axioms

Article

On Local Unique Solvability for a Class of Nonlinear Identification Problems

Vladimir E. Fedorov *, Marina V. Plekhanova and Daria V. Melekhina

Mathematical Analysis Department, Chelyabinsk State University, 129, Kashirin Brothers St., Chelyabinsk 454001, Russia; mariner79@mail.ru (M.V.P.); melehina.d.v@csu.ru (D.V.M.)
* Correspondence: kar@csu.ru; Tel.: +7-351-799-7235

Abstract: Nonlinear identification problems for evolution differential equations, solved with respect to the highest-order Dzhrbashyan–Nersesyan fractional derivative, are studied. An equation of the considered class contains a linear unbounded operator, which generates analytic resolving families for the corresponding linear homogeneous equation, and a continuous nonlinear operator, which depends on lower-order Dzhrbashyan–Nersesyan derivatives and a depending on time unknown element. The identification problem consists of the equation, Dzhrbashyan–Nersesyan initial value conditions and an abstract overdetermination condition, which is defined by a linear continuous operator. Using the contraction mappings theorem, we prove the unique local solvability of the identification problem. The cases of mild and classical solutions are studied. The obtained abstract results are applied to an investigation of a nonlinear identification problem to a linearized phase field system with time dependent unknown coefficients at Dzhrbashyan–Nersesyan time-derivatives of lower orders.

Keywords: fractional differential equation; Dzhrbashyan–Nersesyan fractional derivative; coefficient inverse problem; identification problem; initial boundary value problem

MSC: 34G20; 35R30; 35R11; 34K29

Citation: Fedorov, V.E.; Plekhanova, M.V.; Melekhina, D.V. On Local Unique Solvability for a Class of Nonlinear Identification Problems. *Axioms* **2023**, *12*, 1013. https://doi.org/10.3390/axioms12111013

Academic Editor: Chris Goodrich

Received: 6 September 2023
Revised: 14 October 2023
Accepted: 25 October 2023
Published: 27 October 2023

1. Introduction

Works on fractional integro-differential calculus in contemporary mathematics are very diverse. They concern both various theoretical aspects (properties of fractional integration and differentiation operators, issues of solvability of new problems to equations with various fractional derivatives and integrals, and much more [1–5]) and applications of fractional calculus methods in various applied problems [6–9]. To the reader's attention, we present an article on the existence of a unique solution of a new class of coefficient inverse problems to equations containing fractional derivatives, also called forecast-control problems [10], or identification problems [11,12]. We are talking about a problem for an equation containing, in addition to an unknown solution function, also unknown functional parameters and overdetermination conditions of a corresponding nature.

Consider the equation

$$\mathcal{D}^{\sigma_n} z(t) = Az(t) + B(t, \mathcal{D}^{\sigma_0} z(t), \mathcal{D}^{\sigma_1} z(t), \dots, \mathcal{D}^{\sigma_{n-1}} z(t), u(t)), \tag{1}$$

endowed by the Dzhrbashyan–Nersesyan initial conditions [1]

$$\mathcal{D}^{\sigma_k} z(0) = z_k, \quad k = 0, 1, \dots, n-1, \tag{2}$$

and by the additional overdetermination condition

$$\Phi z(t) = \Psi(t), \quad t \in [0, T]. \tag{3}$$

Here, $\mathcal{D}^{\sigma_k}$, $k = 0, 1, \ldots, n \in \mathbb{N}$ are the Dzhrbashyan–Nersesyan differentiation operator, which corresponds to the set $\{\alpha_k\}$, $0 < \alpha_k \leq 1$, (see Formula (4) below), A is a linear closed operator with a domain D_A in a Banach space $\mathcal{Z}$, $\mathcal{U}$ is another Banach space, $B : [0, T] \times \mathcal{Z}^n \times \mathcal{U} \to \mathcal{Z}$ is a nonlinear mapping, $\Phi : \mathcal{Z} \to \mathcal{U}$ is a linear bounded operator; initial values $z_k \in D_A$, $k = 0, 1, \ldots, n$, and $\Psi : [0, T] \to \mathcal{U}$ are known. A solution of (1)–(3) is a pair of functions (z, u). The main aim of the work are theorems on the existence and the uniqueness of a mild and a classical local solution (z, u).

Various linear inverse problems for differential equations containing Riemann–Liouville or Gerasimov–Caputo fractional derivatives were studied in papers [12–16]. Unique solvability issues for a nonlinear identification problem of form (1)–(3) with Gerasimov–Caputo derivatives and with a closed operator A, which generates an analytic resolving family of operators for a respective linear homogeneous equation, were investigated in [17].

The notion of the Dzhrbashyan–Nersesyan derivative includes Riemann–Liouville and Gerasimov–Caputo fractional derivatives as particular cases; for a study of various problems with this general fractional derivative, see [1], its English translation [18], in works [19–24]. The unique solvability conditions and a form of a solution for linear inhomogeneous problem with the Dzhrbashyan–Nersesyan derivative (1), (2) ($B \equiv f(t)$) in a Banach space were obtained in work [25] in the case of a bounded operator A, and in paper [26] for a closed operator A from the class $\mathcal{A}_{\{\alpha_k\}}$ of generators of analytic resolving families for a linear Equation (1) ($B \equiv 0$). Problem (2) for quasilinear Equation (1) with known u and with a bounded operator A was researched in [27].

The results of [25] were used for obtaining a theorem on the existence of a unique solution of nonlinear inverse problem (1)–(3) with a linear continuous operator A in [28]. In the present work, we extend these results on the case of a nonlinear identification problem with Dzhrbashyan–Nersesyan derivatives and with $A \in \mathcal{A}_{\{\alpha_k\}}$. It is clear that such results on Problem (1)–(3) with a closed operator A provide much greater possibilities for their application to inverse problems to partial differential equations and systems of them than results on the abstract inverse problem with a bounded operator A.

In the second section, preliminary definitions and statements are given; in particular, a theorem on the existence of a unique solution of Dzhrbashyan–Nersesyan problem (2) to linear Equation (1) ($B \equiv f(t)$) with $A \in \mathcal{A}_{\{\alpha_k\}}$ is presented. The first subsection of the third section contains a proof of the theorem on the existence and the uniqueness of a local mild solution of identification problem (1)–(3). In the second subsection, for the case of continuous mapping B:$[0, T] \times \mathcal{Z}^n \times \mathcal{U} \to D_A$, the existence of a unique local classical solution is proven. A similar result is proven under the additional conditions of Hölder continuity in t of problem data. In the fourth section, obtained general results are used for the investigation of a nonlinear identification problem to modified phase field equations with depending on t unknown coefficients at Dzhrbashyan–Nersesyan time-fractional derivatives of lower orders.

2. Preliminaries

For $h : (0, T] \to \mathcal{Z}$,where $\mathcal{Z}$ is a Banach space, and for $\beta > 0$, we introduce the following notations:

$$D^{-\beta}h(t) := J^{\beta}h(t) := \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s)ds, \quad t \in (0, T]$$

for the Riemann–Liouville fractional integral of the order $\beta > 0$, and $D^{\beta} := D^m J^{m-\beta}$ for the Riemann–Liouville fractional derivative of the order $\beta \in (m-1, m]$, $m \in \mathbb{N}$. For sequence $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, $\alpha_k \in (0, 1]$, $k = 0, 1, \ldots, n \in \mathbb{N}$, we define the Dzhrbashyan–Nersesyan fractional derivatives $\mathcal{D}^{\sigma_k}$, $k = 0, 1, \ldots, n$ by equalities [1]

$$\mathcal{D}^{\sigma_0}z(t) := D^{\alpha_0-1}z(t), \quad \mathcal{D}^{\sigma_k}z(t) := D^{\alpha_k-1}D^{\alpha_{k-1}}D^{\alpha_{k-2}}\ldots D^{\alpha_0}z(t), \quad k = 1, 2, \ldots, n. \quad (4)$$

They are natural general constructions of fractional derivatives (see, e.g., [1,19,20]), they generalize the Riemann–Liouville fractional derivatives ($\alpha_0 \in (0,1)$, $\alpha_k = 1$, $k = 1,2,\ldots,n$) and the Gerasimov–Caputo fractional derivatives ($\alpha_k = 1$, $k = 0,1,\ldots,n-1$, $\alpha_n \in (0,1)$). We also use notations

$$\sigma_k := \sum_{j=0}^{k} \alpha_j - 1, \quad k = 0,1,\ldots,n.$$

For $h{:}\mathbb{R}_+ \to \mathcal{Z}$, we denote by $\mathfrak{L}[h]$ the Laplace transform of this function. The following equalities are known [3,26]:

$$\mathfrak{L}[J^\alpha h](\lambda) = \lambda^{-\alpha}\mathfrak{L}[h](\lambda), \quad \mathfrak{L}[D^\alpha h](\lambda) = \lambda^\alpha \mathfrak{L}[h](\lambda) - \sum_{k=0}^{m-1} D^{\alpha-1-k}h(0)\lambda^k,$$

$$\mathfrak{L}[\mathcal{D}^{\sigma_n} h](\lambda) = \lambda^{\sigma_n}\mathfrak{L}[h](\lambda) - \sum_{k=0}^{n-1} \mathcal{D}^{\sigma_k}h(0)\lambda^{\sigma_n-1-\sigma_k}. \tag{5}$$

We let $\mathcal{L}(\mathcal{Z})$ be the Banach algebra of all linear bounded operators in the Banach space $\mathcal{Z}$ and $\mathcal{C}l(\mathcal{Z})$ be the set of all linear closed densely defined in $\mathcal{Z}$ operators. We consider the domain D_A of an operator $A \in \mathcal{C}l(\mathcal{Z})$ with the graph norm of A as a Banach space due to the closedness of A. We consider the linear inhomogeneous equation

$$\mathcal{D}^{\sigma_n} z(t) = Az(t) + f(t), \quad t \in (0,T] \tag{6}$$

endowed by the Dzhrbashyan–Nersesyan initial conditions [1]

$$\mathcal{D}^{\sigma_k} z(0) = z_k \in D_A, \quad k = 0,1,\ldots,n-1. \tag{7}$$

Here, $f \in C([0,T];\mathcal{Z})$.

