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Spectrum of the Cozero-Divisor Graph Associated to Ring $\mathbb{Z}_n$

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Abstract: Let R be a commutative ring with identity $1 \neq 0$ and let $Z(R)'$ be the set of all non-unit and non-zero elements of ring R. $\Gamma'(R)$ denotes the cozero-divisor graph of R and is an undirected graph with vertex set $Z(R)'$, $w \notin zR$, and $z \notin wR$ if and only if two distinct vertices w and z are adjacent, where qR is the ideal generated by the element q in R. In this article, we investigate the signless Laplacian eigenvalues of the graphs $\Gamma'(\mathbb{Z}_n)$. We also show that the cozero-divisor graph $\Gamma'(\mathbb{Z}_{p_1p_2})$ is a signless Laplacian integral.

Keywords: cozero-divisor graph; signless Laplacian spectrum; ring of integer modulo n

MSC: 15A18; 05C50; 05C25; 05C12

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1. Introduction

The concept of a cozero-divisor graph on commutative rings was introduced in [1]. In [2], the Laplacian eigenvalues of this type of graph is computed. Some applications of this research could be in the following areas: quantum chemistry, the topological theory of aromaticity, counts of random walks, structure-resonance theory, and eigenvector–eigenvalue problems.

Throughout this article, unless otherwise stated, R will denote a commutative ring with identity $1 \neq 0$. wR denotes the ideal generated by an element w in R and it is defined as $wR = \{wa : a \in R\}$. $Z(R)'$ is the set of all non-unit and non-zero elements of ring R.

We denote the graph $G = (V, E)$, where V is the vertex set and E denotes the edge set of graph G. The symbol $y_1 \sim y_2$ indicates that y_1 is adjacent to y_2 in a graph G, where y_1 and y_2 are distinct vertices of G. The complete graph is denoted by K_m with m vertices and the set of G vertices that are adjacent to vertex y is known as the vertex's neighbourhood, and it is represented by the symbol $N_G(y)$. The number of edges incident with $y \in V$ is represented by $deg(y)$, which is the *degree* of vertex y, and y is referred to as an *isolated* vertex if $deg(y) = 0$. For each vertex y if $deg(y) = k$, then G is *k-regular*. Now, let $\varphi_1, \varphi_2, \ldots, \varphi_k$ be the distinct eigenvalues of a square matrix B with multiplicities $v_1, v_2, \ldots, v_k$, respectively, then $\sigma(B)$ denotes the *spectrum* of B and is defined by

$$\sigma(B) = \left\{ \begin{array}{cccc} \varphi_1 & \varphi_2 & \cdots & \varphi_k \\ v_1 & v_2 & \cdots & v_k \end{array} \right\}.$$

The square matrix $A(G)$ of G is the *adjacency* matrix of G and is given by

$$A(G) = (a_{rs}) = \begin{cases} 1, & v_r \sim v_s \text{ in } G, \\ 0, & otherwise. \end{cases}$$

The *Laplacian* matrix $L(G)$ of a graph G is defined as

$$L(G) = (l_{rs}) = \begin{cases} deg(v_r), & r = s, \\ -1, & r \neq s \text{ and } v_r \sim v_s, \\ 0, & otherwise. \end{cases}$$

Let $Deg(G)$ be the diagonal matrix of vertex degrees given by $Deg(G) = diag(x_1, x_2, \dots, x_n)$, where $x_i = deg(v_i)$. The Laplacian matrix of a graph G is defined as

$$L(G) = Deg(G) - A(G)$$

and the *signless Laplacian* matrix of a graph G is defined as

$$SL(G) = Deg(G) + A(G).$$

If all the signless Laplacian eigenvalues of a graph G are integers, then G is said to be a signless Laplacian *integral*. The spectrum of signless Laplacian matrix and Laplacian matrix is known as the signless Laplacian spectrum and the Laplacian spectrum of the graph G, respectively. The details of adjacency and the signless Laplacian spectrum can be found in [3–6].

On commutative rings, Afkhami et al. [1] introduced the concept of a cozero-divisor graph. $\Gamma'(R)$ denotes the cozero-divisor graph of R, which is an undirected graph with a vertex set $Z(R)'$, $w \notin zR$, and $z \notin wR$ if and only if two distinct vertices w and z are adjacent. For more details on the cozero-divisor graph see, for example, [1,7,8] where further references can be found.

Parveen et al. [2] calculated the Laplacian eigenvalues of the graph $\Gamma'(\mathbb{Z}_n)$ for $n = p^{n_1}q^{n_2}$, where p, q are distinct primes and $n_1, n_2 \in \mathbb{N}$. In this article, we find the signless Laplacian eigenvalues of the graphs $\Gamma'(\mathbb{Z}_n)$ for different values of n. In Section 2, we recall several basic notions that are used to prove our main conclusions. In Section 3, we look at the signless Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$, where $n = p_1p_2, p_1^2p_2, p_1p_2^m, p_1^{m_1}p_2^{m_2}$.

2. Preliminaries

We begin our discussions with the definition of a generalized join graph and some known results that are used to prove the main results.

Definition 1. *Let $\{v_1, v_2, \dots, v_m\}$ be the vertex set of graph $G(V, E)$ of order m and let m_k be the order of the disjoint graphs $F_k(V_k, E_k)$, $1 \leq k \leq m$. The graphs $F_1, F_2, \dots, F_m$ formed the generalized join graph $G[F_1, F_2, \dots, F_m]$ and whenever i and j are adjacent in G, joined each vertex of F_i to every vertex of F_j.*

We write $x \nmid n$ to denote that x does not divide n and (x, n) denotes the gcd of x and n. For a positive integer n, the number of positive divisors of n is given by $\tau(n)$. An integer x divides n for $1 < x < n$ if and only if x is a proper divisor of n. Euler's phi function $\phi(n) = \{\kappa \in \mathbb{Z}_+ |\ \kappa \leq n \text{ and } (\kappa, n) = 1\}$. The *prime decomposition* of n is $n = {r_1}^{s_1}{r_2}^{s_2} \cdots {r_k}^{s_k}$, where $s_1, s_2, \dots, s_k$ are positive integers and $r_1, r_2, \dots, r_k$ are distinct primes.

