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Common Fixed Point of (ψ, β, L)-Generalized Contractive Mapping in Partially Ordered b-Metric Spaces

Binghua Jiang [1], Huaping Huang [2,*] and Stojan Radenović [3]

[1] School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China
[2] School of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing 404020, China
[3] Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia
* Correspondence: huaping@sanxiau.edu.cn

Abstract: The purpose of this paper is to attain the existence of coincidences and common fixed points in four mappings satisfying (ψ, β, L)-generalized contractive conditions in the framework of partially ordered b-metric spaces. The main results presented in this paper generalize some recent results in the existing literature. Furthermore, a nontrivial example is presented to support the obtained results.

Keywords: coincidence point; partially ordered b-metric space; weakly compatible; partially weakly increasing

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1. Introduction

It is world-renowned that the Banach contraction principle (for short, BCP) (see [1]) occupies a significant role in different fields of basic mathematics, applied mathematics and other subjects, and it has been generalized and improved in many aspects. Numerous generalizations and improvements come forth by changing metric spaces into general abstract metric spaces (see [2,3]). In [4], Vulpe et al. introduced b-metric space (sometimes so-called metric-type space, see [5]) as a new generalization of usual metric space. He provided the generalized BCP in b-metric space. From then on, a large number of papers have considered fixed point theory and its applications or variational methods for single-valued and multi-valued operators in b-metric spaces (the reader may see [5–10] and the related references therein). The mappings satisfying certain contractive conditions can be utilized to establish the existence of solutions to all kinds of operator equations such as integral equations, differential equations and fractional differential equations. Beg and Abbas (see [11]) obtained common fixed point theorems by extending a weakly contractive condition into two mappings. Abbas et al. (see [12]) investigated common fixed points for four mappings satisfying generalized weakly contractive conditions in complete partially ordered metric spaces. Esmaicy et al. (see [13]) initiated coincidence point results for four mappings in partially ordered metric spaces and used their results to seek the common solution of two integral equations. Recently, Abbas et al. (see [14]) acquired coincidence and common fixed points for four mappings satisfying generalized (ψ, β)-contractive conditions in complete partially ordered metric spaces.

Based on the previous work, throughout this paper, we aim to established coincidence and common fixed points for four mappings under generalized (ψ, β, L)-contractive conditions in complete partially ordered b-metric spaces. Our results make great progress in extending, unifying and generalizing the corresponding results in [14–16].

2. Preliminaries

In this paper, unless there are special statements, we always denote $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}^*$ as the set of all real numbers, the set of all non-negative real numbers, the set of all non-negative integers and the set of all positive integers, respectively.

First, we give some basic definitions and results which will be needed in what follows.

Definition 1 ([17]). *Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$ a mapping satisfying*

(1) $d(\xi,\eta) = 0$ if and only if $\xi = \eta$;
(2) $d(\xi,\eta) = d(\eta,\xi)$ for all $\xi,\eta \in X$;
(3) $d(\xi,\eta) \le s[d(\xi,\zeta) + d(\zeta,\eta)]$ for all $\xi,\eta,\zeta \in X$,
where $s \ge 1$ is a given real number, d is then said to be a b-metric on X, and (X,d) is said to be a b-metric space. If $(X,\preceq)$ is still a partially ordered set, then $(X,\preceq,d)$ is said to be a partially ordered b-metric space.

Otherwise, for more notions such as b-convergence, b-completeness, b-Cauchy sequence and b-continuity in b-metric spaces, the reader may refer to [1,4–11,13,15–28] and the references mentioned therein.

In general, b-metric is not continuous; kindly see the following examples.

Example 1 ([16]). *Let $X = \mathbb{N}^* \cup \{\infty\}$; define a mapping $d : X \times X \to \mathbb{R}^+$ using*

$$d(\xi,\eta) = \begin{cases} 0, & \text{when } \xi = \eta; \\ \left|\frac{1}{\xi} - \frac{1}{\eta}\right|, & \text{when one of } \xi,\eta \text{ is even and another is distinctly even or infinity;} \\ 5, & \text{when one of } \xi,\eta \text{ is odd and another is distinctly odd or infinity;} \\ 2, & \text{otherwise.} \end{cases}$$

It is easy to see that when

$$d(\xi,\zeta) \le \frac{5}{2}[d(\xi,\eta) + d(\eta,\zeta)] \ \ (\xi,\eta,\zeta \in X),$$

then (X,d) is a b-metric space with coefficient $s = \frac{5}{2}$. Put $\xi_n = 2n$ $(n \in \mathbb{N})$, then

$$d(\xi_n,\infty) = \frac{1}{2n} \to 0 \ \ (n \to \infty),$$

so $\xi_n \to \infty$ $(n \to \infty)$, but $d(\xi_n,1) = 2 \nrightarrow 5 = d(\infty,1)$ $(n \to \infty)$. That is to say, the b-metric is not continuous.

Example 2 ([6]). *Let $X = \mathbb{R}$ and $\alpha > 1$ be a constant. Define a mapping $d : X \times X \to \mathbb{R}^+$ using*

$$d(\xi,\eta) = \begin{cases} |\xi - \eta|, & \xi\eta \ne 0, \\ \alpha|\xi - \eta|, & \xi\eta = 0, \end{cases} \quad \text{for all } \xi,\eta \in X.$$

Then, (X,d) is a b-metric space with coefficient $s = \alpha$, but the b-metric d is not continuous.

Definition 2 ([20,27,28]). *Let $(X,\preceq,d)$ be a partially ordered b-metric space and f,g,h be self-mappings on X such that $f(X) \cup g(X) \subseteq h(X)$.*

(1) If $\xi,\eta \in X$, $\xi \preceq \eta$ or $\eta \preceq \xi$ holds, then the elements ξ,η are called comparable;
(2) If $f\xi \preceq gf\xi$ for all $\xi \in X$, then the pair (f,g) is called partially weakly increasing;
(3) If $f\xi \preceq g\eta$ for all $\eta \in h^{-1}(f\xi)$, then the pair (f,g) is called partially weakly increasing with respect to h;
(4) If $\lim_{n\to\infty} d(fg\xi_n, gf\xi_n) = 0$, whenever $\{\xi_n\}$ is a sequence in X such that $\lim_{n\to\infty} f\xi_n = \lim_{n\to\infty} g\xi_n = t$ for some $t \in X$, then the pair (f,g) is called compatible;

(5) *If $w = f\xi = g\xi$ for some ξ in X, then ξ is called a coincidence point of f and g, and w is called a point of coincidence of f and g.*
(6) *If f and g commute at their coincidence points, i.e., $fg\xi = gf\xi$, where $f\xi = g\xi$, then the pair (f, g) is called weakly compatible;*
(7) *If $\xi \preceq f\xi$ for each $\xi \in X$, then f is called dominating. If $f\xi \preceq \xi$ for each $\xi \in X$, then f is called dominated.*

Example 3 ([14])**.** *Let $f : X \to X$ be a mapping defined by $f\xi = \xi^{\frac{1}{4}}$, where $X = [0, 1]$ is endowed with usual ordering. Clearly, $\xi \leq \xi^{\frac{1}{4}} = f\xi$ for all $\xi \in X$; thus, f is a dominating map.*

Example 4 ([14])**.** *Let $f : X \to X$ be a mapping defined by $f\xi = \frac{1}{\xi+1}$, where $X = [1, +\infty)$ is endowed with usual ordering. Clearly, $f\xi = \frac{1}{\xi+1} \leq \xi$ for all $\xi \in X$; hence, f is a dominated map.*

An assertion similar to the following lemma was used (and proved) in the course of proofs of several fixed point results in various articles.

