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ω -Limit Sets of Impulsive Semigroups for Hyperbolic Equations

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Abstract: In this paper, we investigate the qualitative behavior of an evolutionary problem consisting of a hyperbolic dissipative equation whose trajectories undergo instantaneous impulsive discontinuities at the moments when the energy functional reaches a certain threshold value. The novelty of the current study is that we consider the case in which the entire infinite-dimensional phase vector undergoes an impulsive disturbance. This substantially broadens the existing results, which admit discontinuities for only a finite subset of phase coordinates. Under fairly general conditions on the system parameters, we prove that such a problem generates an impulsive dynamical system in the natural phase space, and its trajectories have nonempty compact ω -limit sets.

Keywords: evolution problem; hyperbolic dissipative equation; impulsive dynamical system

MSC: 35R12; 37L05



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1. Introduction

Evolution equations are primary mathematical models for a variety of processes in physics, fluid dynamics, chemical engineering, material science, and geophysics [1–3]. Often, solutions to evolution equations undergo very fast (almost instantaneous) changes, and it is convenient to model these abrupt changes as discontinuities or impulsive jumps [4,5]. There are several mathematical frameworks available in the literature for the proper modeling and analysis of these processes, namely impulsive differential equations, discontinuous dynamical systems [6–8], hybrid dynamical systems [9–11], and differential equations with Dirac delta functions [12]. We refer the interested reader to [13], which provides a concise overview and comparison of these frameworks. More recent studies of discontinuous dynamics include results for delay [14–16], fractional order [17–19], stochastic equations [20–22], fuzzy differential systems [23,24], and applications of impulses for stabilization and control purposes [25–27].

This paper is devoted to the study of an important class of evolutionary systems characterized by the presence of impulsive disturbances when the system trajectory reaches a predefined subset in the phase space, often termed impulsive dynamical systems [28,29]. The systematic study of these systems began relatively recently and primarily focused on systems defined in the finite-dimensional phase spaces, e.g., on systems defined in Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$ [30–34] and the so-called multi-frequency systems defined in the product of a torus and Euclidean space $\mathcal{T}_m \times \mathbb{R}^n$, $n, m \in \mathbb{N}$ [35–38]. The results regarding the limit behavior of infinite-dimensional impulsive dynamical systems can be found in [39–45]. However, in both the parabolic and hyperbolic cases, the impulsive parameters are “finite-dimensional” in nature, i.e., only a finite number of coordinates of the phase vector is subjected to an impulsive disturbance. The novelty of the current study is that we consider the case when the entire infinite-dimensional phase vector undergoes an impulsive disturbance at the moment when the system’s energy functional reaches a certain

threshold value. To characterize the long-term asymptotic behavior of the considered hyperbolic problem, first, we formally show that the problem generates an impulsive dynamical system in the natural phase space. Next, we prove the main result of the current paper regarding the limit behavior of the considered impulsive problem, namely that its trajectories have nonempty compact ω -limit sets.

The rest of this paper is organized as follows. In Section 2, we formulate the considered impulsive problem, together with the basic assumption on the system. The formal construction of the corresponding impulsive dynamical systems is presented in Section 3. The main theorem on the limit regimes of the constructed impulsive dynamical system is stated and proved in Section 4. Finally, some conclusions in Section 5 complete the paper.

2. Problem Statement

Let a triple of Hilbert spaces $V \subset H \subset V^*$ with compact dense embeddings be given. Additionally, let $\|\cdot\|$ and $(\cdot, \cdot)$ stand for the norm and the scalar product in H , respectively; $A : V \rightarrow V^*$ be a linear, continuous, self-adjoint, coercive operator; $\|u\|_V := \langle A^{\frac{1}{2}}u, u \rangle$ be the norm in V ; and $\langle \cdot, \cdot \rangle$ denote the scalar product in V .

Consider an evolution problem

$$\begin{cases} \frac{d^2y}{dt^2} + 2\beta \frac{dy}{dt} + Ay = 0, \\ y|_{t=0} = y_0 \in V, \\ y_t|_{t=0} = y_1 \in V. \end{cases} \quad (1)$$

It is known [46] that problem (1) in the phase space $X = V \times H$ generates a continuous semigroup $G : \mathbb{R}_+ \times X \rightarrow X$ so that for any $z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in X$

$$\begin{aligned} G(t, z_0) &= z(t) = \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix} \\ &= e^{-\beta t} \sum_{j=1}^{\infty} \begin{pmatrix} (y_0, \varphi_j) \cos \omega_j t + (\beta(y_0, \varphi_j) + (y_1, \varphi_j)) \frac{1}{\omega_j} \sin \omega_j t \\ (y_1, \varphi_j) \cos \omega_j t - (\lambda_j^2(y_0, \varphi_j) + \beta(y_1, \varphi_j)) \frac{1}{\omega_j} \sin \omega_j t \end{pmatrix}, \end{aligned} \quad (2)$$

where $\omega_j = \sqrt{\lambda_j^2 - \beta^2}$, $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$ are solutions to the spectral problem

$$A\varphi_j = \lambda_j \varphi_j, \quad j \geq 1,$$

$\{\varphi_j\}_{j=1}^{\infty}$ is the orthonormal basis in H , $0 < \lambda_1 \leq \lambda_2 \leq \dots$, $\lambda_j \rightarrow +\infty$, $j \rightarrow \infty$, and without loss of generality, we assume that $\lambda_1 > \beta$.

In what follows, we augment system (1) with impulsive discontinuities of the state trajectories. To this end, let us introduce the functional $\Psi : X \rightarrow \mathbb{R}_+$ defined by

$$\Psi(z) = \|z\|_X^2 = \|u\|_V^2 + \|v\|^2 \quad (3)$$

for any $z = \begin{pmatrix} u \\ v \end{pmatrix} \in X$. This energy-like functional is used to determine the moments of impulsive jumps in the system.

