a^s\ , \tag{44}$$

provided that $g\,a^{-s(p-2)/2} \to g\,a^{-s}$ by properly changing g.

8.2.3. Schrödinger's Representation and Equation

We are now in position to suggest the important affine quantization of standard classical models such as

$$H = \int\{\tfrac{1}{2}\,[\kappa(x)^2/\varphi(x)^2 + (\overrightarrow{\nabla}\varphi(x))^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p\ \}\ d^sx\ , \tag{45}$$

followed by the usual rules leading to

$$\begin{aligned} \int\ \{\tfrac{1}{2}[&\hat{\kappa}(x)(\varphi(x)^{-2})\hat{\kappa}(x) \\ &+(\overrightarrow{\nabla}\varphi(x))^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p\}\ d^sx\ \Psi(\varphi) = E\ \Psi(\varphi)\ . \end{aligned} \tag{46}$$

As like other Characteristic Functions, such as were used for the ultralocal models, we note that for any normalized wave function, such as $\Pi_x\,W(\varphi(x))/\varphi(x)^{1/2}$ in the Hilbert space, a Fourier transformation leads to

$$C_W(f) = \exp\{-b\int d^sx\int[1 - e^{i\,f(x)\varphi(x)}]\,|W(\varphi(x))|^2\,d\varphi(x)/|\varphi(x)|\}\ . \tag{47}$$

9. How to Quantize Einstein's Gravity

If the reader thinks that canonical quantization is the best way to quantize Einstein's gravity, the reader should read this chapter carefully.

9.1. Gravity and AQ, Using Basic Operators

In order to quantize gravity it is important to render a valid quantization of the Arnowitt, Deser, and Misner classical Hamiltonian [11]. We first choose our new classical variables which include what we also call the dilation field $\pi_b^a(x) \equiv \pi^{ac}(x)\,g_{bc}(x)$ (summed on c) along with the metric field $g_{ab}(x)$. We do not need to impose any restriction on the metric field because physics already requires that $ds(x)^2 = g_{ab}(x)\ dx^a\ dx^b > 0$ provided that $\Sigma_{a=1}^3(dx^a)^2 > 0$. The metric can also be diagonalized by non-physical, orthogonal matrices, and then it includes only $g_{11}(x)$, $g_{22}(x)$, & $g_{33}(x)$, each of which must be strictly positive as required by physics (the reader should compare the three diagonalized positive metric variables with $q > 0$, which then requires an affine quantization for the half-harmonic oscillator, and also then appreciate the need for such a quantization that lead to positive results).

Next we present the ADM classical Hamiltonian in our chosen affine variables, which, introducing $g(x) \equiv \det[g_{ab}(x)] > 0$, leads to

$$H(\pi, g) = \int \{g(x)^{-1/2}[\pi^a_b(x)\pi^b_a(x) - \tfrac{1}{2}\pi^a_a(x)\pi^b_b(x)] \\ + g(x)^{1/2}\, {}^{(3)}\!R(x)\}\; d^3x\,, \tag{48}$$

where ${}^{(3)}\!R(x)$ is the Ricci scalar for three spatial coordinates and which contains all of the derivatives of the metric field. Already, this version of the classical Hamiltonian contains reasons that restrict $g(x)$ to $0 < g(x) < \infty$, $0 \leq |\pi^a_b(x)| < \infty$, and $0 \leq |{}^{(3)}\!R(x)| < \infty$, which, like the previous field theory examples, and lead to no integral-infinities for the gravity story.

Finally, we introduce the dilation gravity operator $\hat{\pi}^a_b(x) = [\hat{\pi}^{ac}(x)^\dagger\, \hat{g}_{bc}(x) + \hat{g}_{bc}(x)\, \hat{\pi}^{ac}(x)]/2$ along with $\hat{g}_{ab}(x) > 0$, and adopting Schrödinger's representation and equation, we are led to

$$\mathcal{H}'(\hat{\pi}, g) = \int \{\; [\, \hat{\pi}^a_b(x)\, g(x)^{-1/2}\, \hat{\pi}^b_a(x) - \tfrac{1}{2}\hat{\pi}^a_a(x)\, g(x)^{-1/2}\, \hat{\pi}^b_b(x)\,] \\ + g(x)^{1/2}\; {}^{(3)}\!R(x)\; \}\; d^3x\,. \tag{49}$$

And now, as before, we close with Schrödinger's equation

$$i\hbar\, \partial\, \Psi(g,t)/\partial t = \mathcal{H}'(\hat{\pi}, g)\; \Psi(g,t)\,, \tag{50}$$

which offers the necessary ingredients for the foundation of a valid quantization of the classical Hamiltonian, which is an important part of the full story.