Lemma 1 ([26])**.** *Every sequence $\{\xi_n\}_{n\in\mathbb{N}}$ from a b-metric space (X, d) with the property that there exists $\gamma \in [0, 1)$ such that*

$$d(\xi_n, \xi_{n+1}) \leq \gamma d(\xi_{n-1}, \xi_n)$$

for every $n \in \mathbb{N}$ is b-Cauchy.

Lemma 2. *Let (X, d) be a b-metric space with $s \geq 1$ and $\{\eta_n\}$ a sequence in X such that $\lim\limits_{n\to\infty} d(\eta_{n+1}, \eta_n) = 0$. If $\{\eta_{2n}\}$ or $\{\eta_{2n-1}\}$ is a b-Cauchy sequence, then $\{\eta_n\}$ is a b-Cauchy sequence in X.*

Proof. We only prove the case that $\{\eta_{2n}\}$ is a b-Cauchy sequence. The another case can be proved similarly.

In view of $\lim\limits_{n\to\infty} d(\eta_{n+1}, \eta_n) = 0$, then for every $\varepsilon > 0$, there is a natural number N_1 such that, for all $n \geq N_1$,

$$d(\eta_{n+1}, \eta_n) < \frac{\varepsilon}{3s^2}. \tag{1}$$

Since $\{\eta_{2n}\}$ is a b-Cauchy sequence, then for the above $\varepsilon > 0$ there is a natural number N_2 such that, for all $n \geq N_2$ and any $p \in \mathbb{N}$, one has

$$d(\eta_{2n}, \eta_{2n+2p}) < \frac{\varepsilon}{3s^2}. \tag{2}$$

Now, let $N = \max\{N_1, N_2\}$. We shall claim that, for all $n > N$, it satisfies $d(\eta_n, \eta_{n+p}) < \varepsilon$. We complete the proof with four cases.

(c1) If n and p are even numbers, then, by (2), it follows that

$$d(\eta_n, \eta_{n+p}) < \frac{\varepsilon}{3s^2} < \varepsilon.$$

(c2) If n is an odd number and $n + p$ is an even number, then, by (1) and (2), one has

$$d(\eta_n, \eta_{n+p}) \leq s[d(\eta_n, \eta_{n+1}) + d(\eta_{n+1}, \eta_{n+p})] < s\left(\frac{\varepsilon}{3s^2} + \frac{\varepsilon}{3s^2}\right) < \varepsilon.$$

(c3) If n is an even number and $n + p$ is an odd number, then, by (1) and (2), it is easy to see that

$$d(\eta_n, \eta_{n+p}) \leq s[d(\eta_n, \eta_{n+p+1}) + d(\eta_{n+p}, \eta_{n+p+1})] < s\left(\frac{\varepsilon}{3s^2} + \frac{\varepsilon}{3s^2}\right) < \varepsilon.$$

(c4) If n and $n+p$ all are odd numbers, then, by (1) and (2), we have

$$\begin{aligned} d(\eta_n, \eta_{n+p}) &\leq s[d(\eta_n, \eta_{n+1}) + d(\eta_{n+1}, \eta_{n+p})] \\ &\leq sd(\eta_n, \eta_{n+1}) + s^2[d(\eta_{n+1}, \eta_{n+p+1}) + d(\eta_{n+p}, \eta_{n+p+1})] \\ &\leq s \cdot \frac{\varepsilon}{3s^2} + s^2 \cdot \left(\frac{\varepsilon}{3s^2} + \frac{\varepsilon}{3s^2}\right) \\ &< \varepsilon. \end{aligned}$$

Hence, $\{\eta_n\}$ is a b-Cauchy sequence. □

3. Main Results

In this section, we improve and generalize some common fixed point theorems from several references in several sides.

Throughout this paper, let P be the family of all functions $\beta : [0, +\infty) \to [0, 1)$. Let Ψ be the family of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition that ψ is continuous, nondecreasing and $\psi(t) = 0$ if and only if $t = 0$. In this case, ψ is called altering distance function.

Let $(X, \preceq, d)$ be a partially ordered b-metric space with $s > 1$ and $f, g, S, T : X \to X$ be mappings. If there exist $\beta \in P$ and $\psi \in \Psi$ and a constant $L \geq 0$ such that for every two comparable elements $\xi, \eta \in X$, it satisfies

$$\psi(s^\varepsilon d(f\xi, g\eta)) \leq \beta(M(\xi, \eta))\psi\big(\max\{d(S\xi, T\eta), d(S\xi, f\xi), d(T\eta, g\eta)\}\big) + L\psi(N(\xi, \eta)), \tag{3}$$

where

$$M(\xi, \eta) = \psi\left(\max\left\{d(S\xi, T\eta), d(S\xi, f\xi), d(T\eta, g\eta), \frac{d(S\xi, g\eta) + d(T\eta, f\xi)}{2s}\right\}\right), \tag{4}$$

$$N(\xi, \eta) = \min\{d(S\xi, T\eta), d(S\xi, g\eta), d(T\eta, f\xi), d(g\eta, T\eta), d(\xi, g\eta)\}, \tag{5}$$

and $\varepsilon > 0$ is a constant, then (f, g) is said to be a (ψ, β, L)-ordered contractive pair with respect to S and T.

Theorem 1. *Let f, g, S and T be self-mappings on a partially ordered b-complete b-metric space $(X, \preceq, d)$ with $s > 1$. Let $fX \subseteq TX$ and $gX \subseteq SX$. Suppose that (f, g) of dominating maps is a (ψ, β, L)-ordered contractive pair with respect to dominated maps S and T. If $\{\xi_n\}$ is a nondecreasing sequence with $\xi_n \preceq \eta_n$ for all n and $\eta_n \to u$ implies $\xi_n \preceq u$ and either*

(i) f or S is b-continuous, (f, S) are compatible and (g, T) are weakly compatible;

(ii) g or T is b-continuous, (g, T) are compatible and (f, S) are weakly compatible, then the mappings f, g, S and T possess a common fixed point in X. Moreover, the set of common points of f, g, S and T is well ordered if f, g, S and T have a unique common fixed point.

Proof. Let ξ_0 be an arbitrary point of X. Similar to [14], we construct a sequence $\{\eta_n\}$ in X such that $\eta_{2n-1} = T\xi_{2n-1} = f\xi_{2n-2}$ and $\eta_{2n} = S\xi_{2n} = g\xi_{2n-1}$. Since f, g are dominating maps and S, T are dominated maps, it follows that $\xi_{2n-2} \preceq f\xi_{2n-2} = T\xi_{2n-1} \preceq \xi_{2n-1}$ and $\xi_{2n-1} \preceq g\xi_{2n-1} = S\xi_{2n} \preceq \xi_{2n}$. Thus, for all $n \geq 1$, we have $\xi_n \preceq \xi_{n+1}$.

