Lax Extensions of Conical I-Semifilter Monads

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Abstract: For a quantale I, the unit interval endowed with a continuous triangular norm, we introduce the canonical, op-canonical and Kleisli extensions of the conical I-semifilter monad to I-Rel. It is proved that the op-canonical extension coincides with the Kleisli extension.

Key Words: lax extension; conical I-semifilter monad; Kleisli extension

MSC: 18C15; 18D20

1. Introduction and Preliminaries

Monoidal topology [1] provides a unification of settings to describe some important mathematical structures as (T, Q, Ê)-algebras (lax algebras for short) in which Q is a quantale and T is a monad on Set with a lax extension Ê to the category Q-Rel of sets and Q-relations.

Examples include:

• Metric spaces can be described as (I, P+, I)-algebras [2].
• Topological spaces can be characterized as (β, 2, β)-algebras [3,4].
• Approach spaces [5] can be viewed as (β, P+, β)-algebras [6].

Here, 2 denotes the two-element quantale, P+ = ([0, ∞], 0, +) is the Lawvere quantale, I is the identity monad with the identity extension, and β is the ultrafilter monad with the Barr extensions β to 2-Rel (Rel for short) and I-Rel, respectively.

To study many-valued topologies within the monoidal topology framework, it is of importance to determine the counterpart of the filter monad in the many-valued context and investigate its lax extensions. Extensive studies have been conducted to develop many-valued filter monads and their lax extensions, including the β-valued filter monad [7], the T-filter monad with its Kleisli extension to Rel [8], and the saturated prefilter monad with its Kleisli extension to Rel [9]. The lax algebras for the latter two are both CNS spaces, which are a kind of many-value topological spaces introduced in [10].

Lax extensions offer rich topological structures. For example, as demonstrated in [11], there are two lax extensions of the filter monad F to Q-Rel: the canonical one ˆ F and the op-canonical one ˆ F. When Q = 2, the lax algebras with respect to the canonical extension are closure spaces, while those associated with the op-canonical extension are topological spaces. When Q = P+, the lax algebras with respect to the canonical extension are closedness spaces, while those for the op-canonical extension are approach spaces.

The approach adopted in this paper is motivated by an observation that the filter monad is the discrete restriction of two composite monads on Ord: up-set-ideal monad IdeUp and the down-set-filter monad FilDn. Furthermore, the canonical (op-canonical) lax extension of the filter monad can be induced from the lax extension of IdeUp (FilDn) to Dist.

In Section 2, we introduce the composite monads Cβ P and Cβ P and show that the discrete restriction of them are the conical I-semifilter monad [12], where C is the monad of I-distributors generated by a forward Cauchy net that plays the role of the ordered-ideal monad Ide. The canonical and op-canonical extensions of the conical I-semifilter monad...
to $\mathbf{l}$-$\text{Rel}$ are also presented in this section. Section 3 focuses on the Kleisli extension of the conical $\mathbf{l}$-semifilter monad to $\mathbf{l}$-$\text{Rel}$. The lax algebras for the Kleisli extension to $\mathbf{l}$-$\text{Rel}$ are same to those for the Kleisli extension to $\text{Rel}$.

In the remainder of this section, we introduce the many-valued context in which we work, including the quantale $\mathbf{l}$, $\mathbf{l}$-relations and $\mathbf{l}$-categories.

### 1.1. Monads

A monad on a category $\mathcal{A}$ is a triple $T = (T, m, e)$, where $T: \mathcal{A} \to \mathcal{A}$ is an endfunctor and $m: T^2 \to T, e: \text{id}_{\mathcal{A}} \to T$ are natural transformations such that

$$m \cdot eT = m \cdot Te = \text{id}_{\mathcal{A}} \quad \text{and} \quad m \cdot mT = m \cdot Tm.$$  

Sometimes, we simply write $T$ for $(T, m, e)$ if no confusion arises.

Given two monads $\mathcal{T} = (T, m, e)$ and $\mathcal{S} = (S, n, d)$, a morphism $\sigma: \mathcal{T} \to \mathcal{S}$ of monads is a natural transformation $\sigma: T \to S$ such that

$$d = \sigma \cdot e \quad \text{and} \quad \sigma \cdot m = n \cdot (\sigma \ast \sigma),$$

where $\ast$ is the horizontal composition of natural transformations.

We let $(T, m, e)$ be a monad on $\mathcal{A}$. A submonad of $(T, m, e)$ is a monad $(S, n, d)$ with a monad morphism $i: (S, n, d) \to (T, m, e)$ such that every component $i_X$ is monic. In this case, $i: S \to T$ is called the inclusion transformation. To keep notations simple, we write $(S, m, e)$ for submonad $(S, n, d)$.

Given monad $\mathcal{T} = (T, m, e)$ on $\mathcal{A}$, an Eilenberg–Moore algebra for $\mathcal{T}$ (or $\mathcal{S}$-algebra for short) is a pair $(X, a)$ consisting of an $\mathcal{A}$-object $X$ and an $\mathcal{A}$-morphism $a: TX \to X$ subject to the following:

$$a \cdot e_X = 1_X \quad \text{and} \quad a \cdot m_X = a \cdot Ta.$$

$(TX, m_X)$ is obviously a $\mathcal{T}$-algebra, which is called the free $\mathcal{T}$-algebra on $X$.

A $\mathcal{T}$-homomorphism $f: (X, a) \to (X', a')$ of $\mathcal{T}$-algebras is an $\mathcal{A}$-morphism $f: X \to X'$ such that $a' \cdot Tf = f \cdot a$. $\mathcal{T}$-algebras and $\mathcal{T}$-homomorphisms assemble into a category $\mathcal{A}^\mathcal{T}$, which is called the Eilenberg–Moore category of $\mathcal{T}$.

Given a monad morphism $\sigma: \mathcal{S} \to \mathcal{T}$, there exists a functor $K_\sigma: \text{Set}^{\mathcal{T}} \to \text{Set}^{\mathcal{S}}$ induced by $\sigma$, which is identical on morphisms, sends the $\mathcal{T}$-algebra $(X, a)$ to the $\mathcal{S}$-algebra $(X, a \cdot \sigma_X)$, and makes the diagram

$$\begin{array}{ccc}
\mathcal{A}^\mathcal{S} & \xrightarrow{K_\sigma} & \mathcal{A}^\mathcal{T} \\
\downarrow G_\mathcal{S} & & \downarrow G_\mathcal{T} \\
\mathcal{A} & \xleftarrow{G_\mathcal{T}} & \mathcal{A}
\end{array}$$

commute, where $G_\mathcal{T}, G_\mathcal{S}$ are forgetful functors.

