Article
Common Fixed Point Results on a Double-Controlled Metric Space for Generalized Rational-Type Contractions with Application

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Abstract: In this manuscript, we prove several common fixed point theorems for generalized rational-type contraction mappings under several conditions in the context of double-controlled metric spaces. Further, we utilize a double-controlled metric space equipped with a graph to prove rational-type common fixed point theorems. Furthermore, we establish non-trivial examples to show the validity of the main results. These results improve and generalize already known results. At the end, we solve the Fredholm-type integral equation by utilizing the main results.

Keywords: rational-type contractions; fixed point theorems; existence and uniqueness; integral equations

1. Introduction

The concept of fixed points has been utilized on a global scale in many disciplines of research and engineering [1,2]. Fixed point results were used to show that there are solutions to ordinary boundary value problems and fractional boundary value problems with integral-type boundary conditions, as described by Karapinar et al. [3]. In 1906, Frechet [4] presented the notion of metric spaces (MSs). MS techniques have been used for decades in a variety of applications, including image classification, protein classification, and Internet search engines. In order to extend metric techniques like Banach’s theorem to non-Hausdorff topologies, Matthews [5] introduced a symmetric generalized metric for such topologies. Mustafa and Sims [6] established the notion of MS Banach, and proved fixed point results for Reich contractions in $bv(s)$-MS.

In 1993, Czerwik [9] established the concept of b-MS as a generalization of MS. He used a constant $s \geq 1$ on the right side of the triangle inequality; if we consider $s = 1$ then b-MS becomes an MS. In 2017, Kamran et al. [10] presented the notion of an extended b-MS as a generalization of b-MS. They replaced a constant $s \geq 1$ with a function $\sigma : \Xi \times \Xi \to [1, \infty)$ and provided a Banach version of contraction mapping in the framework of complex-valued MSs. Mitrovic and Radenovic [8] established the notion of $bv(s)$-metric space as a generalization of MS, rectangular MS, b-MS, rectangular b-MS, v-generalized MS Banach, and proved fixed point results for Reich contractions in $bv(s)$-MS.

In 1993, Czerwik [9] established the concept of b-MS as a generalization of MS. He used a constant $s \geq 1$ on the right side of the triangle inequality; if we consider $s = 1$ then b-MS becomes an MS. In 2017, Kamran et al. [10] presented the notion of an extended b-MS as a generalization of b-MS. They replaced a constant $s \geq 1$ with a function $\sigma : \Xi \times \Xi \to [1, \infty)$ and proved a Banach version of contraction mapping in the framework of extended b-MS. Mustafa et al. [12] used the idea of extended b-MS and presented the notion of extended rectangular b-MS. Lateef [13] proved the fixed point
theorem for Kannan-type contraction mapping in the context of CMSs. Abuloha et al. [14] derived several fixed point results for CMSs by utilizing the class of functions denoted by $\psi_{\gamma}$. For more related results in the setting of controlled-type MSs, see [15,16].

Abdeljawad et al. [17] presented double-controlled MS (DCMS) as a generalization of controlled MS by utilizing two functions $\sigma, \gamma : \Xi \times \Xi \to [1, \infty)$, with both terms separately on the right side of triangular inequality as follows:

$$d(\varphi, \omega) = \alpha(\varphi, \omega)\beta(\varphi, \omega) + \gamma(\varphi, \omega)d(\omega, \omega)$$

for all $\varphi, \omega, \omega \in \Xi$.

Lateef [18] proved Fisher type fixed point results in controlled metric spaces. Farhan et al. [19] proved numerous fixed point results for $(a, F)$-contraction and Reich-type contraction in the setting of DCMSs and partially ordered DCMSs. The authors in [20–24] generalized the notion of DCMSs by utilizing intuitionistic fuzzy sets and neutrosophic sets, and proved fixed point theorems with several applications. Latif [25] introduced the concept of neutrosophic delta–beta-connected topological spaces. Touqeer and Rasool [26] derived several fixed point results for CMSs by utilizing the class of functions denoted by $\psi_{\gamma}$.

In this paper, we prove some common fixed point theorems for generalized rational-type contraction mappings and rational-type contractions equipped with graphs in the setting of DCMSs. Further, we solve the integral equation by using the main results. Our results generalize several existing results in [11,18,20].

2. Preliminaries

This section contains some basic definitions.

Definition 1 ([9]). Suppose $\Xi \neq \emptyset$ and $s \geq 1$ is any real number. The pair $(\Xi, \Delta, s)$ is called $b$-MS if a function $\Delta : \Xi \times \Xi \to [0, \infty)$, verifying the following axioms:

(bMS1) $\Delta(\varphi, \omega) \geq 0$ and $\Delta(\varphi, \omega) = 0$ if and only if $\varphi = \omega$,
(bMS2) $\Delta(\varphi, \omega) = \Delta(\omega, \varphi)$,
(bMS3) $\Delta(\varphi, \omega) \leq s[\Delta(\varphi, \omega) + \Delta(\omega, \omega)]$,
for all $\varphi, \omega, \omega \in \Xi$.

Kamran et al. [10] established the following notion of extended b-MS in 2017.

Definition 2 ([10]). Assume that $\Xi \neq \emptyset$ and $\sigma : \Xi \times \Xi \to [1, \infty)$. The triplet $(\Xi, \Delta, \sigma)$ is called extended $b$-MS if a function $\Delta : \Xi \times \Xi \to [0, \infty)$, verifying the following axioms:

(EbMS1) $\Delta(\varphi, \omega) \geq 0$ and $\Delta(\varphi, \omega) = 0$ if and only if $\varphi = \omega$,
(EbMS2) $\Delta(\varphi, \omega) = \Delta(\omega, \varphi)$,
(EbMS3) $\Delta(\varphi, \omega) \leq \sigma(\varphi, \omega)[\Delta(\varphi, \omega) + \Delta(\omega, \omega)]$,
for all $\varphi, \omega, \omega \in \Xi$.


Definition 3 ([11]). Assume that $\Xi \neq \emptyset$ and $\sigma : \Xi \times \Xi \to [1, \infty)$. The pair $(\Xi, \Delta_{\sigma})$ is called a CMS if a function $\Delta : \Xi \times \Xi \to [0, \infty)$, verifying the following axioms:

(CMS1) $\Delta_{\sigma}(\varphi, \omega) \geq 0$ and $\Delta_{\sigma}(\varphi, \omega) = 0$ if and only if $\varphi = \omega$,
(CMS2) $\Delta_{\sigma}(\varphi, \omega) = \Delta_{\sigma}(\omega, \varphi)$,
(CMS3) $\Delta_{\sigma}(\varphi, \omega) \leq \sigma(\varphi, \omega)\Delta_{\sigma}(\varphi, \omega) + \sigma(\varphi, \omega)\Delta_{\sigma}(\omega, \omega)$,
for all $\varphi, \omega, \omega \in \Xi$. 

$$\Delta_{\sigma}(\varphi, \omega) \leq \sigma(\varphi, \omega)\Delta_{\sigma}(\varphi, \omega) + \sigma(\varphi, \omega)\Delta_{\sigma}(\omega, \omega).$$
Definition 4 ([16]). Assume that $\mathbb{E} \neq \emptyset$ and $\sigma, \gamma : \mathbb{E} \times \mathbb{E} \to [1, \infty)$. The quadruple $(\mathbb{E}, \Delta, \sigma, \gamma)$ is called double CMS (DCMS) if a function $\Delta : \mathbb{E} \times \mathbb{E} \to (0, \infty)$, verifying the following axioms:

(DCMS1) $\Delta(\omega, \varepsilon) \geq 0$ and $\Delta(\omega, 0) = 0$ if and only if $\omega = \varepsilon$,
(DCMS2) $\Delta(\omega, \varepsilon) = \Delta(\varepsilon, \omega)$,
(DCMS3) $\Delta(\omega, \omega) \leq \sigma(\omega, \varepsilon)\Delta(\omega, \varepsilon) + \gamma(\varepsilon, \omega)\Delta(\varepsilon, \omega)$,

for all $\omega, \varepsilon, \omega \in \mathbb{E}$.

