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Multiple Positive Solutions for a System of Fractional Order BVP with p-Laplacian Operators and Parameters

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Abstract: In this paper, we investigate the existence of positive solutions to a system of fractional differential equations that include the (r_1, r_2, r_3)-Laplacian operator, three-point boundary conditions, and various fractional derivatives. We use a combination of techniques, including cone expansion and compression of the functional type, and the Leggett–Williams fixed point theorem, to prove the existence of positive solutions. Finally, we provide two examples to illustrate our main results.

Keywords: fractional derivative; positive solutions; boundary value problems; p-Laplacian; parameters

MSC: 34B09; 34B15; 34B18; 34B27

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1. Introduction

Nonlinear fractional systems are a rapidly growing field of research, motivated by the desire to use their unique properties to solve real-world problems. These systems are ubiquitous in nature, from the viscoelastic behavior of polymers to the anomalous diffusion of particles in fluids. Fractional calculus provides a powerful framework for modeling and understanding these complex systems [1–3]. Recent advances in fractional calculus (FC) have led to the development of new theories and models with far-reaching implications in diverse scientific disciplines. In biomathematics, fractional calculus is used to model the dynamics of populations, the spread of diseases, and the transport of nutrients through biological tissues [4]. In viscoelasticity, fractional calculus is used to model the behavior of materials such as polymers and rubber, which exhibit both elastic and viscous properties [5]. In non-Newtonian fluid mechanics, fractional calculus is used to model the behavior of fluids that do not obey Newton's law of viscosity, such as blood and polymers [6]. In the characterization of anomalous diffusion, fractional calculus is used to model the transport of particles in fluids that do not obey the classical diffusion equation, such as proteins in cells and pollutants in the environment [7].

The development of new fractional calculus-based technologies has the potential to revolutionize many aspects of our lives. For example, fractional calculus could be used to design new materials with improved properties, develop more efficient control systems, and create new diagnostic tools and treatments for diseases [8]. Here are some specific examples of how fractional calculus is being used to solve real-world problems. Bioengineering: Fractional calculus is used to model the dynamics of complex biological systems such as the immune system and the brain. This knowledge can be used to develop new diagnostic tools and treatments for diseases [9,10]. Materials science: Fractional calculus is used to design new materials with improved properties, such as polymers that are stronger and more durable, and rubbers that are more elastic and less prone to fatigue [11,12]. Control engineering: Fractional calculus is used to develop more efficient control systems for a variety of applications, including robotics, aircraft, and power plants [13,14]. Fractional calculus is a powerful tool with the potential to revolutionize many different fields. As research in this area continues to grow, we can expect to see even more innovative and groundbreaking applications of fractional calculus in the years to come.

The study of positive solutions of fractional order boundary value problems (FBVPs) is a rapidly growing field of research, and new results are being developed all the time [15–27]. This research has important applications in a variety of fields, including biology, physics, engineering, and finance. For example, positive solutions of FBVPs can be used to model the dynamics of populations of cells or bacteria [9], the transport of heat or diffusion of particles in a medium [11], the design of control systems [13], and the pricing of options [14]. Recently, Samadi et al. [28] studied the following system of nonlinear differential equations consisting of the Caputo fractional order derivatives of the form

$$\begin{cases} {}^{C}D^{\rho_1}\big(\phi_{p_1}\big({}^{C}D^{\mu_1}\sigma_1(\omega)\big)\big) = \kappa_1(\omega,\sigma_1(\omega),\sigma_2(\omega)), \quad \omega \in [c_1,d_1], \\ {}^{C}D^{\rho_2}\big(\phi_{p_2}\big({}^{C}D^{\mu_2}\sigma_2(\omega)\big)\big) = \kappa_2(\omega,\sigma_1(\omega),\sigma_2(\omega)), \quad \omega \in [c_1,d_1], \\ \sigma_1(c_1) = {}^{C}D^{\mu_1}\sigma_1(c_1) = 0,\ \sigma_1(d_1) = \sum_{i=1}^{m}\lambda_i\sigma_2(\eta_i), \\ \sigma_2(c_1) = {}^{C}D^{\mu_2}\sigma_2(c_1) = 0,\ \sigma_2(d_1) = \sum_{i=1}^{m}\lambda_i\sigma_1(\eta_i), \end{cases} \tag{1}$$

and established the existence and uniqueness of solutions to (1) by aplying fixed point theory. Motivated by [28], in this study, our primary objective is to investigate the following system of fractional differential equations that incorporate (r_1, r_2, r_3)-Laplacian operators. We aim to provide a comprehensive analysis of these equations, considering their potential implications and applications:

$$\begin{cases} -\mathcal{D}^{p_1}_{h^+}\big(\phi_{r_1}\big(\mathcal{D}^{q_1}_{h^+}\varpi(\xi)\big)\big) = f_1(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi \in (h,k), \\ -\mathcal{D}^{p_2}_{h^+}\big(\phi_{r_2}\big(\mathcal{D}^{q_2}_{h^+}\vartheta(\xi)\big)\big) = f_2(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi \in (h,k), \\ -\mathcal{D}^{p_3}_{h^+}\big(\phi_{r_3}\big(\mathcal{D}^{q_3}_{h^+}\omega(\xi)\big)\big) = f_3(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi \in (h,k), \end{cases} \tag{2}$$

where h and k are real numbers with $h < k$. The operators $\mathcal{D}h^{+^{qi}}, \mathcal{D}h^{+^{pi}}, \mathcal{D}h^{+^{\alpha i}}$ correspond to standard Riemann–Liouville fractional order derivatives. Additionally, $q_i \in (1,2]$, p_i, $\alpha_i \in (0,1]$, and $\phi_{r_i}(\zeta) = |\zeta|^{r_i-1}\zeta$, with $r_i > 1$, and $\phi_{r_i}^{-1} = \phi_{\varphi_i}$, where $\frac{1}{\varphi_i} + \frac{1}{r_i} = 1$ for $i = 1,2,3$.

The boundary conditions for this system are given as

$$\begin{cases} \varpi(h) = 0,\ \ \phi_{r_1}\big(\mathcal{D}^{q_1}_{h^+}\varpi(h)\big) = 0,\ \ \mu_1\,\mathcal{D}^{\alpha_1}_{h^+}\varpi(k) = \psi_1 + \lambda_1\,\mathcal{D}^{\alpha_1}_{h^+}\varpi(\eta_1), \\ \vartheta(h) = 0,\ \ \phi_{r_2}\big(\mathcal{D}^{q_2}_{h^+}\vartheta(h)\big) = 0,\ \ \mu_2\,\mathcal{D}^{\alpha_2}_{h^+}\vartheta(k) = \psi_2 + \lambda_2\,\mathcal{D}^{\alpha_2}_{h^+}\vartheta(\eta_2), \\ \omega(h) = 0,\ \ \phi_{r_3}\big(\mathcal{D}^{q_3}_{h^+}\omega(h)\big) = 0,\ \ \mu_3\,\mathcal{D}^{\alpha_3}_{h^+}\omega(k) = \psi_3 + \lambda_3\,\mathcal{D}^{\alpha_3}_{h^+}\omega(\eta_3). \end{cases} \tag{3}$$

Here, μ_i, λ_i are positive constants, and η_i are real numbers within the interval (h,k). It is essential that the conditions $\mu_i(k-h)^{q_i-\alpha_i-1} > \lambda_i(\eta_i-h)^{q_i-\alpha_i-1}$ hold for all $i = 1,2,3$.

To ensure the existence of positive solutions to Systems (2) and (3), we make the following assumptions:

(B1) The functions f_1, f_2, and f_3 are continuous on the specified domains.
(B2) The parameters α_i, q_i, μ_i, λ_i, and η_i satisfy certain inequalities, ensuring the conditions required for the existence of solutions.
(B3) We introduce positive constants Φ_1, Φ_2, Φ_3, Θ_1, Θ_2, and Θ_3 with the constraint that $\frac{1}{\Phi_1} + \frac{1}{\Phi_2} + \frac{1}{\Phi_3} + \frac{1}{\Theta_1} + \frac{1}{\Theta_2} + \frac{1}{\Theta_3} \le 1$.

The study of fractional differential equations is a rapidly expanding field with numerous applications in various domains. Our paper provides essential conditions for functions f_1, f_2, and f_3, as well as intervals for parameters (ψ_1, ψ_2, ψ_3), guaranteeing the existence of at least one and three positive solutions for specified boundary value Problems (2) and (3). A positive solution is defined as a triplet of functions $(\varpi(\xi), \vartheta(\xi), \omega(\xi))$ in space $(\mathcal{C}[h,k],[0,\infty))^3$ that satisfies (2) and (3) with non-negative values for all $\xi \in [h,k]$, and where $(\varpi(\xi),\vartheta(\xi),\omega(\xi))$ is not equal to $(0,0,0)$. Tripled fractional order systems in boundary value problems offer unique advantages in various scientific and engineering applications. Here are some notable applications:

- Tripled fractional order systems can be used to design more sustainable and efficient energy systems. For example, these models could be used to optimize the operation of renewable energy sources such as solar and wind power, and to develop more efficient storage and transmission technologies [29].
- These systems can also be used to develop new and improved diagnostic and therapeutic tools for a variety of diseases. For example, fractional order models of physiological systems could be used to design more effective drug delivery systems and to develop new treatments for chronic diseases such as cancer and diabetes [30].
- In addition, tripled fractional order systems can be used to create new and innovative materials with unique properties. For example, these models could be used to design materials with improved thermal conductivity, electrical conductivity, and mechanical strength [31].