For more information on monads, we refer to [13,14]. Monads are useful for encoding general algebraic structures. The monograph by Plotkin [15] offers a comprehensive exploration of the algebraic aspects of database theory. Therefore, further research on the application of monads in the theory of databases is warranted.

### Power-Enriched Monads

The powerset monad $\mathcal{P}$ is given by the covariant powerset functor $P: \text{Set} \to \text{Set}$ and two natural transformations:

$$\{-\}_X: X \to PX, x \mapsto \{x\},$$

$$\bigcup_X: P^2 X \to PX, \mathcal{A} \mapsto \bigcup \mathcal{A}.$$  

The Eilenberg–Moore category of the powerset monad is isomorphic to the category $\text{Sup}$ of complete lattices and sup-maps.
We consider monad $\mathbb{T}$ on $\text{Set}$ equipped with monad morphism $\sigma: P \to \mathbb{T}$. By the functor $K_\sigma: \text{Set}^\mathbb{T} \to \text{Set}^P$, every $\mathbb{T}$-algebra $(X,a)$ carries an order making $X$ a complete lattice, and every morphism of $\mathbb{T}$-algebras is a sup-map. In particular, endowed with the order induced by the free $\mathbb{T}$-algebra structure on $X$, every set $TX$ becomes a complete lattice.

If, for any sets $X,Y$, the map
\[
(\cdot)^\mathbb{T}: \text{Set}(X,TY) \to \text{Set}(TX,TY), \quad f \mapsto m_Y \cdot Tf
\]
is monotone, where the hom-sets $\text{Set}(-,TY)$ are ordered pointwise, then we refer to $(\mathbb{T},\sigma)$ as a power-enriched monad. Morphism $\sigma: (\mathbb{T},\sigma_1) \to (\mathbb{S},\sigma'_1)$ of power-enriched monads is monad morphism $\sigma: \mathbb{T} \to \mathbb{S}$ such that $\sigma'_1 = \sigma \cdot \sigma_1$.

1.2. t-Settings
1.2.1. Continuous Triangular Norms

A triangular norm [16] (t-norm for short) is a binary operation $\&$ on the unit interval $I$ subject to the following:
- $\&$ is commutative;
- $\&$ is associative;
- $a \& (-)$ is monotone for any $a \in I$;
- $a \& 1 = a$ for any $a \in I$.

A t-norm $\&$ is called continuous if map $\&: I^2 \to I$ is continuous with respect to the standard topologies. We denote by $l = (I,\&)$ the unit interval $I$ endowed with a continuous t-norm $\&$.

Example 1. There are three basic continuous t-norms.

1. The Gödel t-norm $a \& b = a \land b$.
2. The product t-norm $a \& b = a \times b$.
3. The Łukasiewicz t-norm $a \& b = \max\{0, a + b - 1\}$.

For each $a \in I$, since $a \& (-): I \to I$ preserves arbitrary joints, then there exists a map $a \to (-): I \to I$ which is right adjoint to $a \& (-)$ and is determined by
\[
a \& b \leq c \iff b \leq a \to c.
\]

A continuous t-norm is said to satisfy condition (S); if it satisfies that, for each $a \in (0,1)$, map $a \to (-)$ is continuous on the interval $[0,a)$.

The following proposition includes some basic properties of continuous t-norms.

**Proposition 1.** For any $a, b, c \in I$ and $\{a_i\} \subset I$,

1. $a \& (a \to b) \leq b$;
2. $1 \to a = a$;
3. $a \to b = 1 \iff a \leq b$;
4. $(a \& b) \to c = a \to (b \to c)$;
5. $a \to (\lor_i a_i) = \lor_i (a \to a_i)$;
6. $(\lor_i a_i) \to a = \lor_i (a_i \to a)$.

The reasons why we work with the particular quantale $l$ include:

i. Some important many-valued topological structures are considered as topologies valued in $l = (I,\&)$ with $\&$ being certain t-norms. For example, fuzzy topologies can be seen as topologies valued in $(I,\land,1)$, and since $(I,\times,1)$ is isomorphic to the Lawvere quantale $P_+$, approach spaces can be considered as topological spaces valued in $(I,\times,1)$.
ii. Many results about topologies valued in $Q$ rely on the structure of $Q$; due to the celebrated ordinal sum decomposition theorem [16,17], the structure of $I$ is clear.

1.2.2. I-Relations

An I-relation $r: X \to Y$ is a map $r: X \times Y \to I$. The composition of $r: X \to Y, s: Y \to Z$ is an I-relation $(s \cdot r): X \to Z$ given by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \& s(y, z).$$

Sets and I-relations assemble into a category

$I$-Rel.

Since the composition of I-relations preserves arbitrary joins in each variable, for each $r: X \to Y$ and set $Z$, there are two maps $(-) \circ r: I$-Rel$(X, Z) \to I$-Rel$(Y, Z)$ and $r \to (-): I$-Rel$(Z, Y) \to I$-Rel$(Z, X)$ determined by

$$r \cdot t \leq s \iff t \leq s \circ r;$$
$$t' \circ r \leq s \iff t' \leq r \circ s$$

for any $t \in I$-Rel$(Y, Z)$ and $t' \in I$-Rel$(Z, X)$.

For each $r: X \to Y$, there is an I-relation $r^{op}: X \to Y$ given by $r^{op}(y, x) = r(x, y)$. For each map $f: X \to Y$, graph $f_0: X \to Y$ of $f$ is given by

$$f_0(x, y) = \begin{cases} 1, & f(x) = y; \\ 0, & f(x) \neq y. \end{cases}$$

And the cograph $f^\circ$ of $f$ is given by $f^\circ = (f_0)^{op}$. There are two functors:

$$(-)^0: \text{Set} \to I$-$\text{Rel}$ and $(-)^{op}: \text{Set} \to I$-$\text{Rel}^{op}$.

1.2.3. Lax Extensions to I-Rel

We let $(T, m, e)$ be a monad on Set. A lax extension [18] of $(T, m, e)$ to I-Rel is a triple $\hat{T} = (T, m, e)$, where $\hat{T}$ is given by a family of maps

$$\hat{T}_{X,Y}: I$-$\text{Rel}(X, Y) \to I$-$\text{Rel}(TX, TY)$$

subject to the following conditions:

1. Every $\hat{T}_{X,Y}$ is monotone;
2. $\hat{T}r \cdot \hat{T}s \leq \hat{T}(r \cdot s)$;
3. $(Tf)_a \leq T(f_0)$ and $(Tf)^\circ \leq T(f^\circ)$;
4. $s \cdot e_X^\circ \leq e_Y^\circ \cdot \hat{T}s$;
5. $\hat{T}Ts \cdot m_X^\circ \leq m_Y^\circ \cdot \hat{T}s$

for any sets $X, Y, Z$, I-relations $s: X \to Y, r: Y \to Z$ and every map $f: X \to Y$.