Let the proper divisors of n be $w_1, w_2, \dots, w_q$. For $1 \leq r \leq q$, consider the sets

$$A_{w_r} = \{x \in \mathbb{Z}_n : (x, n) = w_r\}.$$

Moreover, we see that $A_{w_r} \cap A_{w_s} = \phi$, when $r \neq s$. This implies that the sets $A_{w_1}, A_{w_2}, \dots, A_{w_q}$ are pairwise disjoints and partition the vertex set of $\Gamma'(\mathbb{Z}_n)$ as

$$V(\Gamma'(\mathbb{Z}_n)) = A_{w_1} \cup A_{w_2} \cup \dots \cup A_{w_q}.$$

The next lemma shows the cardinality of A_{w_r}.

Lemma 1 ([6], Proposition 2.1). *Let w_r be the divisor of n. Then, $|A_{w_r}| = \phi(\frac{n}{w_r})$, $1 \leq r \leq q$.*

Lemma 2 ([2], Lemma 3.3). *Let $y \in A_{w_r}, z \in A_{w_s}$, where $y, z \in \{1, 2, \ldots, q\}$. Then, $y \sim z$ in $\Gamma'(\mathbb{Z}_n)$ if and only if $w_r \nmid w_s$ and $w_s \nmid w_r$.*

Let $w_1, w_2, \ldots, w_q$ be the distinct proper divisors of n and let δ_n be the simple graph with vertex set $\{w_1, w_2, \ldots, w_q\}$. Two distinct vertices w_i and w_j of graph δ_n are adjacent if and only if $w_i \nmid w_j$ and $w_j \nmid w_i$. If $n = {p_1}^{n_1} {p_2}^{n_2} \cdots {p_r}^{n_r}$ is a prime decomposition of n, then the order of graph δ_n is given by

$$|V(\delta_n)| = \prod_{i=1}^{r} (n_i + 1) - 2.$$

Lemma 3 ([2], Corollary 3.4). *Let w_r be the proper divisor of the positive integer n. Then, the following holds:*

(i) For $r \in \{1, 2, \ldots, q\}$, the induced subgraph $\Gamma'(A_{w_r})$ of $\Gamma'(\mathbb{Z}_n)$ on the vertex set A_{w_r} is isomorphic to $\overline{K}_{\phi(\frac{n}{w_r})}$.

(ii) For $r, s \in \{1, 2, \ldots, q\}$ with $r \neq s$, a vertex of A_{w_r} is adjacent to either all or none of the vertices of A_{w_s} in $\Gamma'(\mathbb{Z}_n)$.

The above lemma shows that the induced subgraph $\Gamma'(A_{w_r})$ of $\Gamma'(\mathbb{Z}_n)$ is an empty graph. The next lemma says that $\Gamma'(\mathbb{Z}_n)$ is a generalized join of complements of complete graphs.

Lemma 4 ([2], Lemma 3.6). *Let $\Gamma'(A_{w_r})$ be the induced subgraph of $\Gamma'(\mathbb{Z}_n)$ on the vertex set A_{w_r} for $1 \leq r \leq q$. Then, $\Gamma'(\mathbb{Z}_n) = \delta_n[\Gamma'(A_{w_1}), \Gamma'(A_{w_2}), \ldots, \Gamma'(A_{w_q})]$.*

The following result gives the signless Laplacian spectrum of the generalized join graph.

Theorem 1 ([9], Theorem 2.1). *Let K be a graph with $V(K) = \{u_1, u_2, \ldots, u_n\}$ and H_r's be k_r-regular graphs of order h_r with signless Laplacian eigenvalues $\lambda_{r1} \geq \lambda_{r2} \geq \cdots \geq \lambda_{rh_r}$, where $r = 1, 2, \ldots, n$. If $G = K[H_1, H_2, \ldots, H_n]$, then the signless Laplacian spectrum of G can be computed as follows:*

$$\sigma_{SL}(G) = \left(\bigcup_{r=1}^{n} \left(N_r + \left(\sigma_{SL}(H_r) \setminus \{2k_r\} \right) \right) \right) \bigcup \sigma(Y(K)),$$

where

$$N_r = \begin{cases} \sum\limits_{v_s \in N_K(v_r)} h_s, & N_K(v_r) \neq \phi, \\ 0, & \text{otherwise} \end{cases}$$

and

$$Y(K) = (y_{s,t})_{n \times n} = \begin{cases} 2k_s + N_s, & s = t, \\ \sqrt{h_s h_t}, & v_s \sim v_t \in E(K), \\ 0 & \text{otherwise}. \end{cases} \quad (1)$$

The number N_r and the matrix $Y(K)$ are only dependent on the graph K.

Let W be a weighted graph by assigning $|V(W)| = h_s$ to the vertex v_s of graph W and let s vary from 1 to n. Consider the matrix $Z(W) = (z_{s,t})_{n \times n}$, where

$$z_{s,t} = \begin{cases} \sum\limits_{v_s \sim v_t} h_s, & s = t, \\ h_s, & s \neq t \text{ and } v_s \sim v_t, \\ 0 & \text{otherwise}. \end{cases}$$

The vertex-weighted signless Laplacian matrix of W is $Z(W)$. It can be seen that the matrices $Y(K)$ and $Z(W)$ are similar, and hence $\sigma(Y(K)) = \sigma(Z(W))$.

