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On Kemeny Optimization Scheme for Fuzzy Set of Relations

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Abstract: The present paper investigated the aggregation of individual preferences into a group fuzzy preference relation for a fuzzy set of decision-makers (DMs). This aggregation is based on the Kemeny optimization scheme. It was proven that this group relation is a Type-2 fuzzy relation (T2FR). The decomposition approach was used to analyze the group T2FR. It is shown that the group T2FR can be decomposed according to secondary membership grades into a finite collection of Type-1 fuzzy relations. Each of them is a group fuzzy relation for a crisp set of DMs, which is the corresponding a-cut of the original fuzzy set of DMs. Illustrative examples are given.

Keywords: Kemeny distance; group decision-making problem; fuzzy preference relation; Type-2 fuzzy relation

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1. Introduction

In the classical formulation of the group decision-making (GDM) problem [1,2], there is a set of alternatives X and a group of decision-makers (DMs), which express their opinions about the alternatives to reach a common decision. In a fuzzy formulation, the problem is to order the alternatives from the best to the worst by associating them with some preference degrees in the interval [0, 1]. There are some different ways [3] to set preferences:

- By utility values;
- By ordering the alternatives from the best to the worst;
- By using a preference relation.

In the latter case, the DM’s preferences for X are given by a preference matrix. The elements of the matrix indicate the degree or the intensity of preference of one alternative over the other. There are different types of preference relations, which depend on the domain used to evaluate a preference intensity:

1. Fuzzy preference relations (FPRs) [4,5]. These are characterized by the fact that the elements of the preference matrix directly express the preference degree or the preference intensity of the alternative x_i over x_k. The equality p_{ik} = 0.5 indicates an indiscernibility between the alternatives x_i and x_k. The equality p_{ik} = 1 means that the alternative x_i is absolutely preferred to x_k. The inequality p_{ik} > 0.5 indicates that the alternative x_i is preferred to x_k. In addition, the strict consistency of an FPR requires the equality p_{ij} + p_{jk} = 1 for all i and k, which guarantees reflexivity and anti-symmetry. The property of additive transitivity is modeled (see, for instance [6]) by the equalities p_{ik} = p_{ij} + p_{jk} − 0.5 for all i, j, and k.

2. Multiplicative preference relations [7]. Each element p_{ik} ∈ [1/9, 9] of the preference matrix represents the ratio of the preference intensities of the alternatives x_i and x_k. This is interpreted as x_i times p_{ik} better than x_k; the equality p_{ik} = 1 indicates an
indifference between $x_i$ and $x_k$; the equality $p_{ik} = 9$ indicates that $x_i$ is unanimously preferred to $x_k$, and the case $p_{ik} \in \{2, \ldots, 8\}$ indicates intermediate evaluations. Usually, the property of multiplicative reciprocity $p_{ik} \cdot p_{ki} = 1$ for all $i$ and $k$ is also assumed. The property of multiplicative transitivity means that $p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji}$ for all $i, j, k$.

3. Linguistic preference relations [8–10] are characterized by the fact that each value $p_{ik}$ in the preference matrix represents the degree of a linguistic preference or the linguistic intensity of a preference of the alternative $x_i$ over $x_k$.

Once the DM’s preferences have been obtained, the next step in the GDM problem usually consists of the aggregation of the preferences into a group fuzzy preference relation and making a final decision. The choice of the aggregation function [11–13] plays a significant role in this problem. In [3,14,15], two groups of GDM problems were singled out: homogeneous and heterogeneous. If the DMs are of equal importance, then the GDM problem is homogeneous. The GDM problem is heterogeneous if the opinions of the DMs are important to varying degrees. Usually [14], the importance of a DM is defined by assigning an appropriate weight. There are many families of aggregation methods that use weights. Among them, the best known one is a weighted arithmetic mean. This method easily generalizes to the family of quasi-arithmetic means by transforming the weight values using a monotonic function of a weighted sum. In particular, when choosing a logarithmic function, the geometric mean is obtained, which is often applied in practice with multiplicative preference relations. The next important class of weighted averaging functions used in decision-making are ordered weighted averaging operators (OWAs) [16]. These operators also use weights, but they are assigned based on a prior ordering. The main advantage of the above aggregation methods is the ease of their implementation. However, using these methods leads to the following problems:

- The resulting FPR may not match the opinion of the majority of DMs; this happens when a significantly high or significantly low rating from one expert can greatly bias the final rating;
- It is impossible to construct a group FPR with specified properties, which may differ from the properties of individual FPRs;
- The dependence of a group FPR on which a subset of DMs (generally fuzzy) is involved in decision-making may remain outside the scope of attention.