Morphism $\sigma: (S, n, d) \to (\hat{T}, m, e)$ of lax extensions is a monad morphism $\sigma: (S, n, d) \to (T, m, e)$ such that $\hat{S}r \leq (\sigma_Y)^\circ \cdot \hat{T}r \cdot (\sigma_X)_0$ for any I-relation $r: X \to Y$.

We let $\sigma: S \to T$ be a monad morphism and $\hat{T}$ a lax extension of $T$ to I-Rel. There is a lax extension of $S$ given by

$$\hat{S}r = (\sigma_Y)^\circ \cdot \hat{T}r \cdot (\sigma_X)_0$$

for any I-relation $r: X \to Y$. This lax extension $\hat{S}$ is called the initial extension of $S$ induced by $\sigma$. 
1.2.4. l-Categories

An l-category \([2,19]\) is a pair \((X, r)\) consisting of a set \(X\) and a transitive and reflexive l-relation \(r\), that is,

\[ r(x, y) & r(y, z) \leq r(x, z) \quad \text{and} \quad r(x, x) = 1 \]

for all \(x, y, z \in X\). For convenience, we simply use \(X\) to denote an l-category \((X, r)\) and use \(X(-, -)\) to denote \(r(-, -)\).

For every l-category \(X\), the l-relation \(X^{op}(x, y) = X(y, x)\) also gives an l-category, which is called the dual of \(X\).

**Example 2.**

1. The singleton \(\{\ast\}\) set endowed with \((\text{id})_\ast\) is obviously an l-category.
2. The set \(I^X\) can be made an l-category via

\[ \text{sub}_X(\mu, v) = \bigwedge_{x \in X} \mu(x) \to v(x). \]

An l-functor \(f: X \to Y\) is a map \(f: X \to Y\) between l-categories such that

\[ X(x, y) \leq Y(f(x), f(y)) \]

for all \(x, y \in X\). If the converse of the above inequality also holds, we refer to this l-functor as fully faithful. l-Functors \(f: X \to Y, g: Y \to X\) are called an adjunction \(f \dashv g\) if

\[ Y(f(x), y) = X(x, g(y)) \]

for any \(x \in X, y \in Y\). In this case, we say \(f\) is left adjoint to \(g\).

**Example 3.** Given an l-relation \(r: X \to Y\), there is an adjunction \(r_\lor \dashv r_\land\) in which \(r_\land, r_\lor\) are given by

\[ r_\land: I^X \to I^Y, \mu \mapsto \bigwedge_{x \in X} r(x, -) \to \mu(x); \]
\[ r_\lor: I^Y \to I^X, v \mapsto \bigvee_{y \in Y} r(-, y) & v(y). \]

l-categories and l-functors assemble into a category

\(l\text{-Cat}\).

The forgetful functor \(\sigma: l\text{-Cat} \to \text{Set}\) admits a left adjoint:

\[ d: \text{Set} \to l\text{-Cat}, \quad X \mapsto (X, 1_X^X). \]

A locally small category is ordered if every hom-set carries an order such that the composition maps are monotone. A functor \(F: A \to B\) between ordered categories is called a 2-functor if every \(F_{A,B}: A(A, B) \to B(FA, FB)\) is monotone. A monad on an ordered category is called a 2-monad if the endfunctor is a 2-functor.

The underlying order of an l-category \(X\) is given by

\[ x \leq_X y \iff X(x, y) = 1. \]

An l-category \(X\) is called separated if its underlying order is a partial order. l-Cat is an ordered category with \(l\text{-Cat}(X, Y)\) carrying the pointwise order.

Given an l-category \(X\) and \(p \in I, x \in X\), the tensor of \((p, x)\) is an element \(p \otimes x\) of \(X\) such that \(X(p \otimes x, -) = p \to X(x, -)\); the cotensor of \((p, x)\) is an element \(p \mapsto x\) of \(X\) such that \(X(-, p \mapsto x) = p \to X(-, x)\).
An l-category $X$ is called tensored (cotensored) if it fulfills that the tensor $p \otimes x$ (cotensor $p \to x$) exists for all $p \in I, x \in X$.

**Proposition 2 ([20]).** The following statements are equivalent:

1. $X$ is tensored, $(X, \leq_X)$ is complete, and

   $$X(\bigvee_i x_i, y) = \bigwedge_i X(x_i, y)$$

   for all $\{x_i\}_i \subseteq X, y \in X$;

2. $X$ is cotensored, $(X, \leq_X)$ is complete, and

   $$X(x, \bigwedge_i y_i) = \bigvee_i X(x, y_i)$$

   for all $\{y_i\}_i \subseteq X, x \in X$.

An l-category is called complete if it satisfies the equivalent conditions stated above. For a complete l-category, we have $p \otimes (\_ - p \to \_)$.

**Example 4.** The l-category $(I^X, \text{sub}_X)$ is complete and separated. For any $p \in I, \mu \in I^X$, the cotensor of $(p, \mu)$ is given by $p \to \mu$.

The following proposition is useful in ensuring the existence of adjunctions.

**Proposition 3 ([20]).** We let $f : X \to Y, g : Y \to X$ be l-functors between l-categories. Then, $f \dashv g$ is an adjunction if and only if $f \dashv g : (Y, \leq_Y) \to (X, \leq_X)$ is an adjunction.