As before, it may be necessary to introduce some version of regularization for these equations, but these same equations point the way to proceed. In that effort, note that although $\hat{\pi}^{ac}(x)^\dagger \neq \hat{\pi}^{ac}(x)$ it can be helpful to know that $\hat{\pi}^{ac}(x)^\dagger\, g_{bc}(x) = \hat{\pi}^{ac}(x)\, g_{bc}(x)$.

A full quantization of gravity must deal with first and likely second order constraints, which are designed to reduce the overall Hilbert space to secure a final quantization. This project is not the proper place to finalize a quantization of gravity, but several of the author's articles have been designed to go further toward the final steps [12].

Additional Aspects of Quantum Gravity

This section is relevant to follow sections which lead toward a path integration. These topics involve constraints required in the ADM approach. The present story, told just above, follows in the pattern of establishing a Schrödinger equation using his representation, has been the rule in discussing prior examples, e.g., the half-harmonic oscillator, quantum field theories over multiple powers of the interaction term, ultralocal examples of fields and gravity, and covariant field theories.

Now, in the forthcoming section, we offer a careful treatment of constraints and their analysis, which is prominent in gravity and needs its own analysis.

9.2. *Gravity and AQ, Using Path Integration*

We first recall the Arnowitt, Deser, and Misner version of the classical Hamiltonian, seen in [11], as originally expressed in the standard classical variables, namely the momentum, $\pi^{ab}(x)$, the metric, $g_{cd}(x)$, the metric determent, $g(x) = \det[g_{ab}(x)]$, and ${}^{(3)}R(x)$, which is the Ricci scalar for three spatial variables. Now, the ADM classical Hamiltonian is essentially given by

$$\begin{aligned} H(\pi, g) = \textstyle\int \{ g(x)^{-1/2} [\pi^{ac}(x) g_{bc}(x) \pi^{bd}(x) g_{ad}(x) \\ -\tfrac{1}{2}\, \pi^{ac}(x) g_{ac}(x) \pi^{bd}(x) g_{bd}(x)] \\ + g(x)^{1/2}\, {}^{(3)}R(x) \}\, d^3x\,, \end{aligned} \tag{51}$$

9.2.1. Introducing the Favored Classical Variables

The ingredients in providing a path integration of gravity include proper coherent states, the Fubini–Study metric which turns out to be affine in nature, and affine-like Wiener measures are used for the quantizing of the classical Hamiltonian. While that effort is only part of the story, it is an important portion to ensure that the quantum Hamiltonian is a bonafide self-adjoined operator.

According to the ADM classical Hamiltonian, it can also be expressed in affine-like variables, as we did in the previous chapter, namely by introducing, in some papers of this author, the 'momentric' (a name that is the combination of *momen*tum and me*tric*) and, instead, this item is now called the 'dilation variable' becoming $\pi^a_b(x)$ $(\equiv \pi^{ac}(x)\, g_{bc}(x))$, along with the metric $g_{ab}(x)$. The essential physical requirement is that $g_{ab}(x) > 0$, which means that $ds(x)^2 = g_{ab}(x)\; dx^a\; dx^b > 0$, provided that $\Sigma_a (dx^a)^2 > 0$.

Now, the classical Hamiltonian, expressed in affine classical variables, is again given by

$$\begin{aligned} H \equiv \textstyle\int H(x)\; d^3x = \int \{ g(x)^{-1/2} [\pi^a_b(x) \pi^b_a(x) - \tfrac{1}{2}\, \pi^a_a(x) \pi^b_b(x)] \\ + g(x)^{1/2}\, {}^{(3)}R(x) \}\; d^3x\,. \end{aligned} \tag{52}$$