The desire to overcome these disadvantages at the cost of increasing computational complexity leads to the concept of penalty-based aggregation functions [17–20]. Different distance functions can be used to construct a penalty and, in turn, to define an aggregation function based on the penalty. In this article, we used the Kemeny distance [21]. This was due to the fact that the Kemeny optimization scheme is widely used in practice, has remarkable properties, and does not exhibit a pathological behavior inherent in other methods; for example, it has consistency in group judgment when eliminating alternatives [22–24]. Note that the approach that we propose here for solving the GDM problem can also be applied to other distance functions.

The weights used in aggregation methods are interpreted quite well by DMs. Naturally, the more competent the expert, the greater the relative weight that can be assigned to this person. The only question is, “How many times greater in comparison with other experts?” Sometimes, this question is difficult to answer. A detailed review of weight-based methods can find in [25]. In [26,27], the idea was proposed to interpret the weights of the DMs as the values of the membership function of some fuzzy set (FS) describing some property of the DMs’ group, for instance “Experience”, “Conservatism”, etc. [28]. Applying a fuzzy logic approach to some humanitarian problems shows much clearer results compared to traditional statistical models [29].

Thus, we concluded that, in some cases, it is easier for a moderator of a GDM procedure to set not the relative weights of the DMs, but the value of their membership degrees to a certain FS according to the perception of the set of these experts by the moderator. With this at hand, natural questions arise: “Why one should use these degrees of membership as
weights?” and “Perhaps we should consider a different formulation of the GDM problem with a fuzzy set of DMs and use the methodology of FS theory to solve this problem?” The validity of these questions should be illustrated by an example. If we set the FS “Young people”, we do not use the values of the membership degrees of people’s age groups as the weights, but we operate with this set for our purposes according to the FS theory.

In this article, we focused on the investigating the problem “What is the group fuzzy preference relation if different DMs participate in a group decision-making with different membership degrees to an FS?” The goal of the article is to demonstrate that an FS of DMS generates a group Type-2 fuzzy relation (T2FR) with constant secondary grades (these grades do not depend on the primary degrees of membership).

Although, in general, a T2FR is a rather complicated mathematical object, T2FRs with constant secondary grades are simple enough for practical use. This fact allowed us to decompose this set by secondary grades into a collection of the corresponding fuzzy relations. To complete the introduction, we note that the present work continues the line of research in the field of mathematical operations with a fuzzy set of operands [30–33] and used the FS methodology [34].

2. Materials and Methods

There are two important preliminaries for the present work. These are the concepts of a T2FR and T2FRs with constant secondary grades.

2.1. Type-2 Fuzzy Relations

The Type-2 FSs and relations were introduced by Zadeh [34] and developed by Mizumoto and Tanaka [35] and Karnik and Mendel [36] as a generalization of a Type-1 FS (that is, an ordinary fuzzy set). A detailed overview of the applications of fuzzy sets and Type-2 fuzzy sets, demonstrating how these concepts have been applied in different fields, one can find in [37]. Similar to a classical Type-1 fuzzy relation (T1FR) (that is, an ordinary fuzzy relation), one can introduce the notion of a T2FR on a universal set X as a Type-2 FS on the Cartesian product X × X. Unlike a T1FR, the membership degree of elements in a T2FR is an FS on [0, 1]. A T1FR on a finite set X = {x1, . . . , xM} of alternatives can be represented by an m × m square matrix \( R = (r_{ik})_{i,k\in M} \), where \( M = \{1, \ldots, m\} \) is the set of alternative indices. For each \( i, k \in M \), the element \( r_{ik} \in [0, 1] \) characterizes the preference degree of the alternative \( x_i \in X \) over \( x_k \in X \). Therefore, a generalization of the matrix representation to the case of a T2FR leads to a specification of this relation in the form of the square matrix \( \tilde{R} = (\tilde{r}_{ik})_{i,k\in M} \) with elements that are FSs \( \tilde{r}_{ik} = \{(r_{ik}, \mu_{\tilde{r}_{ik}}(r_{ik})) : r_{ik} \in U_{\tilde{r}_{ik}}\} \) (in particular, they are fuzzy numbers) on [0, 1] with the MFs \( \mu_{\tilde{r}_{ik}}(r_{ik}) \), \( i, k \in M \), respectively. Here, for all \( i, k \in M \), \( r_{ik} \) is the primary membership degree of the pair \( (x_i, x_k) \) of the alternatives \( x_i \in X \) and \( x_k \in X \) to the T2FR \( \tilde{R} \); the value \( \mu_{\tilde{r}_{ik}}(r_{ik}) \) is called the secondary grade of \( r_{ik} \); \( U_{\tilde{r}_{ik}} \subseteq [0, 1] \) is the set of primary membership degrees \( r_{ik} \) of the pair \( (x_i, x_k) \in X \times X \) of alternatives to the T2FR \( \tilde{R} \). We interpret the set \( U_{\tilde{r}_{ik}} \) as the support of the FS \( \tilde{r}_{ik} \), that is \( U_{\tilde{r}_{ik}} = \text{supp}(\tilde{r}_{ik}) \).