Without loss of generality, we assume $d(\eta_{2n}, \eta_{2n+1}) > 0$ for every n. If not, then $\eta_{2n_0} = \eta_{2n_0+1}$ for some $n_0 \in \mathbb{N}$. From (3) to (5), we obtain

$$\begin{aligned} &\psi(s^\varepsilon d(\eta_{2n_0+1}, \eta_{2n_0+2})) = \psi(s^\varepsilon d(f\xi_{2n_0}, g\xi_{2n_0+1})) \\ &\leq \beta(M(\xi_{2n_0}, \xi_{2n_0+1})) \times \psi(\max\{d(S\xi_{2n_0}, T\xi_{2n_0+1}), d(S\xi_{2n_0}, f\xi_{2n_0}), d(T\xi_{2n_0+1}, g\xi_{2n_0+1})\}) \\ &\quad + L\psi(N(\xi_{2n_0}, \xi_{2n_0+1})) \\ &= \beta(M(\xi_{2n_0}, \xi_{2n_0+1})) \times \psi(\max\{d(\eta_{2n_0}, \eta_{2n_0+1}), d(\eta_{2n_0}, \eta_{2n_0+1}), d(\eta_{2n_0+1}, \eta_{2n_0+2})\}) \end{aligned}$$

$$
\begin{aligned}
&+ L\psi(N(\xi_{2n_0}, \xi_{2n_0+1})) \\
&= \beta(M(\xi_{2n_0}, \xi_{2n_0+1})) \times \psi(\max\{0, 0, d(\eta_{2n_0+1}, \eta_{2n_0+2})\}) \\
&= \beta(M(\xi_{2n_0}, \xi_{2n_0+1}))\psi(d(\eta_{2n_0+1}, \eta_{2n_0+2})) \\
&\leq \psi(d(\eta_{2n_0+1}, \eta_{2n_0+2})),
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
&N(\xi_{2n_0}, \xi_{2n_0+1})) \\
&= \min\{d(S\xi_{2n_0}, T\xi_{2n_0+1}), d(S\xi_{2n_0}, g\xi_{2n_0+1}), d(T\xi_{2n_0+1}, f\xi_{2n_0}), d(g\xi_{2n_0+1}, T\xi_{2n_0+1}), d(\xi_{2n_0}, g\xi_{2n_0+1})\} \\
&= \min\{d(\eta_{2n_0}, \eta_{2n_0+1}), d(\eta_{2n_0}, \eta_{2n_0+2}), d(\eta_{2n_0+1}, \eta_{2n_0+1}), d(\eta_{2n_0+2}, \eta_{2n_0+1}), d(\xi_{2n_0}, \eta_{2n_0+2})\} \\
&= \min\{0, d(\eta_{2n_0}, \eta_{2n_0+2}), 0, d(\eta_{2n_0+2}, \eta_{2n_0+1}), d(\xi_{2n_0}, \eta_{2n_0+2})\} \\
&= 0,
\end{aligned}
$$

then

$$\psi(N(\xi_{2n_0}, \xi_{2n_0+1})) = 0.$$

Thus, by the monotonicity of function ψ, it is valid from (6) that

$$s^{\varepsilon} d(\eta_{2n_0+1}, \eta_{2n_0+2}) \leq d(\eta_{2n_0+1}, \eta_{2n_0+2}),$$

that is,

$$d(\eta_{2n_0+1}, \eta_{2n_0+2}) \leq \frac{1}{s^{\varepsilon}} d(\eta_{2n_0+1}, \eta_{2n_0+2}),$$

so $d(\eta_{2n_0+1}, \eta_{2n_0+2}) = 0$ (because $s^{\varepsilon} > 1$). Hence, $\eta_{2n_0+1} = \eta_{2n_0+2}$.

Following similar arguments, we obtain $\eta_{2n_0+2} = \eta_{2n_0+3}$. Thus, $\{\eta_n\}$ becomes a constant sequence, and η_{2n_0} is the common fixed point of f, g, S and T. In this case, the conclusion we need to prove is clear.

Now, we take $d(\eta_{2n}, \eta_{2n+1}) > 0$ for each n. As ξ_{2n} and ξ_{2n+1} are comparable, by inequality (3) we have

$$
\begin{aligned}
&\psi(s^{\varepsilon} d(\eta_{2n+1}, \eta_{2n+2})) = \psi(s^{\varepsilon} d(f\xi_{2n}, g\xi_{2n+1})) \\
&\leq \beta(M(\xi_{2n}, \xi_{2n+1})) \times \psi(\max\{d(S\xi_{2n}, T\xi_{2n+1}), d(S\xi_{2n}, f\xi_{2n}), d(T\xi_{2n+1}, g\xi_{2n+1})\}) \\
&\quad + L\psi(N(\xi_{2n}, \xi_{2n+1})) \\
&= \beta(M(\xi_{2n}, \xi_{2n+1})) \times \psi(\max\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2})\}) \\
&\quad + L\psi(N(\xi_{2n}, \xi_{2n+1})) \\
&= \beta(M(\xi_{2n}, \xi_{2n+1})) \times \psi(\max\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2})\}), \\
&< \psi(\max\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2})\}),
\end{aligned}
$$

which follows immediately from the monotonicity of function ψ that

$$s^{\varepsilon} d(\eta_{2n+1}, \eta_{2n+2}) < \max\{d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n+1}, \eta_{2n+2})\}. \tag{7}$$

Now, if

$$d(\eta_{2n}, \eta_{2n+1}) \leq d(\eta_{2n+1}, \eta_{2n+2}),$$

then by using (7) we have

$$s^{\varepsilon} d(\eta_{2n+1}, \eta_{2n+2}) < d(\eta_{2n+1}, \eta_{2n+2}),$$

which leads to a contradiction because of $s^{\varepsilon} > 1$. Therefore, $d(\eta_{2n}, \eta_{2n+1}) > d(\eta_{2n+1}, \eta_{2n+2})$. In this case, we have

$$s^{\varepsilon} d(\eta_{2n+1}, \eta_{2n+2}) < d(\eta_{2n}, \eta_{2n+1}),$$

which implies that

$$d(\eta_{2n+1}, \eta_{2n+2}) < \frac{1}{s^{\varepsilon}} d(\eta_{2n}, \eta_{2n+1}). \tag{8}$$

Again, by inequality (3) we have

$$\begin{aligned}
&\psi(s^{\varepsilon} d(\eta_{2n+1}, \eta_{2n})) = \psi(s^{\varepsilon} d(f\xi_{2n}, g\xi_{2n-1})) \\
&\leq \beta(M(\xi_{2n}, \xi_{2n-1})) \times \psi(\max\{d(S\xi_{2n}, T\xi_{2n-1}), d(S\xi_{2n}, f\xi_{2n}), d(T\xi_{2n-1}, g\xi_{2n-1})\}) \\
&\quad + L\psi(N(\xi_{2n}, \xi_{2n-1})) \\
&= \beta(M(\xi_{2n}, \xi_{2n-1})) \times \psi(\max\{d(\eta_{2n}, \eta_{2n-1}), d(\eta_{2n}, \eta_{2n+1}), d(\eta_{2n-1}, \eta_{2n})\}) \\
&\quad + L\psi(N(\xi_{2n}, \xi_{2n-1})) \\
&= \beta(M(\xi_{2n}, \xi_{2n-1})) \times \psi(\max\{d(\eta_{2n}, \eta_{2n-1}), d(\eta_{2n}, \eta_{2n+1})\}) \\
&< \psi(\max\{d(\eta_{2n}, \eta_{2n-1}), d(\eta_{2n}, \eta_{2n+1})\}),
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
&N(\xi_{2n}, \xi_{2n-1})) \\
&= \min\{d(S\xi_{2n}, T\xi_{2n-1}), d(S\xi_{2n}, g\xi_{2n-1}), d(T\xi_{2n-1}, f\xi_{2n}), d(g\xi_{2n-1}, T\xi_{2n-1}), d(\xi_{2n}, g\xi_{2n-1})\} \\
&= \min\{d(\eta_{2n}, \eta_{2n-1}), d(\eta_{2n}, \eta_{2n}), d(\eta_{2n-1}, \eta_{2n+1}), d(\eta_{2n}, \eta_{2n-1}), d(\xi_{2n}, \eta_{2n})\} \\
&= \min\{d(\eta_{2n}, \eta_{2n-1}), 0, d(\eta_{2n-1}, \eta_{2n+1}), d(\eta_{2n}, \eta_{2n-1}), d(\xi_{2n}, \eta_{2n})\} \\
&= 0.
\end{aligned}$$