The impulsive problem is formulated as follows. If at some point in time, $t > 0$ and the corresponding solution $z = \begin{pmatrix} y \\ y_t \end{pmatrix}$ is such that the functional (3) reaches the value $\Psi_0 \in \mathbb{R}_+$, i.e., $\Psi(z(t)) = \Psi_0$, then the state z is being instantaneously transferred to a new position z^+ defined by

$$z^+ = \varphi(z) + \alpha, \quad (4)$$

where $\alpha \in X$, $\varphi : X \rightarrow X$ are given. Note that formulation (4) is natural in many mechanical problems where there is an instantaneous change in velocity [13,47], i.e., impulsive action (4) causes discontinuity not only in y but also in y_t .

In the current paper, we prove that under certain mild conditions on the system parameters, problems (1), (3), and (4) generate an impulsive dynamical system in X , which we denote as

$$\tilde{G} : \mathbb{R}_+ \times X \rightarrow X$$

(see formal Definition 1 below) and for each $z_0 \in X$, the ω -limit set is nonempty, compact, and the limit relation

$$\text{dist}_X(\tilde{G}(t, z_0), \tilde{\omega}(z_0)) \rightarrow 0, \quad t \rightarrow \infty$$

holds true.

3. ω -Limit Set for Impulsive Dynamical Systems

Following the work in [48], we describe the general construction of the impulsive dynamical system. Suppose that a continuous semigroup $G : \mathbb{R}_+ \times X \rightarrow X$ is given on the phase space X . The trajectories of the semigroup, when they reach a predefined subset $M \subset X$ (the so-called impulsive set), are instantaneously transferred according to the map I (the so-called impulsive map) to a new position

$$z^+ := Iz.$$

For the correctness of such a construction, the following conditions must be satisfied

$G : \mathbb{R}_+ \times X \rightarrow X$ is a continuous semigroup,

$$\text{i.e., for all } z \in X \text{ and } t, s \geq 0 : \quad G(0, z) = z, \quad G(t + s, z) = G(t, G(s, z)), \quad (5)$$

map $(t, z) \mapsto G(t, z)$ is continuous on $\mathbb{R}_+ \times X$;

$$M \text{ is a closed set, } M \cap IM = \emptyset; \quad (6)$$

$$\forall z \in M \exists \tau = \tau(z) > 0 \forall t \in (0, \tau) : G(t, z) \notin M. \quad (7)$$

Under the conditions (5)–(7), it is known [40] that if for $z \in X$,

$$M^+(z) := \left(\bigcup_{t>0} G(t, z) \right) \cap M \neq \emptyset,$$

then, there exists $\tilde{s} := \tilde{s}(z) > 0$, such that

$$\forall t \in (0, \tilde{s}) : G(t, z) \notin M, \quad G(\tilde{s}, z) \in M. \quad (8)$$

Using the introduced notation z^+ , $M^+(z)$, $\tilde{s}$, the impulsive trajectory $\tilde{G}(\cdot, z_0)$ starting at $z_0 \in X$ is constructed as follows:

- If $M^+(z_0) = \emptyset$, then

$$\tilde{G}(t, z_0) = G(t, z_0), \quad t \geq 0;$$

- If $M^+(z_0) \neq \emptyset$, then for $s_0 := \tilde{s}(z_0)$, let us denote $z_1^+ := G(s_0, z_0)$, so that

$$\tilde{G}(t, z_0) = \begin{cases} G(t, z_0), & t \in [0, s_0), \\ z_1^+, & t = s_0; \end{cases}$$

- If $M^+(z_1^+) = \emptyset$, then

$$\tilde{G}(t, z_0) = G(t - s_0, z_1^+), \quad t \geq s_0;$$

- If $M^+(z_1^+) \neq \emptyset$, then for $s_1 := \tilde{s}(z_1^+)$, let us denote $z_1 := G(s_1, z_1^+)$, so that

$$\tilde{G}(t, z_0) = \begin{cases} G(t - s_0, z_1^+), & t \in [s_0, s_0 + s_1], \\ z_2^+, & t = s_0 + s_1; \end{cases}$$

and so on. By continuing this process, we can obtain a finite or infinite number of impulsive points

$$z_{n+1}^+ = IG(s_n, z_n^+), \quad z_0^+ := z_0, \quad n \geq 0;$$

and the corresponding sequence of time moments

$$T_{n+1} := \sum_{k=0}^n s_k, \quad T_0 := 0, \quad n \geq 0.$$

At the same time, $\tilde{G}$ is given by the formula

$$\tilde{G}(t, z_0) = \begin{cases} G(t - T_n, z_n^+), & t \in [T_n, T_{n+1}), \\ z_{n+1}^+, & t = T_{n+1}. \end{cases} \quad (9)$$

It should be noted that the Zeno (or beating) phenomenon can occur in such a system, which is characterized by infinitely many impulsive jumps over a finite interval of time [49,50]. Such behavior is often undesirable, especially for systems that are mathematical models of real physical processes or in control engineering applications in which impulses are assigned to the control actuation. The classically defined trajectories for such systems do exist on a finite interval only, and their prolongation beyond the Zeno point is discussed in [50–52].

Since we are interested in the behavior of (9) when $t \rightarrow \infty$, we make the following assumption:

$$\begin{aligned} &\text{for each } z_0 \in X, \text{ there are either no impulsive points,} \\ &\text{or their number is finite, or } T_n \rightarrow \infty, \quad n \rightarrow \infty. \end{aligned} \quad (10)$$

Condition (10) guarantees that for an arbitrary $z_0 \in X$, the function $t \mapsto \tilde{G}(t, z_0)$ is defined on $[0, +\infty)$.

Definition 1. The mapping $\tilde{G} : \mathbb{R}_+ \times X \rightarrow X$ constructed above is called an **impulsive dynamical system**. We say that the triple $\{V, M, I\}$ generates an impulsive dynamical system if conditions (5)–(7) and (10) are satisfied.