A solution to Problem (6), (7) is function $z \in C((0,T];D_A)$, such that $\mathcal{D}^{\sigma_k} z \in C([0,T];\mathcal{Z})$, $k = 0,1,\ldots,n-1$, $\mathcal{D}^{\sigma_n} z \in C((0,T];\mathcal{Z})$. Equalities (6) for all $t \in (0,T]$ and (7) are fulfilled.

IWe itroduce notations $\rho(A) := \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(\mathcal{Z})\}$, $R_\lambda(A) := (\lambda I - A)^{-1}$ for $\lambda \in \mathbb{C}$.

Definition 1. *We denote by $\mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $\alpha_k \in (0,1]$, $k = 0,1,\ldots,n$, a class of operators $A \in \mathcal{C}l(\mathcal{Z})$, such that*

(i) $\lambda^{\sigma_n} \in \rho(A)$ for all $\lambda \in S_{\theta_0,a_0} := \{\mu \in \mathbb{C} : |\arg(\mu - a_0)| < \theta_0, \mu \neq a_0\}$;

(ii) for any $\theta \in (\pi/2, \theta_0)$, $a > a_0$, there exists such a constant $K(\theta,a) > 0$, that for all $\lambda \in S_{\theta,a}$

$$\|R_{\lambda^{\sigma_n}}(A)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta,a)}{|\lambda - a|^{\alpha_0}|\lambda|^{\sigma_n - \alpha_0}}.$$

If $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, then we define for $t > 0$ operators

$$Y_\beta(t) = \frac{1}{2\pi i}\int_\Gamma \lambda^\beta R_{\lambda^{\sigma_n}}(A) e^{\lambda t} d\lambda, \quad \beta \in \mathbb{R}.$$

Here, for some $\delta > 0$, $a > a_0$, $\theta \in (\pi/2, \theta_0)$ $\Gamma := \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_\pm := \{\lambda \in \mathbb{C} : \lambda = a + re^{\pm i\theta}, r \in (\delta, \infty)\}$, $\Gamma_0 := \{\lambda \in \mathbb{C} : \lambda = a + \delta e^{i\varphi}, \varphi \in (-\theta, \theta)\}$. For brevity, we use the notations of frequently used operators $Z_k(t) := Y_{\sigma_n - \sigma_k - 1}(t)$, $k = 0,1,\ldots,n-1$.

We denote also

$$\mathcal{A}_{\{\alpha_k\}} = \bigcup_{\substack{\theta_0 \in (\pi/2,\pi] \\ a_0 \geq 0}} \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0).$$

We let $C^\gamma([0,T];\mathcal{Z})$ be the set of all Hölder continuous functions from $[0,T]$ to $\mathcal{Z}$ with the power $\gamma \in (0,1]$.

Theorem 1 ([26]). *We let* $\alpha_k \in (0,1]$, $k = 0,1,\ldots,n$, $\alpha_0 + \alpha_n > 1$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $z_k \in D_A$, $k = 0,1,\ldots,n-1$, $f \in C([0,T];D_A) \cup C^\gamma([0,T];\mathcal{Z})$, $\gamma \in (0,1]$. *Then, Problem* (6), (7) *has a unique solution. It has the form*

$$z(t) = \sum_{k=0}^{n-1} Z_k(t) z_k + \int_0^t Y_0(t-s) f(s) ds.$$

Remark 1. *We note that* $Z_k(t)z_k$ *is a unique solution of the initial value problem* $\mathcal{D}^{\sigma_k} z(0) = z_k$, $\mathcal{D}^{\sigma_l} z(0) = 0$, $l \in \{0,1,\ldots,n-1\} \setminus \{k\}$ *to equation* $\mathcal{D}^{\sigma_n} z(t) = Az(t)$*. In addition, the unique solution of Problem* (6), (7) *with zero initial conditions* $\mathcal{D}^{\sigma_l} z(0) = 0$, $l \in \{0,1,\ldots,n-1\}$, *is*

$$\int_0^t Y_0(t-s) f(s) ds.$$

Remark 2. *In the proof of Lemma 1 in* [26], *it was shown that for some* $C > 0$ *and for all* $t \in (0,T]$ $\|Y_\beta(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ct^{\sigma_n - 1 - \beta}$. *Hence, for* $j,k \in \{0,1,\ldots,n-1\}$, $k > j$, $\|\mathcal{D}^{\sigma_j} Z_k(t)\|_{\mathcal{L}(\mathcal{Z})} = \|Y_{\sigma_n - \sigma_k - 1 + \sigma_j}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ct^{\sigma_k - \sigma_j}$. *In addition, for* $j,k \in \{0,1,\ldots,n-1\}$, $k < j$, $z_k \in D_A$, *in the proof of Theorem 3 in* [26], *the following relations were proven:*

$$\mathfrak{L}[\mathcal{D}^{\sigma_j} Z_k(t) z_k] = \lambda^{\sigma_n - \sigma_k - 1 + \sigma_j} R_{\lambda^{\sigma_n}}(A) z_k - \lambda^{\sigma_j - \sigma_k - 1} z_k = \lambda^{\sigma_j - \sigma_k - 1} R_{\lambda^{\sigma_n}}(A) A z_k,$$

$$\|\lambda^{\sigma_j - \sigma_k - 1} R_{\lambda^{\sigma_n}}(A) A z_k\|_{\mathcal{Z}} \leq \frac{K\|Az_k\|_{\mathcal{Z}}}{|\lambda|^{\sigma_n + 1 + \sigma_k - \sigma_j}} \leq \frac{K\|Az_k\|_{\mathcal{Z}}}{|\lambda|^{\alpha_0 + \alpha_1 + \cdots + \alpha_{k-1} + \alpha_k + \alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_{n-1} + \alpha_n}},$$

consequently, $\|\mathcal{D}_j^{\sigma} Z_k(t) z_k\|_{\mathcal{Z}} \leq C_1 t^{\alpha_0 + \alpha_1 + \cdots + \alpha_{k-1} + \alpha_k + \alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_{n-1} + \alpha_n - 1} \leq Ct^{\alpha_0 + \alpha_n - 1}$,

$$\|\mathcal{D}^{\sigma_k} Z_k(t) z_k - z_k\|_{\mathcal{Z}} \leq \frac{1}{2\pi} \int_\Gamma \|\lambda^{-1} R_{\lambda^{\sigma_n}}(A) A z_k\|_{\mathcal{Z}} |d\lambda| \leq Ct^{\alpha_0 + \alpha_n - 1}, \quad t \in (0,T].$$

3. Local Solvability of Identification Problem

3.1. Mild Solution

We take Banach spaces $\mathcal{Z}$ and $\mathcal{U}$, an open set Z in $\mathbb{R} \times \mathcal{Z}^n$, a nonlinear mapping $B : Z \times \mathcal{U} \to \mathcal{Z}$, a linear operator $\Phi \in \mathcal{L}(\mathcal{Z};\mathcal{U})$ and a function $\Psi : [0,T] \to \mathcal{U}$. The purpose of Problem (1)–(3) is to find $z : [0,T] \to \mathcal{Z}$, $u : [0,T] \to \mathcal{U}$ from Relations (1)–(3).

We denote $\overline{y} = (y_0, y_1, \ldots, y_{n-1})$ and formulate several conditions.

$(\mathcal{A})$ The operator $B : Z \times \mathcal{U} \to \mathcal{Z}$ can be presented as

$$B(t, \overline{y}, u) = B_1(t, \overline{y}) + B_2(t, \overline{y}, u), \quad (t, \overline{y}, u) \in Z \times \mathcal{U}.$$

For $\overline{a} = (a_0, a_1, \ldots, a_{n-1}) \in \mathcal{Z}^n$, $R, T > 0$, we use the following notations:

$$S_{\mathcal{Z}^n}(\overline{a}, R) = \{\overline{y} \in \mathcal{Z}^n : \|y_j - a_j\|_{\mathcal{Z}} < R, \ j = 0,1,\ldots,n-1\},$$
$$S_{\mathcal{Z}^n}(\overline{a}, R, T) = [0,T] \times S_{\mathcal{Z}^n}(\overline{a}, R).$$

For sufficiently smooth Ψ, we define

$$v_0 := \mathcal{D}^{\sigma_n} \Psi(0) - \overline{\Phi A} z_0 - \Phi B_1(0, z_0, z_1, \ldots, z_{n-1}). \tag{8}$$

Here, the line above ΦA means the closure of this operator.

We let the next conditions $(\mathcal{B})$–$(\mathcal{F})$ be satisfied:

$(\mathcal{B})$ the equation $\Phi B_2(0, y_0, y_1, \ldots, y_{n-1}, u) = v_0$ with unknown u has a unique solution $u_0 \in \mathcal{U}$;

$(\mathcal{C})$ there is an operator $B_3 : [0,T] \times \mathcal{U}^{n+1} \to \mathcal{U}$, for which

$$\Phi B_2(t,\overline{y},u) = B_3(t,\Phi y_0,\Phi y_1,\dots,\Phi y_{n-1},u), \quad (t,\overline{y},u) \in Z \times \mathcal{U};$$

$(\mathcal{D})$ there is such a constant, $R > 0$, that for every $t \in [0,T]$, the mapping $v = B_3(t,\mathcal{D}^{\sigma_0}\Psi(t),\mathcal{D}^{\sigma_1}\Psi(t),\dots,\mathcal{D}^{\sigma_{n-1}}\Psi(t),u)$ with u in $S_{\mathcal{U}}(u_0,R)$ has an inverse mapping $u = F(t,v)$;

$(\mathcal{E})$ there is such a constant, $R > 0$, that operator F is continuous with respect to the totality of the variables (t,v) on the set $S_{\mathcal{U}}(u_0,R,T)$ and is Lipschitz continuous in v;

$(\mathcal{F})$ there is $R > 0$ such that mappings $B_1(t,\overline{y})$ and $B_2(t,\overline{y},u)$ are continuous with respect to the totality of the variables on set $S_{\mathcal{Z}^n\times\mathcal{U}}((z_0,z_1,\dots,z_{n-1},u_0),R,T)$ and are Lipschitz continuous in $(\overline{y},u)$.