By (9) and the monotonicity of function ψ, we have

$$s^{\varepsilon} d(\eta_{2n+1}, \eta_{2n}) < \max\{d(\eta_{2n}, \eta_{2n-1}), d(\eta_{2n}, \eta_{2n+1})\}. \tag{10}$$

Now, if

$$d(\eta_{2n}, \eta_{2n-1}) \leq d(\eta_{2n}, \eta_{2n+1}),$$

then by using (10) we have

$$s^{\varepsilon} d(\eta_{2n}, \eta_{2n+1}) < d(\eta_{2n}, \eta_{2n+1}),$$

which leads to a contradiction because of $s^{\varepsilon} > 1$. Therefore, $d(\eta_{2n}, \eta_{2n-1}) > d(\eta_{2n}, \eta_{2n+1})$. In this case, we have

$$s^{\varepsilon} d(\eta_{2n}, \eta_{2n+1}) < d(\eta_{2n-1}, \eta_{2n}),$$

which implies that

$$d(\eta_{2n}, \eta_{2n+1}) < \frac{1}{s^{\varepsilon}} d(\eta_{2n-1}, \eta_{2n}). \tag{11}$$

Making full use of (8) and (11), we have

$$d(\eta_n, \eta_{n+1}) < \frac{1}{s^{\varepsilon}} d(\eta_{n-1}, \eta_n).$$

Accordingly, by Lemma 1, we claim that $\{\eta_n\}$ is a b-Cauchy sequence. Since $(X, \preceq, d)$ is b-complete, then there exists a point ζ in X such that $\{\eta_n\}$ converges to ζ.

We first suppose that (i) holds. Assume that S is b-continuous. Because (f, S) are compatible, we have

$$\lim_{n\to\infty} d(fS\xi_{2n+2}, Sf\xi_{2n+2}) = 0,$$

that is,

$$\lim_{n\to\infty} d(f\eta_{2n+2}, S\eta_{2n+3}) = 0.$$

As a result, by the b-continuity of S, we speculate that

$$d(f\eta_{2n+2}, S\zeta) \le s[d(f\eta_{2n+2}, S\eta_{2n+3}) + d(S\eta_{2n+3}, S\zeta)] \to 0 \ \ (n \to \infty),$$

which implies that

$$\lim_{n\to\infty} d(f\eta_{2n+2}, S\zeta) = 0. \tag{12}$$

Now, we show that $\zeta = S\zeta$. If not, that is, $d(S\zeta, \zeta) > 0$. As $\xi_{2n+1} \preceq g\xi_{2n+1} = S\xi_{2n+2}$, from inequality (3), we have

$$\begin{aligned}
&\psi(s^{\varepsilon} d(f\eta_{2n+2}, \eta_{2n+2})) \\
&= \psi(s^{\varepsilon} d(fS\xi_{2n+2}, g\xi_{2n+1})) \\
&\le \beta(M(S\xi_{2n+2}, \xi_{2n+1})) \times \psi(\max\{d(SS\xi_{2n+2}, T\xi_{2n+1}), \\
&\quad d(SS\xi_{2n+2}, fS\xi_{2n+2}), d(T\xi_{2n+1}, g\xi_{2n+1})\} \\
&\quad + L\psi(N(S\xi_{2n+2}, \xi_{2n+1})) \\
&< \psi(\max\{d(SS\xi_{2n+2}, T\xi_{2n+1}), d(SS\xi_{2n+2}, fS\xi_{2n+2}), d(T\xi_{2n+1}, g\xi_{2n+1})\}) \\
&= \psi(\max\{d(S\eta_{2n+2}, \eta_{2n+1}), d(S\eta_{2n+2}, f\eta_{2n+2}), d(\eta_{2n+1}, \eta_{2n+2})\}),
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
N(S\xi_{2n+2}, \xi_{2n+1})) &= \min\{d(SS\xi_{2n+2}, T\xi_{2n+1}), d(SS\xi_{2n+2}, g\xi_{2n+1}), d(T\xi_{2n+1}, fS\xi_{2n+2}), \\
&\quad d(g\xi_{2n+1}, T\xi_{2n+1}), d(S\xi_{2n+2}, g\xi_{2n+1})\} \\
&= \min\{d(S\eta_{2n+2}, \eta_{2n+1}), d(S\eta_{2n+2}, \eta_{2n+2}), d(\eta_{2n+1}, f\eta_{2n+2}), \\
&\quad d(\eta_{2n+2}, \eta_{2n+1}), d(\eta_{2n+2}, \eta_{2n+2})\} \\
&= 0.
\end{aligned}$$

By (13) and the monotonicity of function ψ, we have

$$s^{\varepsilon} d(f\eta_{2n+2}, \eta_{2n+2}) < \max\{d(S\eta_{2n+2}, \eta_{2n+1}), d(S\eta_{2n+2}, f\eta_{2n+2}), d(\eta_{2n+1}, \eta_{2n+2})\}. \tag{14}$$

Combing (12) and the b-continuity of function S, we obtain

$$d(S\eta_{2n+2}, f\eta_{2n+2}) \le s[d(S\eta_{2n+2}, S\zeta) + d(S\zeta, f\eta_{2n+2})] \to s(0+0) = 0 \ (n \to \infty),$$

which means that

$$\lim_{n\to\infty} d(S\eta_{2n+2}, f\eta_{2n+2}) = 0. \tag{15}$$

Moreover, since $\{\eta_n\}$ is a b-Cauchy sequence, we have

$$\lim_{n\to\infty} d(\eta_{2n+1}, \eta_{2n+2}) = 0. \tag{16}$$

By (15) and (16), there exists $N_1 \in \mathbb{N}$ such that, for all $n > N_1$, one has

$$d(S\eta_{2n+2}, \eta_{2n+1}) > d(S\eta_{2n+2}, f\eta_{2n+2}), \quad d(S\eta_{2n+2}, \eta_{2n+1}) > d(\eta_{2n+1}, \eta_{2n+2}). \tag{17}$$

Via (14) and (17), it is easy to see that

$$s^{\varepsilon} d(f\eta_{2n+2}, \eta_{2n+2}) < d(S\eta_{2n+2}, \eta_{2n+1}) \ \ (n > N_1). \tag{18}$$