Overall, we believe that tripled fractional order systems have the potential to make a significant impact on a wide range of industries and scientific disciplines. For further insights into the applications of fractional calculus in various fields and related literature on positive solutions with different boundary conditions, we recommend reading the referenced books [32–34] and exploring the cited papers [35–53].

This paper is organized into four sections. Section 2 introduces the foundational concepts and key lemmas essential to our main results. Section 3 employs various methodological approaches, including cone expansion and compression of functional type, and the Leggett–Williams fixed point theorem, to present our main results. Section 4 provides two illustrative examples that demonstrate the application and relevance of our main results. Finally, Section 5 provides concluding remarks and highlights potential directions for future research.

2. Preliminaries

In this section, we provide some definitions and important lemmas related to fractional calculus theory, which are readily available in the current literature [1,2].

Definition 1. *For a function f given on the interval $[h,k]$, the α^{th} Riemann–Liouville fractional order derivative of f is defined by*

$$(D^{\alpha}_{h^+}f)(\xi) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d\xi}\right)^{\alpha}\int_h^{\xi}(\xi-s)^{n-\alpha-1}f(s)ds;$$

here, $n=[\alpha]+1$ and $[\alpha]$ denote the integral part of α.

Definition 2. *The functional (arbitrary) order integral of the function $f \in L^1\big([h,k],R_+\big)$ of order $\alpha \in R_+$ is defined by*

$$(I^{\alpha}_{h^+}f)(\xi) = \frac{1}{\Gamma(\alpha)}\int_h^{\xi}(\xi-s)^{\alpha-1}f(s)ds,$$

where Γ is the Gamma function.

Lemma 1. *Assume that $\mathcal{D}^{\sigma}_{h^+} \in L^1[h,k]$ with a fractional derivative of order $\sigma>0$; then,*

$$\mathcal{I}^{\sigma}_{h^+}\mathcal{D}^{\sigma}_{h^+}u(t) = u(\xi)+c_1(\xi-h)^{\sigma-1}+c_2(\xi-h)^{\sigma-2}+\cdots\cdots+c_n(\xi-h)^{\sigma-n}$$

for some $c_i \in R$, $i=1,2,3,\cdots,n$ where n is the smallest integer greater than or equal to σ.

Definition 3. *Let φ be a cone in the real Banach space $\mathcal{S}$. Map $\sigma : \varphi \to [0,\infty)$ is said to be nonnegative continuous concave functional on φ if σ is continuous and $\sigma(\lambda u + (1-\lambda)v) \geq \lambda\sigma(u) + (1-\lambda)\sigma(v)$ for all $u, v \in \varphi$ and $\lambda \in [0,1]$.*

Definition 4. *Let φ be a cone in the real Banach space $\mathcal{S}$. Map $\rho : \varphi \to [0,\infty)$ is said to be nonnegative continuous convex functional on φ if ρ is continuous and $\rho(\lambda u + (1-\lambda)v) \leq \lambda\rho(u) + (1-\lambda)\rho(v)$ for all $u, v \in \varphi$ and $\lambda \in [0,1]$.*

Rule $\mathbf{S_1}$: Let κ be a cone in a Banach space $\mathcal{D}$ and x be a bounded open subset of $\mathcal{D}$ and $0 \in x$. Then, a continuous functional $\sigma : \kappa \to [0,\infty)$ is said to satisfy Rule S_1 if one of the following conditions hold:

(i) σ is convex, $\sigma(0) = 0$, $\sigma(t) \neq 0$ if $t \neq 0$ and $\inf_{t\in\kappa\cap\partial x} \sigma(t) > 0$,
(ii) σ is sublinear, $\sigma(0) = 0$, $\sigma(t) \neq 0$ if $t \neq 0$ and $\inf_{t\in\kappa\cap\partial x} \sigma(t) > 0$,
(iii) σ is concave and unbounded.

Rule $\mathbf{S_2}$: Let κ be a cone in a Banach space $\mathcal{D}$ and x be a bounded open subset of $\mathcal{D}$ and $0 \in x$. Then, a continuous functional $\rho : \kappa \to [0,\infty)$ is said to satisfy Rule $\mathcal{S}_2$ if one of the following conditions hold:

(i) ρ is convex, $\rho(0) = 0$, $\rho(t) \neq 0$ if $t \neq 0$,
(ii) ρ is sublinear, $\rho(0) = 0$, $\rho(t) \neq 0$ if $t \neq 0$,
(iii) $\rho(t+s) \geq \rho(t) + \rho(s)$ for all $t, s \in \kappa$, $\rho(0) = 0$, $\rho(t) \neq 0$ if $t \neq 0$.

Theorem 1 ([35]). *Consider two bounded open subsets, Ω_1 and Ω_2, within a Banach space denoted as $\mathcal{D}$. It is assumed that 0 belongs to Ω_1, and Ω_1 is a subset of Ω_2. Furthermore, let κ represent a cone within the same Banach space $\mathcal{D}$. Introduce operator $\mathcal{L}$, which maps from $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$ to κ and is characterized as completely continuous. Alongside this, two non-negative continuous functionals, σ and ρ, are defined on κ. The main result is contingent upon one of the following two conditions being satisfied:*

(a) σ adheres to Rule $\mathcal{S}1$ with $\sigma(\mathcal{L}t) \geq \sigma(t)$ for all t belonging to $\kappa \cap \partial\Omega 1$, and ρ adheres to Rule $\mathcal{S}_2$ with $\rho(\mathcal{L}t) \leq \rho(t)$ for all t in $\kappa \cap \partial\Omega 2$.

(b) Conversely, ρ follows Rule $\mathcal{S}_2$ with $\rho(\mathcal{L}t) \leq \rho(t)$ for all t in $\kappa \cap \partial\Omega 1$, and σ conforms to Rule $\mathcal{S}_1$ with $\sigma(\mathcal{L}t) \geq \sigma(t)$ for all t in $\kappa \cap \partial\Omega 2$.

In either case, the conclusion is that the operator $\mathcal{L}$ possesses at least one fixed point within the set $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 2 (Leggett–Williams [36]). *Let p, q, r and s be positive real numbers, let κ be a cone in a real Banach space $\mathcal{D}$, $\kappa_s = \{t \in \kappa : \|t\| < s\}$, ψ be a nonnegative continuous concave functional on κ such that $\psi(t) \leq \|t\|$, $\forall t \in \overline{\kappa_s}$ and $\kappa(\psi, q, r) = \{t \in \kappa; q \leq \psi(t), \|t\| \leq r\}$. Suppose that $\mathcal{L} : \overline{\kappa_s} \to \overline{\kappa_s}$ is a completely continuous operator and there exist constants $0 < p < q < r \leq s$ such that*

(i) $\{t \in \kappa(\psi, q, r) \mid \psi(t) > q\} \neq \emptyset$ and $\psi(\mathcal{L}t) > q$ for $t \in \kappa(\psi, q, r)$,
(ii) $\|\mathcal{L}t\| < p$ for $\|t\| \leq p$,
(iii) $\psi(\mathcal{L}t) > q$ for $t \in \kappa(\psi, q, s)$ with $\|\mathcal{L}t\| > r$.

Then, $\mathcal{L}$ has at least three fixed points t_1, t_2 and t_3 in $\overline{\kappa_s}$ satisfying $\|t_1\| < p, q < \psi(t_2), p < \|t_3\|$ and $\psi(t_3) < q$.

In what follows, we calculate the Green's function associate with (2) and (3). We consider the homogeneous boundary value problem

$$-\mathcal{D}^{q_1}_{h^+}\varpi(\xi) = 0, \;\; \xi \in (h,k), \tag{4}$$

$$\varpi(h) = 0; \;\; \mu_1\, \mathcal{D}^{\alpha_1}_{h^+}\varpi(k) = \psi_1 + \lambda_1\, \mathcal{D}^{\alpha_1}_{h^+}\varpi(\eta_1). \tag{5}$$

Lemma 2. *Let $\Delta_1 \neq 0$. If $x(\xi) \in \mathcal{C}[h,k]$ and $1 < q_1 \leq 2$; then, the boundary value problem,*

$$\mathcal{D}_{h^+}^{q_1}\varpi(\xi) + x(\xi) = 0, \;\; h < \xi < k, \tag{6}$$

satisfying Boundary condition (5), has a unique solution

$$\varpi(\xi) = \int_h^k \mathcal{H}_1(\xi,\zeta)x(\zeta)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1}, \; \xi \in [h,k],$$

where $\mathcal{H}_1(\xi,\zeta)$ is the Green's function for the BVP (6) and (5) and is given by

$$\mathcal{H}_1(\xi,\zeta) = h_1(\xi,\zeta) + \frac{\lambda_1(\xi-h)^{q_1-1}}{\mathcal{N}_1}h_2(\eta_1,\zeta).$$