Using the form of a classical solution from Theorem 1, we can introduce the notion of a mild solution.

Definition 2. *Pair $(z,u) \in C([0,T];\mathcal{Z}) \times C([0,T];\mathcal{U})$ for which $\mathcal{D}^{\sigma_j}z \in C([0,T];\mathcal{Z})$ for $j = 0,1,\dots,n-1$, the inclusion $(\mathcal{D}^{\sigma_0}z(t),\mathcal{D}^{\sigma_1}z(t),\dots,\mathcal{D}^{\sigma_{n-1}}z(t)) \in Z$ and Condition (3) are valid for all $t \in [0,T]$, and equality*

$$z(t) = \sum_{k=0}^{n-1} Z_k(t)z_k + \int_0^t Y_0(t-s)B(s,\mathcal{D}^{\sigma_0}z(s),\dots,\mathcal{D}^{\sigma_{n-1}}z(s),u(s))ds \tag{9}$$

is fulfilled for all $t \in (0,T]$. It is called a mild solution of identification problem (1)–(3) on $[0,T]$.

Lemma 1. *We let $\alpha_k \in (0,1]$, $k = 0,1,\dots,n$, $\alpha_0+\alpha_n > 1$, $\theta_0 \in (\pi/2,\pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0,a_0)$, $\Phi,\overline{\Phi A} \in \mathcal{L}(\mathcal{Z};\mathcal{U})$, $z_k \in \mathcal{Z}$, $f_k(t) := \Phi Z_k(t)z_k$, $k = 0,1,\dots,n-1$. Then, $\mathcal{D}^{\sigma_n}f_k(t) = \overline{\Phi A}Z_k(t)z_k \in C([0,T];\mathcal{U})$, $k = 0,1,\dots,n-1$.*

Proof. For some $\lambda \in \rho(A)$, we take $y_k = R_\lambda(A)z_k \in D_A$, $k = 0,1,\dots,n-1$. Then, by Remark 1, $\mathcal{D}^{\sigma_n}Z_k(t)y_k = AZ_k(t)y_k$, $f_k(t) = \Phi Z_k(t)(\lambda I - A)y_k = \lambda\Phi Z_k(t)y_k - \Phi AZ_k(t)y_k$,

$$\mathcal{D}^{\sigma_n}f_k(t) = \lambda\Phi\mathcal{D}^{\sigma_n}Z_k(t)y_k - \overline{\Phi A}\mathcal{D}^{\sigma_n}Z_k(t)y_k =$$

$$= \lambda\overline{\Phi A}Z_k(t)y_k - \overline{\Phi A}Z_k(t)Ay_k = \overline{\Phi A}Z_k(t)z_k \in C([0,T];\mathcal{U}).$$

□

Lemma 2. *We let $\alpha_k \in (0,1]$, $k = 0,1,\dots,n$, $\alpha_0+\alpha_n > 1$, $\theta_0 \in (\pi/2,\pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0,a_0)$, $g \in C([0,T];\mathcal{Z})$, $\Phi,\overline{\Phi A} \in \mathcal{L}(\mathcal{Z};\mathcal{U})$,*

$$h(t) := \Phi\int_0^t Y_0(t-s)g(s)ds.$$

Then,

$$\mathcal{D}^{\sigma_n}h(t) = \overline{\Phi A}\int_0^t Y_0(t-s)g(s)ds + \Phi g(t) \in C([0,T];\mathcal{U}).$$

Proof. For $\lambda \in \rho(A)$, we put $f(t) = R_\lambda(A)g(t) \in D_A$, $t \in [0,T]$. Hence, $Af(t) = \lambda R_\lambda(A)g(t) - g(t) \in C([0,T];\mathcal{Z})$; consequently, $f \in C([0,T];D_A)$. Therefore,

$$h(t) = \Phi\int_0^t Y_0(t-s)(\lambda I - A)f(s)ds = (\lambda\Phi - \overline{\Phi A})\int_0^t Y_0(t-s)f(s)ds,$$

and, by Remark 1, we obtain

$$\mathcal{D}^{\sigma_n} h(t) = (\lambda\Phi - \overline{\Phi A})\mathcal{D}^{\sigma_n} \int_0^t Y_0(t-s)f(s)ds$$

$$= (\lambda\Phi - \overline{\Phi A})\left(A\int_0^t Y_0(t-s)f(s)ds + f(t)\right)$$

$$= \overline{\Phi A}\left(\lambda \int_0^t Y_0(t-s)f(s)ds - A\int_0^t Y_0(t-s)f(s)ds\right) + (\lambda\Phi - \overline{\Phi A})f(t)$$

$$= \overline{\Phi A}\int_0^t Y_0(t-s)g(s)ds + (\lambda\Phi - \Phi A)f(t) = \overline{\Phi A}\int_0^t Y_0(t-s)g(s)ds + \Phi g(t).$$

Here, $\overline{\Phi A}f(t) = \Phi A f(t)$, since $f(t) \in D_A$. □

Theorem 2. *We let* $\alpha_0 = 1$, $\alpha_k \in (0,1]$, $k = 1,2,\ldots,n$, $\theta_0 \in (\pi/2,\pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0,a_0)$, $z_k \in D_A$, $k = 0,1,\ldots,n-1$, $(0,z_0,z_1,\ldots,z_{n-1}) \in Z$, $\Phi, \overline{\Phi A} \in \mathcal{L}(\mathcal{Z};\mathcal{U})$, $\mathcal{D}^{\sigma_k}\Psi \in C([0,T];\mathcal{U})$, $k = 0,1,\ldots,n$, $\Phi z_0 = \Psi(0)$, *and we let conditions* $(\mathcal{A})$–$(\mathcal{F})$ *be fulfilled. Then, for some* $T_1 \in (0,T]$, *inverse problem* (1)–(3) *has a unique mild solution* $(z,u) \in C([0,T_1];\mathcal{Z}) \times C([0,T_1];\mathcal{U})$ *on segment* $[0,T_1]$.

Proof. We take $z_k \in D_A$, $k = 0,1,\ldots,n-1$; then, we have $\mathcal{D}^{\sigma_j} Z_k(t)z_k = Y_{\sigma_n-\sigma_k-1+\sigma_j}(t)z_k \in C([0,T];\mathcal{L}(\mathcal{Z}))$ as it is shown in the proof of Theorem 3 [26]. Lemma 1 in [26] implies that

$$\mathcal{D}^{\sigma_j}\int_0^t Y_0(t-s)g(s)ds = \int_0^t Y_{\sigma_j}(t-s)g(s)ds, \quad j = 0,1,\ldots,n-1. \tag{10}$$

Therefore, due to Remark 2 and Equation (9), we have correlations for $j = 0,1,\ldots,n-1$

$$\mathcal{D}^{\sigma_j} z(t) = \sum_{k=0}^{m-1} Y_{\sigma_n-\sigma_k-1+\sigma_j}(t)z_k + \int_0^t Y_{\sigma_j}(t-s)B(s,\mathcal{D}^{\sigma_0}z(s),\ldots,\mathcal{D}^{\sigma_{n-1}}z(s),u(s))ds. \tag{11}$$

For a mild solution (z,u), Equality (9) is valid; then, Condition (3) implies that

$$\sum_{k=0}^{m-1}\Phi Z_k(t)z_k + \Phi\int_0^t Y_0(t-s)B(s,\mathcal{D}^{\sigma_0}z(s),\mathcal{D}^{\sigma_1}z(s),\ldots,\mathcal{D}^{\sigma_{n-1}}z(s),u(s))ds = \Psi(t).$$

Therefore, due to Lemmas 1 and 2, we have

$$\mathcal{D}^{\sigma_n}\sum_{k=0}^{m-1}\Phi Z_k(t)z_k + \mathcal{D}^{\sigma_n}\Phi\int_0^t Y_0(t-s)B(s,\mathcal{D}^{\sigma_0}z(s),\mathcal{D}^{\sigma_1}z(s),\ldots,\mathcal{D}^{\sigma_{n-1}}z(s),u(s))ds$$

$$= \overline{\Phi A}\sum_{k=0}^{m-1} Z_k(t)z_k + \overline{\Phi A}\int_0^t Y_0(t-s)B(s,\mathcal{D}^{\sigma_0}z(s),\mathcal{D}^{\sigma_1}z(s),\ldots,\mathcal{D}^{\sigma_{n-1}}z(s),u(s))ds$$

$$+\Phi B(t,\mathcal{D}^{\sigma_0}z(t),\mathcal{D}^{\sigma_1}z(t),\ldots,\mathcal{D}^{\sigma_{n-1}}z(t),u(t)) = \mathcal{D}^{\sigma_n}\Psi(t).$$