3. Main Results

In the result section, we shall prove the main results of this paper. Let $x_1, x_2, \ldots, x_k$ be the proper divisors of n. For $1 \leq t \leq k$, we assign the weight $\phi(\frac{n}{x_t}) = |A_{x_t}|$ to the vertex x_t of the graph δ_n. Define the integer

$$N_{x_s} = \sum_{x_t \in N_{\delta_n}(x_s)} \phi\left(\frac{n}{x_t}\right).$$

Then, the vertex-weighted signless Laplacian matrix $Z(\delta_n)$ of δ_n is given by

$$Z(\delta_n) = z_{s,t} = \begin{cases} \sum\limits_{x_t \in N_{\delta_n}(x_s)} \phi\left(\frac{n}{x_t}\right), & s = t, \\ \phi\left(\frac{n}{x_t}\right), & s \neq t \text{ and } x_s \sim x_t \text{ in } \delta_n, \\ 0 & \textit{otherwise}. \end{cases} \tag{2}$$

Our main result gives the signless Laplacian spectrum of the cozero-divisor graph of $\Gamma'(\mathbb{Z}_n)$.

Theorem 2. *Let the proper divisors of n be $x_1, x_2, \ldots, x_k$. Then, the signless Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$ can be calculated as*

$$\sigma_{SL}(\Gamma'(\mathbb{Z}_n)) = \left(\bigcup_{r=1}^{k}\left(N_{x_r} + \left(\sigma_{SL}(\Gamma'(A_{x_r})) \setminus \{2k_r\}\right)\right)\right) \bigcup \sigma(Z(\delta_n)),$$

where $\Gamma'(A_{x_r})$ are k_r regular graphs and $N_{x_r} + \left(\sigma_{SL}(\Gamma'(A_{x_r})) \setminus \{2k_r\}\right)$ represents that N_{x_r} is added to each element of the multiset $\left(\sigma_{SL}(\Gamma'(A_{x_r})) \setminus \{2k_r\}\right)$.

Proof. In view of Lemma 4, we have

$$\Gamma'(\mathbb{Z}_n) = \delta_n[\Gamma'(A_{x_1}), \Gamma'(A_{x_2}), \ldots, \Gamma'(A_{x_k})].$$

Thus, by using the relation $\sigma(Y(K)) = \sigma(Z(W))$ and consequence of Theorem 1, the result holds. □

By Lemma 3, $\Gamma'(A_{x_r})$ is isomorphic to $\overline{K}_{\phi(\frac{n}{x_r})}$ for $r \in \{1, 2, \ldots, k\}$. Thus, by Theorem 2, $n - \phi(n) - 1$ signless Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$ exists, out of which $n - \phi(n) - 1 - k$ are known. The remaining k signless Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_n)$ are the roots of the characteristic polynomial of the matrix $Z(\delta_n)$ given in (2).

Proposition 1. *The signless Laplacian spectrum of $\Gamma'(\mathbb{Z}_{p_1 p_2})$, where p_1 and p_2 are distinct primes, is given by*

$$\sigma_{SL}(\Gamma'(\mathbb{Z}_{p_1 p_2})) = \left\{\begin{matrix} 0 & p_1 - 1 & p_2 - 1 & p_1 + p_2 - 2 \\ 1 & p_2 - 2 & p_1 - 2 & 1 \end{matrix}\right\}.$$

Proof. Let $n = p_1 p_2$, where p_1 and p_2 are distinct primes. Here, we see that p_1 and p_2 are the proper divisors of n. So, $\delta_{p_1 p_2}: p_1 \sim p_2$ and by Lemma 4, we have $\Gamma'(\mathbb{Z}_{p_1 p_2}) = \delta_{p_1 p_2}[\Gamma'(A_{p_1}), \Gamma'(A_{p_2})]$. In view of Lemma 3, we have

$$\Gamma'(\mathbb{Z}_{p_1 p_2}) = \delta_{p_1 p_2}[\overline{K}_{\phi(p_2)}, \overline{K}_{\phi(p_1)}].$$

Moreover, $N_{p_1} = p_1 - 1$ and $N_{p_2} = p_2 - 1$. So, by the consequence of Theorem 2, the signless Laplacian spectrum of $\Gamma'(\mathbb{Z}_{p_1 p_2})$ is given by

$$\begin{aligned}\sigma_{SL}(\Gamma'(\mathbb{Z}_{p_1p_2})) =& \Big(N_{p_1} + \Big(\sigma_{SL}(\Gamma'(A_{p_1})) \setminus \{2k_1\}\Big)\Big) \\ & \bigcup \Big(N_{p_2} + \Big(\sigma_{SL}(\Gamma'(A_{p_2})) \setminus \{2k_2\}\Big)\Big) \bigcup \sigma(Z(\delta_n)) \\ =& \left\{ \begin{array}{cc} p_1 - 1 & p_2 - 1 \\ p_2 - 2 & p_1 - 2 \end{array} \right\} \bigcup \sigma(Z(\delta_n)).\end{aligned}$$

Now, from (2), the matrix $Z(\delta_n)$ is given by

$$Z(\delta_n) = \begin{bmatrix} p_1 - 1 & p_1 - 1 \\ p_2 - 1 & p_2 - 1 \end{bmatrix},$$

which has characteristic polynomial $x^2 - (p_1 + p_2 - 2)x$ and eigenvalues 0 and $p_1 + p_2 - 2$. □

Proposition 2. *The signless Laplacian spectrum of* $\Gamma'(\mathbb{Z}_{p_1^2 p_2})$*, where* p_1 *and* p_2 *are distinct primes, consists of the eigenvalues*

$$\left\{ \begin{array}{cccc} p_1^2 - p_1 & p_1^2 - 1 & p_1 p_2 - p_1 & p_2 - 1 \\ p_1 p_2 - p_1 - p_2 & p_2 - 2 & p_1^2 - p_1 - 1 & p_1 - 2 \end{array} \right\}.$$

The other signless Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_{p_1^2 p_2})$ *are the roots of the characteristic polynomial of the matrix (3).*