Using the triangular inequality of the b-metric (12) and the b-continuity of function S, we obtain

$$\begin{aligned}
&\left|d(f\eta_{2n+2},\eta_{2n+2})-d(S\eta_{2n+2},\eta_{2n+1})\right| \\
&=\big|[d(f\eta_{2n+2},\eta_{2n+2})-sd(\eta_{2n+2},S\zeta)]+[sd(\eta_{2n+2},S\zeta)-d(S\zeta,\zeta)] \\
&\quad+[d(S\zeta,\zeta)-sd(S\eta_{n+2},\zeta)]+[sd(S\eta_{n+2},\zeta)-d(S\eta_{n+2},\eta_{2n+1})]\big| \\
&\leq\left|d(f\eta_{2n+2},\eta_{2n+2})-sd(\eta_{2n+2},S\zeta)\right|+\left|sd(\eta_{2n+2},S\zeta)-d(S\zeta,\zeta)\right| \\
&\quad+\left|d(S\zeta,\zeta)-sd(S\eta_{n+2},\zeta)\right|+\left|sd(S\eta_{n+2},\zeta)-d(S\eta_{n+2},\eta_{2n+1})\right| \\
&\leq sd(f\eta_{2n+2},S\zeta)+sd(\eta_{2n+2},\zeta)+sd(S\zeta,S\eta_{n+2})+sd(\zeta,\eta_{2n+1}) \\
&\to 0\ \ (n\to\infty),
\end{aligned}$$

which establishes that

$$\limsup_{n\to\infty} d(f\eta_{2n+2},\eta_{2n+2})=\limsup_{n\to\infty} d(S\eta_{2n+2},\eta_{2n+1}). \tag{19}$$

Consider (18) and (19), they lead to a contradiction because of $s^{\varepsilon}>1$. Consequently, $S\zeta=\zeta$.

Now, by virtue of $\xi_{2n+1}\preceq g\xi_{2n+1}$ and $g\xi_{2n+1}\to\zeta$ as $n\to\infty$, $\xi_{2n+1}\preceq\zeta$. It suffices to prove $\zeta=f\zeta$. As a matter of fact, firstly, notice that

$$\begin{aligned}
&\left|d(f\zeta,\eta_{n+2})-d(\zeta,f\zeta)\right| \\
&=\left|[d(f\zeta,\eta_{n+2})-sd(f\zeta,\eta_{n+1})]+[sd(f\zeta,\eta_{n+1})-d(\zeta,f\zeta)]\right| \\
&\leq\left|d(f\zeta,\eta_{n+2})-sd(f\zeta,\eta_{n+1})\right|+\left|sd(f\zeta,\eta_{n+1})-d(\zeta,f\zeta)\right| \\
&\leq sd(\eta_{n+2},\eta_{n+1})+sd(\eta_{n+1},\zeta)\to 0\ \ (n\to\infty),
\end{aligned}$$

which follows that

$$\limsup_{n\to\infty} d(f\zeta,\eta_{n+2})=d(\zeta,f\zeta). \tag{20}$$

Secondly, by inequality (3), we have

$$\begin{aligned}
&\psi(s^{\varepsilon}d(f\zeta,\eta_{2n+2})) \\
&=\psi(s^{\varepsilon}d(f\zeta,g\xi_{2n+1})) \\
&\leq\beta(M(\zeta,\xi_{2n+1}))\times\psi(\max\{d(S\zeta,T\xi_{2n+1}),d(S\zeta,f\zeta),d(T\xi_{2n+1},g\xi_{2n+1})\}) \\
&\quad+L\psi(N(\zeta,\xi_{2n+1})) \\
&=\beta(M(\zeta,\xi_{2n+1}))\times\psi(\max\{d(\zeta,T\xi_{2n+1}),d(\zeta,f\zeta),d(T\xi_{2n+1},g\xi_{2n+1})\}) \\
&\quad+L\psi(N(\zeta,\xi_{2n+1})) \\
&<\psi(\max\{d(\zeta,\eta_{2n+1}),d(\zeta,f\zeta),d(\eta_{2n+1},\eta_{2n+2})\}) \\
&\quad+L\psi(N(\zeta,\xi_{2n+1})),
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\psi(N(\zeta,\xi_{2n+1}))=\psi(\min\{&d(S\zeta,T\xi_{2n+1}),d(S\zeta,g\xi_{2n+1}),d(T\xi_{2n+1},f\zeta),\\
&d(g\xi_{2n+1},T\xi_{2n+1}),d(\zeta,g\xi_{2n+1})\})
\end{aligned}$$

tends to

$$\psi\left(\min\left\{\limsup_{n\to\infty} d(S\zeta,T\xi_{2n+1}),\limsup_{n\to\infty} d(S\zeta,g\xi_{2n+1}),\limsup_{n\to\infty} d(T\xi_{2n+1},f\zeta),0,0\right\}\right)=\psi(0)=0$$

as $n\to\infty$. By taking the upper limit as $n\to\infty$ from (21), we obtain

$$s^{\varepsilon}\limsup_{n\to\infty} d(f\zeta,\eta_{2n+2})\leq d(f\zeta,\zeta),$$

which is a contradiction with $s^{\varepsilon} > 1$ and (20) unless $\limsup_{n\to\infty} d(f\zeta, \eta_{2n+2}) = 0$. As a result, by

$$\frac{1}{s}d(f\zeta,\zeta) \le d(f\zeta,\eta_{2n+2}) + d(\eta_{2n+2},\zeta),$$

we have

$$\frac{1}{s}d(f\zeta,\zeta) \le \limsup_{n\to\infty}[d(f\zeta,\eta_{2n+2}) + d(\eta_{2n+2},\zeta)] = 0,$$

which implies that $d(f\zeta,\zeta) = 0$. Hence, $f\zeta = \zeta$.

In view of $f(X) \subseteq T(X)$, then there exists a point $w \in X$ such that $\zeta = f\zeta = Tw$. Now, we show $d(Tw, gw) = 0$. If not, i.e., $d(Tw, gw) > 0$. Since $\zeta \preceq f\zeta = Tw \preceq w$ implies $\zeta \preceq w$, from inequality (3), we have

$$\begin{aligned}
&\psi(s^{\varepsilon}d(Tw,gw)) \\
&= \psi(s^{\varepsilon}d(f\zeta,gw)) \\
&\le \beta(M(\zeta,w)) \times \psi(\max\{d(S\zeta,Tw), d(S\zeta,f\zeta), d(Tw,gw)\}) \\
&\quad + L\psi(N(\zeta,w)) \\
&= \beta(M(\zeta,w)) \times \psi(\max\{0,0,d(Tw,gw)\}) \\
&\quad + L\psi(\min\{d(S\zeta,Tw), d(S\zeta,gw), d(Tw,f\zeta), d(gw,Tw), d(\zeta,gw)\})) \\
&= \beta(M(\zeta,w)) \times \psi(d(Tw,gw)) \\
&\quad + L\psi(\min\{0, d(\zeta,gw), 0, d(gw,\zeta), d(\zeta,gw)\})) \\
&= \beta(M(\zeta,w))\psi(d(Tw,gw)) \\
&< \psi(d(Tw,gw)).
\end{aligned}$$