9.2.2. The Gravity Coherent States

The principal operators $\hat{\pi}^a_b(x) = [\hat{\pi}^{ac}(x)^\dagger \hat{g}_{bc}(x) + \hat{g}_{bc}(x) \hat{\pi}^{ac}(x)]/2$ $(= \hat{\pi}^a_b(x)^\dagger)$ and $\hat{g}_{ab}(x) = \hat{g}_{ab}(x)^\dagger > 0$ offer a closed set of commutation relations given by

$$\begin{aligned} [\hat{\pi}^a_b(x), \hat{\pi}^c_d(y)] &= i\,\tfrac{1}{2}\,\hbar\,\delta^3(x,y)\,[\delta^a_d\,\hat{\pi}^c_b(x) - \delta^c_b\,\hat{\pi}^a_d(x)]\,, \\ [\hat{g}_{ab}(x), \hat{\pi}^c_d(y)] &= i\,\tfrac{1}{2}\,\hbar\,\delta^3(x,y)\,[\delta^c_a\,\hat{g}_{bd}(x) + \delta^c_b\,\hat{g}_{ad}(x)]\,, \\ [\hat{g}_{ab}(x), \hat{g}_{cd}(y)] &= 0\,. \end{aligned} \tag{53}$$

We now choose the basic affine operators to build our coherent states for gravity, specifically

$$|\pi;\eta\rangle = e^{(i/\hbar)\int \pi^{ab}(x)\,\hat{g}_{ab}(x)\,d^3x}\; e^{-(i/\hbar)\int \eta^a_b(x)\,\hat{\pi}^b_a(x)\,d^3x}\,|\beta\rangle\;\; [= |\pi;g\rangle]\,. \tag{54}$$

Note: The last item in this equation is the new name of these vectors hereafter.

A new fiducial vector, also named $|\beta\rangle$ but now different, has been chosen now in connection with the relation $[e^{\eta(x)}]_{ab} \equiv g_{ab}(x) > 0$, while $-\infty < \{\eta(x)\} < \infty$, and which enters the coherent states as shown, using $|\beta\rangle$ as the new fiducial vector that is affine-like, and obeys $[(\hat{g}_{ab}(x) - \delta_{ab}\mathbb{1}) + i\hat{\pi}^c_d(x)/\beta(x)\hbar]|\beta\rangle = 0$. It follows that $\langle\beta|\hat{g}_{cd}(x)|\beta\rangle = \delta_{cd}$ and $\langle\beta|\hat{\pi}^c_d(x)|\beta\rangle = 0$, which leads to the form given by

$$\begin{aligned} \langle\pi;g|\hat{g}_{ab}(x)|\pi;g\rangle = [e^{\eta(x)/2}]^c_a\;\langle\beta|\hat{g}_{cd}(x)|\beta\rangle\;[e^{\eta(x)/2}]^d_b \\ = [e^{\eta(x)}]_{ab} = g_{ab}(x) > 0\,. \end{aligned} \tag{55}$$

In addition, we introduce the inner product of two graviy coherent states, which is given by

$$\begin{aligned} &\langle\pi'';g''|\pi';g'\rangle = \exp\Big\{-2\int\beta(x)\,d^3x \\ &\times \ln\Big\{\,\det\Big\{\frac{[g''^{ab}(x) + g'^{ab}(x)] + i/(2\,\beta(x)\,\hbar)[\pi''^{ab}(x) - \pi'^{ab}(x)]}{\det[g''^{ab}(x)]^{1/2}\;\det[g'^{ab}(x)]^{1/2}}\Big\}\Big\}\Big\}. \end{aligned} \tag{56}$$

Finally, for some C, we find the Fubini–Study gravity metric to be

$$\begin{aligned} d\sigma_g^2 &= C\hbar[\,\|\, d|\pi;g\rangle\|^2 - |\langle\pi;g|\; d|\pi;g\rangle|^2] \\ &= \textstyle\int [\,\beta(x)\hbar)^{-1}\,(g_{ab}(x)\,d\pi^{ab}(x))^2 + (\beta(x)\hbar)\,(g^{ab}(x)\,dg_{ab}(x))^2\,]\;d^3x\;, \end{aligned} \tag{57}$$

which is seen to imitate an affine metric, leading to a constant negative curvature, as well, and that will provide a genuine Wiener-like measure for a path integration. In no way could we transform this metric into a proper Cartesian form, as was carried out for the half-harmonic oscillator. That is because there is no physically proper Cartesian metric for the variables $\pi^{ab}(x)$ and $g_{cd}(x)$.