Proof. Let $m = p_1^2 p_2$, where p_1 and p_2 are distinct primes. The proper divisors of m are p_1, p_1^2, p_2, and $p_1 p_2$. So, $\delta_{p_1^2 p_2}: p_1 \sim\ p_2 \sim p_1^2 \sim p_1 p_2$ and by Lemma 4, we have $\Gamma'(\mathbb{Z}_{p_1^2 p_2}) = \delta_{p_1^2 p_2}[\Gamma'(A_{p_1}), \Gamma'(A_{p_1^2}), \Gamma'(A_{p_2}), \Gamma'(A_{p_1 p_2})]$. In view of result Lemma 3, we have

$$\Gamma'(\mathbb{Z}_{p_1^2 p_2}) = \delta_{p_1^2 p_2}[\overline{K}_{\phi(p_1 p_2)}, \overline{K}_{\phi(p_2)}, \overline{K}_{\phi(p_1^2)}, \overline{K}_{\phi(p_1)}].$$

Moreover, the values of N_{x_s} are as follows

$$N_{p_1} = p_1^2 - p_1, N_{p_1^2} = p_1^2 - 1, N_{p_2} = p_1 p_2 - p_1 \text{ and } N_{p_1 p_2} = p_2 - 1.$$

In view of Theorem 2, the signless Laplacian spectrum of $\Gamma'(\mathbb{Z}_{p_1^2 p_2})$ is given by

$$\begin{aligned}\sigma_{SL}(\Gamma'(\mathbb{Z}_{p_1^2 p_2})) =& \Big(N_{p_1} + \Big(\sigma_{SL}(\Gamma'(A_{p_1})) \setminus \{2k_1\}\Big)\Big) \bigcup \Big(N_{p_1^2} + \Big(\sigma_{SL}(\Gamma'(A_{p_1^2})) \setminus \{2k_2\}\Big)\Big) \\ & \bigcup \Big(N_{p_2} + \Big(\sigma_{SL}(\Gamma'(A_{p_2})) \setminus \{2k_3\}\Big)\Big) \bigcup \Big(N_{p_1 p_2} + \Big(\sigma_{SL}(\Gamma'(A_{p_1 p_2})) \setminus \{2k_4\}\Big)\Big) \\ & \bigcup \sigma(Z(\delta_n)) \\ =& \left\{ \begin{array}{cccc} p_1^2 - p_1 & p_1^2 - 1 & p_1 p_2 - p_1 & p_2 - 1 \\ p_1 p_2 - p_1 - p_2 & p_2 - 2 & p_1^2 - p_1 - 1 & p_1 - 2 \end{array} \right\} \bigcup \sigma(Z(\delta_n)).\end{aligned}$$

Now, from (2), the matrix $Z(\delta_n)$ is given by

$$Z(\delta_n) = \begin{bmatrix} p_1^2 - p_1 & 0 & p_1^2 - p_1 & 0 \\ 0 & p_1^2 - 1 & p_1^2 - p_1 & p_1 - 1 \\ p_1 p_2 - p_1 - p_2 + 1 & p_2 - 1 & p_1 p_2 - p_1 & 0 \\ 0 & p_2 - 1 & 0 & p_2 - 1 \end{bmatrix}. \quad (3)$$

□

Example 1. *The signless Laplacian spectrum of the cozero-divisor graph* $\Gamma'(\mathbb{Z}_{45})$ *consists of the eigenvalues*

$$\left\{ \begin{array}{cccccccc} 6 & 8 & 12 & 4 & 0 & 3.035 & 8.623 & 18.342 \\ 7 & 3 & 5 & 1 & 1 & 1 & 1 & 1 \end{array} \right\}.$$

Proof. In this example we find the signless Laplacian spectrum of the cozero-divisor graph of $\mathbb{Z}_{45}$. Here $n = 45$ is of the form $3^2.5$ i.e., $p_1 = 3$ and $p_2 = 5$. So, by Proposition 2 we can easily verify the spectrum of the cozero-divisor graph of $\mathbb{Z}_{45}$. On the other hand, the approximate eigenvalues $\{0, 3.035, 8.623, 18.342\}$ are calculated from the matrix (4), as follows

$$Z(\delta_{45}) = \begin{bmatrix} 6 & 0 & 6 & 0 \\ 0 & 8 & 6 & 2 \\ 8 & 4 & 12 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix}. \tag{4}$$

□

Now, we calculate the signless Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_{p_1 p_2^m})$, which are the second main results of this paper.

Theorem 3. *The signless Laplacian spectrum of* $\Gamma'(\mathbb{Z}_{p_1 p_2^m})$*, where* p_1, p_2 *are distinct primes and* m *is a positive integer, consists of the eigenvalues*

$$\left\{ \begin{array}{cccccc} \sum_{z=1}^{m} \phi(p_1 p_2^{m-z}) & \phi(p_2^m) & \sum_{z=0}^{1} \phi(p_2^{m-z}) & \cdots & \sum_{z=0}^{m-1} \phi(p_2^{m-z}) & \sum_{z=2}^{m} \phi(p_1 p_2^{m-z}) \\ \phi(p_2^m) - 1 & \phi(p_1 p_2^{m-1}) - 1 & \phi(p_1 p_2^{m-2}) - 1 & \cdots & \phi(p_1) - 1 & \phi(p_2^{m-1}) - 1 \end{array} \right.$$
$$\left. \begin{array}{cc} \cdots & \phi(p_1) \\ \cdots & \phi(p_2) - 1 \end{array} \right\}.$$

The roots of the characteristic polynomial of the matrix (5) are the other signless Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_{p_1 p_2^m})$.