Hence, by the monotonicity of function ψ, we have

$$s^{\varepsilon}d(Tw,gw) < d(Tw,gw),$$

which leads to a contradiction with $s^{\varepsilon} > 1$. As a result, $Tw = gw$. On account of the fact that g and T are weakly compatible, we obtain

$$g\zeta = gf\zeta = gTw = Tgw = Tf\zeta = T\zeta.$$

Thus, ζ is a coincidence point of g and T. Next, we show that $\zeta = g\zeta$. As $\xi_{2n} \preceq f\xi_{2n}$ and $f\xi_{2n} \to \zeta$ $(n \to \infty)$ implies that $\xi_{2n} \preceq \zeta$, from (3) we have

$$\begin{aligned}
&\psi(s^{\varepsilon}d(\eta_{2n+1},g\zeta)) = \psi(s^{\varepsilon}d(f\xi_{2n},g\zeta)) \\
&\le \beta(M(\xi_{2n},\zeta)) \times \psi(\max\{d(S\xi_{2n},T\zeta), d(S\xi_{2n},f\xi_{2n}), d(T\zeta,g\zeta)\}) + L\psi(N(\xi_{2n},\zeta)) \\
&< \psi(\max\{d(S\xi_{2n},T\zeta), d(S\xi_{2n},f\xi_{2n}), d(T\zeta,g\zeta)\}) \\
&\quad + L\psi(\min\{d(S\xi_{2n},T\zeta), d(S\xi_{2n},g\zeta), d(T\zeta,f\xi_{2n}), d(g\zeta,T\zeta), d(\xi_{2n},g\zeta)\}) \\
&= \psi(\max\{d(S\xi_{2n},T\zeta), d(S\xi_{2n},f\xi_{2n}), 0\}) \\
&\quad + L\psi(\min\{d(S\xi_{2n},T\zeta), d(S\xi_{2n},g\zeta), d(T\zeta,f\xi_{2n}), 0, d(\xi_{2n},g\zeta)\}) \\
&= \psi(\max\{d(S\xi_{2n},T\zeta), d(S\xi_{2n},f\xi_{2n})\}) + L\psi(0) \\
&= \psi(\max\{d(\eta_{2n},g\zeta), d(\eta_{2n},\eta_{2n+1})\}),
\end{aligned}$$

which follows immediately from the monotonicity of function ψ that

$$s^{\varepsilon}d(f\eta_{2n+1},g\zeta) < \max\{d(\eta_{2n},g\zeta), d(\eta_{2n},\eta_{2n+1})\}. \tag{22}$$

Since $\lim_{n\to\infty} d(\eta_{2n}, \eta_{2n+1}) = 0$, then there exists $N_2 \in \mathbb{N}$ such that, for all $n > N_2$, one has

$$d(\eta_{2n}, g\zeta) > d(\eta_{2n}, \eta_{2n+1}). \tag{23}$$

Considering (22) and (23), we acquire

$$s^{\varepsilon} d(\eta_{2n+1}, g\zeta) < d(\eta_{2n}, g\zeta) \ \ (n > N_2).$$

Thus, it implies

$$s^{\varepsilon} \limsup_{n\to\infty} d(\eta_{2n+1}, g\zeta) \le \limsup_{n\to\infty} d(\eta_{2n}, g\zeta). \tag{24}$$

By virtue of

$$\begin{aligned}
&\left|d(\eta_{2n+1}, g\zeta) - d(\eta_{2n}, g\zeta)\right| \\
&= \left|[d(\eta_{2n+1}, g\zeta) - sd(g\zeta, \eta_{2n-1})] + [sd(g\zeta, \eta_{2n-1}) - d(\eta_{2n}, g\zeta)]\right| \\
&\le \left|d(\eta_{2n+1}, g\zeta) - sd(g\zeta, \eta_{2n-1})\right| + \left|sd(g\zeta, \eta_{2n-1}) - d(\eta_{2n}, g\zeta)\right| \\
&\le sd(\eta_{2n+1}, \eta_{2n-1}) + sd(\eta_{2n-1}, \eta_{2n}) \to 0 \ \ (n \to \infty),
\end{aligned}$$

which follows that

$$\limsup_{n\to\infty} d(\eta_{2n+1}, g\zeta) = \limsup_{n\to\infty} d(\eta_{2n}, g\zeta).$$

Consequently, (24) is a contradiction with $s^{\varepsilon} > 1$ unless $\limsup_{n\to\infty} d(\eta_{2n+1}, g\zeta) = 0$. That is to say, $\limsup_{n\to\infty} d(\eta_{2n+1}, g\zeta) = 0$. Now, by

$$\frac{1}{s} d(g\zeta, \zeta) \le d(g\zeta, \eta_{2n+1}) + d(\eta_{2n+1}, \zeta),$$

we claim that

$$\frac{1}{s} d(g\zeta, \zeta) \le \limsup_{n\to\infty} d(g\zeta, \eta_{2n+1}) + \limsup_{n\to\infty} d(\eta_{2n+1}, \zeta) = 0,$$

which follows that $d(g\zeta, \zeta) = 0$. Thus, $g\zeta = \zeta$.

To sum up, $f\zeta = g\zeta = S\zeta = T\zeta = \zeta$. In other words, ζ is a common fixed point of f, g, S and T. The proof is similar when f is b-continuous.

Similarly, the result follows when (ii) holds. Now, suppose that the set of common fixed points of f, g, S and T is well ordered; we will show that the common fixed point of f, g, S and T is unique. Indeed, assume, on the contrary, that $fq = gq = Sq = Tq = q$ and $fr = gr = Sr = Tr = r$ but $d(q, r) > 0$. By the given assumption, we replace ξ with q and η with r in (3). Then,

$$\begin{aligned}
&\psi(s^{\varepsilon} d(q, r)) = \psi(s^{\varepsilon} d(fq, gr)) \\
&\le \beta(M(q, r)) \times \psi(\max\{d(Sq, Tr), d(Sq, fq), d(Tr, gr)\}) \\
&\quad + L\psi(N(q, r)) \\
&< \psi(d(q, r)).
\end{aligned}$$

Consequently, we claim that $s^{\varepsilon} d(q, r) < d(q, r)$, a contradiction. As a result, $q = r$ (because $s^{\varepsilon} > 1$). In reverse, if f, g, S and T are single, then it is well ordered. □

Remark 1. *Theorem 1 greatly generalizes Theorem 12 of [14] from several sides. On the one hand, Theorem 1 refers to the conclusion in the setting of b-metric spaces, whereas Theorem 12 of [14] considered the result in usual metric space. It is well-known that b-metric space is a sharp generalization of usual metric space since the given b-metric usually is not necessarily continuous,* *but the usual metric must be a continuous function. Therefore, Theorem 1 is more important than Theorem 12 of [14]. On the other hand, as compared with Theorem 12 of [14], function β from Theorem 1 satisfies the simpler condition. In addition, the proof of Theorem 1 is more straightforward than the one of Theorem 12 of [14].*

Remark 2. *We use Lemma 2 to prove that the constructed sequence is a b-Cauchy sequence instead of using Lemma 11 from [14] or Lemma 1 from [16]. This is a straightforward improvement because Lemma 2 is easily understood for most of our readers. Otherwise, by Lemma 2, we also can prove that the sequence is a b-Cauchy sequence but, due to the complicated process, we omit it in this paper. In addition, in order to overcome the discontinuity of b-metric, we use a new method to prove our theorem. This is a great innovation.*

Example 5. *Let $X = \{1,2,3,4\}$ be a partially ordered set defined as $\xi \preceq \eta$ if and only if $\xi \geq \eta$ and $d(\xi,\eta) = |\xi - \eta|^2$ for all $\xi, \eta \in X$. Then, (X,d) is a b-complete b-metric space with $s = \frac{9}{5}$. Define ordered self-mappings f, g, S, and T on X using*

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}, \ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{pmatrix},$$

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 4 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 4 \end{pmatrix}.$$

It is easy to verify that the mappings f and g are dominating and S and T are dominated. Take $\psi(\xi) = \ln(\xi + 1)$ and $\beta = \frac{\psi(\xi)}{\xi}$; then, the mappings f, g, S and T satisfy all the conditions given in Theorem 1 with $\varepsilon = \frac{1}{4}$. Moreover, one is the unique common fixed point of f, g, S, and T.