Definition 5. Let $(\mathbb{E}, \Delta, \sigma, \gamma)$ be a DCMS and $\{\omega_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{E}$. Then a sequence $\{\omega_n\}$ is said to be convergent to $\omega \in \mathbb{E}$ if, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ with $N = N(\epsilon)$ such that $\Delta(\omega_n, \omega) < \epsilon$ for all $n \geq N$. Further, we can write

$$\lim_{n \to \infty} \omega_n = \omega.$$ 

a sequence $\{\omega_n\}$ is called Cauchy if, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ with $N = N(\epsilon)$ such that $\Delta(\omega_m, \omega_n) < \epsilon$ for all $m, n \geq N$.

The DCMS $(\mathbb{E}, \Delta, \sigma, \gamma)$ is said to be complete if every Cauchy sequence is convergent in $\mathbb{E}$.

3. Main Result

In this section, we will prove common fixed point results in DCMS.

Theorem 1. Suppose $(\mathbb{E}, \Delta, \sigma, \gamma)$ is a complete DCMS, $T_1, T_2 : \mathbb{E} \to \mathbb{E}$ and there exists $k_1, k_2 : \mathbb{E} \times \mathbb{E} \to [0, 1)$ such that

(I) $k_1(T_2T_1\omega, \varepsilon) \leq k_1(\omega, \varepsilon)$ and $k_1(\omega, T_1T_2\varepsilon) \leq k_1(\omega, \varepsilon)$,

(II) $k_2(T_2T_1\omega, \varepsilon) \leq k_2(\omega, \varepsilon)$ and $k_2(\omega, T_1T_2\varepsilon) \leq k_2(\omega, \varepsilon)$,

(III) $k_1(\omega, \varepsilon) + k_2(\omega, \varepsilon) < 1$,

(IV) $\Delta(T_1\omega, T_2\varepsilon) \leq k_1(\omega, \varepsilon)\Delta(\omega, \varepsilon) + k_2(\omega, \varepsilon)\Delta(\omega, \varepsilon) + \frac{\Delta(\omega, T_1\omega)\Delta(\omega, T_2\varepsilon)}{1 + \Delta(\omega, \varepsilon)}$, for all $\omega, \varepsilon \in \mathbb{E}$. (1)

for $\omega_0 \in \mathbb{E}$, a sequence $\{\omega_j\}_{j \geq 0}$ is defined as $\omega_{j+1} = T_1\omega_j$ and $\omega_{j+2} = T_2\omega_{j+1}$ for every $j \geq 0$. Suppose that

$$\sup_{m \geq 1} \lim_{n \to \infty} \frac{\sigma(\omega_{i+1}, \omega_{i+2})\gamma(\omega_{i+1}, \omega_m)}{\sigma(\omega_i, \omega_{i+1})} < \frac{1}{k},$$

where $\frac{k_1(\omega_{i+1}, \omega_m)}{k_2(\omega_{i+1}, \omega_m)} = k$. Moreover, assume that $\lim_{j \to +\infty} \sigma(\omega_j, \omega_j)$, $\lim_{j \to +\infty} \sigma(\omega_j, \omega_{j+1})$, $\lim_{j \to +\infty} \gamma(\omega_j, \omega_j)$ and $\lim_{j \to +\infty} \gamma(\omega_j, \omega_{j+1})$ exist and are finite, then there exists a unique fixed point $\omega^* \in \mathbb{E}$ such that $T_1\omega^* = T_2\omega^* = \omega^*$.

Proof. Let $\omega_0 \in \mathbb{E}$. We construct $\{\omega_j\}$ in $\mathbb{E}$ by $\omega_{j+1} = T_1\omega_j$ and $\omega_{j+2} = T_2\omega_{j+1}$ for each $j \geq 0$. From Assumption (1), we obtain

$$\Delta(\omega_{j+1}, \omega_{j+2}) = \Delta(T_1\omega_j, T_2\omega_{j+1})$$
\[ \Delta(\omega_{2j+1}, \omega_{2j+2}) \leq k_1(\omega_{2j}, \omega_{2j+1}) \Delta(\omega_{2j}, \omega_{2j+1}) + k_2(\omega_{2j}, \omega_{2j+1}) \frac{\Delta(\omega_{2j}, T_1 \omega_{2j+1}) \Delta(\omega_{2j+1}, T_2 \omega_{2j+1})}{1 + \Delta(\omega_{2j}, \omega_{2j+1})} \]

\[ \leq k_1(T_1 T_2 \omega_{2j-2}, \omega_{2j+1}) \Delta(\omega_{2j+1}, \omega_{2j+1}) + k_2(T_1 T_2 \omega_{2j-2}, \omega_{2j+1}) \frac{\Delta(\omega_{2j+1}, T_1 \omega_{2j+1})}{1 + \Delta(\omega_{2j}, \omega_{2j+1})} \]

Similarly, \[ \Delta(\omega_{2j+2}, \omega_{2j+3}) = \Delta(T_2 \omega_{2j+1}, T_1 \omega_{2j+2}) = \Delta(T_1 \omega_{2j+2}, T_2 \omega_{2j+1}) \]

\[ \leq k_1(\omega_{2j+2}, \omega_{2j+1}) \Delta(\omega_{2j+2}, \omega_{2j+1}) + k_2(\omega_{2j+2}, \omega_{2j+1}) \frac{\Delta(\omega_{2j+1}, T_1 \omega_{2j+1})}{1 + \Delta(\omega_{2j+2}, \omega_{2j+1})} \]

This implies that

\[ \Delta(\omega_{2j+1}, \omega_{2j+2}) \leq \left( \frac{k_1(\omega_0, \omega_1)}{1 - k_2(\omega_0, \omega_1)} \right) \Delta(\omega_{2j}, \omega_{2j+1}). \]

Similarly,

\[ \Delta(\omega_{2j+2}, \omega_{2j+3}) = \Delta(T_2 \omega_{2j+1}, T_1 \omega_{2j+2}) = \Delta(T_1 \omega_{2j+2}, T_2 \omega_{2j+1}) \]

\[ \leq k_1(\omega_{2j+1}, \omega_{2j+1}) \Delta(\omega_{2j+2}, \omega_{2j+1}) + k_2(\omega_{2j+2}, \omega_{2j+1}) \frac{\Delta(\omega_{2j+1}, T_1 \omega_{2j+1})}{1 + \Delta(\omega_{2j+2}, \omega_{2j+1})} \]

\[ \leq k_1(\omega_{2j+2}, \omega_{2j+1}) \Delta(\omega_{2j+2}, \omega_{2j+1}) + k_2(\omega_{2j+2}, \omega_{2j+1}) \frac{\Delta(\omega_{2j+1}, T_1 \omega_{2j+1})}{1 + \Delta(\omega_{2j+2}, \omega_{2j+1})} \]

This implies that

\[ \Delta(\omega_{2j+1}, \omega_{2j+2}) \leq \left( \frac{k_1(\omega_0, \omega_1)}{1 - k_2(\omega_0, \omega_1)} \right) \Delta(\omega_{2j}, \omega_{2j+1}). \]

By pursuing in this direction, we obtain

\[ \Delta(\omega_j, \omega_{j+1}) \leq k \Delta(\omega_{j-1}, \omega_j) = k^2 \Delta(\omega_{j-2}, \omega_{j-1}) \leq \ldots \leq k^j \Delta(\omega_0, \omega_1). \]

Thus,

\[ \Delta(\omega_j, \omega_{j+1}) \leq k^j \Delta(\omega_0, \omega_1). \] (3)
Now for \( j < m \), for all \( j, m \in \mathbb{N} \), we deduce that
\[
\Delta(\omega_j, \omega_m) \leq \sigma(\omega_j, \omega_{j+1}) \Delta(\omega_j, \omega_{j+1}) + \gamma(\omega_{j+1}, \omega_m) \Delta(\omega_{j+1}, \omega_m)
\]
\[
\leq \sigma(\omega_j, \omega_{j+1}) \Delta(\omega_j, \omega_{j+1}) + \gamma(\omega_{j+1}, \omega_m) \sigma(\omega_{j+1}, \omega_{j+2}) \Delta(\omega_{j+1}, \omega_{j+2})
\]
\[
+ \gamma(\omega_{j+1}, \omega_m) \gamma(\omega_{j+1}, \omega_{j+2}) \Delta(\omega_{j+1}, \omega_{j+2})
\]
\[
= \ldots \leq \sigma(\omega_j, \omega_{j+1}) \Delta(\omega_j, \omega_{j+1}) + \sum_{i=j+1}^{m-2} \left( \prod_{i=j+1}^{i} \gamma(\omega_i, \omega_m) \right) \sigma(\omega_j, \omega_{j+1}) \Delta(\omega_j, \omega_{j+1})
\]
\[
+ \prod_{i=j+1}^{m-1} \gamma(\omega_i, \omega_m) \Delta(\omega_{m-1}, \omega_m).
\]