Proof. Let $n = p_1 p_2^m$, where p_1, p_2 are distinct primes and m is a positive integer. The proper divisors of n are $p_1, p_2, p_2^2, \ldots, p_2^m, p_1 p_2, p_1 p_2^2, \ldots, p_1 p_2^{m-1}$.
We have the following adjacency relations by the definition of δ_n,

$$\begin{aligned} p_1 &\sim p_2^w, \ 1 \le w \le m. \\ p_2^w &\sim p_1 p_2^{z-1}, \ 1 \le w \le m, \ 1 \le z \le w. \\ p_1 p_2^w &\sim p_2^{w+z}, \ 1 \le w \le m-1, \ 1 \le z \le m-w. \end{aligned}$$

By using Lemma 4, we have

$$\Gamma'(\mathbb{Z}_{p_1 p_2^m}) = \delta_{p_1 p_2^m}[\Gamma'(A_{p_1}), \Gamma'(A_{p_2}), \Gamma'(A_{p_2^2}), \ldots, \Gamma'(A_{p_2^m}), \Gamma'(A_{p_1 p_2}), \ldots, \Gamma'(A_{p_1 p_2^{m-1}})].$$

By applying Lemma 3, we can write as

$$\Gamma'(\mathbb{Z}_{p_1 p_2^m}) = \delta_{p_1 p_2^m}[\overline{K}_{\phi(p_2^m)}, \overline{K}_{\phi(p_1 p_2^{m-1})}, \overline{K}_{\phi(p_1 p_2^{m-2})}, \ldots, \overline{K}_{\phi(p_1)}, \overline{K}_{\phi(p_2^{m-1})}, \ldots, \overline{K}_{\phi(p_2)}].$$

It also follows that

$$N_{p_1} = \sum_{z=1}^{m} \phi(p_1 p_2^{m-z}), \ N_{p_2} = \phi(p_2^m), \ N_{p_2^2} = \sum_{z=0}^{1} \phi(p_2^{m-z}) = \phi(p_2^m) + \phi(p_2^{m-1}), \ldots,$$
$$N_{p_2^m} = \sum_{z=0}^{m-1} \phi(p_2^{m-z}), N_{p_1 p_2} = \sum_{z=2}^{m} \phi(p_1 p_2^{m-z}), \ldots, N_{p_1 p_2^{m-1}} = \phi(p_1).$$

By the consequence of Theorem 2, the signless Laplacian spectrum of $\Gamma'(\mathbb{Z}_{p_1p_2^m})$ is given by

$$\begin{aligned}\sigma_{SL}(\Gamma'(\mathbb{Z}_{p_1p_2^m})) =& \Big(N_{p_1} + \Big(\sigma_{SL}(\Gamma'(A_{p_1})) \setminus \{2k\}\Big)\Big) \bigcup \Big(N_{p_2} + \Big(\sigma_{SL}(\Gamma'(A_{p_2})) \setminus \{2k\}\Big)\Big) \\ & \bigcup \Big(N_{p_2^2} + \Big(\sigma_{SL}(\Gamma'(A_{p_2^2})) \setminus \{2k\}\Big)\Big) \\ & \bigcup \cdots \bigcup \Big(N_{p_2^m} + \Big(\sigma_{SL}(\Gamma'(A_{p_2^m})) \setminus \{2k\}\Big)\Big) \\ & \bigcup \Big(N_{p_1p_2} + \Big(\sigma_{SL}(\Gamma'(A_{p_1p_2})) \setminus \{2k\}\Big)\Big) \\ & \bigcup \cdots \bigcup \Big(N_{p_1p_2^{m-1}} + \Big(\sigma_{SL}(\Gamma'(A_{p_1p_2^{m-1}})) \setminus \{2k\}\Big)\Big) \bigcup \sigma(Z(\delta_n)).\end{aligned}$$

$$\sigma_{SL}(\Gamma'(\mathbb{Z}_{p_1p_2^m})) = \left\{ \begin{matrix} \sum\limits_{z=1}^{m} \phi(p_1p_2^{m-z}) & \phi(p_2^m) & \sum\limits_{z=0}^{1} \phi(p_2^{m-z}) & \cdots & \sum\limits_{z=0}^{m-1} \phi(p_2^{m-z}) \\ \phi(p_2^m) - 1 & \phi(p_1p_2^{m-1}) - 1 & \phi(p_1p_2^{m-2}) - 1 & \cdots & \phi(p_1) - 1 \end{matrix} \right.$$
$$\left. \begin{matrix} \sum\limits_{z=2}^{m} \phi(p_1p_2^{m-z}) & \cdots & \phi(p_1) \\ \phi(p_2^{m-1}) - 1 & \cdots & \phi(p_2) - 1 \end{matrix} \right\} \bigcup \sigma(Z(\delta_n)).$$

The roots of the characteristic polynomial of the matrix $Z(\delta_n)$ given in (5) are the remaining $2m$ signless Laplacian eigenvalues of $\Gamma'(\mathbb{Z}_{p_1p_2^m})$.

$$Z(\delta_n) = \begin{bmatrix} N_{p_1} & \phi(p_1p_2^{m-1}) & \phi(p_1p_2^{m-2}) & \cdots & \phi(p_1) & 0 & \cdots & 0 \\ \phi(p_2^m) & N_{p_2} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \phi(p_2^m) & 0 & N_{p_2^2} & \cdots & 0 & \phi(p_2^{m-1}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi(p_1) & 0 & 0 & \cdots & N_{p_2^m} & \phi(p_2^{m-1}) & \cdots & \phi(p_2^m) \\ 0 & 0 & \phi(p_2^{m-1}) & \cdots & \phi(p_1) & N_{p_1p_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \phi(p_1) & 0 & 0\cdots & N_{p_1p_2^{m-1}} \end{bmatrix}. \quad (5)$$

□

For distinct primes p_1 and p_2, our last result gives the signless Laplacian spectrum of $\Gamma'(\mathbb{Z}_n)$, where $n = p_1^{m_1}p_2^{m_2}$.