Theorem 2. *Let f, g, S and T be self-mappings on a partially ordered b-complete b-metric space $(X, \preceq, d)$ with $s > 1$. Let $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. Suppose that the mappings f, g, S and T are b-continuous, the pairs (f,S) and (g,T) are compatible, and the pairs (f,g) and (g,f) are partially weakly increasing with respect to T and S, respectively. Assume that (f,g) is a (ψ, β, L)-ordered contractive pair with respect to S and T for each $\xi, \eta \in X$, for which $S\xi$ and $T\eta$ are comparable. Then, the pairs (f,S) and (g,T) have a coincidence point $\zeta \in X$. Moreover, ζ is a coincidence point of the mappings f, g, S and T provided that $S\zeta$ and $T\zeta$ are comparable.*

Proof. Choose $\xi_0 \in X$. Define a sequence $\{\eta_n\}$ in X that satisfies $\eta_{2n} = f\xi_{2n} = T\xi_{2n+1}$ and $\eta_{2n+1} = g\xi_{2n+1} = S\xi_{2n+2}$ for all $n \in \mathbb{N}$. By the hypothesis, it is easy to see that $\eta_n \preceq \eta_{n+1}$ for all $n \geq 1$. We start the proof with two steps.

Step 1. We prove that

$$d(\eta_{n+1}, \eta_{n+2}) \leq \lambda d(\eta_n, \eta_{n+1}), \tag{25}$$

for each $n \in \mathbb{N}$, where $\lambda \in [0,1)$ is a constant.

Firstly, we assume $\eta_n \neq \eta_{n+1}$ for each $n \in \mathbb{N}$. Since $S\xi_{2n} = \eta_{2n-1} = g\xi_{2n-1}$ and $T\xi_{2n+1} = \eta_{2n} = f\xi_{2n}$ are comparable, then, via (3), it has

$$\begin{aligned}
&\psi(s^\varepsilon d(\eta_{2n}, \eta_{2n+1})) \\
&= \psi(s^\varepsilon d(f\xi_{2n}, g\xi_{2n+1})) \\
&\leq \beta(M(\xi_{2n}, \xi_{2n+1})) \times \psi(\max\{d(S\xi_{2n}, T\xi_{2n+1}), d(S\xi_{2n}, f\xi_{2n}), d(T\xi_{2n+1}, g\xi_{2n+1})\}) \\
&\quad + L\psi(\min\{d(S\xi_{2n}, T\xi_{2n+1}), d(S\xi_{2n}, g\xi_{2n+1}), d(T\xi_{2n+1}, f\xi_{2n}), d(g\xi_{2n+1}, T\xi_{2n+1}), d(\xi_{2n}, g\xi_{2n+1})\}) \\
&< \psi(\max\{d(\eta_{2n-1}, \eta_{2n}), d(\eta_{2n-1}, \eta_{2n}), d(\eta_{2n}, \eta_{2n+1})\}) \\
&\quad + L\psi(\min\{d(\eta_{2n-1}, \eta_{2n}), d(\eta_{2n-1}, \eta_{2n+1}), d(\eta_{2n}, \eta_{2n}), d(\eta_{2n+1}, \eta_{2n}), d(\xi_{2n}, \eta_{2n+1})\})) \\
&= \psi(\max\{d(\eta_{2n-1}, \eta_{2n}), d(\eta_{2n}, \eta_{2n+1})\}),
\end{aligned}$$

which follows from the monotonicity of function ψ that

$$s^{\varepsilon}d(\eta_{2n},\eta_{2n+1}) < \max\{d(\eta_{2n-1},\eta_{2n}),d(\eta_{2n},\eta_{2n+1})\}. \tag{26}$$

If $d(\eta_{2n-1},\eta_{2n}) \leq d(\eta_{2n},\eta_{2n+1})$, then by (26) it means that

$$s^{\varepsilon}d(\eta_{2n},\eta_{2n+1}) < d(\eta_{2n},\eta_{2n+1}),$$

which leads to a contradiction (because $s^{\varepsilon} > 1$). Then,

$$s^{\varepsilon}d(\eta_{2n},\eta_{2n+1}) < d(\eta_{2n-1},\eta_{2n}). \tag{27}$$

Again, since $S\xi_{2n+2} = \eta_{2n+1} = g\xi_{2n+1}$ and $T\xi_{2n+1} = \eta_{2n} = f\xi_{2n}$ are comparable, then, by (3), it implies that

$$\begin{aligned}
&\psi(s^{\varepsilon}d(\eta_{2n+1},\eta_{2n+2})) = \psi(s^{\varepsilon}d(f\xi_{2n+2},g\xi_{2n+1}))\\
&\leq \beta(M(\xi_{2n+2},\xi_{2n+1})) \times \psi(\max\{d(S\xi_{2n+2},T\xi_{2n+1}),d(S\xi_{2n+2},f\xi_{2n+2}),d(T\xi_{2n+1},g\xi_{2n+1})\})\\
&\quad + L\psi(\min\{d(S\xi_{2n+2},T\xi_{2n+1}),d(S\xi_{2n+2},g\xi_{2n+1}),d(T\xi_{2n+1},f\xi_{2n+2}),\\
&\quad\; d(g\xi_{2n+1},T\xi_{2n+1}),d(\xi_{2n+2},g\xi_{2n+1})\})\\
&< \psi(\max\{d(\eta_{2n+1},\eta_{2n}),d(\eta_{2n+1},\eta_{2n+2}),d(\eta_{2n},\eta_{2n+1})\})\\
&\quad + L\psi(\min\{d(\eta_{2n+1},\eta_{2n}),d(\eta_{2n+1},\eta_{2n+1}),d(\eta_{2n},\eta_{2n+2}),d(\eta_{2n+1},\eta_{2n}),d(\xi_{2n+2},\eta_{2n+1})\}))\\
&= \psi(\max\{d(\eta_{2n+1},\eta_{2n+2}),d(\eta_{2n},\eta_{2n+1})\}),
\end{aligned}$$

which follows from the monotonicity of function ψ that

$$s^{\varepsilon}d(\eta_{2n+1},\eta_{2n+2}) < \max\{d(\eta_{2n+1},\eta_{2n+2}),d(\eta_{2n},\eta_{2n+1})\}. \tag{28}$$

If $d(\eta_{2n},\eta_{2n+1}) \leq d(\eta_{2n+1},\eta_{2n+2})$, then, by means of (28), it establishes

$$s^{\varepsilon}d(\eta_{2n+1},\eta_{2n+2}) < d(\eta_{2n+1},\eta_{2n+2}),$$

which leads to a contradiction (because $s^{\varepsilon} > 1$). Thus,

$$s^{\varepsilon}d(\eta_{2n+1},\eta_{2n+2}) < d(\eta_{2n},\eta_{2n+1}). \tag{29}$$

Now, by using (27) and (29), we obtain (25), where $\lambda = \frac{1}{s^{\varepsilon}} \in [0,1)$.