2. The Lax Extensions from the Laxly Extended Monads on l-Cat

2.1. l-Distributors

Given two l-categories, $X$ and $Y$, an l-distributor [2] $r : X \leftrightarrow Y$ is an l-relation such that

$$r \cdot X \leq r \quad \text{and} \quad Y \cdot r \leq r.$$

If an l-distributor $r : X \leftrightarrow Y$ is dummy in one variable, that is $X = \{\_\}$ or $Y = \{\_\}$, then we simply write $r(x)$ for $r(x, \_)$ or $r(\_, x)$. l-categories and l-distributors give rise to an ordered category

$I$-$\text{Dist}.$

The forgetful functor $\circ : I$-$\text{Dist} \to I$-$\text{Rel}$ admits a left adjoint:

$$d : I$-$\text{Rel} \to I$-$\text{Dist}, \quad dX = (X, 1_X^0), \quad r \mapsto r.$$

There are two 2-functors $(-)_* : I$-$\text{Cat} \to I$-$\text{Dist}^{op}$ and $(-)^* : I$-$\text{Cat} \to I$-$\text{Dist}^{op}$ defined on objects and morphisms by

$$(X)_* = X, \quad (f : X \to Y) \mapsto (f_* = (Y \cdot f_0)) : X \leftrightarrow Y;$$

$$(X)^* = X, \quad (f : X \to Y) \mapsto (f^* = (f^0 \cdot Y)) : Y \leftrightarrow X.$$}

We denote the set of l-distributors from an l-category $X$ to $\{\_\}$ by $\text{PX}$. Then, the set $\text{PX}$ can be made an l-category via

$$\text{PX}(\mu, v) = v \leftarrow \mu = \text{sub}_X(\mu, v).$$

Furthermore, $\text{P}$ can be made a 2-functor from $I$-$\text{Dist}^{op}$ to $I$-$\text{Cat}$ via

$$(r : X \leftrightarrow Y) \mapsto (P(r) : \mu \mapsto \mu \cdot r).$$
It is routine to check that $(-)^*$ is left adjoint to $P$. The induced 2-monad $(P, s, y)$ on $l$-Cat is called the presheaf monad.

Similarly, taking the $l$-distributors of type $\{\ast\} \Rightarrow X$ also gives rise to a 2-functor $P^\dagger: \text{Dist}^{\op} \to \text{lCat}$:

$$X \mapsto P^\dagger X, \quad (r: X \Rightarrow Y) \mapsto (P^\dagger(r): \mu \mapsto r \cdot \mu),$$

in which

$$P^\dagger X(\mu, v) = v \circ \mu = \text{sub}_{X}^{\op}(\mu, v)$$

for any $\mu, v \in P^\dagger X$. The functor $(-)\ast$ is left adjoint to $P^\dagger$. The induced 2-monad $(P^\dagger, s^\dagger, y^\dagger)$ on $l$-Cat is called the copresheaf monad.

The following lemmas present some basic properties of $l$-distributors.

**Lemma 1** (Yoneda Lemma). For any $v \in P^\dagger X, \mu \in PX$, we have

$$(y_X)_\ast(-, \mu) = \mu \quad \text{and} \quad (y^\dagger_X)_\ast(v, -) = v.$$  

**Lemma 2.** We let $f: X \to Y, g: Z \to Y$ be $l$-functors. For any $\mu \in PZ, v \in P^\dagger X, \phi \in P^\dagger PX$, and $\psi \in PP^\dagger Z$, we have the following statements:

1. $(Pg)^* \cdot (Pf)_\ast(-, \mu) = y_{PZ}(\mu \cdot g^* \cdot f_\ast)$;
2. $(P^\dagger g)^* \cdot (P^\dagger f)_\ast(v, -) = y^\dagger_{PZ}(g^* \cdot f_\ast \cdot v)$;
3. $(Pg)^* \cdot (Pf)_\ast \cdot \phi = \phi(- \cdot g^* \cdot f_\ast)$;
4. $\psi \cdot (P^\dagger g)^* \cdot (P^\dagger f)_\ast = \psi(g^* \cdot f_\ast \cdot -)$.

2.2. Composite Monads on $l$-Cat

We let $T = (T, m, e)$ and $S = (S, n, d)$ be monads. A distributive law of $T$ over $S$ is a natural transformation $\sigma: TS \to ST$ subject to some conditions. A composite monad of $T$ and $S$ is a monad $(ST, m \circ d \ast e)$ such that $Se: S \to ST, dT: T \to ST$ are monad morphisms and $m$ satisfies that $m \cdot (SeT) = \text{id}_{ST}$. A distributive law $\sigma$ yields a composite monad

$$(ST, (n \ast m) \cdot SeT, d \ast e).$$

This correspondence is bijective. Details can be found in [21].

A saturated class of weights is a submonad $A$ of the presheaf monad $P$. It is easy to check that it also offers a submonad $A^\dagger$ of $P^\dagger$ by $A^\dagger X = (AX^{\op})^{\op}$ for any $X$.

A distributive law $\sigma: P^\dagger A \to A^\dagger P$ of $P^\dagger$ over $A$ also offers a distributive law of $P$ over $A^\dagger$ whose components are given by

$$\sigma^\dagger_X: PA^\dagger X = (P^\dagger AX^{\op})^{\op} \xrightarrow{\sigma^\dagger_X^{op}} (AP^\dagger X^{\op})^{op} = A^\dagger PX.$$  

One example of distributive laws is that the copresheaf monad distributes over the presheaf monad.

**Proposition 4 ([22]).** There is a distributive law of $P^\dagger$ over $P$, which offers the double presheaf 2-monad $PP^\dagger$ on $l$-Cat.

We let $X$ be an $l$-category. A forward Cauchy net [23] on $X$ is a net $\{x_i\}_{i \in D}$ such that

$$\bigvee_{i \in D} \bigwedge_{k \geq i} X(x_i, x_k) = 1.$$  

A forward Cauchy net generates an $l$-distributor $\mu: X \Rightarrow \{\ast\}$:

$$\mu = \bigvee_{i \in D} \bigwedge_{j \geq i} X(-, x_j).$$
Example 5. A directed set $D$ of $(X, \leq_X)$ is a forward Cauchy net $\{x_i\}_{i \in D}$ on $X$. The l-distributor generated by $D$ is

$$\bigvee_{d \in D} X(-, d).$$

We denote by $C_X$ the set of all l-distributors $\mu : X \ni \{\ast\}$ generated by forward Cauchy nets. The proof of that $C$ is a saturated class of weights can be found in [24].

With a lax extension $\hat{T}$ (Theorem 8.5 in [26])

(1) Every $\hat{T}$ defines a lax extension of $I$ for any $\sigma$.

2.3. The Lax Extensions of Composite Monads to $I$

We let $\mathcal{X}$ be a complete separated l-category. For every $\phi \in C_X$, we have that $D = \{x \in X \mid \phi(x) = 1\}$ is a directed set on $(X, \leq_X)$ and

$$\phi = \bigvee_{d \in D} X(-, d).$$

The existence of a distributive law of $P^\dagger$ over $C$ depends on the structure of quantale $I$.

Proposition 5 (Theorem 6.4 in [25]). There is a distributive law of $P^\dagger$ over $C$ if and only if the continuous t-norm satisfies the condition (S).