Theorem 4. *The signless Laplacian spectrum of $\Gamma'(\mathbb{Z}_{p_1^{m_1}p_2^{m_2}})$ for distinct primes p_1, p_2 and m_1, m_2 are positive integers consisting of eigenvalues*

$$\sum_{y=1}^{m_2} \phi(p_1^{m_1}p_2^{m_2-y}),$$
$$\sum_{y=1}^{m_2} \phi(p_1^{m_1}p_2^{m_2-y}) + \sum_{y=1}^{m_2} \phi(p_1^{m_1-1}p_2^{m_2-y}),$$
$$\vdots$$

$$\sum_{y=1}^{m_2}\phi(p_1^{m_1}p_2^{m_2-y})+\sum_{y=1}^{m_2}\phi(p_1^{m_1-1}p_2^{m_2-y})+\cdots+\sum_{y=1}^{m_2}\phi(p_1p_2^{m_2-y}),$$
$$\sum_{y=1}^{m_1}\phi(p_1^{m_1-y}p_2^{m_2}),$$
$$\vdots$$
$$\sum_{y=1}^{m_1}\phi(p_1^{m_1-y}p_2^{m_2})+\sum_{y=1}^{m_1}\phi(p_1^{m_1-y}p_2^{m_2-1})+\cdots+\sum_{y=1}^{m_1}\phi(p_1^{m_1-y}p_2),$$
$$\sum_{y=2}^{m_1}\phi(p_1^{m_1-y}p_2^{m_2})+\sum_{y=2}^{m_2}\phi(p_1^{m_1}p_2^{m_2-y}),$$
$$\vdots$$
$$\sum_{y=2}^{m_2}\phi(p_1^{m_1}p_2^{m_2-y})+\sum_{y=1}^{m_1}\phi(p_1^{m_1-y}p_2^{m_2-2})+\sum_{y=1}^{m_1}\phi(p_1^{m_1-y}p_2^{m_2-3})+\cdots+\sum_{y=1}^{m_1-1}\phi(p_1^{m_1-y}),$$
$$\sum_{y=2}^{m_1}\phi(p_1^{m_1-y}p_2^{m_2})+\sum_{y=3}^{m_2}\phi(p_1^{m_1}p_2^{m_2-y})+\sum_{y=2}^{m_1}\phi(p_1^{m_1-y}p_2^{m_2-1}),$$
$$\vdots$$
$$\sum_{y=2}^{m_1}\phi(p_1^{m_1-y}p_2^{m_2})+\sum_{y=1}^{m_2}\phi(p_1^{m_1-2}p_2^{m_2-y})+\sum_{y=1}^{m_2}\phi(p_1^{m_1-3}p_2^{m_2-y})+\cdots+\sum_{y=1}^{m_2-1}\phi(p_2^{m_2-y}),$$
$$\vdots$$
$$\sum_{y=1}^{m_2-1}\phi(p_2^{m_2-y})+\phi(p_2^{m_2}),$$

with multiplicities

$$\phi(p_1^{m_1-1}p_2^{m_2})-1,\phi(p_1^{m_1-2}p_2^{m_2})-1,\ldots,\phi(p_2^{m_2})-1,\phi(p_1^{m_1}p_2^{m_2-1})-1,\ldots,\phi(p_1^{m_1})-1,$$
$$\phi(p_1^{m_1-1}p_2^{m_2-1})-1,\ldots,\phi(p_2^{m_2-1})-1,\phi(p_1^{m_1-1}p_2^{m_2-2})-1,\ldots,\phi(p_1^{m_1-1})-1,\ldots,\phi(p_1)-1,$$

respectively, and the zeros of the characteristic polynomial of the matrix $Z(\delta_n)$ *given in* (2) *are the* $m_1m_2+m_1+m_2-1$ *signless Laplacian eigenvalues of* $\Gamma'(\mathbb{Z}_{p_1^{m_1}p_2^{m_2}})$.

Proof. Let $n = p_1^{m_1}p_2^{m_2}$, where p_1 and p_2 are distinct primes. The proper divisor of n are $p_1, p_1^2, \ldots, p_1^{m_1}, p_2, p_2^2, \ldots, p_2^{m_2}, p_1p_2, p_1^2p_2, \ldots, p_1^{m_1}p_2, p_1p_2^2, p_1^2p_2^2, \ldots, p_1^{m_1}p_2^2, \ldots, p_1p_2^{m_2}, p_1^2p_2^{m_2}, \ldots, p_1^{m_1-1}p_2^{m_2}$. We have the following adjacency relations by the definition of δ_n,

$$p_1^y \sim p_2^z \text{ for all } y,z.$$
$$p_1^y \sim p_1^w p_2^x \text{ for } y>w \text{ and } x>0.$$
$$p_2^z \sim p_1^y p_2^x \text{ for } z>x \text{ and } y>0.$$
$$p_1^w p_1^x \sim p_1^s p_2^t \text{ if either } w>s, x<t \text{ or } x>t, w<s.$$

Apply Lemma 4, we have

$$\Gamma'(\mathbb{Z}_{p_1^{m_1}p_2^{m_2}}) = \delta_{p_1^{m_1}p_2^{m_2}}[\Gamma'(A_{p_1}),\Gamma'(A_{p_1^2}),\ldots,\Gamma'(A_{p_1^{m_1}}),\Gamma'(A_{p_2}),\Gamma'(A_{p_2^2}),\ldots,\Gamma'(A_{p_2^{m_2}}),$$
$$\Gamma'(A_{p_1p_2}),\Gamma'(A_{p_1^2p_2}),\ldots,\Gamma'(A_{p_1^{m_1}p_2}),\ldots,\Gamma'(A_{p_1p_2^{m_2}}),\Gamma'(A_{p_1^2p_2^{m_2}}),$$
$$\ldots,\Gamma'(A_{p_1^{m_1-1}p_2^{m_2}})].$$