It is known that under conditions (5)–(7) and (10), the mapping $\tilde{G} : \mathbb{R}_+ \times X \rightarrow X$ is a semigroup whose trajectories are continuous from the right. In addition, from the construction of $\tilde{G}$, it holds that for arbitrary $z_0 \in X$ and $t > 0$,

$$\tilde{G}(t, z_0) \cap M = \emptyset.$$

The main object of study in this paper is the ω -limit set:

$$\tilde{\omega}(z_0) = \left\{ \xi \in X \mid \exists \{t_n\}_{n=1}^\infty : t_n \nearrow \infty, \xi = \lim_{n \rightarrow \infty} \tilde{G}(t_n, z_0) \right\}. \quad (11)$$

Lemma 1. Let $\{V, M, I\}$ generate an impulsive dynamical system $\tilde{G}$, and for any $z_0 \in X$, let the following conditions be fulfilled:

1. The set

$$\tilde{\gamma} := \bigcup_{t \geq 0} \tilde{G}(t, z_0)$$

- is bounded;
2. For each $z \in \tilde{\gamma}$:

$$G(t, z) = G_1(t, z) + G_2(t, z),$$

where $\{G_1(t, z), t \geq 0, z \in \tilde{\gamma}\}$ is precompact and

$$\sup_{z \in \tilde{\gamma}} G_2(t, z) \rightarrow 0, t \rightarrow \infty.$$

3. If $\tilde{\gamma}$ has an infinite number of impulsive points $\{z_n^+\}_{n \geq 0}$, then the set $\{z_n^+\}_{n \geq 0}$ is precompact. Then, the set $\tilde{\omega}(z_0) \neq \emptyset$ is compact and

$$\text{dist}_X(\tilde{G}(t, z_0), \tilde{\omega}(z_0)) \rightarrow 0, t \rightarrow \infty.$$

Proof of Lemma 1. According to [40], the nonempty ω -limit set of $\tilde{G}(t, z_0)$ for any initial condition $z_0 \in X$ exists if the sequence

$$\{\xi_m = \tilde{G}(t_m, z_0), t_m \nearrow \infty\}$$

is precompact. In what follows, we show that under the conditions of Lemma 1, the sequence $\{\xi_m\}$ is indeed precompact. To this end, we consider three possible cases:

Case 1: If $M^+(z_0) = \emptyset$, then

$$\xi_m = \tilde{G}(t_m, z_0) = G(t_m, z_0) = G_1(t_m, z_0) + G_2(t_m, z_0)$$

and the precompactness of $\{\xi_m\}$ follows from condition 2.

Case 2: If $M^+(z_0) \neq \emptyset$ but the number of impulsive points is finite $\{z_1^+, \dots, z_p^+\}$, then for sufficiently large m , we have that $t_m > T_p + 1$ and, therefore,

$$\xi_m = \tilde{G}(t_m, z_0) = G(t_m - T_p, z_p^+) = G_1(t_m - T_p, z_p^+) + G_2(t_m - T_p, z_p^+).$$

Since $z_p^+ \in \tilde{\gamma}$, the precompactness of $\{\xi_m\}$ follows from condition 2.

Case 3: Finally, let $M^+(z_0) \neq \emptyset$, and let there be an infinite number of impulsive points $\{z_n^+\}$. Let $\{T_n\}_{n \geq 1}$ be the corresponding moments of the impulsive disturbance, $T_n \rightarrow \infty, n \rightarrow \infty$. Then, for $t_m \rightarrow \infty$, for each $m \geq 1$, there exists $n(m)$ such that

$$t_m \in [T_{n(m)}, T_{n(m)+1}), n(m) \rightarrow \infty, m \rightarrow \infty.$$

Therefore, according to (9)

$$\xi_m = \tilde{G}(t_m, z_0) = G(t_m - T_{n(m)}, z_{n(m)}^+).$$

If $t_m - T_{n(m)} \rightarrow \infty$, then

$$\xi_m \in G_1(t_m - T_{n(m)}, \tilde{\gamma}) + G_2(t_m - T_{n(m)}, \tilde{\gamma}),$$

where, due to condition 2, $\{G_1(t_m - T_{n(m)}, \tilde{\gamma})\}$ is precompact and

$$\|G_2(t_m - T_{n(m)}, z_{n(m)}^+)\| \leq \sup_{z \in \tilde{\gamma}} \|G_2(t_m - T_{n(m)}, z)\| \rightarrow 0, m \rightarrow \infty,$$

which implies the precompactness of $\{\xi_m\}$. If at least along a subsequence

$$t_m - T_{n(m)} \rightarrow \tau \geq 0,$$

then the precompactness of

$$\left\{ \xi_m = G(t_m - T_{n(m)}, z_{n(m)}^+) \right\}$$

follows from condition 3 and the continuity of the semigroup G . $\square$

Remark 1. The fulfillment of condition 1 can be guaranteed under the following conditions

$$\begin{aligned} \exists C_1, C_2 \geq 0 \quad \exists \delta > 0 \quad \forall z \in \tilde{\gamma} \quad \forall t \geq 0 \\ \|G(t, z)\|_X \leq \|z\|_X e^{-\delta t} + C_1, \end{aligned} \quad (12)$$

$$\|Iz\|_X \leq \|z\|_X + C_2, \quad (13)$$

and if $C\{s_k\}_{k \geq 0}$ are the distances between the impulses along the $\tilde{\gamma}$, then

$$\bar{s} := \inf_{k \geq 0} s_k > 0. \quad (14)$$

Indeed, in this case, we have an estimate for the impulsive points

$$\begin{aligned} \|z_n^+\|_X &\leq \|z_0\|_X e^{-\delta n \bar{s}} + (C_1 + C_2) \left(1 + e^{-\delta \bar{s}} + \dots + e^{-(n-1)\delta \bar{s}} \right) \\ &\leq \|z_0\|_X + \frac{C_1 + C_2}{1 - e^{-\delta \bar{s}}}. \end{aligned} \quad (15)$$

By combining (12) and (15), we obtain that for arbitrary $t \geq 0$

$$\|\tilde{G}(t, z_0)\|_X \leq \|z_0\|_X + \frac{C_1 + C_2}{1 - e^{-\delta \bar{s}}} + C_1,$$

which means that condition 1 is fulfilled.