**Remark 1.** The primary membership degree is deemed as the degree of manifestation of some property, which determines the given fuzzy relation. According to [38], we interpret the secondary grade as the degree of truth of the corresponding primary degree of this property.

According to [39], similar to Type-2 FSs, we define embedded T2FRs and T1FRs for any matrix T2FR \( \tilde{R} = (\tilde{r}_{ik})_{i,k\in M} \) on a finite set \( X = \{x_1, \ldots, x_M\} \) with the MFs \( \mu_{\tilde{r}_{ik}}(r_{ik}) \), \( i, k \in M \). Assume that \( r_{ik} \in U_{\tilde{r}_{ik}} \) is a unique primary degree of membership for each pair \( (x_i, x_k) \) of the alternatives \( x_i \in X \) and \( x_k \in X \), that is \( U_{\tilde{r}_{ik}} = \{r_{ik}\} \), \( i, k \in M \). We introduce the embedded matrix T2FR \( \tilde{R}^2 \) in the T2FR \( \tilde{R} \) in the form \( \tilde{R}^2 = (\tilde{r}^2_{ik})_{i,k\in M} \), where \( \tilde{r}^2_{ik} = \{(r_{ik}, \mu_{\tilde{r}^2_{ik}}(r_{ik}))\} \) is a singleton FS of preference degrees with \( \mu_{\tilde{r}^2_{ik}}(r_{ik}) = \mu_{\tilde{r}_{ik}}(r_{ik}) \) for each \( i, k \in M \).

A matrix \( R^2 = (r^2_{ik})_{i,k\in M} \) is called an embedded matrix T1FR in the T2FR \( \tilde{R} \). The notions of embedded T2FRs and T1FRs are an important research tool for T2FRs. Similar to Type-2 FSs, in Wavy-Slice-Representation [39], any T2FR \( \tilde{R} \) can be represented as the
collection \( \tilde{R} = \{ \tilde{R}^2 : \tilde{R}^2 \in E \} \), where \( E \) is the set of all possible embedded T2FRs in the T2FR \( \tilde{R} \).

**Remark 2.** Following [39], each element of the Type-2 fuzzy collection \( \tilde{R} \) is interpreted as a subset. Thus, the collection is represented as the classical union of its elements in the sense of FSs.

### 2.2. T2FRs with Constant Secondary Grades

We need a special case of a T2FR. It is defined in this section along the lines of the construction of T2FSs with constant secondary grades [32,33]. Let \( A \) be a finite set of all possible positive values \( \mu_{r_{ik}}(r_{ik}), r_{ik} \in U_{ik}, i, k \in M \) of secondary grades for the matrix T2FR \( \tilde{R} = (\tilde{r}_{ik})_{i,k \in M} \) on a finite set \( X = \{x_1, \ldots, x_m\} \).

**Definition 1.** We say that an embedded matrix T2FR \( \tilde{R}^2 \in \tilde{R} \) has the constant secondary grade \( \alpha \in A \) if, for each pair \((x_i, x_j) \in X \times X\), the unique primary degree \( r_{ik} \in U_{ik} = \{r_{ik}\} \) exists, for which \( \mu_{r_{ik}}(r_{ik}) = \alpha, i, k \in M \).

In other words, the matrix \( \tilde{R}^2_\alpha = (\tilde{r}^2_\alpha)_{i,k \in M} \) consists of singleton FSs \( \tilde{r}^2_\alpha = \{(r_{ik}, \alpha)\}, i, k \in M \) with the degree of membership being \( \alpha \).

**Remark 3.** There is the unique embedded T1FR \( R^1_\alpha = (r_{ik})_{i,k \in M} \) in the embedded T2FR \( \tilde{R}^2 \), whence a representation in the form

\[
\tilde{R}^2_\alpha = \{(R^1_\alpha, \alpha)\}. 
\]

According to Remark 1, the embedded matrix T2FR \( \tilde{R}^2_\alpha \) can be specified by the matrix \( R^1_\alpha = (r_{ik})_{i,k \in M} \) with the degree of truth being \( \alpha \). The notation (1) should be understood as \( R^2_\alpha = (\tilde{r}_{ik})_{i,k \in M} \), where \( \tilde{r}_{ik} = \{(r_{ik}, \alpha)\}, i, k \in M \) are the singleton FSs of preference degrees \( r_{ik}, i, k \in M \), respectively, with the degree of membership being equal to \( \alpha \).