Due to condition $(\mathcal{A})$, we can write the last equality as

$$\Phi B_2(t, \mathcal{D}^{\sigma_0}z(t), \ldots, \mathcal{D}^{\sigma_{n-1}}z(t), u(t)) = \mathcal{D}^{\sigma_n}\Psi(t) - \overline{\Phi A}\sum_{k=0}^{m-1} Z_k(t)z_k$$
$$-\overline{\Phi A}\int_0^t Y_0(t-s)B(s, \mathcal{D}^{\sigma_0}z(s), \mathcal{D}^{\sigma_1}z(s), \ldots, \mathcal{D}^{\sigma_{n-1}}z(s), u(s))ds$$
$$-\Phi B_1(t, \mathcal{D}^{\sigma_0}z(t), \mathcal{D}^{\sigma_1}z(t), \ldots, \mathcal{D}^{\sigma_{n-1}}z(t)). \tag{12}$$

Then, under assumptions $(\mathcal{C})$, $(\mathcal{D})$, Equation (12) implies equality

$$u(t) = F(t, v(t)), \tag{13}$$

where

$$v(t) = \mathcal{D}^{\sigma_n}\Psi(t) - \overline{\Phi A}\sum_{k=0}^{m-1} Z_k(t)z_k - \Phi B_1(t, \mathcal{D}^{\sigma_0}z(t), \mathcal{D}^{\sigma_1}z(t), \ldots, \mathcal{D}^{\sigma_{n-1}}z(t))$$
$$-\overline{\Phi A}\int_0^t Y_0(t-s)B(s, \mathcal{D}^{\sigma_0}z(s), \mathcal{D}^{\sigma_1}z(s), \ldots, \mathcal{D}^{\sigma_{n-1}}z(s), u(s))ds. \tag{14}$$

Thus, we obtained a nonlinear system of equations (11) with $j = 0, 1, \ldots, n-1$ and Equation (13) for unknown functions $y_0 := \mathcal{D}^{\sigma_0}z$, $y_1 := \mathcal{D}^{\sigma_1}z$, …, $y_{n-1} := \mathcal{D}^{\sigma_{n-1}}z$, u.

We consider set

$$\mathfrak{M}_T = \{(\overline{y}, u) \in C([0,T]; \mathcal{Z}^n \times \mathcal{U}) : \|y_j(t) - z_j\|_{\mathcal{Z}} \le R, j = 0, 1, \ldots, n-1,$$
$$\|u(t) - u_0\|_{\mathcal{U}} \le R\}$$

with metrics $d((\overline{x}, u), (\overline{y}, w)) = \|(\overline{x}, u) - (\overline{y}, w)\|_{C([0,T];\mathcal{Z}^n \times \mathcal{U})}$ and mapping $H = (H^0, H^1, \ldots, H^n)$, which is defined by equalities

$$H^j(y_0, y_1 \ldots, y_{n-1}, u) = \sum_{k=0}^{n-1} Y_{\sigma_n - \sigma_k - 1 + \sigma_j}(t)z_k$$
$$+\int_0^t Y_{\sigma_j}(t-s)B(s, y_0(s), y_1(s), \ldots, y_{n-1}(s), u(s))ds, \quad j = 0, 1, \ldots, n-1,$$

$$H^n(y_0, y_1, \ldots, y_{n-1}, u) = F\Bigg(t, \mathcal{D}^{\sigma_n}\Psi(t) - \overline{\Phi A}\sum_{k=0}^{n-1} Z_k(t)z_k$$
$$-\Phi B_1(t, y_0(t), y_1(t), \ldots, y_{n-1}(t))$$
$$-\int_0^t \overline{\Phi A}Y_0(t-s)B(s, y_0(s), y_1(s), \ldots, y_{n-1}(s), u(s))ds\Bigg).$$

Thus, Problem (1)–(3) are represented in the form of system

$$\begin{gathered} y_0(t) = H^0(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)), \\ y_1(t) = H^1(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)), \\ \ldots, \\ y_{n-1}(t) = H^{n-1}(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)), \\ u(t) = H^n(y_0(t), y_1(t), \ldots, y_{n-1}(t), u(t)). \end{gathered} \tag{15}$$

We take $t = 0$ in (15); then, we have $H^j(z_0, z_1, \ldots, z_{n-1}, u_0) = z_j$ for $j = 0, 1, \ldots, n-1$,

$$H^n(z_0, z_1, \ldots, z_{n-1}, u_0) = F(0, v_0) = u_0.$$

Due to the assumptions of Theorem, if $(y_0, y_1, \dots, y_{n-1}, u) \in \mathfrak{M}_T$, then

$$H(y_0(t), y_1(t), \dots, y_{n-1}(t), u(t))$$

is continuous with respect to t on $[0, T]$. Taking into account Remark 2 and assumptions $(\mathcal{E})$, $(\mathcal{F})$, we have, for $t \in (0, T]$,

$$\|H^j(y_0(t), y_1(t), \dots, y_{n-1}(t), u(t)) - z_j\|_{\mathcal{Z}}$$

$$\leq \sum_{\substack{k=0\\k\neq j}}^{n-1} \|Y_{\sigma_n - \sigma_k - 1 + \sigma_j}(t) z_k\|_{\mathcal{Z}} + \|Y_{\sigma_n - 1}(t) z_k - z_k\|_{\mathcal{Z}}$$

$$+ \int_0^t \|Y_{\sigma_j}(t-s)\|_{\mathcal{L}(\mathcal{Z})} \|B(s, y_0(s), \dots, y_{n-1}(s), u(s)) - B(0, z_0, z_1, \dots, z_{n-1}, u_0)\|_{\mathcal{Z}} ds$$

$$+ \int_0^t \|Y_{\sigma_j}(t-s)\|_{\mathcal{L}(\mathcal{Z})} \|B(0, z_0, z_1, \dots, z_{n-1}, u_0)\|_{\mathcal{Z}} ds$$

$$\leq nCt^{\alpha_0 + \alpha_n - 1} + C_1 t^{\sigma_n - \sigma_j} \left(\sum_{k=0}^{n-1} \|y_k(t) - z_k\|_{\mathcal{Z}} + \|u(t) - u_0\|_{\mathcal{U}} \right)$$

$$+ C_1 t^{\sigma_n - \sigma_j} \|B(0, z_0, z_1, \dots, z_{n-1}, u_0)\|_{\mathcal{Z}} \leq C_2 t^{\alpha_n} (1 + (n+1)R + C_3), \ j = 0, 1, \dots, n-1,$$

$$\|H^n(y_0(t), y_1(t), \dots, y_{n-1}(t), u(t)) - u_0\|_{\mathcal{U}}$$

$$\leq \|F(t, v(t)) - F(t, v_0)\|_{\mathcal{U}} + \|F(t, v_0) - F(0, v_0)\|_{\mathcal{U}}$$

$$\leq l\|v(t) - v_0\|_{\mathcal{U}} + \|F(t, v_0) - F(0, v_0)\|_{\mathcal{U}} \leq \|F(t, v_0) - F(0, v_0)\|_{\mathcal{U}}$$

$$+ l\|\mathcal{D}^{\sigma_n} \Psi(t) - \mathcal{D}^{\sigma_n} \Psi(0)\|_{\mathcal{U}} + l\|\overline{\Phi A}\|_{\mathcal{L}(\mathcal{Z};\mathcal{U})} \|Z_0(t) z_0 - z_0\|_{\mathcal{Z}} + l \left\| \overline{\Phi A} \sum_{k=1}^{n-1} Z_k(t) z_k \right\|_{\mathcal{U}}$$

$$+ l\|\Phi B_1(t, y_0(t), \dots, y_{n-1}(t)) - \Phi B_1(0, z_0, \dots, z_{n-1})\|_{\mathcal{U}}$$

$$+ l \int_0^t \|\overline{\Phi A} Y_0(t-s) B(0, z_0, z_1, \dots, z_{n-1}, u_0)\|_{\mathcal{U}} ds$$

$$+ l \int_0^t \|\overline{\Phi A} Y_0(t-s)(B(s, y_0(s), y_1(s), \dots, y_{n-1}(s), u(s)) - B(0, z_0, z_1, \dots, z_{n-1}, u_0))\|_{\mathcal{U}} ds$$

$$\leq \|F(t, v_0) - F(0, v_0)\|_{\mathcal{U}} + l\|\mathcal{D}^{\sigma_n} \Psi(t) - \mathcal{D}^{\sigma_n} \Psi(0)\|_{\mathcal{U}}$$

$$+ C_4 (1 + t^{\alpha_n}) \sum_{k=0}^{n-1} \|y_k(t) - z_k\|_{\mathcal{Z}} + C_4 t^{\alpha_n}$$

$$\leq \|F(t, v_0) - F(0, v_0)\|_{\mathcal{U}} + l\|\mathcal{D}^{\sigma_n} \Psi(t) - \mathcal{D}^{\sigma_n} \Psi(0)\|_{\mathcal{U}}$$

$$+ t^{\alpha_n} (1 + t^{\alpha_n}) n C_2 C_4 (1 + (n+1)R + C_3) + C_4 t^{\alpha_n} \leq C_5 t^{\alpha_n},$$

where l is a Lipschitz constant for F, and v_0 and $v(t)$ are defined by (8) and (14). Here and further, various constants the value of which is not important are denoted by symbols C_1, C_2, and so on. Thus, for a sufficiently small $T_1 \in (0, T]$, H acts from the set $\mathfrak{M}_{T_1}$ into itself.