Here, $\Delta_1 = \Gamma(q_1)\mathcal{N}_1 \neq 0$; $\mathcal{N}_1 = \mu_1(k-h)^{q_1-\alpha_1-1} - \lambda_1(\eta_1-h)^{q_1-\alpha_1-1}$ and

$$\begin{aligned} h_1(\xi,\zeta) &= \frac{1}{\Gamma(q_1)}\begin{cases} \frac{(\xi-h)^{q_1-1}(k-\zeta)^{q_1-\alpha_1-1}}{(k-h)^{q_1-\alpha_1-1}} - (\xi-\zeta)^{q_1-1}, & h \leq \zeta \leq \xi \leq k, \\ \frac{(\xi-h)^{q_1-1}(k-\zeta)^{q_1-\alpha_1-1}}{(k-h)^{q_1-\alpha_1-1}}, & h \leq \xi \leq \zeta \leq k, \end{cases} \\ h_2(\xi,\zeta) &= \frac{1}{\Gamma(q_1)}\begin{cases} \frac{(\xi-h)^{q_1-\alpha_1-1}(k-\zeta)^{q_1-\alpha_1-1}}{(k-h)^{q_1-\alpha_1-1}} - (\xi-\zeta)^{q_1-\alpha_1-1}, & h \leq \zeta \leq \xi \leq k, \\ \frac{(\xi-h)^{q_1-\alpha_1-1}(k-\zeta)^{q_1-\alpha_1-1}}{(k-h)^{q_1-\alpha_1-1}}, & h \leq \xi \leq \zeta \leq k. \end{cases} \end{aligned} \tag{7}$$

Proof. Assume that $\varpi \in C^{[q_1]+1}[h,k]$ is a solution of fractional order Boundary value problem (6) and (5) and is uniquely expressed by

$$\varpi(\xi) = -\int_h^\xi \frac{(\xi-\zeta)^{q_1-1}}{\Gamma(q_1)}x(\zeta)d\zeta + c_1(\xi-h)^{q_1-1} + c_2(\xi-h)^{q_1-2}.$$

In view of Condition (5), we can obtain $c_2 = 0$ and

$$c_1 = \frac{1}{\Delta_1}\left[\mu_1\int_h^k (k-h)^{q_1-\alpha_1-1}x(\zeta)d\zeta - \lambda_1\int_h^{\eta_1}(\eta_1-\zeta)^{q_1-\alpha_1-1}x(\zeta)d\zeta\right] + \frac{\psi_1\Gamma(q_1-\alpha_1)}{\Delta_1}.$$

Hence, we have

$$\begin{aligned} \varpi(\xi) &= \frac{-1}{\Gamma(\alpha_1)}\int_h^\xi(\xi-\zeta)^{q_1-1}x(\zeta)d\zeta + \frac{(\xi-h)^{q_1-1}}{\Delta_1}\int_h^k(k-\zeta)^{q_1-\mu_1-1}x(\zeta)d\zeta \\ &\quad - \frac{\lambda_1(\xi-h)^{q_1-1}}{\Delta_1}\int_h^{\eta_1}(\eta_1-\zeta)^{q_1-\alpha_1-1}x(\zeta)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1} \\ &= \frac{-1}{\Gamma(q_1)}\int_h^\xi(\xi-\zeta)^{q_1-1}x(\zeta)d\zeta + \frac{(\xi-h)^{q_1-1}}{\Gamma(q_1)}\int_h^k\frac{(k-\zeta)^{q_1-\alpha_1-1}}{(k-h)^{q_1-\alpha_1-1}}x(\zeta)d\zeta \\ &\quad + \frac{\lambda_1(\xi-h)^{q_1-1}}{N}\int_h^k\frac{(\eta_1-h)^{q_1-\alpha_1-1}(k-\zeta)^{q-1-\alpha_1-1}}{\Gamma(q_1)(k-h)^{q_1-\alpha_1-1}}x(\zeta)d\zeta \\ &\quad - \frac{\lambda_1(\xi-h)^{q_1-1}}{N}\int_h^{\eta_1}\frac{(\eta_1-\zeta)^{q_1-\alpha_1-1}}{\Gamma(q_1)}x(\zeta)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1} \\ &= \int_h^k\mathcal{H}_1(\xi,\zeta)x(\zeta)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1}. \end{aligned}$$

□

Lemma 3. *Let $1 < q_1 \leq 2, \; 0 < p_1 \leq 1$. Then, the FBVP*

$$\begin{cases} \mathcal{D}_{h^+}^{p_1}\left(\phi_{r_1}(\mathcal{D}_{h^+}^{q_1}\varpi(\xi))\right) + f_1(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)) = 0, \; h < \xi < k, \\ \varpi(h) = 0, \;\; \mathcal{D}_{h^+}^{q_1}\varpi(h) = 0, \;\; \mu_1\,\mathcal{D}_{h^+}^{\alpha_1}\varpi(k) = \psi_1 + \lambda_1\,\mathcal{D}_{h^+}^{\alpha_1}\varpi(\eta_1) \end{cases} \tag{8}$$

has a unique solution

$$\varpi(\xi) = \int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)} f_1\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1},\ \xi \in [h,k].$$

Proof. It follows from Lemma 1 and $0 < p_1 \le 1$ that

$$\phi_{r_1}\big(\mathcal{D}_{h^+}^{q_1}\varpi(\xi)\big) = -\int_h^\xi \frac{(\xi-\tau)^{p_1-1}}{\Gamma(p_1)} f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau + c_1(\xi-h)^{p_1-1}.$$

By $\mathcal{D}_{h^+}^{q_1}\varpi(h) = 0$, we have $c_1 = 0$. Therefore,

$$\mathcal{D}_{h^+}^{q_1}\varpi(\xi) + \phi_{\varphi_1}\Big(\int_h^t \frac{(\xi-\tau)^{p_1-1}}{\Gamma(p_1)} f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big) = 0.$$

Thus, the BVP (8) is equal to the following problem:

$$\mathcal{D}_{h^+}^{q_1}\varpi(\xi) + \phi_{\varphi_1}\Big(\int_h^\xi \frac{(\xi-\tau)^{p_1-1}}{\Gamma(p_1)} f_1(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big) = 0;\ \ h < \xi < k;$$

$$\varpi(h) = 0;\ \ \mu_1\, \mathcal{D}_{h^+}^{\alpha_1}\varpi(k) = \psi_1 + \lambda_1\, \mathcal{D}_{h^+}^{\alpha_1}\varpi(\eta_1).$$

By Lemma 2, Boundary value problem (8) has a unique solution:

$$\varpi(\xi) = \int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)} f_1\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1},\ \xi \in [h,k].$$

□

Lemma 4 ([43]). *Suppose that condition* $(B2)$ *holds; then, Green's function* $\mathcal{H}_1$ *has the following properties:*
(i) $\mathcal{H}_1(\xi,\zeta) \ge 0$ *for all* $(\xi,\zeta) \in (h,k)\times(h,k)$,
(ii) $\mathcal{H}_1(\xi,\zeta) \le \mathcal{H}_1(k,\zeta)$, *for all* $(\xi,\zeta) \in [h,k]\times[h,k]$,
(iii) $\mathcal{H}_1(\xi,\zeta) \ge \left(\frac{1}{4}\right)^{q_1-1}\mathcal{H}_1(k,\zeta)$, *for all* $(\xi,\zeta) \in \mathcal{I}\times(h,k)$, *where* $\mathcal{I} = \left[\frac{3h+k}{4}, \frac{h+3k}{4}\right]$.

Remark 1. *In a similar manner , the results of Green's function* $\mathcal{H}_2(\xi,\zeta)$ *and* $\mathcal{H}_3(\xi,\zeta)$ *for the homogeneous BVP corresponding to the fractional differential equation are obtained. Consider the following condition:*

$$\mathcal{H}_i(\xi,\zeta) \ge \aleph\mathcal{H}_i(k,\zeta) \ \text{ for all } (\xi,\zeta) \in \mathcal{I}\times(h,k); i = 1,2,3,$$

where $\mathcal{I} = \left[\frac{3h+k}{4}, \frac{h+3k}{4}\right]$ *and* $\aleph = \min\left\{\left(\frac{1}{4}\right)^{q_1-1}, \left(\frac{1}{4}\right)^{q_2-1}, \left(\frac{1}{4}\right)^{q_3-1}\right\}$.

We consider the Banach space $\mathcal{X} = \mathcal{C}[h,k]$ with the supremum norm $\|\cdot\|$ and the Banach space $\mathcal{Y} = \mathcal{X}\times\mathcal{X}\times\mathcal{X}$ with the norm $\|(\varpi,\vartheta,\omega)\| = \|\varpi\| + \|\vartheta\| + \|\omega\|$. We define the cone

$$\mathcal{P} = \big\{(\varpi,\vartheta,\omega) \in \mathcal{Y} : \varpi(\xi),\vartheta(\xi),\omega(\xi) \ge 0, \forall\xi \in [h,k], \text{ and } \min_{\xi\in\mathcal{I}}[\varpi(\xi)+\vartheta(\xi)+\omega(\xi)] \ge \aleph\ \|\ (\varpi,\vartheta,\omega)\ \|\ \big\},$$

where $\mathcal{I} = \left[\frac{3h+k}{4}, \frac{h+3k}{4}\right]$ and $\aleph = \min\left\{\left(\frac{1}{4}\right)^{q_1-1}, \left(\frac{1}{4}\right)^{q_2-1}, \left(\frac{1}{4}\right)^{q_3-1}\right\}$.