### 3. Formulation of the Problem and Main Idea

#### 3.1. The Group FPR

We denote by \( M = \{1, \ldots, m\} \) the set of alternative indices and by \( N = \{1, \ldots, n\} \) the set (group) of DMs, where \( m \geq 2 \) and \( n \geq 2 \) are the cardinalities of the sets \( M \) and \( N \), respectively. We investigated the problem of constructing a group matrix FPR \( P = (p_{jk})_{i,j \in M} \) on a finite set \( X = \{x_1, \ldots, x_m\} \) of alternatives. Information on the pairwise comparison of alternatives by each DM \( j \in N \) is represented in the form of matrix FPRs \( p_{jk} = (p_{jk})_{i,j \in M} \). The element \( p_{jk} \) directly expresses the preference of the alternative \( x_i \) over \( x_k, i, k \in M \) for the DM \( j \in N \). We denote by \( \Pi = \{P = (p_{jk})_{i,j \in M} : p_{jk} \in [0,1], p_{jk} + p_{kl} = 1, i, k, l \in M \} \) the set of all matrices of preference degrees for which the conditions of strict consistency hold [4,5]. If necessary, another FPR model can also be used. For instance, the additive transitivity constraints \( p_{ik} = p_{kl} + p_{lk} - 0.5 \) for all \( i, k, l \in M \) can be added. Following the idea of constructing a group matrix FPR \( P = (p_{jk})_{i,j \in M} \) that is as similar as possible to the preference relations \( P^j = (p^j_{jk})_{i,j \in M} \) of DMs \( j \in N \), we use the Manhattan distance:

\[
d(P, P^j) = \sum_{i,j \in M} |p_{ik} - p^j_{ik}|
\]

between matrices for the construction of an aggregation function [34]. The sum of distances from all relations \( P^j, j \in N \) to a group relation \( P \) defines the objective function of an optimization problem:

\[
d^{\text{min}} = \min_{P \in \Pi} \sum_{j \in N} d(P, P^j) = \min_{P \in \Pi} \sum_{j \in N} \sum_{i,k \in M} |p_{ik} - p^j_{ik}|. 
\]
This formulation of the problem generalizes the Kemeny median [21] to the case of FPRs. The minimum $d_{ik}^{min}$ of the Kemeny distance characterizes the degree of inconsistency between the group FPR and the individual FPRs of DMs. The original formulation of the Kemeny median is difficult to solve because of the presence of binary optimization variables, whereas in the present formulation, the values of the variables $p_{ik}, i, k \in M$ are numbers from the closed interval $[0,1]$.

In addition, the problem (2) can be decomposed into the collection of simpler problems:

$$d_{ik}^{min} = \min_{p_{i} \in [0,1]} \sum_{j \in N} |p_{ik} - p_{ij}|; i < k, i, k \in M,$$

so that $d_{ik}^{min} = 2 \sum_{i,k \in M} d_{ik}^{min}$ and $p_{ii} = 0.5, i \in M$. For other FPR models, for instance those having additive transitivity constraints, such a decomposition is not possible.

### 3.2. Fuzzy Set of Decision-Makers

Assume that $\tilde{N} = \{(j, \mu_{S}(j)) : j \in N\}$ is a fuzzy set of DMs with the MF $\mu_{S}(j), j \in N$. The following question arises: “What is the group FPR on $X$ in the case of the fuzzy set $\tilde{N}$ of DMs with the corresponding individual matrix FPRs $P' = (p_{ik}')_{i,k \in M}, j \in N$?” In other words: “What is the group FPR on $X$ if the DMs $j \in N$ participate in decision-making with the corresponding degrees of membership $\mu_{S}(j), j \in N$?” We denote by $\tilde{P} = (\tilde{p}_{ik})_{i,k \in M}$ the matrix of this group FPR.

First, we generalize Problem (2) for the case of an arbitrary subset $J \subseteq N, J \neq \emptyset$ of DMs and represent (2) in a convenient form:

$$d^{min}(J) = \min \sum_{j \in J} \sum_{i,k \in M} y_{ik}^{j},$$

s.t.

$$y_{ik}^{j} = |p_{ik} - p_{ij}^{j}|; i, k \in M; j \in J; (p_{ik})_{i,k \in M} \in \Pi.$$

We denote by $P(J) = (p_{ik}(J))_{i,k \in M}$ the preference matrix, which is a solution to Problem (3). By using the same approach as for (2), Problem (3) can be decomposed into a collection of simpler problems. For the convenience of further presentation, the set $J \subseteq N$ of DMs is indicated hereinafter as a parameter in (3).