We denote for $i = 1, 2$, $j = 0, 1, \ldots, n-1$ $y_j^i(t) = H^j(y_0^i(t), y_1^i(t), \ldots, y_{n-1}^i(t), u^i(t))$, $u^i(t) = H^n(y_0^i(t), y_1^i(t), \ldots, y_{n-1}^i(t), u^i(t))$; then, for $k = 0, 1, \ldots, n-1$, the Lipschitz condition for B implies that

$$\|y_k^1(t) - y_k^2(t)\|_{\mathcal{Z}} \leq C_1 T^{\alpha_n} \left(\sum_{j=0}^{n-1} \sup_{s\in[0,T]} \|y_j^1(s) - y_j^2(s)\|_{\mathcal{Z}} + \sup_{s\in[0,T]} \|u^1(s) - u^2(s)\|_{\mathcal{U}} \right),$$

$$\|u^1(t) - u^2(t)\|_{\mathcal{U}} \leq C_2 \sum_{j=0}^{n-1} \sup_{s\in[0,T]} \|y_j^1(s) - y_j^2(s)\|_{\mathcal{Z}}$$

$$+C_2 T^{\alpha_n} \left(\sum_{j=0}^{n-1} \sup_{s\in[0,T]} \|y_j^1(s) - y_j^2(s)\|_{\mathcal{Z}} + \sup_{s\in[0,T]} \|u^1(s) - u^2(s)\|_{\mathcal{U}} \right)$$

$$\leq (nC_1 + 1) C_2 T^{\alpha_n} \left(\sum_{j=0}^{n-1} \sup_{s\in[0,T]} \|y_j^1(s) - y_j^2(s)\|_{\mathcal{Z}} + \sup_{s\in[0,T]} \|u^1(s) - u^2(s)\|_{\mathcal{U}} \right).$$

Hence, for a small enough $T_1 > 0$, H is a contraction operator on $\mathfrak{M}_{T_1}$ and has in the complete metric space $\mathfrak{M}_{T_1}$ a unique fixed point $(y_0^0, y_1^0, \ldots, y_{n-1}^0, u^0)$.

Since $\alpha_0 = 1$, we have

$$y_0^0(t) = \mathcal{D}^{\sigma_0} y_0^0(t) = \sum_{k=0}^{m-1} Z_k(t) z_k + \int_0^t Y_0(t-s) B(s, y_0^0(s), \ldots, y_{n-1}^0(s), u^0(s)) ds.$$

Hence, pair $(y_0^0(t), u^0(t))$ is a mild solution of (1)–(3).

Each of the two mild solutions (z^0, u^0) and (z^1, u^1) corresponds to a fixed point $(z^0, \mathcal{D}^{\sigma_1} z^0, \ldots, \mathcal{D}^{\sigma_{n-1}} z^0, u^0)$ and $(z^1, \mathcal{D}^{\sigma_1} z^1, \ldots, \mathcal{D}^{\sigma_{n-1}} z^1, u^1)$ of the mapping H. The uniqueness of a fixed point for H in $\mathfrak{M}_{T_1}$ with a small enough $T_1 > 0$ implies that $z^0(t) = z^1(t)$, $u^0(t) = u^1(t)$ for $t \in [0, T_1]$. $\square$

3.2. Classical Solution

Definition 3. *A classical solution of Problem* (1)–(3) *on* $[0, T]$ *is pair* $(z, u) \in C((0, T]; D_A) \times C([0, T]; \mathcal{U})$, *for which* $\mathcal{D}^{\sigma_k} z \in C([0, T]; \mathcal{Z})$, $k = 0, 1, \ldots, n-1$, $\mathcal{D}^{\sigma_n} z \in C((0, T]; \mathcal{Z})$. *Inclusion* $(t, \mathcal{D}^{\sigma_0} z(t), \mathcal{D}^{\sigma_1} z(t), \ldots, \mathcal{D}^{\sigma_{n-1}} z(t)) \in Z$ *and Condition* (3) *for* $t \in [0, T]$, *Equality* (1) *for* $t \in (0, T]$ *and Condition* (2) *are fulfilled.*

To obtain the unique solvability theorem in the sense of classical solution for the identification problem, we first use an additional condition $B \in C(Z; D_A)$.

Theorem 3. *We let* $\alpha_0 = 1$, $\alpha_k \in (0, 1]$, $k = 1, 2, \ldots, n$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $z_k \in D_A$, $k = 0, 1, \ldots, n-1$, $(0, z_0, z_1, \ldots, z_{n-1}) \in Z$, $\Phi, \overline{\Phi A} \in \mathcal{L}(\mathcal{Z}; \mathcal{U})$, $\mathcal{D}^{\sigma_k} \Psi \in C([0, T]; \mathcal{U})$, $k = 0, 1, \ldots, n-1$, $\Phi z_0 = \Psi(0)$, *and conditions* $(\mathcal{A})$–$(\mathcal{F})$ *be satisfied,* $B \in C(Z; D_A)$. *Then, Problem* (1)–(3) *has a unique classical solution* (z, u) *on segment* $[0, T_1]$ *with some* $T_1 \in (0, T]$.

Proof. By Theorem 2, Problem (1)–(3) has a mild solution $(z(t), u^0(t))$ on a small enough segment $[0, T_1]$. Then, $f(t) := B(t, \mathcal{D}^{\sigma_0} z(t), \mathcal{D}^{\sigma_1} z(t), \ldots, \mathcal{D}^{\sigma_{n-1}} z(t))$ is contained in class $C([0, T]; D_A)$, and by Theorem 1, z is a classical solution of direct Problem (1), (2) with a given u^0. $\square$

But condition $B \in C(Z; D_A)$ is not often met in applications, so we replace it with conditions for additional smoothness of B. For this aim, we replace assumptions $(\mathcal{E})$ and $(\mathcal{F})$ with stronger conditions:

$(\mathcal{E}_1)$ there is such a constant, $R > 0$, that F is continuous with respect to the totality of the variables (t, v) on the set $S_{\mathcal{U}}(u_0, R, T)$, which is Lipschitz continuous in v and is Hölder continuous with a power of $\gamma \in (0, 1]$ in t;

$(\mathcal{F}_1)$ there exists $R > 0$, such that mappings $B_1(t, \overline{y})$ and $B_2(t, \overline{y}, u)$ are continuous with respect to the totality of the variables on the set $S_{\mathcal{Z}^n \times \mathcal{U}}((z_0, z_1, \dots, z_{n-1}, u_0), R, T)$, which is Lipschitz continuous in $(\overline{y}, u)$ and is Hölder continuous with a power of $\gamma \in (0, 1]$ in t.

Lemma 3. *We let $\alpha \in (0, 1)$, $h, D^\alpha h \in C([0, T]\mathcal{Z})$; then,*

$$\exists C > 0 \ \ \forall s, t \in [0, T] \ \ \|h(s) - h(t)\|_{\mathcal{Z}} \leq C \|D^\alpha h\|_{C([0,T];\mathcal{Z})} |s - t|^\alpha.$$

Proof. We take $0 \leq s < t \leq T$; hence,

$$h(t) = D^1 J^1 h(t) = D^1 J^\alpha J^{1-\alpha} h(t) = J^\alpha D^1 J^{1-\alpha} h(t) = J^\alpha D^\alpha h(t),$$

since $J^{1-\alpha} h(0) = 0$. Then,

$$h(t) - h(s) = J^\alpha D^\alpha h(t) - J^\alpha D^\alpha h(s)$$

$$= \int_0^s \frac{(t-\tau)^{\alpha-1} - (s-\tau)^{\alpha-1}}{\Gamma(\alpha)} D^\alpha h(\tau) d\tau + \int_s^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} D^\alpha h(\tau) d\tau,$$

$$\|h(t) - h(s)\|_{\mathcal{Z}} \leq \|D^\alpha h\|_{C([0,T];\mathcal{Z})} \left(\frac{(t-s)^\alpha}{\Gamma(\alpha+1)} + \left| \int_0^s \int_s^t \frac{(r-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} dr d\tau \right| \right)$$

$$= \frac{\|D^\alpha h\|_{C([0,T];\mathcal{Z})}}{\Gamma(\alpha+1)} \left((t-s)^\alpha + \alpha \left| \int_s^t (r^{\alpha-1} - (r-s)^{\alpha-1}) dr \right| \right)$$

$$\leq \frac{\|D^\alpha h\|_{C([0,T];\mathcal{Z})}}{\Gamma(\alpha+1)} \left((t-s)^\alpha + \alpha \int_s^t (r-s)^{\alpha-1} dr \right) = \frac{2\|D^\alpha h\|_{C([0,T];\mathcal{Z})}}{\Gamma(\alpha+1)} (t-s)^\alpha.$$

□

Theorem 4. *We let $\alpha_0 = 1$, $\alpha_k \in (0, 1]$, $k = 1, 2, \dots, n$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$, $z_k \in D_A$, $k = 0, 1, \dots, n-1$, $(0, z_0, z_1, \dots, z_{n-1}) \in Z$, $\Phi, \overline{\Phi A} \in \mathcal{L}(\mathcal{Z}; \mathcal{U})$, $\mathcal{D}^{\sigma_k} \Psi \in C([0, T]; \mathcal{U})$, $k = 0, 1, \dots, n-1$, $\mathcal{D}^{\sigma_n} \Psi \in C^\gamma([0, T]; \mathcal{U})$, $\gamma \in (0, 1]$, $\Phi z_0 = \Psi(0)$, and conditions $(\mathcal{A})$–$(\mathcal{D})$, $(\mathcal{E}_1)$, $(\mathcal{F}_1)$ be satisfied. Then, Problem (1)–(3) has a unique classical solution (z, u) on segment $[0, T_1]$ with some $T_1 \in (0, T]$.*

Proof. As before, $(z(t), u^0(t))$ is a mild solution of (1)–(3) on some small enough segment $[0, T_1]$, which exists by Theorem 2. We take $\alpha \in (0, \min\{\alpha_k : k = 0, 1, \dots, n\})$, $\alpha \leq \gamma$. Then, due to Remark 2, for $k, j \in \{0, 1, \dots, n-1\}$, $k > j$, we have $D^\alpha D^{\alpha_j} Z_k(t) = D^\alpha Y_{\sigma_n - \sigma_k - 1 + \sigma_j}(t) = Y_{\sigma_n - \sigma_k - 1 + \sigma_j + \alpha}(t)$ by Formula (5),

$$\|D^\alpha D^{\alpha_j} Z_k(t)\|_{\mathcal{L}(\mathcal{Z})} = \|Y_{\sigma_n - \sigma_k - 1 + \sigma_j + \alpha}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C t^{\sigma_k - \sigma_j - \alpha}, \quad t \in [0, T],$$