This further implies that
\[
\Delta(\omega_j, \omega_m) \leq \sigma(\omega_j, \omega_{j+1}) \Delta(\omega_j, \omega_{j+1}) + \sum_{i=j+1}^{m-2} \left( \prod_{i=j+1}^{i} \gamma(\omega_i, \omega_m) \right) \sigma(\omega_j, \omega_{j+1}) \Delta(\omega_j, \omega_{j+1})
\]
\[
+ \left( \prod_{i=j+1}^{m-1} \gamma(\omega_i, \omega_m) \right) \Delta(\omega_{m-1}, \omega_m).
\]

Let
\[
\psi_\Delta = \sum_{i=j+1}^{m-1} \left( \prod_{i=j+1}^{i} \gamma(\omega_i, \omega_m) \right) \sigma(\omega_i, \omega_{i+1}) \Delta(\omega_0, \omega_1).
\]

From (4), we obtain
\[
\Delta(\omega_j, \omega_m) \leq \sigma(\omega_j, \omega_{j+1}) k^j \Delta(\omega_0, \omega_1) + \sum_{i=j+1}^{m-1} \left( \prod_{i=j+1}^{i} \gamma(\omega_i, \omega_m) \right) \sigma(\omega_i, \omega_{i+1}) k^i \Delta(\omega_0, \omega_1).
\]

Thus
\[
\Delta(\omega_j, \omega_m) \leq \sigma(\omega_j, \omega_{j+1}) k^j \Delta(\omega_0, \omega_1) + \sum_{i=j+1}^{m-1} \left( \prod_{i=j+1}^{i} \gamma(\omega_i, \omega_m) \right) \sigma(\omega_i, \omega_{i+1}) k^i \Delta(\omega_0, \omega_1). \tag{4}
\]

Let
\[
\psi_\Delta = \sum_{i=j+1}^{m-1} \left( \prod_{i=j+1}^{i} \gamma(\omega_i, \omega_m) \right) \sigma(\omega_i, \omega_{i+1}) k^i \Delta(\omega_0, \omega_1).
\]

From (4), we obtain
\[
\Delta(\omega_j, \omega_m) \leq \Delta(\omega_0, \omega_1) [\sigma(\omega_j, \omega_{j+1}) k^j + (\psi_{m-1} - \psi_j)]. \tag{5}
\]

Since \( \sigma(\omega, x) \geq 1 \), and by employing the ratio test \( \lim_{j \to \infty} \psi_j \) exists. Thus \( \{ \psi_j \} \) is a Cauchy sequence.

Therefore, taking \( j, m \to +\infty \) in inequality (5), we obtain
\[
\lim_{j,m \to +\infty} \Delta(\omega_j, \omega_m) = 0. \tag{6}
\]

Hence, \( \{ \omega_j \} \) is a Cauchy sequence in DCMS. As \( (\mathcal{X}, \Delta, \sigma, \gamma) \) is complete, so there exists \( \omega^* \in \mathcal{X} \) such that
\[
\lim_{j \to +\infty} \Delta(\omega_j, \omega^*) = 0. \tag{7}
\]
Hence, $\omega_j \to \omega^*$ as $j \to +\infty$. From conditions (III) and (IV), we deduce that

$$
\Delta(\omega^*, T_1\omega^*) \leq \sigma(\omega^*, \omega_{2j+2}) \Delta(\omega^*, \omega_{2j+2}) + \gamma(\omega_{2j+2}, T_1\omega^*) \Delta(\omega_{2j+2}, T_1\omega^*)
$$

$$
= \sigma(\omega^*, \omega_{2j+2}) \Delta(\omega^*, \omega_{2j+2}) + \gamma(\omega_{2j+2}, T_1\omega^*) \Delta(T_2\omega_{2j+1}, T_1\omega^*)
$$

$$
= \sigma(\omega^*, \omega_{2j+2}) \Delta(\omega^*, \omega_{2j+2}) + \gamma(\omega_{2j+2}, T_1\omega^*) \Delta(T_1\omega^*, T_2\omega_{2j+1})
$$

$$
= \sigma(\omega^*, \omega_{2j+2}) \Delta(\omega^*, \omega_{2j+2}) + \gamma(\omega_{2j+2}, T_1\omega^*) \Delta(T_2\omega_{2j+1}, T_1\omega^*)
$$

Now, we define $\Delta(\omega^*, \omega^*) = \begin{bmatrix} k_1(\omega^*, \omega_{2j+1}) \Delta(\omega^*, \omega_{2j+1}) & +k_2(\omega^*, \omega_{2j+1}) \Delta(\omega^*, \omega_{2j+1}) \end{bmatrix}$.

Letting $j \to +\infty$ and applying Equation (7), which contradicts $\Delta(\omega^*, T_1\omega^*) > 0$. That is, $\Delta(\omega^*, T_1\omega^*) = 0$. We have $\omega^* = T_1\omega^*$. On the same lines, we can examine that $\omega^* = T_2\omega^*$. Hence, $T_1$ and $T_2$ has a common fixed point $\omega^*$.

Now, we examine the uniqueness of the fixed point $\omega^*$. Let another fixed point $\omega' \in \Xi$, which is different from $\omega^*$ such that $T_1\omega' = T_2\omega'$. We have

$$
\Delta(\omega^*, \omega') = \Delta(\omega^*, \omega') \Delta(\omega^*, \omega') + k_2(\omega^*, \omega') \Delta(\omega^*, \omega') \Delta(\omega^*, \omega') \frac{1}{1+\Delta(\omega^*, \omega')}
$$

$$
= k_1\left(\omega^*, \omega'\right) \Delta\left(\omega^*, \omega'\right).
$$

Since $k_1(\omega^*, \omega') \in [0, 1)$, we have $\Delta(\omega^*, \omega') = 0$. Thus, we obtain $\omega^* = \omega'$, which shows that $\omega^*$ is unique. $\Box$

**Example 1.** Let $\Xi = [0, 1]$. Now we define $\Delta : \Xi \times \Xi \to [0, 1]$ by

$$
\Delta(\omega, \kappa) = (\omega + \kappa)^2,
$$

where $\sigma(\omega, \kappa) = \omega + \kappa + 2$ and $\gamma(\omega, \kappa) = \omega^2 + \kappa^2 + 1$ for all $\omega, \kappa \in \Xi$. Now, we define $T_1, T_2 : \Xi \to \Xi$ by $T_1\omega = \frac{\omega}{2}$ and $T_2\omega = \frac{\omega}{3}$, for $\omega \in \mathbb{R}$. Choose $k_1, k_2 : \Xi \times \Xi \to [0, 1)$ by

$$
k_1(\omega, \kappa) = \frac{16 + \omega + \kappa}{144} \text{ and } k_2(\omega, \kappa) = \frac{15 + \omega + \kappa}{144}.
$$

Then, evidently,

$$
k_1(\omega, \kappa) + k_2(\omega, \kappa) < 1.
$$

Now,

$$
k_1(T_2T_1\omega, \kappa) = \frac{1}{9} + \frac{\omega}{1726} + \frac{\kappa}{144} \leq \frac{16 + \omega + \kappa}{144} = k_1(\omega, \kappa),
$$

and

$$
k_1(\omega, T_2T_1 \kappa) = \frac{1}{9} + \frac{\omega}{144} + \frac{\kappa}{1726} \leq \frac{16 + \omega + \kappa}{144} = k_1(\omega, \kappa),
$$

and

$$
k_2(T_2T_1\omega, \kappa) = \frac{5}{46} + \frac{\omega}{1726} + \frac{\kappa}{144} \leq \frac{15 + \omega + \kappa}{144} = k_1(\omega, \kappa),
$$

and

$$
k_2(\omega, T_2T_1 \kappa) = \frac{5}{46} + \frac{\omega}{144} + \frac{\kappa}{1726} \leq \frac{15 + \omega + \kappa}{144} = k_1(\omega, T_2T_1 \kappa).$$
Take $\omega_0 = 0$, so (2) is satisfied. Let $\omega, \kappa \in \Sigma$. Then,

$$
\Delta(T_1 \omega, T_2 \kappa) = \frac{(4\omega + 3\kappa)^2}{144} \leq \frac{(4\omega + 4\kappa)^2}{144}
$$

$$
\leq \frac{16 + \omega + \kappa (\omega + \kappa)^2 + 15 + \omega + \kappa}{144 + (\omega + \kappa)^2}.
$$

Hence, Theorem 1 is fulfilled and $\omega^* = 0 \in \Sigma$ is a fixed point, that is, $T_1 \omega^* = T_2 \omega^* = \omega^*$.