In view of Lemmas 1 and 3 we obtain

$$\begin{aligned}\Gamma'(A_{p_1^y}) =& \overline{K}_{\phi(p_1^{m_1-y}p_2^{m_2})} \text{ for } 1 \leq y \leq m_1,\\ \Gamma'(A_{p_2^z}) =& \overline{K}_{\phi(p_1^{m_1}p_2^{m_2-z})} \text{ for } 1 \leq z \leq m_2,\\ \Gamma'(A_{p_1^y p_2^z}) =& \overline{K}_{\phi(p_1^{m_1-y}p_2^{m_2-z})}.\end{aligned}$$

Moreover, the values of N_{x_s} are as follows

$$\begin{aligned}
N_{p_1} &= \sum_{y=1}^{m_2} \phi(p_1^{m_1} p_2^{m_2-y}),\\
N_{p_1^2} &= \sum_{y=1}^{m_2} \phi(p_1^{m_1} p_2^{m_2-y}) + \sum_{y=1}^{m_2} \phi(p_1^{m_1-1} p_2^{m_2-y}),\\
&\vdots\\
N_{p_1^{m_1}} &= \sum_{y=1}^{m_2} \phi(p_1^{m_1} p_2^{m_2-y}) + \sum_{y=1}^{m_2} \phi(p_1^{m_1-1} p_2^{m_2-y}) + \cdots + \sum_{y=1}^{m_2} \phi(p_1 p_2^{m_2-y}),\\
N_{p_2} &= \sum_{y=1}^{m_1} \phi(p_1^{m_1-y} p_2^{m_2}),\\
&\vdots\\
N_{p_2^{m_2}} &= \sum_{y=1}^{m_1} \phi(p_1^{m_1-y} p_2^{m_2}) + \sum_{y=1}^{m_1} \phi(p_1^{m_1-y} p_2^{m_2-1}) + \cdots + \sum_{y=1}^{m_1} \phi(p_1^{m_1-y} p_2),\\
N_{p_1 p_2} &= \sum_{y=2}^{m_1} \phi(p_1^{m_1-y} p_2^{m_2}) + \sum_{y=2}^{m_2} \phi(p_1^{m_1} p_2^{m_2-y}),\\
&\vdots\\
N_{p_1^{m_1} p_2} &= \sum_{y=2}^{m_2} \phi(p_1^{m_1} p_2^{m_2-y}) + \sum_{y=1}^{m_1} \phi(p_1^{m_1-y} p_2^{m_2-2}) + \sum_{y=1}^{m_1} \phi(p_1^{m_1-y} p_2^{m_2-3}) +\\
&\quad \cdots + \sum_{y=1}^{m_1-1} \phi(p_1^{m_1-y}),\\
N_{p_1 p_2^2} &= \sum_{y=2}^{m_1} \phi(p_1^{m_1-y} p_2^{m_2}) + \sum_{y=3}^{m_2} \phi(p_1^{m_1} p_2^{m_2-y}) + \sum_{y=2}^{m_1} \phi(p_1^{m_1-y} p_2^{m_2-1}),\\
&\vdots\\
N_{p_1 p_2^{m_2}} &= \sum_{y=2}^{m_1} \phi(p_1^{m_1-y} p_2^{m_2}) + \sum_{y=1}^{m_2} \phi(p_1^{m_1-2} p_2^{m_2-y}) + \sum_{y=1}^{m_2} \phi(p_1^{m_1-3} p_2^{m_2-y}) +\\
&\quad \cdots + \sum_{y=1}^{m_2-1} \phi(p_2^{m_2-y}),\\
&\vdots\\
N_{p_1^{m_1-1} p_2^{m_2}} &= \sum_{y=1}^{m_2-1} \phi(p_2^{m_2-y}) + \phi(p_2^{m_2}).
\end{aligned}$$

Thus, by Theorem 2, the signless Laplacian spectrum of $\Gamma'(\mathbb{Z}_{p_1^{m_1} p_2^{m_2}})$ is given by

$$\begin{aligned}
\sigma_{SL}(\Gamma'(\mathbb{Z}_{p_1^{m_1} p_2^{m_2}})) = & \Big(N_{p_1} + \big(\sigma_{SL}(\Gamma'(A_{p_1})) \setminus \{2k\}\big)\Big) \bigcup \Big(N_{p_1^2} + \big(\sigma_{SL}(\Gamma'(A_{p_1^2})) \setminus \{2k\}\big)\Big) \\
& \bigcup \cdots \bigcup \Big(N_{p_1^{m_1}} + \big(\sigma_{SL}(\Gamma'(A_{p_1^{m_1}})) \setminus \{2k\}\big)\Big) \\
& \bigcup \Big(N_{p_2} + \big(\sigma_{SL}(\Gamma'(A_{p_2})) \setminus \{2k\}\big)\Big) \bigcup \Big(N_{p_2^2} + \big(\sigma_{SL}(\Gamma'(A_{p_2^2})) \setminus \{2k\}\big)\Big) \\
& \bigcup \ \cdots \bigcup \Big(N_{p_2^{m_2}} + \big(\sigma_{SL}(\Gamma'(A_{p_2^{m_2}})) \setminus \{2k\}\big)\Big) \\
& \bigcup \Big(N_{p_1 p_2} + \big(\sigma_{SL}(\Gamma'(A_{p_1 p_2})) \setminus \{2k\}\big)\Big) \\
& \bigcup \ \cdots \bigcup \Big(N_{p_1^{m_1} p_2} + \big(\sigma_{SL}(\Gamma'(A_{p_1^{m_1} p_2})) \setminus \{2k\}\big)\Big) \\
& \bigcup \cdots \bigcup \Big(N_{p_1 p_2^{m_2}} + \big(\sigma_{SL}(\Gamma'(A_{p_1 p_2^{m_2}})) \setminus \{2k\}\big)\Big) \\
& \bigcup \ \cdots \bigcup \Big(N_{p_1^{m_1-1} p_2^{m_2}} + \big(\sigma_{SL}(\Gamma'(A_{p_1^{m_1-1} p_2^{m_2}})) \setminus \{2k\}\big)\Big) \bigcup \sigma(Z(\delta_n)).
\end{aligned}$$