We now assume $\eta_{n_0} = \eta_{n_0+1}$ for some $n_0 \in \mathbb{N}$. If $n_0 = 2k-1$, then $\eta_{2k-1} = \eta_{2k}$ implies $\eta_{2k} = \eta_{2k+1}$. As a matter of fact, if $\eta_{2k} \neq \eta_{2k+1}$, i.e., $d(\eta_{2k},\eta_{2k+1}) > 0$, then, by the fact that $\eta_{2k} = f\xi_{2k} = T\xi_{2k+1}$ and $\eta_{2k-1} = g\xi_{2k-1} = S\xi_{2k}$ are comparable, then, via (26), we speculate that

$$s^{\varepsilon}d(\eta_{2k},\eta_{2k+1}) < \max\{d(\eta_{2k-1},\eta_{2k}),d(\eta_{2k},\eta_{2k+1})\} = d(\eta_{2k},\eta_{2k+1}).$$

This is a contradiction (because $s^{\varepsilon} > 1$). Hence, $d(\eta_{2k},\eta_{2k+1}) = 0$, i.e., $\eta_{2k} = \eta_{2k+1}$. If $n_0 = 2k$, then $\eta_{2k} = \eta_{2k+1}$ leads to $\eta_{2k+1} = \eta_{2k+2}$. Actually, if $\eta_{2k+1} \neq \eta_{2k+2}$, then, i.e., $d(\eta_{2k+1},\eta_{2k+2}) > 0$. Since $\eta_{2k+1} = g\xi_{2k+1} = S\xi_{2k+2}$ and $\eta_{2k} = f\xi_{2k} = T\xi_{2k+1}$ are comparable, then, by (28), we claim that

$$s^{\varepsilon}d(\eta_{2k+1},\eta_{2k+2}) < \max\{d(\eta_{2k},\eta_{2k+1}),d(\eta_{2k+1},\eta_{2k+2})\} = d(\eta_{2k+1},\eta_{2k+2}).$$

This is a contradiction (because $s^{\varepsilon} > 1$). Thus, $d(\eta_{2k+1},\eta_{2k+2}) = 0$, i.e., $\eta_{2k+1} = \eta_{2k+2}$. Therefore, the sequence $\{\eta_n\}$ in both cases is equal to a constant for $n \geq n_0$ and so (25) holds.

Step 2. We prove that f, g, S and T have a coincidence point. Taking advantage of (25) and Lemma 1, we say that $\{\eta_n\}$ is a b-Cauchy sequence. Since (X,d) is b-complete, then there is a $\zeta \in X$ satisfying that $\lim_{n\to\infty} \eta_n = \zeta$. We obtain

$$\lim_{n\to\infty} d(S\xi_{2n},\zeta) = \lim_{n\to\infty} d(f\xi_{2n},\zeta) = \lim_{n\to\infty} d(S\xi_{2n+2},\zeta) = \lim_{n\to\infty} d(T\xi_{2n+1},\zeta) = \lim_{n\to\infty} d(g\xi_{2n+1},\zeta) = 0.$$

Since the pairs (f,S) and (g,T) are compatible, then

$$\lim_{n\to\infty} d(Sf\xi_{2n}, fS\xi_{2n}) = \lim_{n\to\infty} d(Tg\xi_{2n+1}, gT\xi_{2n+1}) = 0.$$

On the other hand, owing to the b-continuity of f, g, S and T, we obtain

$$\lim_{n\to\infty} d(Sf\xi_{2n}, S\zeta) = \lim_{n\to\infty} d(fS\xi_{2n}, f\zeta) = 0,$$

$$\lim_{n\to\infty} d(Tg\xi_{2n+1}, T\zeta) = \lim_{n\to\infty} d(gT\xi_{2n+1}, g\zeta) = 0.$$

We now acquire that

$$\begin{aligned}\frac{1}{s}d(S\zeta, f\zeta) &\le d(S\zeta, Sf\xi_{2n}) + d(Sf\xi_{2n}, f\zeta)\\ &\le d(S\zeta, Sf\xi_{2n}) + s[d(Sf\xi_{2n}, fS\xi_{2n}) + d(fS\xi_{2n}, f\zeta)],\end{aligned} \tag{30}$$

and

$$\begin{aligned}\frac{1}{s}d(T\zeta, g\zeta) &\le d(T\zeta, Tg\xi_{2n+1}) + d(Tg\xi_{2n+1}, g\zeta)\\ &\le d(T\zeta, Tg\xi_{2n+1}) + s[d(Tg\xi_{2n+1}, gT\xi_{2n+1}) + d(gT\xi_{2n+1}, g\zeta)].\end{aligned} \tag{31}$$

On taking the limit as $n \to \infty$ from both sides of (30) and (31), we obtain $\frac{1}{s}d(S\zeta, f\zeta) \le 0$ and $\frac{1}{s}d(T\zeta, g\zeta) \le 0$, that is, $f\zeta = S\zeta, g\zeta = T\zeta$.

Since $S\zeta$ and $T\zeta$ are comparable, we prove $f\zeta = g\zeta$. Suppose the contrary, then, by (3), it is valid that

$$\begin{aligned}&\psi(s^{\varepsilon}d(f\zeta, g\zeta)\\ &\le \beta(M(\zeta,\zeta)) \times \psi(\max\{d(S\zeta, T\zeta), d(S\zeta, f\zeta), d(T\zeta, g\zeta)\}) + L\psi(N(\zeta,\zeta))\\ &< \psi(d(S\zeta, T\zeta)) = \psi(d(f\zeta, g\zeta)),\end{aligned}$$

that is to say,

$$s^{\varepsilon}d(f\zeta, g\zeta) < d(f\zeta, g\zeta).$$

This is a contradiction (because $s^{\varepsilon} > 1$). Accordingly, $f\zeta = g\zeta = S\zeta = T\zeta$. □

Remark 3. *According to the main results of [14,15], Theorem 2 makes a general generalization. That is to say, it generalizes Theorem 15 in [14] and Theorem 2.1 in [15]. Similar superiority is discussed in Remarks 1 and 2.*

The following result is a straightforward outcome of Theorem 2.

Corollary 1. *Let f and T be self-mappings on a partially ordered b-complete b-metric space $(X, \preceq, d)$ with $s > 1$. Assume that $f(X) \subseteq T(X)$ and the pair (f,T) is compatible, f and T are b-continuous, and f is partially weakly increasing with respect to T. Assume that f satisfies the following inequality*

$$\psi(s^{\varepsilon}d(f\xi, f\eta)) \le \beta(M(\xi,\eta))\psi\big(\max\{d(T\xi, T\eta), d(T\xi, f\xi), d(T\eta, f\eta)\}\big) + L\psi(N(\xi,\eta)),$$

for every $\xi, \eta \in X$, for which $T\xi$ and $T\eta$ are comparable, where

$$M(\xi,\eta) = \psi\left(\max\left\{d(T\xi, T\eta), d(T\xi, f\xi), d(T\eta, f\eta), \frac{d(T\xi, f\eta) + d(T\eta, f\xi)}{2s}\right\}\right),$$

$$N(\xi,\eta) = \min\{d(T\xi,T\eta), d(T\xi,f\eta), d(T\eta,f\xi), d(f\eta,T\eta), d(\xi,f\eta)\},$$

and $\varepsilon > 0$ *is a constant. Then, the pair* (f,T) *has a coincidence point* $\zeta \in X$*. Moreover,* ζ *is a coincidence point of* f *and* T*.*

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