By setting $T_1 = T_2 = T$ in Theorem 1, we have the following corollary.

**Corollary 1.** Suppose $(\Sigma, \Delta, \sigma, \gamma)$ is a complete DCMS, $T : \Sigma \to \Sigma$ and there exists $k_1, k_2 : \Sigma \times \Sigma \to [0, 1)$ such that

(I) $k_1(T \omega, \kappa) \leq k_1(\omega, \kappa)$ and $k_1(\omega, T \kappa) \leq k_1(\omega, \kappa)$,

(II) $k_2(T \omega, \kappa) \leq k_2(\omega, \kappa)$ and $k_2(\omega, T \kappa) \leq k_2(\omega, \kappa)$,

(III) $k_1(\omega, \kappa) + k_2(\omega, \kappa) < 1$,

(IV) $\Delta(T \omega, T \kappa) \leq k_1(\omega, \kappa) \Delta(\omega, \kappa) + k_2(\omega, \kappa) \Delta(T \omega, T \kappa)$, for all $\omega, \kappa \in \Sigma$.

for $\alpha_0 \in \Sigma$, a sequence $\{\alpha_i\}_{j \geq 0}$ is defined as $\alpha_{j+1} = T \alpha_j$ for each $j \geq 0$. Assume that

$$
\sup_{m \geq 1} \lim_{i \to +\infty} \frac{\sigma(\alpha_{i+1}, \alpha_{i+2}) \gamma(\alpha_{i+1}, \alpha_m)}{\sigma(\alpha_i, \alpha_{i+1})} < \frac{1}{k},
$$

where $\frac{k_1(\alpha_j, \alpha_i)}{k_2(\omega_j, \omega_i)} = k$. Moreover, assume that $\lim_{j \to +\infty} \sigma(\alpha_j, \omega_i)$, $\lim_{j \to +\infty} \sigma(\alpha_j, \omega_j)$, $\lim_{j \to +\infty} \gamma(\alpha_j, \omega_i)$ and $\lim_{j \to +\infty} \gamma(\alpha_j, \omega_j)$ exist and are finite, then there exists a unique point $\alpha^* \in \Sigma$ such that $T \alpha^* = \alpha^*$.

4. Deduced Results

**Theorem 2.** Suppose $(\Sigma, \Delta, \sigma, \gamma)$ is a complete DCMS, $T_1, T_2 : \Sigma \to \Sigma$ and there exists $\tau_1, \tau_2 : \Sigma \to [0, 1)$ such that

(I) $\tau_1(T_1 \omega, \kappa) \leq \tau_1(\omega, \kappa)$ and $\tau_2(T_1 \omega) \leq \tau_1(\omega)$,

(II) $\tau_1(T_2 \omega) \leq \tau_1(\omega)$ and $\tau_2(T_2 \omega) \leq \tau_2(\omega)$,

(III) $(k_1 + k_2) \omega < 1$,

(IV) $\Delta(T_1 \omega, T_2 \kappa) \leq \tau_1(\omega, \kappa) \Delta(\omega, \kappa) + \tau_2(\omega, \kappa) \Delta(T_1 \omega, T_2 \kappa)$, for all $\omega, \kappa \in \Sigma$.

for $\alpha_0 \in \Sigma$, a sequence $\{\alpha_i\}_{j \geq 0}$ is defined as $\alpha_{2j+1} = T_1 \alpha_{2j}$ and $\alpha_{2j+2} = T_2 \alpha_{2j+1}$ for every $j \geq 0$. Let

$$
\sup_{m \geq 1} \lim_{i \to +\infty} \frac{\sigma(\alpha_{i+1}, \alpha_{i+2}) \gamma(\alpha_{i+1}, \alpha_m)}{\sigma(\alpha_i, \alpha_{i+1})} < \frac{1}{\tau},
$$

where $\frac{\tau_1(\alpha_i, \alpha_i)}{\tau_2(\omega_j, \omega_i)} = \tau$. Furthermore, let $\lim_{j \to +\infty} \sigma(\alpha_j, \omega_i)$, $\lim_{j \to +\infty} \sigma(\alpha_j, \omega_j)$, $\lim_{j \to +\infty} \gamma(\alpha_j, \omega_i)$ and $\lim_{j \to +\infty} \gamma(\alpha_j, \omega_j)$ exist and are finite, then there exists a unique point $\alpha^* \in \Sigma$ such that $T_1 \alpha^* = T_2 \alpha^* = \alpha^*$.

**Proof.** Define $k_1, k_2 : \Sigma \times \Sigma \to [0, 1)$ by $\tau_1(\omega, \kappa) \leq \tau_1(\omega)$ and $\tau_2(\omega, \kappa) \leq \tau_2(\omega)$, for all $\omega, \kappa \in \Sigma$.

(i) $k_1(T_2 T_1 \omega, \kappa) = \tau_1(T_2 T_1 \omega) \leq \tau_1(T_1 \omega) \leq \tau_1(\omega) = k_1(\omega, \kappa)$ and $k_1(\omega, T_1 T_2 \kappa) = \tau_1(\omega) = \tau_1(\omega, \kappa)$,
(ii) \( k_1(T_2T_1\omega, \varkappa) = \tau_2(T_2T_1\omega) \leq \tau_2(T_1\omega) \leq \tau_2(\omega) = k_1(\omega, \varkappa) \) and \( k_2(\omega, T_1T_2 \varkappa) = \tau_2(\omega) = \tau_2(\omega, \varkappa) \).

(iii) \( k_1(\omega, \varkappa) + k_2(\omega, \varkappa) = \tau_1(\omega) + \tau_2(\omega) < 1 \).

(iv) \( \Delta(T_1\omega, T_2\varkappa) \leq \tau_1(\omega) \Delta(\omega, \varkappa) + \tau_2(\omega) \frac{\Delta(\omega, T_1\omega) \Delta(\varkappa, T_2\varkappa)}{1 + \Delta(\omega, \varkappa)} \)

\[ \frac{\tau_2(\omega)}{1 - \tau_2(\omega)} = \tau. \]

By Theorem 2, \( T_1 \) and \( T_2 \) have a unique common fixed point. \( \square \)

**Corollary 2.** Suppose \((\Xi, \Delta, \sigma, \gamma)\) be a complete DCMS, \( T: \Xi \to \Xi \) and there exists \( \tau_1, \tau_2: \Xi \to [0, 1) \) such that

(I) \( \tau_1(T\omega, \varkappa) \leq \tau_1(\omega, \varkappa) \) and \( \tau_2(T\omega) \leq \tau_1(\omega), \)

(II) \( (\tau_1 + \tau_2)\omega < 1, \)

(III) \( \Delta(T\omega, T\varkappa) \leq \tau_1(\omega) \Delta(\omega, \varkappa) + \tau_2(\omega) \frac{\Delta(\omega, T\omega) \Delta(\varkappa, T\varkappa)}{1 + \Delta(\omega, \varkappa)} \), for all \( \omega, \varkappa \in \Xi, (2) \)

for \( \omega_0 \in \Xi, \) we set \( \frac{\tau_1(\omega_0)}{1 - \tau_2(\omega_0)} = \tau. \) Assume that

\[ \sup_{m \geq 1} \lim_{i \to -\infty} \frac{\sigma(\omega_{i+1}, \omega_{i+2}) \gamma(\omega_{i+1}, \omega_m)}{\sigma(\omega_i, \omega_{i+1})} < \frac{1}{\tau}, \]

where \( \omega_{i+1} = T\omega_i \) for each \( j \geq 0. \) Moreover, assume that \( \lim_{j \to +\infty} \sigma(\omega_j, \omega), \lim_{j \to +\infty} \sigma(\omega, \omega_j), \)