9.2.3. A Special Measure for the Lagrange Multipliers

To ensure a proper treatment of the operator constraints, we choose a special measure of the Lagrange multipliers, $R(N_a, N)$, guided by the following procedures.

The first step is to unite the several classical constraints by using

$$\begin{aligned} &\textstyle\int\!\!\int e^{i(y^a H_a(x)+yH(x))}\;W(u,y^a,y,g^{ab}(x))\;\Pi_a dy^a\,dy \\ &\qquad = e^{-iu[H_a(x)g^{ab}(x)H_b(x)+H(x)^2]} \\ &\qquad \equiv e^{-iuH_v(x)^2} \end{aligned} \tag{58}$$

with a suitable measure W.

An elementary Fourier transformation (in mathematics, the following function being Fourier transformed is known as (a version of) rect(u) = 1 for $|u| \leq 1$, and 0 for $|u| > 1$) is given by $M\int_{-\delta^2}^{\delta^2} e^{i\epsilon\tau\,uy}\;dy/2 = \sin(u\epsilon\tau\delta^2)/u$, using a suitable M, which then ensures that the inverse Fourier transformation, where ϵ represents a tiny spatial interval and τ represents a tiny time interval, as part of a fully regularized integration in space and time, and u is another part of the Lagrange multipliers, $N_a(n\epsilon)$ and $N(n\epsilon)$, which leads to

$$\begin{aligned} &\lim_{\zeta\to 0^+}\;\lim_{L\to\infty}\int_{-L}^{L} e^{-iu\epsilon\tau\mathcal{H}_v^2(x)}\;\sin(u\epsilon\tau(\delta^2+\zeta))/u\pi\;du \\ &\qquad = \mathbb{E}(\epsilon\tau\mathcal{H}_v(x)^2 \leq \epsilon\tau\delta^2) \\ &\qquad = \mathbb{E}(\mathcal{H}_v(x)^2 \leq \delta^2)\;. \end{aligned} \tag{59}$$

This expression covers all self-adjoined operators, and leads to a self-adjoined $\mathcal{H}_v = \int \mathcal{H}_v(x)\;d^3x$.

Bringing together our present tools lets us first offer a path integral for the gravity overlap of two coherent states, as given by

$$\begin{aligned} \langle\pi'';g''|\pi';g'\rangle &= \lim_{\nu\to\infty}\mathcal{N}_\nu\int \exp[-(i/\hbar)\textstyle\int_0^T\int[(g_{ab}\,\dot\pi^{ab})\;d^3x\,dt] \\ &\times \exp\{-(1/2\nu\hbar)\textstyle\int_0^T\int[(\beta(x)\hbar)^{-1}\,(g_{ab}\,\dot\pi^{ab})^2 + (\beta(x)\hbar)\,(g^{ab}\dot g_{ab})^2]\;d^3x\,dt\} \\ &\qquad\times\Pi_{x,t}\Pi_{a,b}\,d\pi^{ab}(x,t)\,dg_{ab}(x,t) \\ &= \exp\Big\{-2\int\beta(x)\,d^3x \\ &\qquad\times\ln\Big\{\det\Big\{\frac{[g''^{ab}(x)+g'^{ab}(x)] + i/(2\,\beta(x)\,\hbar)[\pi''^{ab}(x)-\pi'^{ab}(x)]}{\det[g''^{ab}(x)]^{1/2}\;\det[g'^{ab}(x)]^{1/2}}\Big\}\Big\}\Big\}\;, \end{aligned} \tag{60}$$

where the second equation indicates what such a path integration has been designed to acheive for its goal.

9.3. The Affine Gravity Path Integration

By adding all the necessary tools, and implicitly having examined a regularized integration version in order to effectively deal with suitable constraint projection terms, we have chosen $\mathbb{E} \equiv \mathbb{E}(\mathcal{H}_v^2 \leq \delta(\hbar)^2)$ for simplicity here, all of which leads us to