$\sigma_k - \sigma_j - \alpha \geq \alpha_k - \alpha > 0$. If $k < j$, $z_k \in D_A$; then, we have, due to (5), $D^\alpha D^{\alpha_j} Z_k(t) z_k = D^\alpha Y_{\sigma_j - \sigma_k - 1}(t) A z_k = Y_{\sigma_j - \sigma_k - 1 + \alpha}(t) A z_k$,

$$\|D^\alpha D^{\alpha_j} Z_k(t) z_k\|_{\mathcal{Z}} = \|Y_{\sigma_j - \sigma_k - 1 + \alpha}(t) A z_k\|_{\mathcal{Z}} \leq C t^{\sigma_n - \sigma_j + \sigma_k - \alpha} \|A z_k\|_{\mathcal{Z}},$$

$\sigma_n - \sigma_j + \sigma_k - \alpha \geq \alpha_n + \alpha_0 - 1 - \alpha = \alpha_n - \alpha > 0$. In addition, for $z_k \in D_A$,

$$\|D^\alpha(\mathcal{D}^{\sigma_k} Z_k(t) z_k - z_k)\|_{\mathcal{Z}} = \|D^\alpha Y_{-1}(t) A z_k\|_{\mathcal{Z}} = \|Y_{\alpha-1}(t) A z_k\|_{\mathcal{Z}} \leq C t^{\sigma_n - \alpha} \|A z_k\|_{\mathcal{Z}},$$

$\sigma_n - \alpha = \alpha_1 + \alpha_2 + \ldots \alpha_n - \alpha > 0$. Thus, $D^\alpha \mathcal{D}^{\sigma_j} Z_k(\cdot) z_k, D^\alpha(\mathcal{D}^{\sigma_k} Z_k(t) z_k - z_k) \in C([0,T];\mathcal{Z})$ for $z_k \in D_A, k, j \in \{0,1,\ldots,n-1\}, k \neq j$; therefore, by Lemma 3, $\mathcal{D}^{\sigma_j} Z_k(\cdot) z_k \in C^\alpha([0,T];\mathcal{Z})$.

Moreover, Equality (10) implies that

$$D^\alpha D^{\alpha_j} \int_0^t Y_0(t-s) B(s, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s), u(s)) ds$$

$$= D^1 \int_0^t Y_{\sigma_j - 1 + \alpha}(t-s) B(s, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s), u(s)) ds$$

$$= \int_0^t Y_{\sigma_j + \alpha}(t-s) B(s, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s), u(s)) ds,$$

$$\left\| D^\alpha D^{\alpha_j} \int_0^t Y_0(t-s) B(s, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s), u(s)) ds \right\|_{\mathcal{Z}}$$

$$\leq \frac{C}{\sigma_n - \sigma_j - \alpha} \max_{s \in [0,t]} \|B(s, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s), u(s))\|_{\mathcal{Z}} t^{\sigma_n - \sigma_j - \alpha}.$$

Here, we use inequalities $\|Y_{\sigma_j - 1 + \alpha}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C t^{\sigma_n - \sigma_j - \alpha}$, $\sigma_n - \sigma_j - \alpha \geq \alpha_n - \alpha > 0$. Hence,

$$D^{\alpha_j} \int_0^t Y_0(t-s) B(s, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s), u(s)) ds \in C^\alpha([0,T];\mathcal{Z})$$

and $D^{\alpha_j} z \in C^\alpha([0,T];\mathcal{Z})$, $j = 0,1,\ldots,n-1$.

We recall that $u^0(t) = F(t, v(t))$, where

$$v(t) = \mathcal{D}^{\sigma_n} \Psi(t) - \overline{\Phi A} \sum_{k=0}^{n-1} Z_k(t) z_k - \Phi B_1(t, \mathcal{D}^{\sigma_0} z(t), \mathcal{D}^{\sigma_1} z(t), \ldots, \mathcal{D}^{\sigma_{n-1}} z(t))$$

$$- \int_0^t \overline{\Phi A} Y_0(t-\tau) B(\tau, \mathcal{D}^{\sigma_0} z(\tau), \mathcal{D}^{\sigma_1} z(\tau), \ldots, \mathcal{D}^{\sigma_{n-1}} z(\tau), u(\tau)) d\tau.$$

Consequently, under conditions $(\mathcal{E}_1)$, $(\mathcal{F}_1)$,

$$u^0(t) - u^0(s) = F(t, v(t)) - F(s, v(s)) = F(t, v(t)) - F(t, v(s)) + F(t, v(s)) - F(s, v(s)),$$

$$\|u^0(t) - u^0(s)\|_{\mathcal{U}} \leq C_1(|s-t|^\alpha + \|v(t) - v(s)\|_{\mathcal{U}}) \leq C_1|s-t|^\alpha +$$

$$+ C_2 \|B_1(t, \mathcal{D}^{\sigma_0} z(t), \mathcal{D}^{\sigma_1} z(t), \ldots, \mathcal{D}^{\sigma_{n-1}} z(t)) - B_1(t, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s))\|_{\mathcal{Z}}$$

$$+ C_2 \|B_1(t, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s)) - B_1(s, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s))\|_{\mathcal{Z}}$$

$$\leq C_3|s-t|^\alpha + C_3 \sum_{k=0}^{n-1} \|\mathcal{D}^{\sigma_k} z(s) - \mathcal{D}^{\sigma_k} z(s)\|_{\mathcal{Z}} \leq C_3|s-t|^\alpha.$$

Therefore, mapping $t \to B(t, \mathcal{D}^{\sigma_0} z(s), \mathcal{D}^{\sigma_1} z(s), \ldots, \mathcal{D}^{\sigma_{n-1}} z(s), u(t))$ satisfies the Hölder condition with power α due to condition $(\mathcal{F}_1)$. Thus, by Theorem 1, z is a solution of direct problem (1), (2) with a given u^0; therefore, (z, u^0) is a classical solution of identification problem (1)–(3). □

4. Time-Fractional Phase Field System of Equations

We let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial\Omega$, $\delta, \nu, \kappa \in \mathbb{R}$, $\alpha_0 = 1$, $\alpha_1, \alpha_2, \dots, \alpha_n \in (0,1)$, $\sigma_k = \alpha_0 + \alpha_1 + \cdots + \alpha_k - 1$, $k = 0,1,\dots,n$; $\Delta = \frac{\partial^2}{\partial^2\xi_1} + \cdots + \frac{\partial^2}{\partial^2\xi_d}$ is the Laplace operator, $\langle\cdot,\cdot\rangle$ is the inner product in space $L_2(\Omega)$, $\eta_l \in H^2(\Omega)$, $l = 1,2,\dots,2n$. We consider problem

$$\mathcal{D}_t^{\sigma_k} x(\xi,0) = x_k(\xi), \quad \mathcal{D}_t^{\sigma_k} y(\xi,0) = y_k(\xi), \quad k = 0,1,\dots,n-1, \quad \xi \in \Omega, \tag{16}$$

$$(1-\delta)x(\xi,t) + \delta\frac{\partial x}{\partial n}(\xi,t) = 0, \ (1-\delta)y(\xi,t) + \delta\frac{\partial y}{\partial n}(\xi,t) = 0, \ (\xi,t) \in \partial\Omega \times [0,T], \tag{17}$$

$$\langle x(\cdot,t), \eta_l\rangle = \phi_l(t), \ \langle y(\cdot,t), \eta_l\rangle = \psi_l(t), \ l = 1,2,\dots,2n, \ t \in [0,T], \tag{18}$$

to the system of equations in $\Omega \times [0,T]$,

$$\mathcal{D}_t^{\sigma_n} x(\xi,t) = \Delta x(\xi,t) - \Delta y(\xi,t) + \sum_{k=0}^{n-1} u_k^1(t)\mathcal{D}_t^{\sigma_k} x(\xi,t) + \sum_{k=0}^{n-1} v_k^1(t)\mathcal{D}_t^{\sigma_k} y(\xi,t) + f(\xi,t), \tag{19}$$

$$\mathcal{D}_t^{\sigma_n} y(\xi,t) = \nu\Delta y(\xi,t) - \kappa y^3(\xi,t) + \sum_{k=0}^{n-1} u_k^2(t)\mathcal{D}_t^{\sigma_k} x(\xi,t) + \sum_{k=0}^{n-1} v_k^2(t)\mathcal{D}_t^{\sigma_k} y(\xi,t) + g(\xi,t) \tag{20}$$

with unknown functions x, y, u_k^i, v_k^i, $k = 0,1,\dots,n-1$, $i = 1,2$. At $n = 1$, $\alpha_1 = 1$, $u_k^1 = v_k^1 = 0$, $k = 0,1,\dots,n-1$, $u_k^2 = v_k^2 = 0$, $k = 1,2,\dots,n-1$, System (19), (20) up to linear replacement $x(\xi,t) = \widetilde{x}(\xi,t) + \frac{l}{2}\widetilde{y}(\xi,t)$, $y(\xi,t) = \frac{l}{2}\widetilde{y}(\xi,t)$, $l \in \mathbb{R}$, and stretching over t coincide with the linearization of phase field equations [29,30].

We put $j \in \mathbb{N}$, $j > d/2$, $\mathcal{Z} = (H^j(\Omega))^2$,

$$A = \begin{pmatrix} \Delta & -\Delta \\ 0 & \nu\Delta \end{pmatrix}, \quad D_A = (H_\delta^{j+2}(\Omega))^2,$$

$$H_\delta^{j+2}(\Omega) := \left\{ h \in H^{j+2}(\Omega) : \left(\delta\frac{\partial}{\partial n} + 1 - \delta\right) h(\xi) = 0, \ \xi \in \partial\Omega \right\}.$$

Therefore, $A \in \mathcal{C}l(\mathcal{Z})$.