\( \lim_{j \to +\infty} \gamma(\omega_j, \omega) \) and \( \lim_{j \to +\infty} \gamma(\omega, \omega_j) \) exist and are finite, then there exists a unique point \( \omega^* \in \Xi \) such that \( T\omega^* = \omega^*. \)

**Proof.** Immediate by considering \( T_1 = T_2 = T \) in the Theorem 2. \( \square \)

**Theorem 3.** Suppose \((\Xi, \Delta, \sigma, \gamma)\) be a complete DCMS, \( T: \Xi \to \Xi \) and there exists \( \tau_1, \tau_2: \Xi \to [0, 1) \) such that

(I) \( \tau_1(T'\omega, \varkappa) \leq \tau_1(\omega, \varkappa) \) and \( \tau_2(T'\omega) \leq \tau_1(\omega), \)

(II) \( (\tau_1 + \tau_2)\omega < 1, \)

(III) \( \Delta(T'\omega, T'\varkappa) \leq \tau_1(\omega) \Delta(\omega, \varkappa) + \tau_2(\omega) \frac{\Delta(\omega, T'\omega) \Delta(\varkappa, T'\varkappa)}{1 + \Delta(\omega, \varkappa)} \), for all \( \omega, \varkappa \in \Xi, (3) \)

for \( \omega_0 \in \Xi, \) we set \( \frac{\tau_1(\omega_0)}{1 - \tau_2(\omega_0)} = \tau. \) Assume that

\[ \sup_{m \geq 1} \lim_{i \to -\infty} \frac{\sigma(\omega_{i+1}, \omega_{i+2}) \gamma(\omega_{i+1}, \omega_m)}{\sigma(\omega_i, \omega_{i+1})} < \frac{1}{\tau}, \]

where \( \omega_{i+1} = T\omega_i \) for all \( j \geq 0. \) Furthermore, let \( \lim_{j \to +\infty} \sigma(\omega_j, \omega), \lim_{j \to +\infty} \sigma(\omega, \omega_j), \lim_{j \to +\infty} \gamma(\omega_j, \omega) \)

and \( \lim_{j \to +\infty} \gamma(\omega, \omega_j) \) exist and are finite, then there exists a unique point \( \omega^* \in \Xi \) such that \( T\omega^* = \omega^*. \)

**Proof.** Using Theorem 1, we obtain \( T'\omega^* = \omega^*. \) Now, as

\[ T' \left( T\omega^* \right) = T \left( T'\omega^* \right) = T\omega^*. \]

So, \( T' \) has a fixed point \( T\omega^*. \) Hence, \( T\omega^* = \omega^*. \) Meanwhile \( T' \) has a unique fixed point, so \( T \) has a fixed point \( \omega^*. \) \( \square \)
Corollary 3. Suppose \((\Sigma, \Delta, \sigma, \gamma)\) is a complete DCMS, \(T_1, T_2 : \Sigma \rightarrow \Sigma\) and there exists \(\gamma_1, \gamma_2 \in [0, 1)\) with \(\gamma_1 + \gamma_2 < 1\) such that

\[
\Delta(T_1\omega, T_2\kappa) \leq \gamma_1 \Delta(\omega, \kappa) + \gamma_2 \frac{\Delta(\omega, T_1\omega) + \Delta(\kappa, T_2\kappa)}{1 + \Delta(\omega, \kappa)}, \quad \text{for all } \omega, \kappa \in \Sigma,
\]

for \(\omega_0 \in \Sigma\), a sequence \(\{\omega_j\}_{j \geq 0}\) is generated as \(\omega_{j+1} = T_1\omega_j\) and \(\omega_{j+2} = T_2\omega_{j+1}\) for every \(j \geq 0\). Let

\[
\sup_{m \geq 1} \lim_{j \to +\infty} \frac{\sigma(\omega_{j+1}, \omega_{j+2}) \gamma(\omega_{j+1}, \omega_m)}{\sigma(\omega_j, \omega_{j+1})} < \frac{1}{\Omega},
\]

where \(\frac{\gamma_2}{\gamma_1} = \Omega\). Moreover, assume that \(\lim_{j \to +\infty} \sigma(\omega_j, \omega), \lim_{j \to +\infty} \sigma(\omega_j, \omega_j), \lim_{j \to +\infty} \gamma(\omega_j, \omega)\) and \(\lim_{j \to +\infty} \gamma(\omega_j, \omega_j)\) exist and are finite, then there exists a unique point \(\omega^* \in \Sigma\) such that \(T_1\omega^* = T_2\omega^* = \omega^*\).

Proof. Immediate by letting \(\gamma_1(.) = \gamma_1\) and \(\gamma_2(.) = \gamma_2\) in Theorem 2. \(\square\)

Corollary 4. Suppose \((\Sigma, \Delta, \sigma, \gamma)\) is a complete DCMS, \(T : \Sigma \rightarrow \Sigma\) and there exist \(\gamma_1, \gamma_2 \in [0, 1)\) with \(\gamma_1 + \gamma_2 < 1\) such that

\[
\Delta(T\omega, T\kappa) \leq \gamma_1 \Delta(\omega, \kappa) + \gamma_2 \frac{\Delta(\omega, T\omega) + \Delta(\kappa, T\kappa)}{1 + \Delta(\omega, \kappa)}, \quad \text{for all } \omega, \kappa \in \Sigma,
\]

for \(\omega_0 \in \Sigma\), we set \(\frac{\gamma_2}{\gamma_1} = \Omega\). Suppose that

\[
\sup_{m \geq 1} \lim_{j \to +\infty} \frac{\sigma(\omega_{j+1}, \omega_{j+2}) \gamma(\omega_{j+1}, \omega_m)}{\sigma(\omega_j, \omega_{j+1})} < \frac{1}{\Omega},
\]

where \(\omega_{j+1} = T\omega_j\) for all \(j \geq 0\). Moreover, assume that \(\lim_{j \to +\infty} \sigma(\omega_j, \omega), \lim_{j \to +\infty} \sigma(\omega_j, \omega_j), \lim_{j \to +\infty} \gamma(\omega_j, \omega)\) and \(\lim_{j \to +\infty} \gamma(\omega_j, \omega_j)\) exist and are finite, then there exists a unique point \(\omega^* \in \Sigma\) such that \(T\omega^* = \omega^*\).

Corollary 5. Suppose \((\Sigma, \Delta, \sigma, \gamma)\) is a complete DCMS, \(T : \Sigma \rightarrow \Sigma\) and there exists \(\gamma_1, \gamma_2 \in [0, 1)\) with \(\gamma_1 + \gamma_2 < 1\) such that

\[
\Delta(T'\omega, T'\kappa) \leq \gamma_1 \Delta(\omega, \kappa) + \gamma_2 \frac{\Delta(\omega, T'\omega) + \Delta(\kappa, T'\kappa)}{1 + \Delta(\omega, \kappa)}, \quad \text{for all } \omega, \kappa \in \Sigma,
\]

for \(\omega_0 \in \Sigma\), we set \(\frac{\gamma_2}{\gamma_1} = \Omega\). Assume that

\[
\sup_{m \geq 1} \lim_{j \to +\infty} \frac{\sigma(\omega_{j+1}, \omega_{j+2}) \gamma(\omega_{j+1}, \omega_m)}{\sigma(\omega_j, \omega_{j+1})} < \frac{1}{\Omega},
\]

where \(\omega_{j+1} = T\omega_j\) for all \(j \geq 0\). Moreover, assume that \(\lim_{j \to +\infty} \sigma(\omega_j, \omega), \lim_{j \to +\infty} \sigma(\omega_j, \omega_j), \lim_{j \to +\infty} \gamma(\omega_j, \omega)\) and \(\lim_{j \to +\infty} \gamma(\omega_j, \omega_j)\) exist and are finite, then there exists a unique point \(\omega^* \in \Sigma\) such that \(T\omega^* = \omega^*\).