In the remainder of this paper, we always assume that the continuous t-norm & satisfies the condition (S).

Lemma 3 (Proposition 4.8 in [25]). We let $X$ be a complete separated l-category. For every $\phi \in C_X$, we have that $D = \{x \in X \mid \phi(x) = 1\}$ is a directed set on $(X, \leq_X)$ and

$$\phi = \bigvee_{d \in D} X(-, d).$$

The existence of a distributive law of $P^\dagger$ over $C$ depends on the structure of quantale $I$.

Theorem 1 (Theorem 8.5 in [26]). We let $T$ be a 2-monad on $I$-Cat. Then,

$$\hat{T}_X : I$Dist$(X, Y) \to I$Dist$(TX, TY)$$

subject to the following conditions:

1. Every $\hat{T}_{X,Y}$ is monotone;
2. $\hat{T}_r \cdot s \leq \hat{T}(r \cdot s)$;
3. $(Tf)_* \leq \hat{T}(f_*)$ and $(Tf)^* \leq \hat{T}(f)^*$;
4. $s \cdot e_T^X \leq e_T^Y \cdot \hat{T}_s$;
5. $\hat{T}_r \cdot m_T^X \leq m_T^Y \cdot \hat{T}_s$

for any l-categories $X, Y, Z$, distributors $s : X \ni Y, r : Y \ni Z$ and every l-functor $f : X \to Y$.

Theorem 1 (Theorem 8.5 in [26]). We let $T$ be a 2-monad on $I$-Cat. Then,

$$\hat{T}_r = (T^\dagger r^\dagger)^* \cdot (TY_X)_s : TX \to TY$$

defines a lax extension of $T$ to $I$-Dist, where $r^\dagger : Y \to PX, y \mapsto r(-, y)$.

We let $A$ be a saturated class of weights and assume that there is a distributive law $\sigma : P^\dagger A \to AP^\dagger$. Then, by Theorem 1, there are lax extensions of the monad $AP^\dagger$ and $A^\dagger P$ given by

$$\overline{AP^\dagger}_r = (AP^\dagger r^\dagger)^* \cdot (AP^\dagger y_X)_s;$$
$$\overline{A^\dagger P}_r = (A^\dagger P^\dagger r^\dagger)^* \cdot (AP^\dagger y_X)_s.$$

In [27], Lai and Tholen introduced a functor $\Gamma$ which maps monads $(T, m, e)$ on $I$-Cat with a lax extension $\hat{T}$ to $I$-Dist to monads on Set with a lax extension to $I$-Rel:

$$\Gamma(T, m, e) = (oTd \cdot omd \cdot oTeTd, oed),$$
$$\Gamma(\hat{T})r = oTd(r).$$
in which $c$ is the counit of the adjunction $d \dashv o$.

It is routine to check that $\Gamma(\mathcal{A}^+ P) = \Gamma(\mathcal{A}P^+)$. We denote this monad by $(U_A, n, d)$.

For the lax extensions, using Lemma 2, we can compute as follows: for any $l$-relation $r: X \to Y, \phi, \psi, \phi' \in \mathcal{A}^+ P X, \psi \in \mathcal{A}^+ P Y, \phi' \in \mathcal{A}^+ P X$,

$$\overline{\mathcal{A}P^+ r}(\phi, \phi') = ((\mathcal{A}^+ P r)^* \cdot (\mathcal{A}^+ P Y))_* (\phi, \phi')$$

$$= P^+ Y ((\mathcal{A}^+ P r)^* \cdot (\mathcal{A}^+ P Y))_* (\phi, \phi')$$

and

$$\overline{\mathcal{A}P^+ r}(\psi, \psi') = ((\mathcal{A}^+ P Y)^* \cdot (\mathcal{A}^+ P r))_* (\psi, \psi')$$

$$= P^+ Y ((\mathcal{A}^+ P Y)^* \cdot (\mathcal{A}^+ P r))_* (\psi, \psi').$$

Thus, we obtain the following result.

**Proposition 6.** We let $\mathcal{A}P^+$ be a composite monad. There are two lax extensions of the monad $(U_A, n, d)$:

$$\overline{U_A r}(\phi, \psi) = \bigwedge_{\mu \in I^X} \phi(\mu) \to \psi((r^* \circ)(\mu)), \quad \text{(canonical)}$$

$$\overline{U_A r}(\phi, \psi) = \bigwedge_{\nu \in I^Y} \psi(\nu) \to \phi(r(\nu)), \quad \text{(op-canonical)}$$

where $r: X \to Y$ is an $l$-relation, $\phi \in U_A X, \psi \in U_A Y$.

### 2.4. The Conical $l$-Semifilter Monad

A conical $l$-semifilter [12] on set $X$ is a function $\phi: I^X \to I$ subject to the following:

- (F1) $\phi(1_X) = 1$;
- (F2) $\phi(\mu \land \nu) = \phi(\mu) \land \phi(\nu)$;
- (F3) $\text{sub}_X(\mu, \nu) \leq \phi(\mu) \to \phi(\nu)$;
- (F4) $\phi = \lor_{\phi(\xi) = 1} \text{sub}_X(\xi, -)$.

**Proposition 7.** The elements of $\mathcal{C}P^+ dX$ are exactly the conical $l$-semifilters.

**Proof.** Given a conical $l$-semifilter $\phi$ on $X$, it follows from (F2) that $\{\mu \mid \phi(\mu) = 1\}$ is a directed set of $P^+ dX$; hence, by (F4), we have $\phi \in \mathcal{C}P^+ dX$.

We let $\phi \in \mathcal{C}P^+ dX$. Since $P^+ dX$ is separated and complete, by Lemma 3, it holds that

$$\phi = \bigvee_{\phi(\nu) = 1} P^+ dX(-, \nu) = \bigvee_{\phi(\nu) = 1} \text{sub}_X(\nu, -).$$

Hence, (F1), (F3) and (F4) are obvious. For (F2),

$$\phi(\mu_1 \land \mu_2) = \bigvee_{\phi(\nu) = 1} (P^+ dX(\mu_1, \nu) \land P^+ dX(\mu_2, \nu)) = \phi(\mu_1) \land \phi(\mu_2),$$

the last equality holds because $\{\nu \mid \phi(\nu) = 1\}$ is directed. \[\square\]

For every set $X, o(y \cdot y^t)_{dX}$ maps $x \in X$ to $P^+ dX(-, y_{dX}(x)) = (-)(x); (o(s \cdot s^t)\cdot oC\sigma P^t d \cdot oC\tau P^t)_{dX}$ maps $\Phi \in U_{C^2} X$ to the conical $l$-semifilter

$$\phi: P^+ dX \to I, \mu \mapsto \Phi(\mu^t),$$
where \( \mu^i \) belongs to \( P^i do CP^i dX \) and maps every \( \psi \in do CP^i dX \) to \( \psi(\mu) \). Therefore, the monad \( \langle U_C, n, d \rangle \) is exactly the conical l-semifilter monad in [12]. We adopt the notation from [12] and denote \( \langle U_C, n, d \rangle \) by \( (CSF, n, d) \).