By $\{\varphi_k : k \in \mathbb{N}\}$, we denote orthonormal in the sense of the inner product $\langle\cdot,\cdot\rangle$ in $L_2(\Omega)$ eigenfunctions of the Laplace operator with domain $H_\delta^{j+2}(\Omega)$, which are numbered in the order of non-increasing eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ taking into account their multiplicities.

Theorem 5. *We let $\alpha_0, \alpha_1, \dots, \alpha_n \in (0,1)$, $\alpha_0 + \alpha_n > 1$, $\sigma_n = 1$, $\nu > 0$, $\delta \in \mathbb{R}$, $j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$ in this case $\sigma(A) = \{\lambda_k, \nu\lambda_k : k \in \mathbb{N}\}$.*

Proof. We take the basis $\{\varphi_k : k \in \mathbb{N}\}$ in $L_2(\Omega)$ and obtain for $\lambda \neq \lambda_k$, $\lambda \neq \nu\lambda_k$, $k \in \mathbb{N}$ operators

$$\lambda^{\sigma_n} I - A = \begin{pmatrix} \lambda I - \Delta & \Delta \\ 0 & \lambda^{\sigma_n} I - \nu\Delta \end{pmatrix},$$

$$(\lambda^{\sigma_n} I - A)^{-1} = \sum_{k=1}^{\infty} \begin{pmatrix} \frac{1}{\lambda-\lambda_k} & -\frac{\lambda_k}{(\lambda-\lambda_k)(\lambda-\nu\lambda_k)} \\ 0 & \frac{1}{\lambda-\nu\lambda_k} \end{pmatrix} \langle\cdot,\varphi_k\rangle\varphi_k.$$

We take arbitrary $\theta_0 \in (\pi/2, \pi)$, $a_0 > \max\{\lambda_k, \nu\lambda_k : k \in \mathbb{N}\}$; then, for any $a \geq a_0$, $\lambda \in S_{\theta_0,a_0}$,

$$\left|\frac{1}{\lambda - \lambda_k}\right| \leq \frac{-1}{|\lambda - a|\sin\theta_0}, \quad \left|\frac{1}{\lambda - \nu\lambda_k}\right| \leq \frac{-1}{|\lambda - a|\sin\theta_0},$$

$$\left|\frac{\lambda_k}{(\lambda-\lambda_k)(\lambda-\nu\lambda_k)}\right| = \left|\frac{1}{(1-\lambda/\lambda_k)(\lambda-\nu\lambda_k)}\right| \le \frac{C_1}{|\lambda-a|\sin^2\theta_0}$$

for some $C_1 > 0$, since for all $\lambda \in S_{\theta_0,a_0}$, $k \in \mathbb{N}$,

$$\left|\frac{1}{1-\lambda/\lambda_k}\right| \le C_1.$$

Therefore, for all $\lambda \in S_{\theta_0,a_0}$,

$$\|R_{\lambda^{\sigma_n}}(A)\|^2_{\mathcal{L}(\mathcal{Z})} \le \frac{C_2}{|\lambda^{\sigma_n}-a|^{\alpha_0}|\lambda|^{\sigma_n-\alpha_0}}$$

and $A \in \mathcal{A}_{\{\alpha_k\}}(\theta_0, a_0)$. □

Theorem 6. *We let $\alpha_0 = 1$, $\alpha_1, \alpha_2, \dots, \alpha_n \in (0,1)$, $\sigma_n = 1$, $\nu > 0$, $\delta, \kappa \in \mathbb{R}$, $j \in \mathbb{N}$, $j > d/2$, $g, h \in C([0,T]; H^j(\Omega))$, $x_k, y_k \in H^{2+j}_\delta(\Omega)$, $k = 0,1,\dots,n-1$, $\eta_l \in H^2(\Omega)$, $\mathcal{D}^{\sigma_k}\phi_l, \mathcal{D}^{\sigma_k}\psi_l \in C([0,T];\mathbb{R})$, $k = 0,1,\dots,n$, $\langle x_0, \eta_l\rangle = \phi_l(0)$, $\langle y_0, \eta_l\rangle = \psi_l(0)$, $l = 1,2,\dots,2n$,*

$$\det\begin{pmatrix} \mathcal{D}^{\sigma_0}\phi_1(t) & \mathcal{D}^{\sigma_0}\psi_1(t) & \dots & \mathcal{D}^{\sigma_{n-1}}\phi_1(t) & \mathcal{D}^{\sigma_{n-1}}\psi_1(t) \\ \mathcal{D}^{\sigma_0}\phi_2(t) & \mathcal{D}^{\sigma_0}\psi_2(t) & \dots & \mathcal{D}^{\sigma_{n-1}}\phi_2(t) & \mathcal{D}^{\sigma_{n-1}}\psi_2(t) \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{D}^{\sigma_0}\phi_{2n}(t) & \mathcal{D}^{\sigma_0}\psi_{2n}(t) & \dots & \mathcal{D}^{\sigma_{n-1}}\phi_{2n}(t) & \mathcal{D}^{\sigma_{n-1}}\psi_{2n}(t) \end{pmatrix} \ne 0 \qquad (21)$$

for all $t \in [0,T]$. Then, Problem (16)–(20) has a unique mild solution.

Proof. We take $\mathcal{U} = \mathbb{R}^{4n}$, $(b_k, c_k) \in \mathcal{Z}$, $k = 0,1,\dots,n-1$, $u^i_k, v^i_k \in \mathbb{R}$, $k = 0,1,\dots,n-1$, $i = 1,2$,

$$B(t, b_0, c_0, \dots, b_{n-1}, c_{n-1}, u^1_0, v^1_0, \dots, u^1_{n-1}, v^1_{n-1}, u^2_0, v^2_0, \dots, u^2_{n-1}, v^2_{n-1})$$

$$= \begin{pmatrix} \sum\limits_{k=0}^{n-1} u^1_k b_k + \sum\limits_{k=0}^{n-1} v^1_k c_k + f(\cdot,t) \\ \sum\limits_{k=0}^{n-1} u^2_k b_k + \sum\limits_{k=0}^{n-1} v^2_k c_k - \kappa c_0^3 + g(\cdot,t) \end{pmatrix}.$$

Hence, $B : \mathcal{Z}^n \times \mathcal{U} \to \mathcal{Z}$,

$$B_1(t, b_0, c_0, \dots, b_{n-1}, c_{n-1}) = \begin{pmatrix} f(\cdot,t) \\ -\kappa c_0^3 + g(\cdot,t) \end{pmatrix}.$$

We have $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_{2n})$, where for $h = (h_1, h_2)$ $\Phi_l h = (\langle h_1, \eta_l\rangle, \langle h_2, \eta_l\rangle)$, $l = 1,2,\dots,2n$. Then,

$$\Phi_l A h = (\langle \Delta h_1 - \Delta h_2, \eta_l\rangle, \langle \nu\Delta h_2, \eta_l\rangle) = (\langle h_1 - h_2, \Delta\eta_l\rangle, \langle h_2, \nu\Delta\eta_l\rangle), \quad l = 1,2,\dots,2n,$$

since $\eta_l \in H^2(\Omega)$, $l = 1,2,\dots,2n$. Hence, $\overline{\Phi A} \in \mathcal{L}(\mathcal{Z};\mathcal{U})$.

Condition $(\mathcal{B})$ is satisfied due to Condition (21) at $t = 0$. In Condition $(\mathcal{C})$, we take for $u_{lk}, v_{lk}, u^i_k, v^i_k \in \mathbb{R}$, $k = 0,1,\dots,n-1$, $l = 1,2,\dots,2n$, $i = 1,2$, $B_3 : \mathbb{R}^{4n(n+1)} \to \mathbb{R}^{4n}$,

$$B_3(u_{10}, v_{10}, \dots, u_{2n\,n-1}, v_{2n\,n-1}, u^1_0, v^1_0, \dots, u^1_{n-1}, v^1_{n-1}, u^2_0, v^2_0, \dots, u^2_{n-1}, v^2_{n-1})$$

$$= \begin{pmatrix} u_{10}u^1_0 + v_{10}v^1_0 + \dots + u_{1\,n-1}u^1_{n-1} + v_{1\,n-1}v^1_{n-1} \\ \dots \\ u_{2n\,0}u^1_0 + v_{2n\,0}v^1_0 + \dots + u_{2n\,n-1}u^1_{n-1} + v_{2n\,n-1}v^1_{n-1} \\ u_{10}u^2_0 + v_{10}v^2_0 + \dots + u_{1\,n-1}u^2_{n-1} + v_{1\,n-1}v^2_{n-1} \\ \dots \\ u_{2n\,0}u^2_0 + v_{2n\,0}v^2_0 + \dots + u_{2n\,n-1}u^2_{n-1} + v_{2n\,n-1}v^2_{n-1} \end{pmatrix}.$$

By Inequality (21), Condition $(\mathcal{D})$ is valid.