Remark 2. Condition 3 can be replaced with the following:

$$\text{if } \{z_n\} \text{ is bounded, then } \{Iz_n\} \text{ is precompact.} \quad (16)$$

Indeed, according to the proof of Lemma 1, it suffices to prove the precompactness of

$$\left\{ \xi_m = G(t_m - T_{n(m)}, z_{n(m)}^+) \right\},$$

where $t_m - T_{n(m)} \rightarrow \tau \geq 0$. In this case, according to (9),

$$z_{n(m)}^+ = IG(s_{n(m)-1}, z_{n(m)-1}^+).$$

As a result of conditions 1 and 2, the sequence

$$\left\{ G(s_{n(m)}, z_{n(m)}^+) \right\}$$

is bounded and, therefore, condition (16) guarantees the precompactness of $\{z_{n(m)}^+\}$.

It is important to note that we cannot expect $\tilde{\omega}(z_0)$ to be stable in any sense since this is not true even in the non-impulsive case. The stability property can be guaranteed for more massive objects, namely for uniform attractors [40]. Nevertheless, we can ensure the invariance of the non-impulsive part of $\tilde{\omega}(z_0)$. For this purpose, it is necessary to impose additional conditions on trajectories starting from the initial data close to $\tilde{\omega}(z_0)$. This is performed in the following lemma.

Lemma 2. Let $\{V, M, I\}$ generate impulsive dynamical system $\tilde{G}$, the conditions of Lemma 1 be fulfilled for $z_0 \in X$, and, in addition

$$I : M \rightarrow X \text{ be continuous;} \quad (17)$$

$$\text{if } \xi \in \tilde{\omega}(z_0) \setminus M, \text{ then for } \xi_n \rightarrow \xi$$

$$\begin{cases} \tilde{s}(\xi) = \infty, & \text{if } \tilde{s}(\xi_n) = \infty \text{ for infinitely many } n, \\ \tilde{s}(\xi_n) \rightarrow \tilde{s}(\xi), & \text{otherwise.} \end{cases} \quad (18)$$

Then, for each $t \geq 0$

$$\tilde{G}(t, \tilde{\omega}(z_0) \setminus M) \subset \tilde{\omega}(z_0) \setminus M. \quad (19)$$

If additionally, for $\xi \in \tilde{\omega}(z_0) \cap M$ and for $\xi_m \rightarrow \xi$, $\xi_m \notin M$,

$$\tilde{s}(\xi_n) = \infty \text{ for infinitely many } n \text{ or } \tilde{s}(\xi_n) \rightarrow 0, \quad (20)$$

then, for arbitrary $t \geq 0$

$$\tilde{G}(t, \tilde{\omega}(z_0)) \supset \tilde{\omega}(z_0) \setminus M. \quad (21)$$

Proof of Lemma 2. We use the following fact, proven in a much more general situation in [40]: Under the conditions of (18), if

$$\xi \in \tilde{\omega}(z_0) \cap M, \quad \xi_m \rightarrow \xi,$$

then for each $t \geq 0$, there exists a sequence $\eta_n \searrow 0$, such that along a subsequence

$$\tilde{G}(t + \eta_n, \xi_n) \rightarrow \tilde{G}(t, \xi), \quad n \rightarrow \infty. \quad (22)$$

Moreover, for a given sequence $\alpha_n \searrow 0$

$$\tilde{G}(\alpha_n, \xi_n) \rightarrow \xi, \quad n \rightarrow \infty. \quad (23)$$

Now, let us prove (19). Let $\xi \in \tilde{\omega}(z_0) \setminus M$, $t > 0$. Then, from (22) for $t > 0$, there is $\eta_n \searrow 0$, such that

$$\tilde{G}(t + \eta_n, \xi_n) = \tilde{G}(t + t_n + \eta_n, z_0) \rightarrow \tilde{G}(t, \xi).$$

But $t + t_n + \eta_n \rightarrow \infty$, so

$$\tilde{G}(t, \xi) \in \omega(z_0) \setminus M,$$

which proves (19).

Let us prove (21). Let

$$\xi \in \tilde{\omega}(z_0) \setminus M, \quad t > 0.$$

Then,

$$\xi = \lim_{n \rightarrow \infty} \xi_n,$$

where $\xi_n = \tilde{G}(t_n, z_0)$. Let us denote

$$y_n = \tilde{G}(t_n - t, z_0).$$

Along a subsequence, $y_n \rightarrow y \in \tilde{\omega}(z_0)$. If $y \notin M$, then due to (22), there is a sequence $\eta_n \searrow 0$, such that

$$\tilde{G}(t + \eta_n, y_n) \rightarrow \tilde{G}(t, y).$$

On the other hand, $\tilde{G}(t, y_n) = \xi_n \rightarrow \xi \notin M$. Therefore, as a result of (22),

$$\tilde{G}(t + \eta_n, y_n) = \tilde{G}(\eta_n, \xi_n) \rightarrow \xi.$$

Hence, $\xi = \tilde{G}(t, y) \in \tilde{G}(t, \tilde{\omega}(z_0) \setminus M)$.

Now, let $y \in M$. If $\tilde{s}(y_n) = \infty$ for infinitely many n , then for all $p \geq 0$:

$$\tilde{G}(p, y_n) = G(p, y_n) \rightarrow G(p, y).$$

So, according to (7), there exists $\tau > 0$, such that for an arbitrary $p \in (0, \tau] : G(p, y) \notin M$. Therefore, if $t < \tau$, then

$$\xi_n = \tilde{G}(t, y_n) = G(t, y_n) \rightarrow \xi = G(t, y) = \tilde{G}(t, y),$$

i.e., $\xi \in \tilde{G}(t, \tilde{\omega}(z_0))$. If $t > \tau$, then since

$$G(\tau, y_n) = \tilde{G}(\tau, y_n) \rightarrow \tilde{G}(\tau, y) = G(\tau, y),$$

$G(\tau, y) \in \tilde{\omega}(z_0)$, and, therefore, due to (18) for all $p \geq 0$, it holds that

$$\tilde{G}(p, G(\tau, y)) = G(p, G(\tau, y)).$$

Hence,

$$\xi_n = \tilde{G}(t - \tau, \tilde{G}(\tau, y_n)) \rightarrow G(t - \tau, G(\tau, y)) = \xi = \tilde{G}(t, y),$$

i.e., $\xi \in \tilde{G}(t, \tilde{\omega}(z_0))$.