We considered a mapping $P : 2^{N} \rightarrow [0,1]^{m^{2}}$, which associates each subset $J \subseteq N$ of DMs with a solution (the matrix $P(J) = (p_{ik}(J))_{i,k \in M}$ of preference degrees) to Problem (3). We note that the inequality $P(J) \neq \emptyset$ holds for each non-empty subset $J \subseteq N$. We generalized Problem (3) for the case of the FS $\tilde{N} = \{(j, \mu_{S}(j)) : j \in N\}$ of DMs. To this end, we represent the FS $\tilde{N}$ by its $a$-cuts $N_{a} = \{j \in N : \mu_{S}(j) \geq a\}, a \in [0,1]$ in the form $\tilde{N} = \bigcup_{a \in [0,1]} (N_{a}, a)$, where, for each $a \in [0,1]$, $(N_{a}, a) = \{(j, \mu_{S}(j)) : j \in N_{a}\}$ is the FS with the MF:

$$\mu_{(N_{a}, a)}(j) = \begin{cases} a, & j \in N_{a}; \\ 0, & \text{otherwise} \end{cases}$$

$j \in N$. According to Zadeh’s extension principle [35], the image of the FS $\tilde{N}$ under the mapping $P : 2^{N} \rightarrow [0,1]^{m^{2}}$ is the FS:

$$P(\tilde{N}) = \{(R, \mu_{P(\tilde{N})}(R)) : R = (r_{ik})_{i,k \in M} \in [0,1]^{m^{2}}\}$$

(4)

on the set $[0,1]^{m^{2}}$ of all possible $m \times m$ square matrices $R = (r_{ik})_{i,k \in M}$ with the elements $r_{ik} \in [0,1], i, k \in M$. The MF of the FS $P(\tilde{N})$ has the form:

$$\mu_{P(\tilde{N})}(R) = \max \{a \in [0,1] : R = (r_{ik})_{i,k \in M} = P(N_{a})\}, R \in \text{supp}(P(\tilde{N})).$$

(5)
where $P(N_a) = (p_{ik}(N_a))_{i,k \in M}$ is the matrix of preference degrees, which is a solution to Problem (3) for the set $J = N_a$ of DMs; $N_a = \{j \in N : \mu_S(j) \geq \alpha\}$ is the $\alpha$-cut, $\alpha \in [0,1]$ of the FS $\tilde{N} = \{(j, \mu_S(j)) : j \in N\}$ of DMs. The set:

$$\text{supp}(P(\tilde{N})) = \{P(N_a) : \alpha \in [0,1]\}$$ (6)

is the support of the FS $P(\tilde{N})$.

**Remark 4.** Let $A = \{\mu_S(j) : j \in N\}$ be the set of membership degree values of the fuzzy set $\tilde{N} = \{(j, \mu_S(j)) : j \in N\}$ of DMs. Note that the cardinality of the set $A$ is $|A| \leq n$. The situation $|A| < n$ can occur if the degrees of membership $\mu_S(j)$ are the same for different indices $j \in N$ of DMs. It is clear that, when obtaining $\alpha$-cuts $N_a = \{j \in N : \mu_S(j) \geq \alpha\}$ of the FS $\tilde{N} = \{(j, \mu_S(j)) : j \in N\}$, we can assume that $\alpha \in A$ rather than $\alpha \in [0,1]$.

According to Remark 4, Formulæ (4)–(6) take the form:

$$P(\tilde{N}) = \{(R, \mu_{P(\tilde{N})}(R)) : R = (r_{ik})_{i,k \in M} \in \text{supp}(P(\tilde{N}))\},$$

$$\mu_{P(\tilde{N})}(R) = \max\{\alpha \in A : R = (r_{ik})_{i,k \in M} = P(N_a)\}, R \in \text{supp}(P(\tilde{N})),$$

$$\text{supp}(P(\tilde{N})) = \{P(N_a) : \alpha \in A\}.$$

Thus, the FS $P(\tilde{N})$ of the FPR matrices is a solution to Problem (3) for the case of the FS $\tilde{N}$ of DMs. Therefore, we concluded that each fixed pair $(x_i, x_k)$ of the alternatives $x_i$ and $x_k$ forms an FS on $[0,1]$ of membership degrees of this pair to the group FPR. Then, according to [36] and the comments in Section 2.1, $P(\tilde{N})$ is the T2FR on $X$.