We consider the coupled system of integral equations

$$\begin{aligned}
\varpi(\xi) &= \int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)} f_1\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1},\ \xi\in[h,k], \\
\vartheta(\xi) &= \int_h^k \mathcal{H}_2(\xi,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)} f_2\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_2\Gamma(q_2-\alpha_2)(\xi-h)^{q_2-1}}{\Delta_2},\ \xi\in[h,k], \\
\omega(\xi) &= \int_h^k \mathcal{H}_3(\xi,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)} f_3\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_3\Gamma(q_3-\alpha_3)(\xi-h)^{q_3-1}}{\Delta_3},\ \xi\in[h,k].
\end{aligned}$$

By Lemma 2, $(\varpi,\vartheta,\omega)\in\mathcal{P}$ is a solution of Boundary value problems (2) and (3) if and only if it is a solution of the system of integral equations.

We define operators $\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3:\mathcal{P}\to\mathcal{X}$ by

$$\begin{aligned}
\mathcal{T}_1(\varpi,\vartheta,\omega)(\xi) &= \int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)} f_1\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1},\ \xi\in[h,k], \\
\mathcal{T}_2(\varpi,\vartheta,\omega)(\xi) &= \int_h^k \mathcal{H}_2(\xi,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)} f_2\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_2\Gamma(q_2-\alpha_2)(\xi-h)^{q_2-1}}{\Delta_2},\ \xi\in[h,k], \\
\mathcal{T}_3(\varpi,\vartheta,\omega)(\xi) &= \int_h^k \mathcal{H}_3(\xi,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)} f_3\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_3\Gamma(q_3-\alpha_3)(\xi-h)^{q_3-1}}{\Delta_3},\ \xi\in[h,k],
\end{aligned}$$

and operator $\mathcal{T}:\mathcal{Y}\to\mathcal{Y}$ as

$$\mathcal{T}(\varpi,\vartheta,\omega) = \big(\mathcal{T}_1(\varpi,\vartheta,\omega),\mathcal{T}_2(\varpi,\vartheta,\omega),\mathcal{T}_3(\varpi,\vartheta,\omega)\big),\ \ (\varpi,\vartheta,\omega)\in\mathcal{Y}.$$

It is clear that the existence of a positive solution to Systems (2) and (3) is equivalent to the existence of a fixed points of operator $\mathcal{T}$.

3. Main Results

In this section, we employ cone expansion and compression of functional type and the Leggett–Williams fixed point theorem to study the existence of positive solutions to Equations (2) and (3).

We denote the following notations for our convenience:

$$\mathcal{D} = \max\left\{\phi_{\varphi_1}\Big(\frac{4^{p_1}\Gamma(p_1+1)}{(k-h)^{p_1}}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_1(k,\zeta)d\zeta,\ \phi_{\varphi_2}\Big(\frac{4^{p_2}\Gamma(p_2+1)}{(k-h)^{p_2}}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_2(k,\zeta)d\zeta,\right.$$
$$\left.\phi_{\varphi_3}\Big(\frac{4^{p_3}\Gamma(p_3+1)}{(k-h)^{p_3}}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}} \mathcal{H}_3(k,\zeta)d\zeta\right\},$$
$$\mathcal{C} = \min\left\{\int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}d\tau\Big)d\zeta,\quad \int_h^k \mathcal{H}_2(k,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)}d\tau\Big)d\zeta,\right.$$
$$\left.\int_h^k \mathcal{H}_3(k,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)}d\tau\Big)d\zeta\right\}.$$

Let us define two continuous functionals α and β on the cone $\mathcal{P}$ by

$$\alpha(\varpi,\vartheta,\omega) = \min_{\zeta\in\mathcal{I}}\big\{|\varpi|+|\vartheta|+|\omega|\big\}\text{and}$$
$$\beta(\varpi,\vartheta,\omega) = \max_{\zeta\in[h,k]}\big\{|\varpi|+|\vartheta|+|\omega|\big\} = \varpi(k)+\vartheta(k)+\omega(k) = \|(\varpi,\vartheta,\omega)\|.$$

It is clear that $\alpha(\varpi,\vartheta,\omega) \le \beta(\varpi,\vartheta,\omega)$, for all $(\varpi,\vartheta,\omega)\in\mathcal{P}$.

Lemma 5. *$\mathcal{T}:\mathcal{P}\to\mathcal{P}$ is completely continuous.*

Proof. We let $(\varpi,\vartheta,\omega)\in\mathcal{P}$. By Lemma 3, we have

$$\|\mathcal{T}_1(\varpi,\vartheta,\omega)\| \le \int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(s-\tau)^{p_1-1}}{\Gamma(p_1)}f_1\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta$$
$$+\frac{\psi_1\Gamma(q_1-\alpha_1)(k-h)^{q_1-1}}{\Delta_1},$$
$$\|\mathcal{T}_2(\varpi,\vartheta,\omega)\| \le \int_h^k \mathcal{H}_2(k,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)}f_2\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))\big)d\tau\Big)d\zeta$$
$$+\frac{\psi_2\Gamma(q_2-\alpha_2)(k-h)^{q_2-1}}{\Delta_2},$$
$$\|\mathcal{T}_3(\varpi,\vartheta,\omega)\| \le \int_h^k \mathcal{H}_3(k,\zeta)\phi_{\varphi_3}\Big(\int_a^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)}f_3\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta$$
$$+\frac{\psi_3\Gamma(q_3-\alpha_3)(k-h)^{q_3-1}}{\Delta_3},$$

and

$$\min_{\xi\in\mathcal{I}}\mathcal{T}_1(\varpi,\vartheta,\omega)(\xi) = \min_{\xi\in\mathcal{I}}\Big[\int_h^k \mathcal{H}_1(\xi,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}f_1\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta$$
$$+\frac{\psi_1\Gamma(q_1-\alpha_1)(\xi-h)^{q_1-1}}{\Delta_1}\Big],$$
$$\ge \Big(\frac{1}{4}\Big)^{q_1-1}\Big[\int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}f_1\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta$$
$$+\frac{\psi_1\Gamma(q_1-\alpha_1)(k-h)^{q_1-1}}{\Delta_1}\Big],$$
$$\ge \aleph\|\mathcal{T}_1(\varpi,\vartheta,\omega)\|.$$

Similarly, $\min_{\xi\in\mathcal{I}} \mathcal{T}_2(\varpi,\vartheta,\omega)(\xi) \geq \aleph\|\mathcal{T}_2(\varpi,\vartheta,\omega)\|$ and $\min_{\xi\in\mathcal{I}} \mathcal{T}_3(\varpi,\vartheta,\omega)(\xi) \geq \aleph\|\mathcal{T}_3(\varpi,\vartheta,\omega)\|$. Therefore,

$$\begin{aligned}
&\min_{\xi\in I}\{\mathcal{T}_1(\varpi,\vartheta,\omega)(\xi)+\mathcal{T}_2(\varpi,\vartheta,\omega)(\xi)+\mathcal{T}_3(\varpi,\vartheta,\omega)(\xi)\}\\
&\quad\geq \aleph\|\mathcal{T}_1(\varpi,\vartheta,\omega)\|+\aleph\|\mathcal{T}_2(\varpi,\vartheta,\omega)\|+\aleph\|\mathcal{T}_3(\varpi,\vartheta,\omega)\|\\
&\quad= \aleph\|(\mathcal{T}_1(\varpi,\vartheta,\omega),\mathcal{T}_2(\varpi,\vartheta,\omega),\mathcal{T}_3(\varpi,\vartheta,\omega))\|\\
&\quad= \aleph\|\mathcal{T}(\varpi,\vartheta,\omega)\|.
\end{aligned}$$

Thus, we obtain $\mathcal{T}(\mathcal{P})\subset\mathcal{P}$. Using standard arguments involving the Arzela–Ascoli theorem, we can easily show that $\mathcal{T}_1,\mathcal{T}_2$ and $\mathcal{T}_3$ are completely continuous operators. Therefore, $\mathcal{T}$ is a completely continuous operator from $\mathcal{P}$ to $\mathcal{P}$. □

Theorem 3. *Assume that conditions $(B1)-(B3)$ hold and suppose that there exist positive real numbers $r,\mathcal{R}$ with $r<\eta\mathcal{R}$ and $\psi_j < \frac{r\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}} \leq \frac{\mathcal{R}\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}$ such that $f_j; j=1,2,3$ satisfying the following conditions:*

(C1) $f_j(\xi,\varpi,\vartheta,\omega)\geq\phi_{r_j}(\frac{1}{3}\frac{r}{\aleph\mathcal{D}})$, $\xi\in\mathcal{I}$ and $(\varpi,\vartheta,\omega)\in[r,\mathcal{R}]$,

(C2) $f_j(\xi,\varpi,\vartheta,\omega)\leq\phi_{r_j}(\frac{1}{\Phi_j}\frac{\mathcal{R}}{\mathcal{C}})$, $\xi\in[h,k]$ and $(\varpi,\vartheta,\omega)\in[0,\mathcal{R}]$.