Now, let $\tau_n := \tilde{s}(y_n) \rightarrow 0$. Then,

$$G(\tau_n, y_n) \rightarrow y, \quad \tilde{G}(\tau_n, y_n) = IG(\tau_n, y_n) \rightarrow Iy.$$

Since

$$\tilde{G}(\tau_n, y_n) = \tilde{G}(t_n - \tau_n, z_0)$$

and $t_n - \tau_n \nearrow \infty$, we conclude that $Iy \in \omega(z_0) \setminus M$. Then, as a result of (22), there exists $\eta_n \searrow 0$, such that

$$\tilde{G}(t + \eta_n, \tilde{G}(\tau, y_n)) = \tilde{G}(\eta_n + \tau_n, \tilde{G}(t, y_n)) \rightarrow \tilde{G}(t, Iy).$$

Since $\eta_n + \tau_n \rightarrow 0$, then due to (23), we obtain

$$\xi = \tilde{G}(t, Iy) \in \tilde{G}(t, \omega(z_0) \setminus M).$$

This completes the proof. $\square$

Remark 3. If we add the following condition to the conditions of Lemma 2:

$$\text{for } t_n \nearrow \infty \text{ along a subsequence } G(t_n, z_0) \rightarrow y \notin M, \quad (24)$$

then for an arbitrary $t \geq 0 : \tilde{G}(t, \tilde{\omega}(z_0) \setminus M) = \tilde{\omega}(z_0) \setminus M$.

Condition (24) means that the ω -limit set of the non-impulsive semiflow G does not intersect M .

4. Limit Regimes of the Impulsive Problem (1), (3), (4)

For problems (1), (3), and (4), the phase space is the Hilbert space $X = V \times H$, on which the solutions of the evolutionary problem (1) generate a continuous semigroup $G : \mathbb{R}_+ \times X \rightarrow X$ according to Formula (2).

The impulsive set M is given by (3) according to the formula

$$M = \left\{ z = \begin{pmatrix} u \\ v \end{pmatrix} \in X \mid \Psi(z) = \Psi_0 \right\}, \quad \Psi_0 > 0.$$

The impulsive map $I : M \rightarrow X$ is given by Formula (4) according to the following rule: for each $z \in M$,

$$Iz = \varphi(z) + \alpha, \quad \alpha \in X, \quad \varphi : M \rightarrow X.$$

Recall that we know from (3) that $\Psi(x) = \|z\|_X^2$ and $\|z\|_X^2 = \|u\|_V^2 + \|v\|^2$. Thus,

$$\Psi(Iz) = \|Iz\|_X^2 = \|\varphi(z) + \alpha\|_X^2.$$

Then, the set M is closed, and for $z = \begin{pmatrix} u \\ v \end{pmatrix}$, $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} : M \rightarrow X$, the following estimate holds true:

$$\begin{aligned} \Psi(Iz) &= \|\varphi_1(z) + \alpha_1\|_V^2 + \|\varphi_2(z) + \alpha_2\|^2 \\ &= \|\varphi_1(z) + \alpha_1\|_V^2 + \underbrace{\|\varphi_1(z)\|_V^2 - \|\varphi_1(z)\|_V^2}_{=0} \\ &\quad + \|\varphi_2(z) + \alpha_2\|^2 + \underbrace{\|\varphi_2(z)\|^2 - \|\varphi_2(z)\|^2}_{=0} \\ &\geq \frac{1}{2}\|\alpha_1\|_V^2 + \frac{1}{2}\|\alpha_2\|^2 - \|\varphi_1(z)\|_V^2 - \|\varphi_2(z)\|^2 \\ &= \frac{1}{2}\|\alpha\|_X^2 - \|\varphi_1(z)\|_V^2 - \|\varphi_2(z)\|^2 \\ &= \frac{1}{2}\|\alpha\|_X^2 - \|\varphi(z)\|_X^2. \end{aligned} \tag{25}$$

We assume that the following conditions are fulfilled

$$\|\varphi(z)\|_X \leq \|z\|_X, \quad \Psi_0 < \frac{1}{4}\|\alpha\|_X^2. \tag{26}$$

Then, from (25), we deduce that for $z \in M$: $\Psi(Iz) > \Psi_0$, and, therefore, $M \cap IM = \emptyset$.

Since for any solution $z(t) = \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix}$ to (1), the equality

$$\|z(t)\|_X^2 = \|z(s)\|_X^2 - 4\beta \int_s^t \|y_t(\tau)\|^2 d\tau \tag{27}$$

holds true for any $t \geq s \geq 0$, then for $z(0) \neq 0$, the function $t \mapsto \|z(t)\|_X^2$ is strictly decreasing on $[0, +\infty)$. Therefore, for $z_0 \in M$ and for $t > 0$: $\|G(t, z_0)\|_X^2 < \Psi_0$, i.e., for an arbitrary $t > 0$: $G(t, z_0) \notin M$. Thus, all conditions (5)–(7) are fulfilled for problems (1), (3), and (4).

In what follows, we use the following estimate derived in [46]: For any

$$0 < \varepsilon \leq \min \left\{ \frac{\beta}{2}, \frac{\lambda_1}{2\beta} \right\}$$

and for any $z_0 \in X$

$$\|G(t, z_0)\|_X \leq \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}} \right)^2 e^{-\frac{\varepsilon}{4}t} \|z_0\|_X \quad \text{for all } t \geq 0. \tag{28}$$

Then, for any initial state $z_0 \in IM$ and for the corresponding solution $z(t) = G(t, z_0)$ we have that (as a result of (26)): $\|z_0\|_X^2 > \Psi_0$, and from (28), there exists $\bar{t} > 0$, such that $\|z(\bar{t})\|_X^2 = \Psi_0$.