Corollary 6. Suppose \((\Sigma, \Delta, \sigma, \gamma)\) is a complete DCMS, \(T : \Sigma \rightarrow \Sigma\) and there exist \(\gamma_1, \gamma_2 \in [0, 1)\) with \(\gamma_1 + \gamma_2 < 1\) such that

\[
\Delta(T'\omega, T'\kappa) \leq \gamma_1 \Delta(\omega, \kappa), \quad \text{for all } \omega, \kappa \in \Sigma,
\]
for \( \omega_0 \in \Xi \), we set \( \frac{\tau_1 - \tau_1}{\tau_1} = \Omega \). Suppose that
\[
\sup_{m \geq 1 \rightarrow +\infty} \lim \frac{\sigma(\omega_{i+1}, \omega_{i+2}) \gamma(\omega_{i+1}, \omega_m)}{\sigma(\omega_i, \omega_{i+1})} < \frac{1}{\Omega},
\]
where \( \omega_{i+1} = T\omega_j \) for all \( j \geq 0 \). Moreover, assume that \( \lim_{j \rightarrow +\infty} \sigma(\omega_j, \omega), \lim_{j \rightarrow +\infty} \sigma(\omega, \omega_j), \lim_{j \rightarrow +\infty} \gamma(\omega_j, \omega) \) and \( \lim_{j \rightarrow +\infty} \gamma(\omega, \omega_j) \) exist and are finite, then there exists a unique point \( \omega^* \in \Xi \) such that \( T\omega^* = \omega^* \).

5. Application in Graphs

Suppose \( (\Xi, \Delta, \sigma, \gamma) \) is a complete DCMS and a directed graph \( G \). Assume \( G^{-1} \) is a graph that we obtain from \( G \) by altering the direction of \( E(G) \). Thus,
\[
E \left( G^{-1} \right) = \{ (\omega, \kappa) \in \Xi \times \Xi : (\kappa, \omega) \in E(G) \}.
\]

**Definition 6.** An arbitrary point \( \omega \in \Xi \) is a common fixed point of \( (T_1, T_2) \), if \( T_1(\omega) = T_2(\omega) = \omega \). By \( \text{CFix}(T_1, T_2) \), we represent the set of all common fixed point of \( (T_1, T_2) \), i.e.,
\[
\text{CFix}(T_1, T_2) = \{ \omega \in \Xi : T_1(\omega) = T_2(\omega) = \omega \}.
\]

**Definition 7.** Suppose that \( T_1, T_2 : \Xi \rightarrow \Xi \) are two mappings on complete DCMS \( (\Xi, \Delta, \sigma, \gamma) \) equipped with a directed graph \( G \). For any \( \omega \in \Xi \), \((T_1, T_2)\) is called a \( G \)-Orbital cyclic pair, if
\[
(\omega, T_1\omega) \in E(G) \Rightarrow (T_1\omega, T_2(T_1\omega)) \in E(G),
(\omega, T_2\omega) \in E(G) \Rightarrow (T_2\omega, T_1(T_2\omega)) \in E(G).
\]

Consider the below sets
\[
\Xi_{T_1} = \{ \omega \in \Xi : (\omega, T_1\omega) \in E(G) \},
\Xi_{T_2} = \{ \omega \in \Xi : (\omega, T_2\omega) \in E(G) \}.
\]

**Remark 1.** If the pair \((T_1, T_2)\) is a \( G \)-Orbital cyclic pair, then \( \Xi_{T_1} \neq \phi \) if and only if \( \Xi_{T_2} \neq \phi \).

**Proof.** Let \( \omega_0 \in \Xi_{T_1} \). Then, \((\omega_0, T_1\omega_0) \in E(G) \Rightarrow (T_1\omega_0, T_2(T_1\omega_0)) \in E(G) \). If \( \omega_1 = T_1\omega_0 \), then we obtain \((T_1, T_2)\omega_1 \in E(G) \), thus \( \Xi_{T_2} \neq \phi \). □

**Theorem 4.** Let \((\Xi, \Delta, \sigma, \gamma)\) be a complete DCMS equipped with a directed graph \( G \) and \( T_1, T_2 : \Xi \rightarrow \Xi \) is a \( G \)-Orbital cyclic pair. Suppose there exist \( \tau_1 \in [0, 1) \) such that
(i) \( \Xi_{T_1} \neq \phi \),
(ii) for all \( \omega \in \Xi_{T_1} \) and \( \kappa \in \Xi_{T_2} \),
\[
\Delta(T_1\omega, T_2\kappa) \leq \tau_1 \max\{\Delta(\omega, \kappa), \Delta(\omega, T_1\omega), \Delta(\kappa, T_2\kappa)\};
\]
(iii) for all \( \{\omega_n\}_{n \in \mathbb{N}} \subset \Xi \), one has \((\omega_n, \omega_{n+1}) \in E(G) \),
\[
\sup_{m \leq 1 \rightarrow +\infty} \lim \frac{\sigma(\omega_{i+1}, \omega_{i+2}) \gamma(\omega_{i+1}, \omega_m)}{\sigma(\omega_i, \omega_{i+1})} < \frac{1}{\tau},
\]
where \( \tau = \frac{\tau_1}{1 - \tau_1} \).
(iv) \(T_1\) and \(T_2\) are continuous, or for all \((\omega_j)_{j \in \mathbb{N}} \subset \mathcal{Z}\), with \(\omega_j \to \omega\) as \(j \to +\infty\), and \((\omega_j, \omega_{j+1}) \in E(\mathcal{G})\) for \(j \in \mathbb{N}\), we have \(\omega \in \mathcal{X}_1 \cap \mathcal{X}_2\). In these conditions, \(C \text{Fix}(T_1, T_2) \neq \emptyset\).

(v) For all \(\omega \in \mathcal{Z}\), we have \(\lim_{j \to +\infty} \sigma(\omega_j, \omega_j), \lim_{j \to +\infty} \gamma(\omega_j, \omega_j)\), \(\lim_{j \to +\infty} \gamma(\omega_{j+1}, \omega_j)\) exists and finite.

(vi) If \((\omega^*, \omega') \in C \text{Fix}(T_1, T_2)\) implies \(\omega^* \in \mathcal{X}_1\) and \(\omega' \in \mathcal{X}_2\), then the pair \((T_1, T_2)\) has a unique common fixed point.

**Proof.** Let \(\omega_0 \in \mathcal{Y}_1\). Thus \((\omega_0, T_1\omega_0) \in E(\mathcal{G})\). As the pair \((T_1, T_2)\) is a \(G\)-Orbital cyclic, we obtain \((T_1\omega_0, T_2T_1\omega_0) \in E(\mathcal{G})\). Construct \(\omega_1 = T_1\omega_0\), we have \((\omega_1, T_2\omega_1) \in E(\mathcal{G})\) and from here \((T_2\omega_1, T_1T_2\omega_1) \in E(\mathcal{G})\). Denoting by \(\omega_2 = T_2\omega_1\), we have \((\omega_2, T_1\omega_2) \in E(\mathcal{G})\). In the same manner, we obtain a sequence \((\omega_j)_{j \in \mathbb{N}}\) with \(\omega_{2j} = T_2\omega_{2j-1}\) and \(\omega_{2j+1} = T_1\omega_{2j}\), such that \((\omega_{2j}, \omega_{2j+1}) \in E(\mathcal{G})\). We assume that \(\omega_j \neq \omega_{j+1}\). If, there exist \(j_0 \in \mathbb{N}\) such that \(\omega_{j_0} = \omega_{j_0+1}\), then from the fact that \(\Delta \subset E(\mathcal{G})\), \((\omega_{j_0}, \omega_{j_0+1}) \in E(\mathcal{G})\) and \(\omega^{*} = \omega_{j_0}\) is a fixed point of \(T_1\). Now, for \(\omega^* \in C \text{Fix}(T_1, T_2)\), we deliberate the two cases for \(j_0\). If \(j_0 = 2j\), then \(\omega_{2j} = \omega_{2j+1} = T_1\omega_{2j}\) and thus, \(\omega_{2j}\) is a fixed point of \(T_1\). Assume that \(\omega_{2j} = \omega_{2j+1} = T_1\omega_{2j}\) but \(\Delta(T_1\omega_{2j}, T_2\omega_{2j+1}) > 0\), and let \(\omega = \omega_{2j} \in \mathcal{X}_1\) and \(\omega = \omega_{2j+1} \in \mathcal{X}_2\). So

\[
0 < \Delta(\omega_{2j+1}, \omega_{2j+2}) = \Delta(T_1\omega_{2j}, T_2\omega_{2j+1}) \\
\leq \tau_1 \max\{\Delta(\omega_{2j}, \omega_{2j+1}), \Delta(\omega_{2j}, T_1\omega_{2j}), \Delta(\omega_{2j+1}, T_2\omega_{2j+1})\}; \\
= \tau_1 \max\{\Delta(\omega_{2j}, \omega_{2j+1})\}; \\
= \tau_1 \Delta(\omega_{2j+1}, \omega_{2j+2}).
\]