Then, the system of fractional order Boundary value problems (2) and (3) has at least one positive and nondecreasing solution $(\varpi^\star,\vartheta^\star,\omega^\star)$ satisfying $r\leq\alpha(\varpi^\star,\vartheta^\star,\omega^\star)$ with $\beta(\varpi^\star,\vartheta^\star,\omega^\star)\leq\mathcal{R}$.

Proof. Let $\Omega_1=\{(\varpi,\vartheta,\omega);\alpha(\varpi,\vartheta,\omega)<r\}$ and $\Omega_2=\{(\varpi,\vartheta,\omega);\beta(\varpi,\vartheta,\omega)<\mathcal{R}\}$. It is easy to see that $0\subset\Omega_1$, and Ω_1,Ω_2 are bounded open subsets of $\mathcal{E}$. Letting $(\varpi,\vartheta,\omega)\in\Omega$, obtain

$$r>\alpha(\varpi,\vartheta,\omega)=\min_{\xi\in I}\{\varpi(\xi)+\vartheta(\xi)+\omega(\xi)\}\geq\aleph\{\|\varpi\|+\|\vartheta\|+\|\omega\|\}=\aleph\beta(\varpi,\vartheta,\omega).$$

Thus, $\mathcal{R}>\frac{r}{\aleph}>\beta(\varpi,\vartheta,\omega)$, i.e $(\varpi,\vartheta,\omega)\in\Omega_2$, so $\Omega_1\subseteq\Omega_2$.

Claim 1: If $(\varpi,\vartheta,\omega)\in\mathcal{P}\cap\partial\Omega_1$, then $\alpha(\mathcal{T}(\varpi,\vartheta,\omega))\geq\alpha(\varpi,\vartheta,\omega)=r$, for $\zeta\in\mathcal{I}$. It follows from $(C1)$ and Lemma 4 that

$$\begin{aligned}
\alpha(\mathcal{T}(\varpi,\vartheta,\omega)) &= \min_{\xi\in\mathcal{I}}\sum_{j=1}^{3}\Big[\int_h^k\mathcal{H}_j(\xi,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta\frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}f_j(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau))d\tau\Big)d\zeta\\
&\quad+\frac{\psi_j\Gamma(q_j-\alpha_j)(\xi-h)^{q_j-1}}{\Delta_j},\Big]\\
&\geq\sum_{j=1}^{3}\Big[\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\Big(\frac{1}{4}\Big)^{q_j-1}\mathcal{H}_j(k,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta\frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}\phi_{r_j}\Big(\frac{1}{3}\frac{r}{\aleph\mathcal{D}}\Big)d\zeta\\
&\quad+\frac{\psi_j\Gamma(q_j-\alpha_j)(\frac{1}{4})^{q_j-1}(k-h)^{q_j-1}}{\Delta_j}\Big]\\
&\geq\sum_{j=1}^{3}\phi_{\varphi_j}\Big(\frac{(k-h)^{p_j}}{4^{p_j}\Gamma(p_j+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\aleph\mathcal{H}_j(k,\zeta)\Big(\frac{1}{3}\frac{r}{\aleph\mathcal{D}}\Big)d\zeta\\
&\geq\frac{r}{3\mathcal{D}}\phi_{\varphi_1}\Big(\frac{(k-h)^{p_1}}{4^{p_1}\Gamma(p_1+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_1(k,\zeta)d\zeta+\frac{r}{3\mathcal{D}}\phi_{\varphi_2}\Big(\frac{(k-h)^{p_2}}{4^{p_2}\Gamma(p_2+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_2(k,\zeta)d\zeta\\
&\quad+\frac{r}{3\mathcal{D}}\phi_{\varphi_3}\Big(\frac{(k-h)^{p_3}}{4^{p_3}\Gamma(p_3+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_3(k,\zeta)d\zeta\\
&=\frac{r}{3}+\frac{r}{3}+\frac{r}{3}=r=\alpha(\varpi,\vartheta,\omega).
\end{aligned}$$

Claim 2: If $(\varpi, \vartheta, \omega) \in \mathcal{P} \cap \partial\Omega_2$, then $\beta(\mathcal{T}(\varpi, \vartheta, \omega)) \leq \beta(\varpi, \vartheta, \omega)$. Then,

$$\Big[\varpi(\zeta) + \vartheta(\zeta) + \omega(\zeta)\Big] \leq \beta(\varpi, \vartheta, \omega) = \mathcal{R},$$

for $\zeta \in [h, k]$. It follows from $(C2)$ and Lemma 4 that

$$\begin{aligned}
\beta(\mathcal{T}(\varpi, \vartheta, \omega)) &= \max_{\xi \in [h,k]} \sum_{j=1}^{3} \Big[\int_h^k \mathcal{H}_j(\xi, \zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta - \tau)^{p_j - 1}}{\Gamma(p_j)} f_j\big(\tau, \varpi(\tau), \vartheta(\tau), \omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_j \Gamma(q_j - \alpha_j)(\xi - h)^{q_j - 1}}{\Delta_j}\Big] \\
&\leq \sum_{j=1}^{3} \Big[\int_h^k \mathcal{H}_j(k, \zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta - \tau)^{p_j - 1}}{\Gamma(p_j)} f_j\big(\tau, \varpi(\tau), \vartheta(\tau), \omega(\tau)\big)d\tau\Big)d\zeta \\
&\quad + \frac{\psi_j \Gamma(q_j - \alpha_j)(k - h)^{q_j - 1}}{\Delta_j}\Big] \\
&< \frac{\mathcal{R}}{\Phi_1 \mathcal{C}} \int_h^k \mathcal{H}_1(k, \zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta - \tau)^{p_1 - 1}}{\Gamma(p_1)} d\tau\Big)d\zeta + \frac{\mathcal{R}}{\Phi_2 \mathcal{C}} \int_h^k \mathcal{H}_2(\xi, \zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta - \tau)^{p_2 - 1}}{\Gamma(p_2)} d\tau\Big)d\zeta \\
&\quad + \frac{\mathcal{R}}{\Phi_3 \mathcal{C}} \int_h^\zeta \mathcal{H}_3(\xi, \zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta - \tau)^{p_3 - 1}}{\Gamma(p_3)} d\tau\Big)d\zeta + \frac{\mathcal{R}}{\Theta_1} + \frac{\mathcal{R}}{\Theta_2} + \frac{\mathcal{R}}{\Theta_3} \\
&= \frac{\mathcal{R}}{\Phi_1} + \frac{\mathcal{R}}{\Phi_2} + \frac{\mathcal{R}}{\Phi_3} + \frac{\mathcal{R}}{\Theta_1} + \frac{\mathcal{R}}{\Theta_2} + \frac{\mathcal{R}}{\Theta_3} \\
&= \mathcal{R}\Big[\frac{1}{\Phi_1} + \frac{1}{\Phi_2} + \frac{1}{\Phi_3} + \frac{1}{\Theta_1} + \frac{1}{\Theta_2} + \frac{1}{\Theta_3}\Big] \leq \mathcal{R} = \beta(\varpi, \vartheta, \omega).
\end{aligned}$$

Clearly, α satisfies (iii) of Rule (S_1) and β satisfies (i) of Rule (S_2). Therefore, condition (a) of Theorem 1 is satisfied, and hence $\mathcal{T}$ has at least one fixed point $(\varpi^\star, \vartheta^\star, \omega^\star) \in \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1)$, i.e, the system of fractional order Boundary value problems (2) and (3) has at least one positive and nondecreasing solution $(\varpi^\star, \vartheta^\star, \omega^\star)$ satisfying $r \leq \alpha(\varpi^\star, \vartheta^\star, \omega^\star)$ and $\beta(\varpi^\star, \vartheta^\star, \omega^\star) \leq \mathcal{R}$. □

Theorem 4. *Assume that conditions* $(B1) - (B3)$ *hold and suppose that there exist positive real numbers* $r, \mathcal{R}$ *with* $r < \mathcal{R}$ *and* $\psi_j < \frac{r\Delta_j}{\Theta_j \Gamma(q_j - \alpha_j)(k-h)^{q_j - 1}} \leq \frac{\mathcal{R}\Delta_j}{\Theta_j \Gamma(q_j - \alpha_j)(k-h)^{q_j - 1}}$ *such that* $f_j; j = 1, 2, 3$ *satisfying the following conditions:*

(C3) $f_j(\xi, \varpi, \vartheta, \omega) \leq \phi_{r_j}\big(\frac{1}{\Phi_j}\frac{r}{\mathcal{D}}\big)$, $\xi \in [h, k]$ *and* $(\varpi, \vartheta, \omega) \in [0, r]$,

(C4) $f_j(\xi, \varpi, \vartheta, \omega) \geq \phi_{r_j}\big(\frac{1}{\Phi_j}\frac{\mathcal{R}}{\aleph\mathcal{C}}\big)$, $\xi \in \mathcal{I}$ *and* $(\varpi, \vartheta, \omega) \in [\mathcal{R}, \frac{\mathcal{R}}{\aleph}]$.

Then, the system of fractional order Boundary value problems (2) *and* (3) *has at least one positive and nondecreasing solution* $(\varpi^\star, \vartheta^\star, \omega^\star)$ *satisfying* $r \leq \beta(\varpi^\star, \vartheta^\star, \omega^\star)$ *with* $\alpha(\varpi^\star, \vartheta^\star, \omega^\star) \leq \mathcal{R}$.