Now, from equality

$$\frac{d}{dt}\|z(t)\|_X^2 = -2\beta\|y_t(t)\|^2,$$

we obtain

$$\|z(t)\|_X^2 = \Psi_0 \geq \|z_0\|_X^2 e^{-2\beta\bar{t}}. \quad (29)$$

From (29), and taking into account (25) and (26), we obtain

$$\Psi_0 \geq \left(\frac{1}{2}\|\alpha\|_X^2 - \Psi_0\right)e^{-2\beta\bar{t}}, \quad \bar{t} \geq \frac{1}{2\beta} \ln\left(\frac{\frac{1}{2}\|\alpha\|_X^2 - \Psi_0}{\Psi_0}\right). \quad (30)$$

Estimate (30) guarantees, in particular, the fulfillment of condition (10). Thus, we have shown that problems (1), (3), and (4) generate an impulsive dynamical system, and each impulsive trajectory has an infinite number of impulsive points.

Now, we are in a position to prove a statement on the existence of nonempty, compact ω -limit sets for the trajectories of the impulsive dynamical system $\tilde{G}$.

Theorem 1. Suppose that for problems (1), (3), and (4), conditions (26) together with

$$\frac{1}{\sqrt{\lambda_1}} < \frac{1}{8\beta} \ln\left(\frac{\|\alpha\|_X^2}{2\Psi_0} - 1\right); \quad (31)$$

$$\varphi : M \rightarrow X \text{ is a compact mapping} \quad (32)$$

hold true. Then, for the corresponding impulsive dynamical system $\tilde{G}$, it holds that for an arbitrary $z_0 \in X$, the ω -limit set $\tilde{\omega}(z_0) \neq \emptyset$, compact and

$$\text{dist}_X(\tilde{G}(t, z_0), \tilde{\omega}(z_0)) \rightarrow 0, \quad t \rightarrow \infty. \quad (33)$$

Proof of Theorem 1. To prove the theorem, it is sufficient to show that all conditions of Lemma 1 hold true.

Let us first show that condition 1 of Lemma 1 holds true under the conditions of the theorem. As a result of estimate (28), we have that if $M^+(z_0) = \emptyset$, then $\tilde{\omega}(z_0) = \{0\}$ and

$$\text{dist}_X(\tilde{G}(t, z_0), \tilde{\omega}(z_0)) \leq \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^2 e^{-\frac{\varepsilon}{4}t} \|z_0\|_X \rightarrow 0, \quad t \rightarrow \infty.$$

If $M^+(z_0) \neq \emptyset$, then $\tilde{G}(t, z_0)$ has an infinite number of impulsive points, and as a result of (28) and (30),

$$z_1^+ = \varphi(G(s_1, z_0)) + \alpha,$$

and the norm can be estimated as

$$\begin{aligned} \|z_1^+\|_X &\leq \|G(s_1, z_0)\|_X + \|\alpha\|_X \\ &\leq \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^2 e^{-\frac{\varepsilon}{4}\bar{t}} \|z_0\|_X + \|\alpha\|_X; \end{aligned}$$

Similarly,

$$z_2^+ = \varphi(G(s_2, z_1^+)) + \alpha,$$

and

$$\begin{aligned}\|z_2^+\|_X &\leq \|G(s_2, z_1^+)\|_X + \|\alpha\|_X \\ &\leq \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^2 e^{-\frac{\varepsilon}{4}\bar{t}} \|z_1^+\|_X + \|\alpha\|_X \\ &\leq \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^4 e^{-\frac{\varepsilon}{2}\bar{t}} \|z_0\|_X + \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^2 e^{-\frac{\varepsilon}{4}\bar{t}} \|\alpha\|_X + \|\alpha\|_X \dots\end{aligned}$$

At the n th step, we obtain for $n \geq 1$

$$\begin{aligned}\|z_n^+\|_X &\leq \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^{2n} e^{-\frac{n\varepsilon}{4}\bar{t}} \|z_0\|_X + \|\alpha\|_X \times \\ &\times \left(\left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^{2(n-1)} e^{-\frac{(n-1)\varepsilon}{4}\bar{t}} + \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^{2(n-2)} e^{-\frac{(n-2)\varepsilon}{4}\bar{t}} + \dots + 1 \right) \quad (34)\end{aligned}$$

From (30) and (31), it follows that

$$\frac{1}{4}\bar{t} > \frac{2}{\sqrt{\lambda_1}}.$$

This means that for small enough ε , the inequality

$$\frac{2}{\sqrt{\lambda_1}} \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right) \leq \frac{\bar{t}}{4} e^{\frac{\varepsilon}{4}\bar{t}}$$

holds true. Therefore, for sufficiently small ε ,

$$\delta := \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^2 e^{-\frac{\varepsilon}{4}\bar{t}} < 1.$$

Then, from (34), for all $n \geq 1$,

$$\|z_n^+\|_X \leq \delta^n \|z_0\|_X + \frac{\|\alpha\|_X}{1 - \delta}, \quad (35)$$

which, together with (28), implies the dissipativeness of (1), (3), and (4). Hence, in particular, for an arbitrary $t \geq 0$:

$$\|\tilde{G}(t, z_0)\| \leq \left(1 + \frac{\varepsilon}{\sqrt{\lambda_1}}\right)^2 \left(\|z_0\|_X + \frac{\|\alpha\|_X}{1 - \delta}\right). \quad (36)$$

Thus, condition 1 of Lemma 1 is fulfilled.

Condition 2 is fulfilled due to (28), with $G_1 \equiv V$ and $G_2 \equiv 0$.

Let us prove condition 3. It follows from (9) that

$$z_n^+ = \varphi(G(s_n, z_{n-1}^+)) + \alpha.$$

From the estimate (36), the sequence $\{z_n^+\}_{n \geq 1}$ is bounded. Then, from (28), the sequence

$$\{G(s_n, z_{n-1}^+)\}_{n \geq 1}$$

is bounded too. Therefore, condition (32) guarantees the precompactness of $\{z_n^+\}_{n \geq 1}$.