The remaining $m_1 m_2 + m_1 + m_2 - 1$ eigenvalues are the roots of the characteristic polynomial of the matrix $Z(\delta_n)$ given in (2). □

Example 2. *The signless Laplacian spectrum of the cozero-divisor graph of $\mathbb{Z}_{30}$ shown in Figure 1 is*

$$\left\{ \begin{array}{ccccccccccc} 7 & 12 & 16 & 5 & 9 & 21 & 2.407 & 3.482 & 7.578 & 11.475 & 17.058 \\ 7 & 3 & 1 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array} \right\}.$$

Proof. Let $n = 30$. The proper divisors of 30 are $2, 3, 5, 6, 10, 15$, and δ_{30}: $3 \sim 5 \sim 2 \sim 3 \sim 10 \sim 6 \sim 15 \sim 2$, $5 \sim 6$, $10 \sim 15$. Now, increasing the divisor sequence to order the vertices and using Lemma 4,

$$\Gamma'(\mathbb{Z}_{30}) = \delta_{30}[\Gamma'(A_2), \Gamma'(A_3), \Gamma'(A_5), \Gamma'(A_6), \Gamma'(A_{10}), \Gamma'(A_{15})],$$

where the simple graph δ_{30} is shown in Figure 2. Using Lemma 3, we have

$$\Gamma'(\mathbb{Z}_{30}) = G_6[\overline{K}_8, \overline{K}_4, \overline{K}_2, \overline{K}_4, \overline{K}_2, \overline{K}_1].$$

The values of N_{x_s} are given by

$$N_1 = 7, N_2 = 12, N_3 = 16, N_4 = 5, N_5 = 9, N_6 = 14.$$

Thus, by Theorem 2 the signless Laplacian spectrum of $\Gamma'(\mathbb{Z}_{30})$ consists of eigenvalues

$$\left\{ \begin{array}{cccccc} 7 & 12 & 16 & 5 & 9 & 14 \\ 7 & 3 & 1 & 3 & 1 & 0 \end{array} \right\}$$

together with the eigenvalues of the matrix $Z(\delta_n)$ given below

$$Z(\delta_n) = \begin{bmatrix} 7 & 2 & 4 & 0 & 0 & 8 \\ 1 & 12 & 4 & 0 & 4 & 0 \\ 1 & 2 & 16 & 2 & 0 & 0 \\ 0 & 0 & 4 & 5 & 4 & 8 \\ 0 & 2 & 0 & 2 & 9 & 8 \\ 1 & 0 & 0 & 2 & 4 & 14 \end{bmatrix} \tag{6}$$

The characteristic polynomial of matrix $Z(\delta_n)$ is given by

$$|A - \lambda I| = \lambda^6 - 63\lambda^5 + 1515\lambda^4 - 17505\lambda^3 + 100640\lambda^2 - 268380\lambda + 261072$$

or

$$|A - \lambda I| = (\lambda - 21)(\lambda^5 - 42\lambda^4 + 633\lambda^3 - 4212\lambda^2 + 12188\lambda - 12432)$$

and approximated eigenvalues of matrix $Z(\delta_n)$ given in (6) are

$$\{21, 2.407, 3.482, 7.578, 11.475, 17.058\}.$$

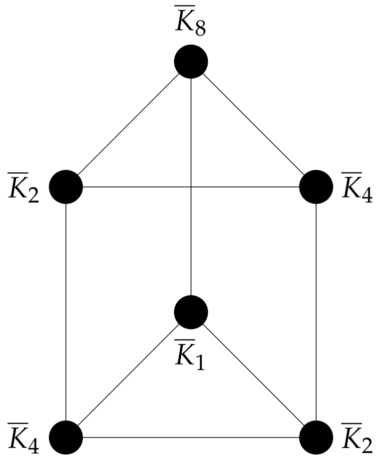

Figure 1. Cozero-divisor graph $\Gamma'(\mathbb{Z}_{30})$.

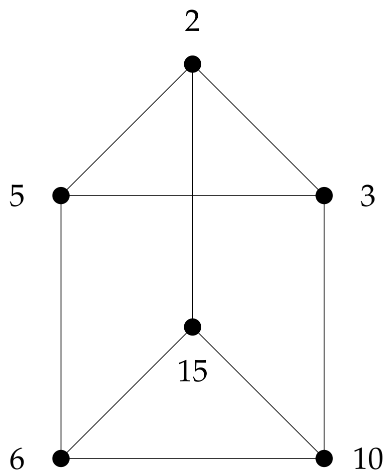

Figure 2. Proper divisor graph δ_{30}.

The nodes of the graph of the cozero-divisor graph $\Gamma'(\mathbb{Z}_{30})$ shown in Figure 1 are the complement of the complete graph, i.e., a node in Figure 1 itself an empty graph, whereas the nodes of the graph of the proper divisor graph δ_{30} shown in Figure 2 are simply the proper divisors of 30. □

4. Conclusions and Further Work

The characteristic polynomial in λ of degree n, where n is the number of atoms, finds its uses in quantum chemistry, the topological theory of aromaticity, counts of random walks, structure-resonance theory, and eigenvector–eigenvalue problems (for more details see [10] and references therein). Our result gives the signless Laplacain spectrum of the cozero-divisor graph of integer modulo n for different values of n by using the generalize join graph of induced subgraphs. One may generalized these results and find the signless Laplacian spectrum when $n = p^M q^N r^P$ for positive integers M, N, and P and $n = pqr$, where p, q, r are distinct primes.

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