Proof. We let $\Omega_3 = \{(\varpi, \vartheta, \omega); \beta(\varpi, \vartheta, \omega) < r\}$ and $\Omega_4 = \{(\varpi, \vartheta, \omega); \alpha(\varpi, \vartheta, \omega) < \mathcal{R}\}$. We have $0 \in \Omega_3$. We set $\Omega_3 \subset \Omega_4$; Ω_3 and Ω_4 are bounded open subsets of $\mathcal{E}$.

Claim 1: If $(\varpi, \vartheta, \omega) \in \mathcal{P} \cap \partial\Omega_3$, then $\beta(\mathcal{T}(\varpi, \vartheta, \omega)) \leq \beta(\varpi, \vartheta, \omega)$. To see this, we let $(\varpi, \vartheta, \omega) \in \mathcal{P} \cap \partial\Omega_3$. Then, $[\varpi(\zeta) + \vartheta(\zeta) + \omega(\zeta)] \leq \beta(\varpi, \vartheta, \omega) = r$, for $\zeta \in [h, k]$. It follows from $(C3)$ and Lemma 4 that

$$\beta(\mathcal{T}(\varpi,\vartheta,\omega)) = \max_{\xi\in[h,k]}\sum_{j=1}^{3}\Big[\int_h^k \mathcal{H}_j(\xi,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} f_j\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta + \frac{\psi_j\Gamma(q_j-\alpha_j)(\xi-h)^{q_j-1}}{\Delta_j}\Big]$$

$$\leq \sum_{j=1}^{3}\Big[\int_h^k \mathcal{H}_j(k,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} f_j\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta + \frac{\psi_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}{\Delta_j}\Big]$$

$$< \frac{r}{\Phi_1\mathcal{D}}\int_h^k \mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}d\tau\Big)d\zeta + \frac{r}{\Phi_2\mathcal{D}}\int_h^k \mathcal{H}_2(k,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)}d\tau\Big)d\zeta$$
$$+ \frac{r}{\Phi_3\mathcal{D}}\int_h^k \mathcal{H}_3(k,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)}d\tau\Big)d\zeta + \frac{r}{\Theta_1} + \frac{r}{\Theta_2} + \frac{r}{\Theta_3}$$
$$= \frac{r}{\Phi_1} + \frac{r}{\Phi_2} + \frac{r}{\Phi_3} + \frac{r}{\Theta_1} + \frac{r}{\Theta_2} + \frac{r}{\Theta_3}$$
$$= r\Big[\frac{1}{\Phi_1} + \frac{1}{\Phi_2} + \frac{1}{\Phi_3} + \frac{1}{\Theta_1} + \frac{1}{\Theta_2} + \frac{1}{\Theta_3}\Big] \leq r = \beta(\varpi,\vartheta,\omega).$$

Claim 2: If $(\varpi,\vartheta,\omega) \in \mathcal{P}\cap\partial\Omega_4$, then $\alpha(\mathcal{T}(\varpi,\vartheta,\omega)) \geq \alpha(\varpi,\vartheta,\omega)$. To see this, we let $(\varpi,\vartheta,\omega) \in \mathcal{P}\cap\partial\Omega_4$. Then, $\frac{\mathcal{R}}{\aleph} = \frac{\alpha(\varpi,\vartheta,\omega)}{\eta} \geq \beta(\varpi,\vartheta,\omega) \geq [\varpi(\zeta)+\vartheta(\zeta)+\omega(\zeta)] \geq \alpha(\varpi,\vartheta,\omega) = \mathcal{R}$ for $\zeta\in\mathcal{I}$. It follows from $(C4)$ and Lemma 4 that

$$\alpha(\mathcal{T}(\varpi,\vartheta,\omega)) = \min_{\xi\in\mathcal{I}}\sum_{j=1}^{3}\Big[\int_h^k \mathcal{H}_j(\xi,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} f_j\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta + \frac{\psi_j\Gamma(q_j-\alpha_j)(\xi-h)^{q_j-1}}{\Delta_j}\Big]$$

$$\geq \sum_{j=1}^{3}\Big[\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\Big(\frac{1}{4}\Big)^{q_j-1}\mathcal{H}_j(k,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}\phi_{r_j}\Big(\frac{1}{3}\frac{\mathcal{R}}{\aleph\mathcal{C}}\Big)d\zeta + \frac{\psi_j\Gamma(q_j-\alpha_j)\big(\frac{1}{4}\big)^{q_j-1}(k-h)^{q_j-1}}{\Delta_j}\Big]$$

$$\geq \sum_{j=1}^{3}\phi_{\varphi_j}\Big(\frac{(k-h)^{p_j}}{4^{p_j}\Gamma(p_j+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\aleph\mathcal{H}_j(k,\zeta)\Big(\frac{1}{3}\frac{\mathcal{R}}{\aleph\mathcal{C}}\Big)d\zeta$$

$$\geq \frac{\mathcal{R}}{3\,\mathcal{C}}\phi_{\varphi_1}\Big(\frac{(k-h)^{p_1}}{4^{p_1}\Gamma(p_1+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_1(k,\zeta)d\zeta + \frac{\mathcal{R}}{3\,\mathcal{C}}\phi_{\varphi_2}\Big(\frac{(k-h)^{p_2}}{4^{p_2}\Gamma(p_2+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_2(k,\zeta)d\zeta$$
$$+ \frac{\mathcal{R}}{3\,\mathcal{C}}\phi_{\varphi_3}\Big(\frac{(k-h)^{p_3}}{4^{p_3}\Gamma(p_3+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_3(k,\zeta)d\zeta$$
$$= \frac{\mathcal{R}}{3} + \frac{\mathcal{R}}{3} + \frac{\mathcal{R}}{3} = \mathcal{R} = \alpha(\varpi,\vartheta,\omega).$$

Clearly, α satisfies (iii) of Rule (S_1) and β satisfies (i) of Rule (S_2). Therefore, condition (a) of Theorem 1 is satisfied, and hence $\mathcal{T}$ has at least one fixed point $(\varpi^\star,\vartheta^\star,\omega^\star) \in \mathcal{P}\cap\Big(\overline{\Omega_4}\setminus\Omega_2\Big)$, i.e, the system of fractional order Boundary value problems (2) and (3) has at least one positive and nondecreasing solution $(\varpi^\star,\vartheta^\star,\omega^\star)$ satisfying $r \leq \beta((\varpi^\star,\vartheta^\star,\omega^\star))$ and $\alpha(\varpi^\star,\vartheta^\star,\omega^\star) \leq \mathcal{R}$. $\square$

Theorem 5. *Assume that* $(B1)-(B3)$ *hold and suppose that there exist* $0<k<l<\aleph d$ *and* $0<\psi_j<\frac{k\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}\leq\frac{d\Delta_j}{\Theta_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}$ *such that* $f_j(j=1,2,3)$ *satisfies the following conditions:*

$(C5) f_j(\xi,\varpi,\vartheta,\omega)<\phi_{r_j}\big(\frac{d}{\Phi_j\mathcal{C}}\big)$*, for all* $\xi\in[h,k]$, $(\varpi,\vartheta,\omega)\in[0,d]$,

$(C6) f_j(\xi,\varpi,\vartheta,\omega)>\phi_{r_j}\big(\frac{l}{3\aleph\mathcal{D}}\big)$*, for all* $\xi\in\mathcal{I}$, $(\varpi,\vartheta,\omega)\in[l,\frac{l}{\aleph}]$,

$(C7) f_j(\xi,\varpi,\vartheta,\omega)<\phi_{r_j}\big(\frac{k}{\Phi_j\mathcal{C}}\big)$*, for all* $\xi\in[h,k]$, $(\varpi,\vartheta,\omega)\in[0,k]$.

Then, Systems (2) *and* (3) *have at least three positive solution* $(\varpi_1,\vartheta_1,\omega_1),(\varpi_2,\vartheta_2,\omega_2)$ *and* $(\varpi_3,\vartheta_3,\omega_3)$ *with* $\varphi(\varpi_1,\vartheta_1,\omega_1)<k$, $l<\psi(\varpi_2,\vartheta_2,\omega_2)<\varphi(\varpi_2,\vartheta_2,\omega_2)<d$, $k<\varphi(\varpi_3,\vartheta_3,\omega_3)<d$ *with* $\psi(\varpi_3,\vartheta_3,\omega_3)<l$.