In summary, all the conditions of Lemma 1 that guarantee the fulfillment of the statement of the theorem are fulfilled. This completes the proof. $\square$

Remark 4. Condition (31) can be removed by requiring the limit $\lim_{k \rightarrow \infty} s_k$ to exist instead.

5. Conclusions

In the present paper, we have studied the qualitative behavior of an evolution problem consisting of a hyperbolic dissipative equation, whose trajectories undergo instantaneous impulsive discontinuities at the moment when the energy functional reaches a specific threshold value. The novel aspect of this area of study is that the entire infinite-dimensional phase vector undergoes impulsive disturbances. Through our analysis and under fairly general conditions on the system parameters, we have demonstrated that this problem generates an impulsive dynamical system in the natural phase space. Furthermore, we have established that the trajectories associated with this system possess nonempty compact ω -limit sets. These results shed light on the asymptotic behavior of solutions to the considered impulsive dynamical system.

In conclusion, the research presented here represents a step toward understanding the behavior of impulsive dynamical systems in infinite-dimensional phase spaces.

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References

1. Kato, T. Nonlinear semigroups and evolution equations. *J. Math. Soc. Jpn.* **1967**, *19*, 508–520. [\[CrossRef\]](#)
2. Ablowitz, M.J.; Kaup, D.J.; Newell, A.C.; Segur, H. Nonlinear-evolution equations of physical significance. *Phys. Rev. Lett.* **1973**, *31*, 125. [\[CrossRef\]](#)
3. Zheng, S. *Nonlinear Evolution Equations*; CRC Press: Boca Raton, FL, USA, 2004.
4. Samoilenko, A.M.; Perestyuk, N. *Impulsive Differential Equations*; World Scientific: Singapore, 1995.
5. Liu, J.H. Nonlinear impulsive evolution equations. *Dyn. Contin. Discret. Impuls. Syst.* **1999**, *6*, 77–85.
6. Samoilenko, A.; Perestyuk, N. *Differential Equations with Impulse Effect*; Visca Skola: Kiev, Ukraine, 1987.
7. Akhmet, M. *Principles of Discontinuous Dynamical Systems*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2010.
8. Lakshmikantham, V.; Simeonov, P.S.; Bainov, D. *Theory of Impulsive Differential Equations*; World Scientific: Singapore, 1989; Volume 6.
9. Goebel, R.; Sanfelice, R.G.; Teel, A.R. Hybrid dynamical systems. *IEEE Control Syst. Mag.* **2009**, *29*, 28–93. [\[CrossRef\]](#)
10. Van Der Schaft, A.J.; Schumacher, H. *An Introduction to Hybrid Dynamical Systems*; Springer: Berlin/Heidelberg, Germany, 2007; Volume 251.
11. Goebel, R.; Sanfelice, R.G.; Teel, A.R. *Hybrid Dynamical Systems. Modeling, Stability, and Robustness*; Princeton University Press: Princeton, NJ, USA, 2012.
12. Dirac, P.A.M. *The Principles of Quantum Mechanics*; Oxford University Press: Oxford, UK, 1981; p. 27.
13. Feketa, P.; Klinshov, V.; Lücken, L. A survey on the modeling of hybrid behaviors: How to account for impulsive jumps properly. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *103*, 105955. [\[CrossRef\]](#)
14. Wang, Z.; Zhang, G.; Sun, Y. Convergence of Collocation Methods for One Class of Impulsive Delay Differential Equations. *Axioms* **2023**, *12*, 700. [\[CrossRef\]](#)
15. Kalidass, M.; Zeng, S.; Yavuz, M. Stability of fractional-order quasi-linear impulsive integro-differential systems with multiple delays. *Axioms* **2022**, *11*, 308. [\[CrossRef\]](#)
16. Liu, W.; Li, P.; Li, X. Impulsive systems with hybrid delayed impulses: Input-to-state stability. *Nonlinear Anal. Hybrid Syst.* **2022**, *46*, 101248. [\[CrossRef\]](#)
17. Fečkan, M.; Kostić, M.; Velinov, D. (ω, ρ) -BVP Solutions of Impulsive Differential Equations of Fractional Order on Banach Spaces. *Mathematics* **2023**, *11*, 3086. [\[CrossRef\]](#)
18. Agarwal, R.P.; Hristova, S. Boundary Value Problem for Impulsive Delay Fractional Differential Equations with Several Generalized Proportional Caputo Fractional Derivatives. *Fractal Fract.* **2023**, *7*, 396. [\[CrossRef\]](#)
19. Hristova, S. Approximate iterative method for initial value problems of impulsive fractional differential equations with generalized proportional fractional derivatives. *AIP Conf. Proc.* **2023**, *2849*, 320005.