4. The Group FPR for a Fuzzy Set of Decision-Makers

4.1. The Group T2FR

The conclusion made in Section 3.2 allows us to introduce the following notion.

**Definition 2.** By the group FPR on the finite set $X = \{x_1, \ldots, x_m\}$ of alternatives for the FS $\tilde{N} = \{(j, \mu_S(j)) : j \in N\}$ of DMs $j \in N = \{1, \ldots, n\}$ is meant the matrix T2FR $P = (p_{ik})_{i,k \in M}$ with elements $p_{ik} = \{(r_{ik}, \mu_{P_{ik}}(r_{ik})) : r_{ik} \in U_{ik}\}$, which are FSs (in particular, fuzzy numbers) on the closed interval $[0,1]$ with the MFs:

$$\mu_{P_{ik}}(r_{ik}) = \begin{cases} \max_{\alpha \in A}\{\alpha : r_{ik} = p_{ik}(N_a)\}, & r_{ik} \in U_{ik}; \\ 0, & r_{ik} \notin U_{ik} \end{cases}$$ (7)

$i, k \in M$, respectively.

Here, $p_{ik}(N_a)$, $i, k \in M$ are the elements of the preference degree matrix $P(N_a) = (p_{ik}(N_a))_{i,k \in M}$, which is a solution to Problem (3) with the set $J = N_a$ of DMs; the value $p_{ik}(N_a)$ is equal to the degree of membership of the pair $(x_i, x_k)$ of the alternatives $x_i$ and $x_k$ to the group FPR for the set $N_a$ of DMs, $i, k \in M$:

$$U_{ik} = \{p_{ik}(N_a) : \alpha \in A\}$$ (8)

is the set of primary membership degrees $r_{ik} \in [0,1]$ with strictly positive secondary grades $\mu_{P_{ik}}(r_{ik})$, and the matrix $U = (U_{ik})_{i,k \in M}$ coincides with the support $\text{supp}(P(\tilde{N}))$ of the FS $P(\tilde{N})$:

$$N_a = \{j \in N : \mu_S(j) \geq \alpha\}$$ (9)

is the $\alpha$-cut of the FS $\tilde{N}$ of DMs $j \in N$ for $\alpha \in A$ and

$$A = \{\mu_S(j) \geq \alpha : j \in N\}$$

is the set of the membership degree values $\mu_S(j)$, $j \in N$ of the FS $\tilde{N}$ of DMs (see Remark 4).
4.2. Decomposition of the Group T2FR

We applied a decomposition approach to represent the group T2FR $\hat{P}$, in a more-convenient form, as a collection of embedded T2FRs with constant secondary grades (Definition 1 in Section 2.2).

**Proposition 1.** The group T2FR $\hat{P}$ for the FS $\mathcal{N} = \{(j, \mu_S(j)) : j \in \mathcal{N}\}$ of DMs can be represented as the collection:

$$\hat{P} = \{(P(N_a), \alpha) : \alpha \in A\}$$  \hspace{1cm} (10)

of the embedded matrix T2FRs:

$$\mathbb{P}^2_{\alpha} = \{(P(N_a), \alpha)\}$$  \hspace{1cm} (11)

with the constant secondary grades $\alpha \in A$, where the embedded matrix T1FR:

$$\mathbb{P}^1_{\alpha} = P(N_a)$$  \hspace{1cm} (12)

is given by the preference degree matrix $P(N_a) = (p_{ik}(N_a))_{i,k \in M}$, which is a solution to Problem (3) with the set $J = N_a$ of DMs for $\alpha \in A$.

**Proof.** According to Definition 2, the matrix T2FR $\hat{P}$ for the FS $\mathcal{N}$ of DMs has the form $\hat{P} = \{(p_{ik})_{i,k \in M}\}$ with the elements $p_{ik} = \{(r_{ik}, \mu_{p_{ik}}(r_{ik})) : r_{ik} \in U_{ik}, i,k \in M\}$. Hence, $\hat{P}_{\alpha} = \{(r_{ik}, \mu_{p_{ik}}(r_{ik})) : r_{ik} \in U_{ik}, i,k \in M\}$ by (7), and thereupon, $\hat{P}_{\alpha} = \{(p_{ik}(N_a), \alpha) : \alpha \in A, i,k \in M\}$ by virtue of (8) and Remark 2. Therefore, we obtain the representation (10) of the group T2FR $\hat{P}$. Each value $\alpha \in A$ in (10) corresponds to the T2FR (denoted by $\mathbb{P}^2_{\alpha} = (p_{ik}(\alpha))_{i,k \in M}$) with the elements $\mathbb{P}^2_{\alpha}(\alpha) = \{(p_{ik}(\alpha), \alpha) : \alpha \in A, i,k \in M\}$. These are singleton FSs with the degree of membership being equal to $\alpha$. Therefore, by Definition 1, $\mathbb{P}^2_{\alpha}(\alpha)$ is an embedded T2FR with the constant secondary grade $\alpha$ in the T2FR $\hat{P}$. The single embedded matrix T1FR $P(N_a) = (p_{ik}(N_a))_{i,k \in M}$ (we denote it by $\mathbb{P}^1_{\alpha}$ according to (12)) corresponds to the embedded T2FR $\mathbb{P}^2_{\alpha}$. In view of Remark 3, the embedded matrix T2FR $\hat{P}_{\alpha}$ is given by (11).  