**Corollary 1.** There are two lax extensions of the conical l-semifilter monad \( (CSF, n, d) \):

\[
\begin{align*}
\text{CSFr}(\phi, \psi) &= \bigwedge_{\mu \in I^X} \phi(\mu) \rightarrow \psi((r^{op})\vee(\mu)), \quad \text{(canonical)} \\
\text{CSFr}(\phi, \psi) &= \bigwedge_{\nu \in I^Y} \psi(\nu) \rightarrow \phi(\nu), \quad \text{(op-canonical)}
\end{align*}
\]

where \( r: X \rightarrow Y \) is an l-relation, \( \phi \in \text{CSFX}, \psi \in \text{CSFY} \).

**Remark 1.** Here, we prove that the continuous l-norm satisfies the condition \( (S) \) is a sufficient condition for conical l-semifilters to give rise to a monad. In fact, it is also a necessary condition; see [12].

3. The Kleisli Extensions of \( (UA, n, d) \)

3.1. The l-Powerset Monad

For each set \( X \), we let \( P_1X = I^X \). Then, \( P_1 \) can be made a functor from \( l\text{-Rel}^{op} \) to \( \text{Set} \) by letting

\[
R_1(r)(\mu) = r(\nu) = \bigvee_{y \in Y} \mu(y) & r(-, y)
\]

for each l-relation \( r: X \rightarrow Y \) and \( \mu \in I^Y \). It is routine to check that \( (-)^{op} \) is left adjoint to \( P_1 \). The induced monad is called the l-powerset monad and is denoted by \( P_1 = (P_1, m, e) \). We spell it out here: for any maps \( f: X \rightarrow Y \) and \( \mu \in R_1X \),

\[
R_1(f)(\mu): y \mapsto \bigvee_{f(x) = y} \mu(x),
\]

\[
e_X: x \mapsto 1_x,
\]

\[
m_X: \phi \mapsto \bigvee_{\mu \in R_1X} \phi(\mu) \& \mu,
\]

where \( 1_A \) is defined as \( 1_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases} \) and \( 1 \) denotes \( 1_X \).

It is easy to check that the l-powerset monad is power-enriched by

\[
\theta_X: P_1X \rightarrow P_1X, \quad A \mapsto 1_A.
\]

It also holds that \( P_1 = \Gamma(P, s, y) = \Gamma(P^i, s^i, y^i) \).

3.2. l-Power-Enriched Monads

An l-power-enriched monad is a pair \( (T, \sigma) \) composed of a monad \( (T, m, e) \) on \( \text{Set} \) and a monad morphism \( \sigma: P_1 \rightarrow T \) such that \( (T, \sigma \cdot \theta) \) is a power-enriched monad. A morphism \( \sigma: (T, \sigma_1) \rightarrow (S, \sigma_2) \) of l-power-enriched monads is a monad morphism \( \sigma: T \rightarrow S \) such that \( \sigma_2 = \sigma \cdot \sigma_1 \).

We let \( AP^i \) be a composite monad. Since there is a monad morphism \( yP^i: P^i \rightarrow AP^i \), by applying the functor \( \Gamma \), we obtain the following Proposition.

**Proposition 8.** The monad \( \langle UA, n, d \rangle \) is l-power-enriched by \( \kappa \) whose components are given by

\[
\kappa_X: P_1X \rightarrow UA X, \quad \mu \mapsto \text{sub}_X(\mu, -).
\]
An l-action in $\text{Sup}$ is a complete lattice $X$ endowed with a map $- \otimes -$: $I \times X \to X$ subject to the following: for any $p, q \in I$ and $x \in X$

(1) $p \otimes -$ and $- \otimes x$ are sup-maps;

(2) $(p \& q) \otimes x = p \otimes (q \otimes x)$ and $1 \otimes x = x$.

A morphism of l-actions is a sup-map $f: X \to Y$ such that $p \otimes_Y f(x) = f(p \otimes_X x)$ for any $p \in I$ and $x \in X$. l-actions in $\text{Sup}$ and their morphisms assemble into a category $\text{Sup}^l$.

It is shown in [28] that $\text{Sup}^l$ is isomorphic to the Eilenberg–Moore category of the l-powerset monad and there exists a functor $\Lambda: \text{Set}^{\mathbb{P}} \to \text{l-Cat}$.

Explicitly, we let $(X, a)$ be a $\mathbb{P}_l$-algebra; by functor $\kappa_\mathbb{P}: \text{Set}^{\mathbb{P}} \to \text{Set}_l$, $X$ can be made a complete lattice. The l-action on $X$ in $\text{Sup}$ is given by

$$- \otimes -: I \times X \to X, \quad (p, x) \mapsto a(p \& 1_x).$$

Conversely, an l-action $(X, - \otimes -)$ yields a $\mathbb{P}_l$-algebra structure as follows:

$$a: \mathbb{P}_l X \to X, \quad \mu \mapsto \bigvee_x \mu(x) \otimes x.$$ 

The functor $\Lambda$ maps a $\mathbb{P}_l$-algebra $(X, a)$ to

$$\Lambda(X, a)(x, y) = a^{-1}(y)(x),$$

where $a^{-1}$: $(X, \leq_X) \to (\mathbb{P}_I X, \leq_{\mathbb{P}_I X})$ is an adjunction. Furthermore, we have the following proposition.

**Proposition 9.** Every l-category $\Lambda(X, a)$ is complete.

**Proof.** For every $p \in I$, since $p \otimes -$ and $a$ are sup-maps, we have the following adjunctions:

$$\begin{align*}
X & \xleftarrow{p \otimes -} X \\
p \otimes - & \xrightarrow{a^{-1}} \mathbb{P}_I X.
\end{align*}$$

To show $X$ is cotensored by $\rightarrow$, we can follow these steps:

$$\begin{align*}
\mu \leq p \rightarrow a^{-1}(x) & \iff p \& \mu \leq a^{-1}(x) \\
& \iff a(p \& \mu) \leq x \\
& \iff \bigvee_t (p \& \mu(t)) \otimes t \leq x \\
& \iff p \otimes \left( \bigvee_t \mu(t) \otimes t \right) \leq x \\
& \iff p \otimes a(\mu) \leq x \\
& \iff a(\mu) \leq p \rightarrow x \\
& \iff \mu \leq a^{-1}(p \rightarrow x). \quad \Box
\end{align*}$$

Thus, the tensor of $\Lambda(X, a)$ is given by its l-action, the cotensor is given by the right adjoint of its l-action. That is the reason why we use the same notations.