20. Xia, M.; Liu, L.; Fang, J.; Zhang, Y. Stability analysis for a class of stochastic differential equations with impulses. *Mathematics* **2023**, *11*, 1541. [[CrossRef](#)]
21. Liu, J.; Wei, W.; Xu, W. An averaging principle for stochastic fractional differential equations driven by fBm involving impulses. *Fractal Fract.* **2022**, *6*, 256. [[CrossRef](#)]
22. Wu, X.; Zheng, W.X.; Tang, Y.; Jin, X. Stability analysis for impulsive stochastic time-varying systems. *IEEE Trans. Autom. Control* **2022**, *68*, 2584–2591. [[CrossRef](#)]
23. Liu, R.; Fečkan, M.; O'Regan, D.; Wang, J. Controllability Results for First Order Impulsive Fuzzy Differential Systems. *Axioms* **2022**, *11*, 471. [[CrossRef](#)]
24. Deng, H.; Li, C.; Wang, Y. Asymptotic stability of non-instantaneous impulsive systems and T-S fuzzy non-instantaneous impulsive control for nonlinear systems. *IET Control Theory Appl.* **2023**, *17*, 1184–1202. [[CrossRef](#)]
25. Yang, T. *Impulsive Control Theory*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2001; Volume 272.
26. Feketa, P.; Bajcinca, N. On robustness of impulsive stabilization. *Automatica* **2019**, *104*, 48–56. [[CrossRef](#)]
27. Li, X.; Rao, R.; Yang, X. Impulsive stabilization on hyper-chaotic financial system under neumann boundary. *Mathematics* **2022**, *10*, 1866. [[CrossRef](#)]
28. Kaul, S. On impulsive semidynamical systems. *J. Math. Anal. Appl.* **1990**, *150*, 120–128. [[CrossRef](#)]
29. Ciesielski, K. On stability in impulsive dynamical systems. *Bull. Pol. Acad. Sci. Math.* **2004**, *259*, 81–91. [[CrossRef](#)]
30. Akhmet, M. Perturbations and Hopf bifurcation of the planar discontinuous dynamical system. *Nonlinear Anal. Theory Methods Appl. Ser. A Theory Methods* **2005**, *60*, 163–178. [[CrossRef](#)]
31. Kirilich, V.; Myshkis, A.; Prokhorenko, M. Oscillations of a diaphragm under the action of pulse forces. *Ukr. Math. J.* **2009**, *61*, 1357–1363. [[CrossRef](#)]
32. Samoilenko, A.; Samoilenko, V.; Sobchuk, V. On periodic solutions of the equation of a nonlinear oscillator with pulse influence. *Ukr. Math. J.* **1999**, *51*, 926–933. [[CrossRef](#)]
33. Asrorov, F.; Sobchuk, V.; Kurylko, O. Finding of bounded solutions to linear impulsive systems. *East.-Eur. J. Enterp. Technol.* **2019**, *6*, 14–20. [[CrossRef](#)]
34. Sobchuk, V.; Asrorov, F.; Perehuda, O.; Sukretna, A.; Laptiev, O.; Lukova-Chuiko, N. The Limited Solutions Method for Telecommunications Network Information Security Models. In Proceedings of the 2021 IEEE 3rd International Conference on Advanced Trends in Information Theory (AITT), Kyiv, Ukraine, 15–17 December 2021; pp. 126–131.
35. Feketa, P.; Perestyuk, Y. Perturbation Theorems for a Multifrequency System with Pulses. *J. Math. Sci.* **2016**, *217*, 515–524. [[CrossRef](#)]
36. Tkachenko, V. The Green function and conditions for the existence of invariant sets of impulse systems. *Ukr. Math. J.* **1989**, *41*, 1187–1190. [[CrossRef](#)]
37. Perestyuk, M.; Feketa, P. Invariant manifolds of one class of systems of impulsive differential equations. *Nonlinear Oscil.* **2010**, *2*, 260–273. [[CrossRef](#)]
38. Feketa, P.; Meurer, T.; Perestyuk, M.; Perestyuk, Y.M. Exponential Stability of Invariant Manifold for a Nonlinear Impulsive Multifrequency System. *J. Math. Sci.* **2020**, *249*, 694–703. [[CrossRef](#)]
39. Bonotto, E.; Bortolan, A.; Carvalho, A.; Czaja, R. Global attractors for impulsive dynamical systems—A precompact approach. *J. Differ. Equ.* **2015**, *259*, 2602–2625. [[CrossRef](#)]
40. Dashkovskiy, S.; Feketa, P.; Kapustyan, O.V.; Romaniuk, I.V. Existence and invariance of global attractors for impulsive parabolic system without uniqueness. In *Modern Mathematics and Mechanics: Fundamentals, Problems and Challenges*; Springer: Berlin/Heidelberg, Germany, 2019; pp. 57–78.
41. Dashkovskiy, S.; Kapustyan, O.; Romaniuk, I. Global attractors of impulsive parabolic inclusions. *Discret. Contin. Dyn. Syst. Ser. B* **2017**, *22*, 1875–1886. [[CrossRef](#)]
42. de Mello Bonotto, E.; Kalita, P. On attractors of generalized semiflows with impulses. *J. Geom. Anal.* **2020**, *30*, 1412–1449. [[CrossRef](#)]
43. Dashkovskiy, S.; Feketa, P.; Kapustyan, O.; Romaniuk, I. Invariance and stability of global attractors for multi-valued impulsive dynamical systems. *J. Math. Anal. Appl.* **2018**, *458*, 193–218. [[CrossRef](#)]
44. Kapustyan, O.; Perestyuk, M.; Romanyuk, I. Stability of global attractors of impulsive infinite-dimensional systems. *Ukr. Math. J.* **2018**, *70*, 30–42. [[CrossRef](#)]
45. Caraballo, T.; Uzal, J.M. Dynamics of Nonautonomous Impulsive Multivalued Processes. *Set-Valued Var. Anal.* **2023**, *31*, 7. [[CrossRef](#)]
46. Temam, R. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed.; Springer: New York, NY, USA, 1997; pp. XXII, 650.
47. Myshkis, A. Vibrations of the string with energy dissipation and impulsive feedback support. *Nonlinear Anal. Theory Methods Appl.* **1996**, *26*, 1271–1278. [[CrossRef](#)]
48. Dashkovskiy, S.; Kapustyan, O.; Gorban, N. Attractors of multivalued impulsive dynamical systems existence and application to reaction-diffusion systems. *Math. Probl. Eng.* **2021**, *2021*, 7385450. [[CrossRef](#)]
49. Samoilenko, A.; Perestyuk, N.; Trofimchuk, S. Generalized solutions of impulse systems and the phenomenon of pulsations. *Ukr. Math. J.* **1991**, *43*, 610–615. [[CrossRef](#)]
50. Dashkovskiy, S.; Feketa, P. Asymptotic properties of Zeno solutions. *Nonlinear Anal. Hybrid Syst.* **2018**, *30*, 256–265. [[CrossRef](#)]

51. Goebel, R.; Sanfelice, R.G. Pointwise asymptotic stability in a hybrid system and well-posed behavior beyond Zeno. *SIAM J. Control Optim.* **2018**, *56*, 1358–1385. [[CrossRef](#)]
52. Dashkovskiy, S.; Feketa, P. Prolongation and stability of Zeno solutions to hybrid dynamical systems. *IFAC-PapersOnLine* **2017**, *50*, 3429–3434. [[CrossRef](#)]

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