This is a contradiction of the fact \(\tau_1 < 1\). Therefore, \(\omega_{2j}\) is a fixed point of \(T_2\). In the same manner, if \(j_0\) is an odd number, then there exists \(\omega^{*} \in \mathcal{Z}\) such that \(T_1\omega^{*} = T_2\omega^{*} = \omega^{*}\). So, we assume that \(\omega_j \neq \omega_{j+1}\) for each \(j \in \mathbb{N}\). Now, we examine that \((\omega_j)_{j \in \mathbb{N}}\) is a Cauchy sequence. We discuss the below two cases:

**Case 1.**

\[
0 < \Delta(\omega_{2j+1}, \omega_{2j+2}) = \Delta(T_1\omega_{2j}, T_2\omega_{2j+1}) \\
\leq \tau_1 \max\{\Delta(\omega_{2j}, \omega_{2j+1}), \Delta(\omega_{2j}, T_1\omega_{2j}), \Delta(\omega_{2j+1}, T_2\omega_{2j+1})\}; \\
= \tau_1 \max\{\Delta(\omega_{2j}, \omega_{2j+1})\};
\]

that is

\[
(1 - \tau_1)\Delta(\omega_{2j+1}, \omega_{2j+2}) \leq \tau_1 \Delta(\omega_{2j}, \omega_{2j+1}),
\]

which implies

\[
\Delta(\omega_{2j+1}, \omega_{2j+2}) \leq \frac{\tau_1}{1 - \tau_1} \Delta(\omega_{2j}, \omega_{2j+1})
\]

**Case 2.** \(\omega = \omega_{2j} \in \mathcal{X}_1\) and \(\omega = \omega_{2j+1} \in \mathcal{X}_2\):

\[
0 < \Delta(\omega_{2j+1}, \omega_{2j}) = \Delta(T_1\omega_{2j}, T_2\omega_{2j+1}) \\
\leq \tau_1 \max\{\Delta(\omega_{2j}, \omega_{2j+1}), \Delta(\omega_{2j}, T_1\omega_{2j}), \Delta(\omega_{2j+1}, T_2\omega_{2j+1})\}; \\
= \tau_1 \max\{\Delta(\omega_{2j}, \omega_{2j+1})\};
\]

\[
\leq \tau_1 \Delta(\omega_{2j+1}, \omega_{2j+2}).
\]
that is
\[(1-\tau_1)\Delta(\omega_{2j+1}, \omega_{2j}) \leq \tau_1 \Delta(\omega_{2j}, \omega_{2j-1}),\]
which implies
\[\Delta(\omega_{2j}, \omega_{2j+1}) \leq \frac{\tau_1}{1-\tau_1}\Delta(\omega_{2j-1}, \omega_{2j}). \tag{11}\]
Since \(\tau = \frac{\tau_1}{1-\tau_1}\), we have
\[\Delta(\omega_{j}, \omega_{j+1}) \leq \tau \Delta(\omega_{j-1}, \omega_{j}). \tag{12}\]

Thus, we have
\[\Delta(\omega_{j}, \omega_{j+1}) \leq \tau \Delta(\omega_{j-1}, \omega_{j}) \leq \tau^2 \Delta(\omega_{j-2}, \omega_{j-1}) \leq \ldots \leq \tau^i \Delta(\omega_0, \omega_1). \]

For each \(j, m \in \mathbb{N}(j < m)\), we have
\[
\begin{align*}
\Delta(\omega_{j}, \omega_{m}) & \leq \sigma(\omega_{j}, \omega_{j+1})\Delta(\omega_{j}, \omega_{j+1}) + \gamma(\omega_{j+1}, \omega_{m})\Delta(\omega_{j+1}, \omega_{m}) \\
& \leq \sigma(\omega_{j}, \omega_{j+1})\Delta(\omega_{j}, \omega_{j+1}) + \gamma(\omega_{j+1}, \omega_{j+2})\sigma(\omega_{j+1}, \omega_{j+2}) \Delta(\omega_{j+1}, \omega_{j+2}) \\
& \quad + \gamma(\omega_{j+1}, \omega_{m})\gamma(\omega_{j+2}, \omega_{m})\Delta(\omega_{j+2}, \omega_{m}) \\
& \leq \ldots \leq \sigma(\omega_{j}, \omega_{j+1})\Delta(\omega_{j}, \omega_{j+1}) + m^{-2} \left( \prod_{i=j+1}^{m} \gamma(\omega_{i}, \omega_{m}) \right) \sigma(\omega_{j}, \omega_{j+1})\Delta(\omega_{j}, \omega_{j+1}) \\
& \quad + \prod_{i=j+1}^{m-1} \gamma(\omega_{i}, \omega_{m})\Delta(\omega_{m1}, \omega_{m}) \tag{13}.
\end{align*}
\]

This further implies that
\[
\begin{align*}
\Delta(\omega_{j}, \omega_{m}) & \leq \sigma(\omega_{j}, \omega_{j+1})\Delta(\omega_{j}, \omega_{j+1}) + \sum_{i=j+1}^{m-2} \left( \prod_{t=j+1}^{i} \gamma(\omega_{t}, \omega_{m}) \right) \sigma(\omega_{j}, \omega_{j+1})\Delta(\omega_{j}, \omega_{j+1}) \\
& \quad + \left( \prod_{i=j+1}^{m-1} \gamma(\omega_{i}, \omega_{m}) \right) \Delta(\omega_{m-1}, \omega_{m}) \\
& \leq \sigma(\omega_{j}, \omega_{j+1})\tau^1 \Delta(\omega_0, \omega_1) + \sum_{i=j+1}^{m-2} \left( \prod_{t=j+1}^{i} \gamma(\omega_{t}, \omega_{m}) \right) \sigma(\omega_{j}, \omega_{j+1})\tau^1 \Delta(\omega_0, \omega_1) \\
& \quad + \left( \prod_{i=j+1}^{m-1} \gamma(\omega_{i}, \omega_{m}) \right) \tau^{m-1} \Delta(\omega_0, \omega_1) \\
& = \sigma(\omega_{j}, \omega_{j+1})\tau^1 \Delta(\omega_0, \omega_1) + \sum_{i=j+1}^{m-1} \left( \prod_{t=j+1}^{i} \gamma(\omega_{t}, \omega_{m}) \right) \sigma(\omega_{j}, \omega_{j+1})\tau^1 \Delta(\omega_0, \omega_1). \\
\end{align*}
\]

Let
\[
\psi_{\nu} = \sum_{i=j+1}^{\nu} \left( \prod_{t=j+1}^{i} \gamma(\omega_{t}, \omega_{m}) \right) \sigma(\omega_{i}, \omega_{i+1})\tau^i \Delta(\omega_0, \omega_1). \]
Then, from (13), we obtain
\[
\Delta(\omega_i, \omega_m) \leq \Delta(\omega_0, \omega_1) \left[ \tau^j \sigma(\omega_j, \omega_{j+1}) + (\psi_{m-1} - \psi_j) \right].
\] (14)

Since \(\sigma(\omega, \kappa) \geq 1\), and by employing the ratio test, then \(\lim_{j \to +\infty} \psi_j\) exists. Clearly, letting \(j, m \to +\infty\) in (14), we obtain
\[
\lim_{j,m \to +\infty} \Delta(\omega_i, \omega_m) = 0.
\] (15)

Hence, \(\{\omega_j\}\) is a Cauchy sequence in \((\Xi, \Delta, \sigma, \gamma)\). So, there exist \(\omega^* \in \Xi\), we have
\[
\lim_{j \to +\infty} \Delta(\omega_j, \omega^*) = 0.
\] (16)

That is \(\omega_j \to \omega^*\) as \(j \to +\infty\). It is obvious that
\[
\lim_{j \to +\infty} \omega_{2j} = \lim_{j \to +\infty} \omega_{2j+1} = \omega^*.
\] (17)

As \(T_1\) and \(T_2\) are continuous, so we have
\[
\omega^* = \lim_{j \to +\infty} \omega_{2j+1} = \lim_{j \to +\infty} T_1(\omega_{2j}) = T_1(\omega^*), \omega^* = \lim_{j \to +\infty} \omega_{2j+2} = \lim_{j \to +\infty} T_2(\omega_{2j+1}) = T_2(\omega^*),
\]