Thus, a group T2FR one can decompose into a collection of corresponding T1FRs, each of which is assigned a corresponding constant secondary grade. In addition, we point out explicitly that Proposition 1 simplifies the construction of the T2FR $\hat{P}$. According to Remark 1, the resulting T2FR $\hat{P}$ can be interpreted as a collection of classic fuzzy matrix preference relations (Type-1) $P(N_a)$ with the degree of truth being equal to $\alpha \in A$.

4.3. Calculation of the Group T2FR

In this section, we consider the algorithm for calculating the group T2FR.

Step 0. We construct the finite set $A = \{\mu_S(j) : j \in \mathcal{N}\}$ of membership degree values of the FS $\mathcal{N} = \{(j, \mu_S(j)) : j \in \mathcal{N}\}$ of DMs and represent $A$ in the form $A = \{a_1, \ldots, a_{|A|}\}$.

Step 1. According to (9), for $\alpha = a_i$, we construct the $\alpha$-cut $N_{a_i} = \{j \in \mathcal{N} : \mu_S(j) \geq \alpha\}$. We solve Problem (3) with the set $J = N_{a_i}$ of DMs. We obtain the minimum distance $d_{\min}(N_{a_i})$ and a solution in the form of the matrix $P(N_{a_i}) = (p_{ik}(N_{a_i}))_{i,k \in M}$ of preference degrees, which is the embedded T1FR, that is $\mathbb{P}^1_{\alpha} = P(N_{a_i})$.

The final step. Once all T1FRs $P(N_{a_i})$, $\alpha \in A$ have been obtained, the resulting T2FS is given by $\hat{P} = \{(P(N_{a_i}), \alpha) : \alpha \in A\}$ by (10).

According to Remark 1, the T2FR $\hat{P}$ can be interpreted as follows. The group matrix T2FR $\hat{P}$ is equal to: the matrix T1FR $P(N_{a_1})$ with the degree of truth being equal to $a_1$; the T1FR $P(N_{a_2})$ with the degree of truth being equal to $a_2$; \ldots; the T1FR $P(N_{a_{|A|}})$ with the degree of truth being equal to $a_{|A|}$.
4.4. Properties of the Group T2FR

Proposition 2 provides some useful properties of the group T2FR.

**Proposition 2.** Assume that the group T2FR $\hat{P}$ is given by (10). Then, the following two properties hold true:

1. For any $\alpha', \alpha'' \in A$, $\alpha' > \alpha''$, $N_{\alpha'} \subset N_{\alpha''}$ and inequality $d^{\min}(N_{\alpha'}) \leq d^{\min}(N_{\alpha''})$, where $d^{\min}(\cdot)$ is the minimum Kemeny distance.

2. If $p_{ik}(N_{\alpha^*})$ is the membership degree of the alternative $x_i \in X$ over $x_k \in X$ for the $\alpha^*$-cut $N_{\alpha^*} = \{j \in N : \mu_{N}(j) \geq \alpha^*\}$, $\alpha^* \in A$ of the FS $\hat{N}$ of DMs, then the pair $(x_i, x_k)$ of alternatives has the primary membership degree $r_{ik} = p_{ik}(N_{\alpha^*})$ to the group T2FR $\hat{P}$ with the secondary grade (the degree of truth) no smaller than $\alpha^*$, that is $\mu_{\hat{P}}(r_{ik}) \geq \alpha^*$, $r_{ik} \in U_{ik}$, $i, k \in M$.

**Proof.** We first prove Item 1. Formula (9) and Remark 4 imply that $N_{\alpha'} \subset N_{\alpha''}$. We used a known fact that the value of the objective function of a mathematical programming problem deteriorates if, and only if, some new constraints are added. Therefore, according to (3), $N_{\alpha'} \subset N_{\alpha''}$ entails $d^{\min}(N_{\alpha'}) \leq d^{\min}(N_{\alpha''})$ whenever $J = N_{\alpha'}$ or $J = N_{\alpha''}$.