$$\begin{aligned}
\langle \pi''; g'' : T | \mathbb{E} | \pi'; g' : 0 \rangle &= \langle \pi''; g'' | \mathbb{E}\, e^{-(i/\hbar)\mathbf{E}\,T} \mathbb{E}\, | \pi'; g' \rangle \\
&= \lim_{\nu \to \infty} \mathcal{N}'_\nu \int \exp\{-(i/\hbar) \textstyle\int_0^T \int [g_{ab}\, \dot{\pi}^{ab} + N^a H_a + NH]\, d^3x\, dt\} \\
\times \exp\{-(1/2\nu\hbar) \textstyle\int_0^T \int [(\beta(x)\hbar)^{-1} &(g_{ab}\, \dot{\pi}^{ab})^2 + (\beta(x)\hbar)\, (g^{ab}\, \dot{g}_{ab})^2]\, d^3x\, dt\} \\
&\times [\Pi_{x,t} \Pi_{a,b}\, d\pi^{ab}(x,t)\, dg_{ab}(x,t)\, \mathcal{D}R(N^a, N)\ . \qquad (61)
\end{aligned}$$

The role of the measure $R(N^a, N)$ is defined so that the operators, $\mathcal{H}_a\ \mathcal{H}$, only support a sample of non-zero eigenvalues, e.g., $\mathbb{E}(\mathcal{H}_v^2 \leq \delta(\hbar)^2)$, where, e.g., $\delta(\hbar)^2 \sim c\,\hbar^2$, or some other tiny value that vanishes if $\hbar \to 0$ if $\mathcal{H}_v^2 \leq \delta(\hbar)^2$ consists only of a continuous spectrum; see [12].

We let the reader choose their own regularization of the last equation to ensure that the ϵ terms are proper, and that the ϵ^2 terms —And higher $\epsilon^K,\ K > 2$, terms as well —Lead to a proper continuum limit. In so doing, the overlap of two gravity coherent states, as above in (60), could be particularly useful.

Several papers by the author offer additional information regarding topics, and additional procedures to use, has been discussed in [Kla-2].

10. Summary, and Outlook

Each Field Problem Needs AQ or CQ, Otherwise, There Can Be Incorrect Results

Could it be the time now to pass from CQ to AQ to solve difficult problems? Perhaps new procedures can help. The passing of years can often lead to the introduction of improved procedures. For example, though history, people first took around the mail on horses , then trains, then cars, then switched to the internet, etc. Likewise, first there was CQ, now AQ, which is added to become CQ & AQ = EQ, known as 'Enhanced Quantization'. It has introduced a huge jump that greatly extends quantization procedures, along with a noteworthy proof that, essentially, AQ $\to$ CQ.

In an artistic sense, CQ represents the beautiful surface of a brick made of pure gold, while AQ represents the wonderful interior of the same gold brick. Hence, moving around within an inside path, AQ points can reach a point on the CQ surface!

While the half-harmonic oscillator using AQ was shown to be valid, the validity of AQ for field theories or for gravity are not as yet proved to be true, and there are opportunities for others to test their out coming. While every attempt to maintain correctness has been made, something may still have been overlooked. The improvement of every step is open to consideration and further recommendation.

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Appendix A. Appendix to Section 5

Any tired reader may skip to the last paragraph.

Our analysis of general dilation variables is given as follows. The quantum kinetic term (now, with $\hbar = 1$) in affine variables is $DF^{-2}D$. This expression, is helped by $F = F(Q), G =$

$G(Q) \equiv 1/F(Q), F(Q)P - PF(Q) = i\,F'(Q)$. and $G(Q)P - PG(Q) = i\,G'(Q)$ leads to $4DG^2D = (PF + FP)GG(PF + FP) = PP + FPGGPF + FPGP + PGPF = PP + (PF + iF')GG(FP - iF') + (PF + iF')GP + PG(FP - iF') = 4PP + 2i(F'GP - PGF') + F'GGF' = 4PP - 2(F'G)' + (F')^2G^2$. Restoring $\hbar$, we have $DG^2D = P^2 + (1/4)\hbar^2[(F')^2G^2 - 2(F'G)']$.

Keeping $\hbar$, and here, using $D = [P^\dagger F(Q) + F(Q)P]/2$, then $DF^{-2}D = P^2 + (1/4)\hbar^2$ $[(\ln(F)')^2 - 2(\ln(F))'']$, where, symbolically, $F'(Q) = dF(Q)/dQ$. For $F(Q) = (Q^2 - b^2)$, we now find the $\hbar$-term to be $\hbar^2(2Q^2 + b^2)/(b^2 - Q^2)^2$.

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