Now letting \(\omega = \omega^* \in \Xi^T\) and \(\kappa = \omega_{2j+2} \in \Xi^T\), we have
\[
0 < \Delta(T_1 \omega^*, \omega_{2j+2}) = \Delta(T_1 \omega^*, T_2 \omega_{2j+1}) \leq \tau_1 \max \{\Delta(\omega^*, \omega_{2j+1}), \Delta(\omega^*, T_1 \omega^*), \Delta(\omega_{2j+1}, T_2 \omega_{2j+1})\},
\]
\[
= \tau_1 \max \{\Delta(\omega^*, \omega_{2j+1}), \Delta(\omega^*, T_1 \omega^*), \Delta(\omega_{2j+1}, \omega_{2j+2})\}.
\]

Taking \(j \to +\infty\) and by (17), it is immediate that \(\Delta(\omega^*, T_1 \omega^*) = 0\). This yields that \(\omega^* = T_1 \omega^*\). Similarly, suppose that \(\omega = \omega_{2j+1} \in \Xi^T\) and \(\kappa = \omega^* \in \Xi^T\), we have
\[
0 < \Delta(\omega_{2j+2}, T_2 \omega^*) = \Delta(T_1 \omega_{2j}, T_2 \omega^*) \leq \tau_1 \max \{\Delta(\omega_{2j}, \omega^*), \Delta(\omega_{2j}, T_1 \omega_{2j}), \Delta(\omega^*, T_2 \omega^*)\},
\]
\[
= \tau_1 \max \{\Delta(\omega_{2j}, \omega^*), \Delta(\omega_{2j}, \omega_{2j+1}), \Delta(\omega^*, T_2 \omega^*)\}.
\]

Taking \(j \to +\infty\) and by applying (17), we obtain \(\Delta(\omega^*, T_2 \omega^*) = 0\). This yields that \(\omega^* = T_1 \omega^*\) are that conclude that ases for \(2\omega^*\). □

**Corollary 7.** Let \((\Xi, \Delta, \sigma, \gamma)\) be a complete DCMS equipped with a directed graph \(G\) and \(T : \Xi \to \Xi\) is a \(G\)-Orbital cyclic. Assume that there exist \(s \tau_1 \in [0, 1)\) such that

(i) \(\Xi^T \neq \phi\),

(ii) for all \(\omega, \kappa \in \Xi^T\),

\[
\Delta(T \omega, T \kappa) \leq \tau_1 \max \{\Delta(\omega, \kappa), \Delta(\omega, T \omega), \Delta(\kappa, T \kappa)\},
\]

(iii) for all \(\{\omega_n\}_{n \in \mathbb{N}} \subseteq \Xi\), one has \((\omega_j, \omega_{j+1}) \in E(G),
\]

\[
\sup_{m \leq \tau_1} \frac{\sigma(\omega^{j+1}, \omega^{j+2}) \gamma(\omega^{j+1}, \omega^m)}{\sigma(\omega^{j}, \omega^{j+1})} < \frac{1}{\tau_1},
\] (18)

where \(\tau = \frac{\tau_1}{1 - \tau_1}\).
(iv) \( T \) is continuous, for all \( (\omega_j)_{j \in \mathbb{N}} \subseteq \Xi, \) with \( \omega_j \rightarrow \omega \) as \( j \rightarrow +\infty, \) and \( (\omega_0, \omega_{j+1}) \in E(G) \) for \( j \in \mathbb{N}, \) we obtain \( \omega \in \Xi^T, \)

(v) for all \( \omega \in \Xi, \) we assume \( \lim_{j \rightarrow +\infty} \sigma(\omega_j, \omega), \lim_{j \rightarrow +\infty} \sigma(\omega, \omega_j), \lim_{j \rightarrow +\infty} \gamma(\omega_j, \omega) \) and \( \lim_{j \rightarrow +\infty} \gamma(\omega, \omega_j) \) exists and are finite.

then \( T \) has a unique fixed point.

**Example 2.** Suppose \( \Xi = \{0, 1, 2, 3, 4\}. \) Define \( \Delta : \Xi \times \Xi \rightarrow [0, +\infty) \) by
\[
\Delta(\omega, \kappa) = |\omega - \kappa|^2
\]
and \( \sigma : \Xi \times \Xi \rightarrow [1, +\infty) \) by
\[
\sigma(\omega, \kappa) = 1 + \omega + \kappa
\]
and
\[
\gamma(\omega, \kappa) = 2 + \omega^2 + \kappa^2
\]
for all \( \omega, \kappa \in \Xi. \) Then, \( (\Xi, \Delta, \sigma, \gamma) \) is a complete DCMS. Now define \( T : \Xi \rightarrow \Xi \) by \( T\omega = 0 \) for \( \omega \in \{0, 1\}, \) and \( T\omega = 1 \) for \( \omega \in \{2, 3\}. \)

Moreover, let a directed graph by \( G = \{(0, 1), (0, 2), (2, 3), (0, 0), (1, 1), (2, 2)(3, 3)\}. \) Then, Corollary 3 is fulfilled with \( \tau_1 = \frac{1}{3} \) and \( T \) has a unique fixed point \( \omega^* = 0. \)

6. Application to Integral Equations

In this part, we examine the solution of the following Fredholm equation:
\[
\omega(t) = \int_0^1 K(t, s, \omega(t))ds,
\]
for all \( t \in [0, 1], \) where \( K(t, s, \omega(t)) \) is a continuous function from \([0, 1] \times [0, 1]\) into \( \mathbb{R}. \)

Suppose \( \Xi = C([0, 1], \mathbb{R}). \) Define \( \Delta : \Xi \times \Xi \rightarrow [1, +\infty) \) by
\[
\Delta(\omega, \kappa) = \sup_{t \in [0, 1]} \left( \frac{|\omega(t)| + |\kappa(t)|}{2} \right).
\]

Then, \( (\Xi, \Delta, \sigma, \gamma) \) is a complete DCMS with \( \sigma(\omega, \kappa) = 1 + \omega + \kappa, \) and \( \gamma(\omega, \kappa) = 2 + \omega^2 + \kappa^2. \)

**Theorem 5.** Assume that

(a) \( |K(t, s, \omega(t))| + |K(t, s, \kappa(t))| \leq \tau_1 \left( \sup_{t \in [0, 1]} |\omega(t)| + |\kappa(t)| \right) (|\omega(t)| + |\kappa(t)|) \)

for some \( \tau_1 \rightarrow \Xi \rightarrow [0, 1); \)

(b) \( K(t, s, \int_0^1 K(t, s, \omega(t))ds) < K(t, s, \omega(t)) \)

for all \( t, s \in [0, 1]. \) Then, there is a unique solution to the integral Equation (19).

**Proof.** Define the mapping \( T : \Xi \rightarrow \Xi \) by
\[
T\omega(t) = \int_0^1 K(t, s, \omega(t))ds.
\]
Then
\[
\Delta(T\omega, T\kappa) = \sup_{t \in [0, 1]} \left( \frac{|\omega(t)| + |\kappa(t)|}{2} \right).
\]
Now
\[
\Delta(T\varphi(t), T\psi(t)) = \frac{|T\varphi(t)| + |T\psi(t)|}{2}
\]
\[
= \frac{\int_0^1 |K(t, s, \varphi(t))| ds + \int_0^1 |K(t, s, \psi(t))| ds}{2}
\]
\[
\leq \int_0^1 \sup_{t \in [0,1]} |\varphi(t)| + |\psi(t)| ds
\]
\[
\leq \tau_1 \left( \sup_{t \in [0,1]} |\varphi(t)| + |\psi(t)| \right) \Delta(\varphi(t), \psi(t)).
\]

Also, we observe that
\[
\sigma(\varphi, \psi) = \frac{1}{\tau_1 \left( \sup_{t \in [0,1]} |\varphi(t)| + |\psi(t)| \right)}.
\]

Hence, all of the requirements for Corollary 6 have been met. As a result, Equation (19) has a unique solution. □

7. Conclusions

In this manuscript, we proved two common fixed point results for generalized rational-type contractions under some conditions in the context of DCMS. Further, we extended and proved rational-type contractions equipped with graphs in DCMS. Several non-trivial examples and an application are presented to show the validity of the main results. The given results are improved and generalized to the existing ones in [11,18,20]. These results can be generalized by utilizing the notions in [19,21–26].


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