**Example 6.** For a composite monad $\mathbb{A} \mathbb{P}_l^+$, since $(\mathbb{A} \mathbb{X}, n_X \cdot \kappa_{\mathbb{A} \mathbb{X}}) = \kappa_\mathbb{X}(\mathbb{A} \mathbb{X}, n_X)$ is a $\mathbb{P}_l$-algebra, $\mathbb{A} \mathbb{X}$ can be made a complete l-category via

$$\mathbb{A}(\phi, \psi) = (n_X \cdot \kappa_{\mathbb{A} \mathbb{X}})^{-1}(\psi)(\phi) = \text{sub}_{\mathbb{I} \mathbb{X}}(\psi, \phi) = \bigwedge_{\mu \in \mathbb{I} \mathbb{X}} \psi(\mu) \rightarrow \phi(\mu).$$
The tensor of \((p, \phi)\) in \(U_A X\) is given by
\[
(n X \cdot \kappa U_A X) (p \& 1_{\phi}) = \bigwedge_{\varphi \in U_A X} (p \& 1_{\phi}(\psi) \to \psi) = p \to \phi.
\]

3.3. Kleisli Extensions

Given an \(l\)-power-enriched category \((\mathcal{T}, \sigma)\), for any \(l\)-relations \(r: X \to Y\), the composite \(\mathbb{P}_l\)-homomorphism
\[
(TY, m_Y) \xrightarrow{T(r \sigma r^\sigma)} (T^2 X, m_{TX}) \xrightarrow{m_X} (TX, m_X)
\]
offers an \(l\)-functor \(r^\sigma: TY \to TX\), where \(r^\sigma: Y \to P_I X, y \mapsto r(-, y)\).

According to Section 4.5 in [18], there is a lax extension \(\hat{T}\) of \(T\) to \(l\)-Rel named the Kleisli extension, which is given by
\[
\hat{T}r(\phi, \psi) = TX(\phi, r^\sigma(\psi))
\]
for any \(\phi \in TX, \psi \in TY\) and every \(l\)-relation \(r: X \to Y\).

**Proposition 10.** For a composite monad \(A P^+\), the Kleisli extension of \((U_A, n, e)\) is given by
\[
\overline{U_A}r(\phi, \psi) = U_A X(\phi, r^\sigma(\psi)) = \bigwedge_{\mu \in I^X} \psi(r_\land(\mu)) \to \phi(\mu),
\]
where \(r: X \to Y\) is an \(l\)-relation, \(\phi \in U_A X, \psi \in U_A Y\).

**Theorem 2.** For the monad \(U_p\), the op-canonical extension to \(l\)-Rel coincides with the Kleisli extension to \(l\)-Rel.

**Proof.** For any \(l\)-relation \(r: X \to Y\) and \(\phi \in U_p X\), by Lemma 2, the \(l\)-distributor
\[
\begin{array}{c}
\star \xrightarrow{\phi} PdX \xrightarrow{(P dX)\cdot \kappa} P^2 dX \xrightarrow{(P dY)\cdot \kappa} PdY
\end{array}
\]
is given by \(\phi(-, \cdot (\mathcal{T})^* (y dX)_*) = \phi(r_\lor(-))\). Thus, mapping \(\phi\) to \(\phi(r_\lor(-))\) is an \(l\)-functor \(f: U_p X \to U_p Y\).

To show the op-canonical extension to \(l\)-Rel coincides with the Kleisli extension to \(l\)-Rel, by Proposition 3, it suffices to show that \(f \dashv r^\sigma: (U_p Y, \leq_{U_p Y}) \to (U_p X, \leq_{U_p X})\) is an adjunction. For any \(\chi \in U_p X, \psi \in U_p Y\), since \(r_\lor \vdash r_\land\) we have
\[
(r^\sigma \cdot f)(\chi) = \chi \cdot r_\lor \cdot r_\land \geq_{U_p X} \chi \quad \text{and} \quad (f \cdot r^\sigma)(\psi) = \psi \cdot r_\land \cdot r_\lor \leq_{U_p Y} \psi.
\]
This completes the proof. \(\square\)

Since
\[
\text{CSFr}(\phi, \psi) = \overline{U_p} r(i_X(\phi), i_Y(\psi)) \quad \text{and} \quad \text{CSFr}(\phi, \psi) = \overline{U_p} r(i_X(\phi), i_Y(\psi))
\]
for any \(\phi \in \text{CSF} X, \psi \in \text{CSF} Y, r: X \to Y\), where \(i : \text{CSF} \to U_p\) is the inclusion transformation, we have the following corollary.

**Corollary 2.** For the conical \(l\)-semifilter monad, the op-canonical extension to \(l\)-Rel coincides with the Kleisli extension to \(l\)-Rel.
Proposition 11. We let $\lambda : (S, \sigma) \to (T, \sigma')$ be a morphism of l-power-enriched monads. Then, $\lambda$ is a morphism of the Kleisli extensions to l-Rel. Furthermore, every component $\lambda_X : SX \to TX$ is fully faithful if and only if the initial extension of $S$ induced by $\lambda$ is the Kleisli extension of $S$.