Due to inequality $j > d/2$, by virtue of the Sobolev embedding theorem, $\mathcal{Z} \subset (C(\overline{\Omega}))^2$. In such a case, a nonlinear operator $N(b_0, c_0) = (0, -\kappa c_0^3)$ acts from $\mathcal{Z}$ into $\mathcal{Z}$ and satisfies the local Lipschitz condition. Therefore, Condition $(\mathcal{F})$ is satisfied by the construction of operators B_1 and $B_2 = B - B_1$. In addition, $F(t, v) = G(t)^{-1}v$, where matrix

$$G(t) = \begin{pmatrix} \mathcal{D}^{\sigma_0}\phi_1(t) & \mathcal{D}^{\sigma_0}\psi_1(t) & \dots & \mathcal{D}^{\sigma_{n-1}}\phi_1(t) & \mathcal{D}^{\sigma_{n-1}}\psi_1(t) \\ \mathcal{D}^{\sigma_0}\phi_2(t) & \mathcal{D}^{\sigma_0}\psi_2(t) & \dots & \mathcal{D}^{\sigma_{n-1}}\phi_2(t) & \mathcal{D}^{\sigma_{n-1}}\psi_2(t) \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{D}^{\sigma_0}\phi_{2n}(t) & \mathcal{D}^{\sigma_0}\psi_{2n}(t) & \dots & \mathcal{D}^{\sigma_{n-1}}\phi_{2n}(t) & \mathcal{D}^{\sigma_{n-1}}\psi_{2n}(t) \end{pmatrix} \in C([0, T]; \mathbb{R}^{2n\times 2n}),$$

hence, due to (21) $G^{-1} \in C([0, T]; \mathbb{R}^{2n\times 2n})$, Condition $(\mathcal{E})$ is fulfilled.

Reference to Theorems 2 and 5 completes the proof. □

By slightly increasing the smoothness of some data, we obtain a theorem on the classical solution of identification problem (16)–(20).

Theorem 7. *We let* $\alpha_0 = 1$, $\alpha_1, \alpha_2, \dots, \alpha_n \in (0, 1)$, $\sigma_n = 1$, $\nu > 0$, $\delta, \kappa \in \mathbb{R}$, $j \in \mathbb{N}$, $j > d/2$, $g, h \in C^\gamma([0, T]; H^j(\Omega))$, $\gamma \in (0, 1]$, $x_k, y_k \in H_\delta^{2+j}(\Omega)$, $k = 0, 1, \dots, n-1$, $\eta_l \in H^2(\Omega)$, $\mathcal{D}^{\sigma_k}\phi_l, \mathcal{D}^{\sigma_k}\psi_l \in C([0, T]; \mathbb{R})$, $k = 0, 1, \dots, n-1$, $\mathcal{D}^{\sigma_n}\phi_l, \mathcal{D}^{\sigma_n}\psi_l \in C^\gamma([0, T]; \mathbb{R})$, $\langle x_0, \eta_l\rangle = \phi_l(0)$, $\langle y_0, \eta_l\rangle = \psi_l(0)$, $l = 1, 2, \dots, 2n$, *Condition* (21) *be fulfilled for all* $t \in [0, T]$, $G^{-1} \in C^\gamma([0, T]; \mathbb{R}^{2n\times 2n})$. *Then, Problem* (16)–(20) *has a unique classical solution.*

Proof. Under the additional condition $g, h \in C^\gamma([0, T]; H^j(\Omega))$, assumption $(\mathcal{E}_1)$ is satisfied. Due to condition $G^{-1} \in C([0, T]; \mathbb{R}^{2n\times 2n})$, assumption $(\mathcal{F}_1)$ is valid. By Theorems 4 and 5, we obtain the required result. □

5. Conclusions

The obtained abstract results of the unique solvability in the sense of mild and classical solutions for a wide new class of identification problems can find their applications in the study of coefficient inverse problems for various evolutionary equations and systems of equations with the general fractional derivative. In particular, it can be useful for parabolic equations, for equations of hydrodynamics, the theory of visoelasticity, etc. Moreover, the unique solvability theorems can be used for the correct formulation of initial boundary value problems in applied research and for the development of numerical methods for solving these problems.

Author Contributions: Conceptualization, V.E.F.; methodology, V.E.F. and M.V.P.; software, D.V.M.; validation, M.V.P. and D.V.M.; formal analysis, D.V.M.; investigation, V.E.F. and M.V.P.; resources, D.V.M.; data curation, D.V.M.; writing—original draft preparation, V.E.F. and M.V.P.; writing—review and editing, V.E.F.; visualization, D.V.M.; supervision, V.E.F. and M.V.P.; project administration, V.E.F.; funding acquisition, M.V.P. All authors have read and agreed to the published version of the manuscript.

Funding: The work is supported by the grant of Russian Science Foundation and of the Government of the Chelyabinsk Region, project number 23-21-10015.

Data Availability Statement: Data sharing not applicable.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analysis, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

References

1. Dzhrbashyan, M.M.; Nersesyan, A.B. Fractional derivatives and Cauchy problem for differential equations of fractional order. *Izv. AN Arm. SSR. Mat.* **1968**, *3*, 3–28. (In Russian) [CrossRef]
2. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach Science Publishers: Philadelphia, PA, USA, 1993.
3. Podlubny, I. *Fractional Differential Equations*; Academic: Boston, MA, USA, 1999.
4. Pskhu, A.V. *Partial Differential Equations of Fractional Order*; Nauka: Moscow, Russia, 2005. (In Russian)
5. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Publishing: Amsterdam, The Netherlands; Boston, MA, USA; Heidelberg, Germany, 2006.
6. Hilfer, R. *Applications of Fractional Calculus in Physics*; WSPC: Singapore, 2000.
7. Nakhushev, A.M. *Fractional Calculus ant Its Applications*; Fizmatlit: Moscow, Russia, 2003. (In Russian)
8. Tarasov, V.E. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles*; Springer: New York, NY, USA, 2011.
9. Uchaykin, V.V. *Fractional Derivatives for Physicists and Engineers*; Higher Education Press: Beijing, China, 2012.
10. Prilepko, A.I.; Orlovsky, D.G.; Vasin, I.A. *Methods for Solving Inverse Problems in Mathematical Physics*; Marcel Dekker, Inc.: New York, NY, USA, 2000.
11. Favini, A.; Lorenzi, A. *Differential Equations. Inverse and Direct Problems*; Chapman and Hall/CRC: New York, NY, USA, 2006.
12. Fedorov, V.E.; Ivanova, N.D. Identification problem for degenerate evolution equations of fractional order. *Fract. Calc. Appl. Anal.* **2017**, *20*, 706–721. [CrossRef]
13. Orlovsky, D.G. Parameter determination in a differential equation of fractional order with Riemann–Liouville fractional derivative in a Hilbert space. *J. Sib. Fed. Univ. Math. Phys.* **2015**, *8*, 55–63. [CrossRef]
14. Orlovsky, D.G. Determination of the parameter of the differential equation of fractional order with the Caputo derivative in Hilbert space. *J. Phys. Conf. Ser.* **2019**, *1205*, 012042. [CrossRef]
15. Kostin, A.B.; Piskarev, S.I. Inverse source problem for the abstract fractional differential equation. *J. Inverse Ill-Posed Probl.* **2021**, *29*, 267–281. [CrossRef]
16. Orlovsky, D.; Piskarev, S. Inverse problem with final overdetermination for time-fractional differential equation in a Banach space. *J. Inverse Ill-Posed Probl.* **2022**, *30*, 221–237. [CrossRef]
17. Fedorov, V.E.; Ivanova, N.D.; Borel, L.V.; Avilovich, A.S. Nonlinear inverse problems for fractional differential equations with sectorial operators. *Lobachevskii J. Math.* **2022**, *43*, 3125–3141. [CrossRef]
18. Dzhrbashyan, M.M.; Nersesyan, A.B. Fractional derivatives and Cauchy problem for differential equations of fractional order. *Fract. Calc. Appl. Anal.* **2020**, *23*, 1810–1836. [CrossRef]
19. Pskhu, A.V. The fundamental solution of a diffusion-wave equation of fractional order. *Izv. Math.* **2009**, *73*, 351–392. [CrossRef]
20. Pskhu, A.V. Fractional diffusion equation with discretely distributed differentiation operator. *Sib. Electron. Math. Rep.* **2016**, *13*, 1078–1098.
21. Pskhu, A.V. Boundary value problem for a first-order partial differential equation with a fractional discretely distributed differentiation operator. *Differ. Equ.* **2016**, *52*, 1610–1623. [CrossRef]
22. Bogatyreva, F.T. Boundary value problems for first order partial differential equation with the Dzhrbashyan–Nersesyan operators. *Chelyabinsk Phys. Math. J.* **2021**, *6*, 403–416.
23. Mamchuev, M.O. Boundary value problem for a system of partial differential equations with the Dzhrbashyan–Nersesyan fractional differentiation operators. *Bull. Karaganda Univ. Math. Ser.* **2022**, *2*, 143–160. [CrossRef]
24. Ahmad, A.; Baleanu, D. On two backward problems with Dzherbashian–Nersesian operator. *AIMS Math.* **2023**, *8*, 887–904. [CrossRef]
25. Fedorov, V.E.; Plekhanova, M.V.; Izhberdeeva, E.M. Initial value problem for linear equations with the Dzhrbashyan–Nersesyan derivative in Banach spaces. *Symmetry* **2021**, *13*, 1058. [CrossRef]
26. Fedorov, V.E.; Plekhanova, M.V.; Izhberdeeva, E.M. Analytic resolving families for equations with the Dzhrbashyan–Nersesyan fractional derivative. *Fractal Fract.* **2022**, *6*, 541. [CrossRef]
27. Plekhanova, M.V.; Izhberdeeva, E.M. Local unique solvability of a quasilinear equation with the Dzhrbashyan–Nersesyan derivatives. *Lobachevskii J. Math.* **2022**, *43*, 1379–1388. [CrossRef]
28. Fedorov, V.E.; Plekhanova, M.V.; Melekhina, D.V. Nonlinear inverse problems for equations with Dzhrbashyan–Nersesyan derivatives. *Fractal Fract.* **2023**, *7*, 464. [CrossRef]
29. Caginalp, G. An analysis of a phase field model of a free boundary. *Arch. Ration. Mech. Anal.* **1986**, *92*, 205–245. [CrossRef]
30. Caginalp, G. Stefan and Hele–Shaw type models as asymptotic limits of the phase-field equations. *Phys. Rev. A* **1989**, *39*, 5887–5896. [CrossRef] [PubMed]