Proof. Firstly, if $(\varpi,\vartheta,\omega)\in\overline{\mathcal{P}_d}$; then, we may assert that $\mathcal{T}:\overline{\mathcal{P}_d}\to\overline{\mathcal{P}_d}$ is a completely continuous operator. To see this, we suppose $(\varpi,\vartheta,\omega)\in\overline{\mathcal{P}_d}$, then $\|(\varpi,\vartheta,\omega)\|\leq d$. It follows from Lemma 4 and (C5) that

$$\begin{aligned}
\|\mathcal{T}(\varpi,\vartheta,\omega)\| &= \max_{\xi\in[h,k]}\big\{\mathcal{T}_1(\varpi,\vartheta,\omega)(\xi)+\mathcal{T}_2(\varpi,\vartheta,\omega)(\xi)+\mathcal{T}_3(\varpi,\vartheta,\omega)(\xi)\big\}\\
&= \max_{\xi\in[h,k]}\sum_{i=1}^{3}\Big[\int_h^k\mathcal{H}_j(\xi,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta\frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}f_j\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta\\
&\quad+\frac{\psi_j\Gamma(q_j-\alpha_j)(\xi-h)^{q_j-1}}{\Delta_j}\Big]\\
&\leq\sum_{j=1}^{3}\Big[\int_h^k\mathcal{H}_j(k,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta\frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}f_j\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta\\
&\qquad+\frac{\psi_j\Gamma(q_j-\alpha_j)(k-h)^{q_j-1}}{\Delta_j}\Big]\\
&<\frac{1}{\Phi_1}\frac{d}{C}\int_h^k\mathcal{H}_1(k,\zeta)\phi_{\varphi_1}\Big(\int_h^\zeta\frac{(\zeta-\tau)^{p_1-1}}{\Gamma(p_1)}d\tau\Big)d\zeta+\frac{1}{\Phi_2}\frac{d}{\mathcal{C}}\int_h^k\mathcal{H}_2(k,\zeta)\phi_{\varphi_2}\Big(\int_h^\zeta\frac{(\zeta-\tau)^{p_2-1}}{\Gamma(p_2)}d\tau\Big)d\zeta\\
&\quad+\frac{1}{\Phi_3}\frac{d}{\mathcal{C}}\int_h^k\mathcal{H}_3(k,\zeta)\phi_{\varphi_3}\Big(\int_h^\zeta\frac{(\zeta-\tau)^{p_3-1}}{\Gamma(p_3)}d\tau\Big)d\zeta+\frac{d}{\Theta_j}+\frac{d}{\Theta_j}+\frac{d}{\Theta_j}\\
&= d\Big[\frac{1}{\Phi_1}+\frac{1}{\Phi_2}+\frac{1}{\Phi_3}+\frac{1}{\Theta_1}+\frac{1}{\Theta_2}+\frac{1}{\Theta_3}\Big]\leq d.
\end{aligned}$$

Therefore, $\mathcal{T}:\overline{\mathcal{P}_d}\to\overline{\mathcal{P}_d}$. This, together with Lemma 5, implies that $\mathcal{T}:\overline{\mathcal{P}_d}\to\overline{\mathcal{P}_d}$ is a completely continuous operator. Similarly, if $(\varpi,\vartheta,\omega)\in\overline{\mathcal{P}_k}$, then, from (C7), it follows that $\|\mathcal{T}(\varpi,\vartheta,\omega)\|<k$. This shows that condition (ii) of Theorem 2 is fulfilled.

Now, we let $\varpi(\xi)+\vartheta(\xi)+\omega(\xi)=\frac{l}{\aleph}$ for $\xi\in[h,k]$. It is easy to verify that $\varpi(\xi)+\vartheta(\xi)+\omega(\xi)=\frac{l}{\aleph}\in\mathcal{P}(\psi,l,\frac{l}{\aleph})$ and $\psi(\varpi,\vartheta,\omega)=\frac{l}{\aleph}>l$, and so $\{(\varpi,\vartheta,\omega)\in\mathcal{P}(\psi,l,\frac{l}{\aleph});\psi(\varpi,\vartheta,\omega)>l\}\neq\emptyset$. Thus, for all $(\varpi,\vartheta,\omega)\in\mathcal{P}(\psi,l,\frac{l}{\aleph})$, we have that $l\leq\varpi(\xi)+\vartheta(\xi)+\omega(\xi)\leq\frac{l}{\aleph}$ for $\xi\in\mathcal{I}$ and $\mathcal{T}(\varpi,\vartheta,\omega)\in\mathcal{P}$, from (C6), we have

$$
\begin{aligned}
&\psi\big(\mathcal{T}(\varpi,\vartheta,\omega)(\xi)\big) = \min_{\xi\in\mathcal{I}}\big\{\mathcal{T}_1(\varpi,\vartheta,\omega)(t) + \mathcal{T}_2(\varpi,\vartheta,\omega)(\xi) + \mathcal{T}_3(\varpi,\vartheta,\omega)(\xi)\big\}\\
&= \min_{\xi\in I}\sum_{i=1}^{3}\Big[\int_h^k \mathcal{H}_j(\xi,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)} f_j\big(\tau,\varpi(\tau),\vartheta(\tau),\omega(\tau)\big)d\tau\Big)d\zeta\\
&\quad + \frac{\psi_j\Gamma(q_j-\alpha_j)(t-h)^{q_j-1}}{\Delta_j}\Big]\\
&\geq \aleph\sum_{i=1}^{3}\Big[\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\Big(\frac{1}{4}\Big)^{q_j-1}\mathcal{H}_j(k,\zeta)\phi_{\varphi_j}\Big(\int_h^\zeta \frac{(\zeta-\tau)^{p_j-1}}{\Gamma(p_j)}\phi_{r_j}\big(\frac{l}{3\aleph\mathcal{D}}\big)\Big)d\zeta\\
&\qquad + \frac{\psi_j\Gamma(q_j-\alpha_j)\big(\frac{1}{4}\big)^{q_j-1}(k-h)^{q_j-1}}{\Delta_j}\Big]\\
&\geq \sum_{j=1}^{3}\phi_{\varphi_j}\Big(\frac{(k-h)^{p_j}}{4^{p_j}\Gamma(p_j+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\aleph\mathcal{H}_j(k,\zeta)\Big(\frac{l}{3\aleph\mathcal{D}}\Big)d\zeta\\
&\geq \frac{l}{3\,\mathcal{D}}\phi_{\varphi_1}\Big(\frac{(k-h)^{p_1}}{4^{p_1}\Gamma(p_1+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_1(k,\zeta)d\zeta + \frac{l}{3\,\mathcal{D}}\phi_{\varphi_2}\Big(\frac{(k-h)^{p_2}}{4^{p_2}\Gamma(p_2+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_2(k,\zeta)d\zeta\\
&\quad + \frac{l}{3\,\mathcal{D}}\phi_{\varphi_3}\Big(\frac{(k-h)^{p_3}}{4^{p_3}\Gamma(p_3+1)}\Big)\int_{\frac{3h+k}{4}}^{\frac{h+3k}{4}}\mathcal{H}_3(k,\zeta)d\zeta\\
&= \frac{l}{3}+\frac{l}{3}+\frac{l}{3} = l.
\end{aligned}
$$

Hence, condition (i) of Theorem 2 is verified. Next, we prove that (iii) of Theorem 2 is satisfied. By Lemma 5, we have $\min_{\xi\in I}|\mathcal{T}_1(\varpi,\vartheta,\omega)(\xi) + \mathcal{T}_2(\varpi,\vartheta,\omega)(\xi) + \mathcal{T}_3(\varpi,\vartheta,\omega)(t)| > \aleph\|\mathcal{T}(\varpi,\vartheta,\omega)\| > d$ for $(\varpi,\vartheta,\omega) \in \mathcal{P}(\psi,l,d)$ with $\|\mathcal{T}(\varpi,\vartheta,\omega)\| > \frac{l}{\aleph}$. To sum up, all the conditions of Theorem 2 are satisfied; then, there exist three positive solutions $(\varpi_1,\vartheta_1,\omega_1)$, $(\varpi_2,\vartheta_2,\omega_2)$ and $(\varpi_3,\vartheta_3,\omega_3)$ with $\varphi(\varpi_1,\vartheta_1,\omega_1) < k,\ l < \psi(\varpi_2,\vartheta_2,\omega_2) < \varphi(\varpi_2,\vartheta_2,\omega_2) < d,\ \ k < \varphi(\varpi_3,\vartheta_3,\omega_3) < d$, and $\psi(\varpi_3,\vartheta_3,\omega_3) < l$. $\square$

4. Examples

We let $h = 1,\ k = 2,\ p_1 = 0.5,\ p_2 = 0.6,\ p_3 = 0.7,\ q_1 = 1.5,\ q_2 = 1.6,\ q_3 = 1.7,\ \alpha_1 = 0.5,\ \alpha_2 = 0.6,\ \alpha_3 = 0.7,\ \eta_1 = 1.5,\ \eta_2 = 1.6,\ \eta_3 = 1.7, \mu_1 = 2,\ \mu_2 = 3,\ \mu_3 = 4,\ \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3,\ r_1 = 2,\ r_2 = 2,\ r_3 = 2.$