**Proof.** We denote $T = (T, m, e)$ and $S = (S, n, d)$. By the commutative diagram

```
\[
\begin{array}{ccc}
S^2X & \xrightarrow{n_X} & SX \\
\downarrow{S(\lambda_X)} & & \downarrow{\lambda_X} \\
STX & \xrightarrow{\lambda_TX} T^2X & \xrightarrow{m_X} TX,
\end{array}
\]
```

$\lambda_X : (SX, n_X) \to (TX, m_X \cdot \lambda_TX)$ is an $S$-homomorphism; hence, it is an l-functor:

$$\hat{\lambda}(\sigma, \beta) = SX(\sigma, \tau^\sigma(\beta)) \leq TX(\lambda_X(\sigma), \lambda_X(\tau^\sigma(\beta))).$$

By the commutative diagram

```
\[
\begin{array}{ccc}
SY & \xrightarrow{S(\tau')} & SPX \\
\downarrow{\lambda_Y} & & \downarrow{\lambda_X} \\
TY & \xrightarrow{T(\tau')} TFX & \xrightarrow{T^2X} TX,
\end{array}
\]
```

we have

$$TX(\lambda_X(\sigma), \lambda_X(\tau^\sigma(\beta))) = TX(\lambda_X(\sigma), \tau^\lambda_Y(\beta)) = \hat{T}(\lambda_X(\sigma), \lambda_Y(\beta)).$$

This completes the proof. \(\square\)

An element of $I^X$ is called bounded if $\wedge \mu > 0$. A conical l-semifilter $\phi$ is called bounded if $\phi(\mu) < 1$ for any unbounded $\mu$. Conical bounded l-semifilters also give rise to a monad $(\text{ConBSF}, n, d)$, and there is a monad morphism $\eta : \text{CSF} \to \text{ConBSF}$

$$\eta_X : \text{CSFX} \to \text{ConBSFX}, \; \phi \mapsto \bigvee_{\phi(\mu) = 1} \text{sub}_X(\mu, -);$$

see [12] for details.

**Example 7.**

1. *The Kleisli extension of the conical l-semifilter monad to l-Rel coincides with the initial extension induced by the inclusion transformation $i : \text{CSF} \to \text{Up}.*

2. *The conical bounded l-semifilter monad is l-power-enriched by $\eta \cdot \kappa$, and $\eta : (\text{CSF}, \kappa) \to (\text{ConBSF}, \eta \cdot \kappa)$ is a morphism of l-power-enriched monads. Since $\kappa$ is not fully faithful, the Kleisli extension CSF does not coincide with the initial extension induced by $\kappa.*

### 3.4. Lax Algebras

Given a lax extension $\hat{T}$ of $T$ to l-Rel, a $(T, l, \hat{T})$-algebra (lax algebra for short) is a pair $(X, a : TX \to X)$ so that

$$(1_X)_o \leq a \cdot (e_X)_o \quad \text{and} \quad a \cdot \hat{T}a \leq a \cdot (m_X)_o.$$

A morphism $f : (X, a) \to (Y, b)$ of lax algebras is a map $f : X \to Y$ subject to

$$f_o \cdot a \leq b \cdot (Tf)_o.$$
Lax algebras and morphisms of lax algebras form a category denoted by

\[(\mathbb{T}, I, \hat{T})\text{-Cat}.
\]

When the involved lax extension is clear, we simply write \( (\mathbb{T}, I)\text{-Cat} \).

Lax extensions \( \hat{T} \) of monad \( \mathbb{T} \) to Rel and lax algebras of \( (\mathbb{T}, 2, \hat{T}) \) are defined in a manner similar to those of lax extensions to \( I\text{-Rel} \) and lax algebras of \( (\mathbb{T}, I, \hat{T}) \). Given an \( I \)-power-enriched monad \( (\mathbb{T}, \sigma) \), it can be extended to Rel via

\[a(Tr)\psi \iff \phi \leq_TX r^\sigma(\psi),\]

which is called the Kleisli extension of \( \mathbb{T} \) to Rel, where \( r \) is a 2-relation and \( r^\sigma \) is defined by treating \( r \) as the \( I \)-relation

\[r(x, y) = \begin{cases} 1, & x r y, \\ 0, & \text{otherwise}. \end{cases}\]

The following proposition affirms that, at the level of lax algebras, there is no distinction between the Kleisli extension to \( I\text{-Rel} \) and the Kleisli extension to Rel.

**Proposition 12** (Proposition 6.1 in [18]). We let \( (X, \sigma) \) be an \( I \)-power-enriched category. Then, there is an isomorphism

\[(\mathbb{T}, I)\text{-Cat} \cong (\mathbb{T}, 2)\text{-Cat},\]

in which the lax extensions are the Kleisli extensions.

In [9], it is proven that

\[(\text{CSF}, 2, \text{CSF})\text{-Cat} \cong \text{CNS},\]

where CNS is the category of CNS spaces. Therefore, we have the following corollary.

**Corollary 3.** There is an isomorphism:

\[(\text{CSF}, I, \text{CSF})\text{-Cat} \cong \text{CNS}.\]

When \& is the product t-norm, the conical bounded \( I \)-semifilter monad is isomorphic to the functional ideal monad, and by [29], we have

\[(\text{ConBSF}, 2, \text{ConBSF})\text{-Cat} \cong \text{App},\]

where App is the category of approach spaces and \( \text{ConBSF} \) is the Kleisli extension to Rel.

Since \( \eta : (\text{CSF}, \kappa) \to (\text{ConBSF}, \eta \cdot \kappa) \) is a morphism of the \( I \)-power-enriched category, by Theorem 11, it is a morphism of the Kleisli extensions. Hence, it induces an algebraic functor as follows:

**Proposition 13.** If \& is the product t-norm, there is a functor \( A_\kappa : \text{CNS} \to \text{App} : \)

\[(X, (-)^\circ) \mapsto (X, \mathfrak{A})\]

that maps a CNS space \( X \) to the approach space \( (X, \mathfrak{A}) \), where the bounded approach system \( \{\mathfrak{A}(x)\}_{x \in X} \) is given by

\[\mathfrak{A}(x) = \{\mu \in [0, \infty]^X \mid \bigvee_{\omega:\omega>0,\omega^\circ(x)=1} \text{sub}_X(\omega, e^{-\mu}) = 1\},\]

in which \( (-)^\circ \) is the interior operator of the CNS space \( X \).

4. Conclusions

In order to find the many-valued version of the filter monad, we begin with the composite monads \( \text{CP}^+, \text{C}^+ \text{P} \) on \( I\text{-Cat} \) and then restrict them to Set to obtain the monad \( U_C \).
This Set-based monad $U_C$ is precisely the conical $I$-semifilter monad. Three lax extensions of the conical $I$-semifilter monad to $I$-Rel are presented: the canonical, op-canonical and Kleisli extensions. We prove that the op-canonical extension coincides with the Kleisli extension. Lax algebras of this extension can be described using relations rather than $I$-relations; hence, they are CNS spaces.

**Problem 1.** When considering the canonical extension of the conical $I$-semifilter monad, what are the lax algebras?

As for the future research direction, exploring the connections between monoidal topology and nonstandard analysis [30,31] is of interest.

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**References**


27. Lai, H.; Tholen, W. Monads on Q-Cat and Their Lax Extensions to Q-Dist. *J. Pure Appl. Algebra* 2018, 222, 2143–2163. [CrossRef]

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