We consider the system of fractional differential equations

$$
\begin{cases}
-\mathcal{D}_{1^+}^{0.5}\big(\phi_2\big(\mathcal{D}_{1^+}^{1.5}\varpi(\xi)\big)\big) = f_1(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi\in(1,2),\\
-\mathcal{D}_{1^+}^{0.6}\big(\phi_2\big(\mathcal{D}_{1^+}^{1.6}\vartheta(\xi)\big)\big) = f_2(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi\in(1,2),\\
-\mathcal{D}_{1^+}^{0.7}\big(\phi_2\big(\mathcal{D}_{1^+}^{1.7}\omega(\xi)\big)\big) = f_3(\xi,\varpi(\xi),\vartheta(\xi),\omega(\xi)),\ \xi\in(1,2),
\end{cases} \tag{9}
$$

$$
\begin{cases}
\varpi(1) = 0;\ \ \phi_2\big(\mathcal{D}_{1^+}^{1.5}\varpi(1)\big) = 0;\ \ 2\,\mathcal{D}_{1^+}^{0.5}\varpi(2) = \psi_1 + 1\,\mathcal{D}_{1^+}^{0.5}\varpi(1.5),\\
\vartheta(1) = 0;\ \ \phi_2\big(\mathcal{D}_{1^+}^{1.6}\vartheta(1)\big) = 0;\ \ 3\,\mathcal{D}_{1^+}^{0.6}\vartheta(2) = \psi_2 + 2\,\mathcal{D}_{1^+}^{0.6}\vartheta(1.6),\\
\omega(1) = 0;\ \ \phi_2\big(\mathcal{D}_{1^+}^{1.7}\omega(1)\big) = 0;\ \ 4\,\mathcal{D}_{1^+}^{0.7}\omega(2) = \psi_3 + 3\,\mathcal{D}_{1^+}^{0.7}\omega(1.7),
\end{cases} \tag{10}
$$

where $\psi_1,\ \psi_2,\ \psi_3$ are parameters. We have $\aleph = 0.378929$; $\Delta_1 = 0.886227 > 0$; $\Delta_2 = 0.893515$; $\Delta_3 = 0.908639$, so Assumption $(A2)$ satisfied. In addition, we found $\mathcal{D} = \max\{2.0,\ 3.446,\ 5.278\} = 5.278$; $\mathcal{C} = \min\{1.858929,\ 2.620029,\ 3.237839\} = 1.858929$.

Example 1. *We consider the functions*

$$f_1(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{18}e^{-(\varpi+\vartheta+\omega)}+\sin\xi, & 0\leq\varpi,\vartheta,\omega<5,\\ \frac{1}{2}(e^{-(\varpi+\vartheta+\omega)}+1)+\frac{1}{3}\log\xi, & 5\leq\varpi,\vartheta,\omega\leq 10,\end{cases}$$

$$f_2(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{3}(e^{-(\varpi+\vartheta+\omega)}+1)+\frac{t}{7}\sin\xi, & 0\leq\varpi,\vartheta,\omega<5,\\ \frac{1}{2}(e^{-\xi}+1)+\frac{1}{\xi+1}(e^{-(\varpi+\vartheta+\omega)}\log\xi), & 5\leq\varpi,\vartheta,\omega\leq 10,\end{cases}$$

$$f_3(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{\xi+1}+\log\xi(e^{-(\varpi+\vartheta+\omega)}+2)^{-1}, & 0\leq\varpi,\vartheta,\omega<5,\\ \frac{1}{\xi+1}(\log\xi+1)+\frac{\xi e^{-\xi}}{\varpi+\vartheta+\omega}+\frac{10}{119}, & 5\leq\varpi,\vartheta,\omega\leq 10.\end{cases}$$

Choosing $r=1$, $\mathcal{R}=10$, *with* $\frac{1}{\Phi_1}=\frac{1}{\Phi_2}=\frac{1}{\Phi_3}=\frac{1}{\Theta_1}=\frac{1}{\Theta_2}=\frac{1}{\Theta_3}=\frac{1}{6}$ *then* $r<\aleph\mathcal{R}$ *and* $f_i(i=1,2,3)$ *fulfilling the following conditions:*

(C1) $f_i(\xi,\varpi,\vartheta,\omega)\geq 0.166668=\phi_{r_i}\big(\frac{1}{3}\frac{r}{\aleph\mathcal{D}}\big)$, $\xi\in[2.25,2.75]$ *and* $(\varpi,\vartheta,\omega)\in[1,10]$

(C2) $f_i(\xi,\varpi,\vartheta,\omega)\leq 0.896574=\phi_{r_i}\big(\frac{1}{\Phi_j}\frac{\mathcal{R}}{\mathcal{C}}\big)$, $\xi\in[1,2]$ *and* $(\varpi,\vartheta,\omega)\in[0,10]$.

Thus, all conditions of Theorem 3 are fulfilled. Hence, for $\psi_1\leq 1.47705$, $\psi_2\leq 1.489192$, $\psi_3\leq 1.514398$, *the system of* (9) *and* (10) *has at least three positive solutions.*

Example 2. *We consider the functions*

$$f_1(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{\xi+25}\log(\varpi+\vartheta+\omega+1)+\frac{e^{-\xi}}{12}, & 0\leq\varpi,\vartheta,\omega\leq 1,\\ \frac{2}{\xi^2+1}+\frac{\log\xi+1}{e^{-\xi}+5}, & 1<\varpi,\vartheta,\omega\leq 10,\\ \frac{1}{5}(\xi+e^{-(\varpi+\vartheta+\omega)})+\frac{1}{9}(\xi+\sin\xi), & 10<\varpi,\vartheta,\omega\leq 20,\end{cases}$$

$$f_2(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{40+\xi^5}e^{(\varpi+\vartheta+\omega)}+\frac{\log\xi}{10}, & 0\leq\varpi,\vartheta,\omega\leq 1,\\ \frac{1}{\xi+1}[\log\xi+e^{-(\varpi+\vartheta+\omega)}]+\frac{e^{-\xi}}{u+v+w}+\frac{9}{24}, & 1<\varpi,\vartheta,\omega\leq 10,\\ \frac{\log\xi+1}{\varpi+\vartheta+\omega}+\frac{e^{\xi}}{\xi^2+1}-\sin\xi, & 10<\varpi,\vartheta,\omega\leq 20,\end{cases}$$

$$f_3(\xi,\varpi,\vartheta,\omega)=\begin{cases}\frac{1}{\xi+6}[e^{-(\varpi+\vartheta+\omega)}\log\xi], & 0\leq\varpi,\vartheta,\omega\leq 1,\\ \frac{1}{2}[e^{-(\varpi+\vartheta+\omega)}+1]+\frac{2}{\xi+1}[e^{-\xi}\sin\xi], & 1<\varpi,\vartheta,\omega\leq 10,\\ \log\xi+\frac{2}{5}(1+\xi)-\frac{2}{\varpi+\vartheta+\omega}, & 10<\varpi,\vartheta,\omega\leq 20.\end{cases}$$

Choosing $k=4$, $l=5$, $d=727.55$, $\frac{1}{\Im_1}=\frac{1}{\Im_2}=\frac{1}{\Im_3}=\frac{1}{\Re_1}=\frac{1}{\Re_2}=\frac{1}{\Re_3}=\frac{1}{6}$ *then* $0<k<l<\aleph d$ *and* $f_i(i=1,2,3)$ *fulfilling the following conditions:*

(C5) $f_i(\xi,\varpi,\vartheta,\omega)<1.793147=\phi_{r_j}\big(\frac{d}{\Phi_j\mathcal{C}}\big)$, *for all* $\xi\in[1,2]$, $(\varpi,\vartheta,\omega)\in[0,20]$,

(C6) $f_i(\xi,\varpi,\vartheta,\omega)>0.333335=\phi_{r_j}\big(\frac{l}{3\aleph\mathcal{D}}\big)$, *for all* $\xi\in[2.25,2.75]$, $(\varpi,\vartheta,\omega)\in[2,5.278034]$,

(C7) $f_i(\xi,\varpi,\vartheta,\omega)<0.089657=\phi_{r_j}\big(\frac{k}{\Phi_j\mathcal{C}}\big)$, *for all* $\xi\in[1,2]$, $(\varpi,\vartheta,\omega)\in[0,1]$.

Thus, all conditions of Theorem 5 are fulfilled. Hence, for $\sigma_1\leq 2.95409$, $\sigma_2\leq 2.978384$, $\sigma_3\leq 3.028796$, *the system of* (9) *and* (10) *has at least three positive solutions.*

5. Conclusions

In this study, we established the existence of positive solutions to a system of three-point Riemann–Liouville fractional order boundary value problems with (r_1,r_2,r_3)-Laplacian operator. Our approach involved employing techniques such as cone expansion and compression of the functional type and the Leggett–Williams fixed point theorem.

Our results have important implications for the field of fractional differential equations and its applications. For example, our results could be used to develop new models of biological systems, physical phenomena, and engineering systems, including fractional multi-energy groups of neutron diffusion equations [8,9,13,54,55]. Additionally, our results could be used to develop new numerical methods for solving fractional differential equations [56].

For future research, we propose to investigate the following directions:

1. Establish necessary conditions for the existence of an infinite number of solutions to the system.
2. Study infinite systems of sequential hybrid fractional order boundary value problems.
3. Extend the idea used in this paper to study fractional difference equations and dynamic equations on time scales.
4. Explore the implications of our results for fractional multi-energy groups of neutron diffusion equations and develop new models and numerical methods for this important application.

We believe that our work has the potential to improve the field of fractional calculus and its applications, especially in the area of fractional multi-energy groups of neutron diffusion equations. Our results could be used to develop more accurate and efficient models of neutron diffusion in nuclear reactors, which could lead to safer and better performing reactors. Additionally, our results could be used to develop new numerical methods for solving fractional multi-energy groups of neutron diffusion equations, which could make it possible to solve these problems faster and more accurately on large computers. We are excited to explore these future research directions and contribute to the advancement of fractional calculus and its applications in nuclear physics and reactor engineering.

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