Next, we intend to check Item 2. Let $\alpha^* \in A$ and $r_{ik} = p_{ik}(N_{\alpha^*})$. Then, $r_{ik} \in U_{ik}$ by (8) and, thereupon,

$$\mu_{\hat{P}}(r_{ik}) = \max\{a \in A : p_{ik}(N_{\alpha^*}) = r_{ik} = p_{ik}(N_{a}) \geq \alpha^* \}$$

for $i, k \in M$ by (7). □

According to Item 1 of Proposition 2, accounting for individual FPRs with smaller degrees of membership to the FS of DMs (which correspond to larger cuts of the FS of DMs) leads to a greater degree of inconsistency between the group FPR and the individual FPRs. Item 2 of Proposition 2 implies that the guaranteed value $\alpha^*$ of the degree of truth of the embedded matrix T1FR $R = (r_{ik})_{i,k \in M}$ in the group T2FR $\hat{P}$ is determined by the $\alpha^*$-cut level, $\alpha^* \in A$ of the FS $\hat{N}$ of DMs, at which the matrix $R$ is a solution (denoted by $P(N_{\alpha^*})$) to Problem (3) with the set $J = N_{\alpha}$ of DMs.

In other words, the individual FPRs of DMs with lower degrees of membership to the FS of DMs lead to a greater degree of inconsistency between the group FPR and the individual FPRs of DMs. In addition, the guaranteed degree of truth of any FPR embedded in a group T2FR is determined by the lowest degree of membership to the FS $\hat{N}$ of those DMs for which this FPR is their group preference.

5. Numerical Examples

In this section, we look at examples of constructing a group T2FR for an FS of DMs. We chose the numbers of alternatives and DMs in these examples being equal to three to more transparently demonstrate our approach and not to use software. Example 1 helps to illustrate the decomposition approach. Example 2 is intended to demonstrate Definition 1 and uses the same input data as Example 1.

**Example 1.** Suppose that the DMs $j \in N = \{1, 2, 3\}$ have the individual matrix FPRs:

$$p^1 = \begin{pmatrix} 0.5 & 0.2 & 0.6 \\ 0.8 & 0.5 & 0.7 \\ 0.4 & 0.3 & 0.5 \end{pmatrix}, \quad p^2 = \begin{pmatrix} 0.5 & 0.3 & 0.4 \\ 0.7 & 0.5 & 0.5 \\ 0.6 & 0.5 & 0.5 \end{pmatrix}, \quad p^3 = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.6 & 0.5 & 0.8 \\ 0.9 & 0.2 & 0.5 \end{pmatrix}$$

on the set $X = \{x_1, x_2, x_3\}$ of alternatives. Let $\bar{N} = \{(1, 0.4), (2, 1), (3, 1)\}$ be an FS on the set $N = \{1, 2, 3\}$ of DMs with the corresponding MF values $\mu_{\bar{N}}(1) = 0.4$ and $\mu_{\bar{N}}(2) = \mu_{\bar{N}}(3) = 1$. We have to construct the group T2FR on $X$ for the FS $\bar{N}$ of DMs using the algorithm from Section 4.3.
Step 0. According to the formula $A = \{ \mu_N(j) : j \in N \}$, the set of membership degrees of the FS $\tilde{N}$ takes the form $A = \{0.4, 1\}$.

Step 1. For $a = a_1 = 0.4$, we construct the $\alpha$-cut $N_{0.4} = \{1, 2, 3\}$ of the FS $\tilde{N}$ of DMs according to (9). We solve Problem (3) with the set $f = N_{0.4} = \{1, 2, 3\}$ of DMs. Since the constraints of Problem (3) take the form:

$$
\begin{align*}
p_{11} &= p_{22} = p_{33} = 0.5, \\
p_{21} &= 1 - p_{12}, \quad p_{31} = 1 - p_{13}, \quad p_{32} = 1 - p_{23}, \\
p_{12}, p_{13}, p_{23} &\in [0, 1],
\end{align*}
$$

Problem (3) reduces to the simpler problem:

$$
d^{\text{min}}(N_{0.4}) = 2 \min_{p_{12}, p_{13}, p_{23} \in [0, 1]} \sum_{j=1}^{3} (|p_{12} - p_{12}'| + |p_{13} - p_{13}'| + |p_{23} - p_{23}'|),
$$

where $p_{ik}', i, k = 1, 2, 3$, $j = 1, 2, 3$ are the corresponding elements of the matrix $P^j$, $j = 1, 2, 3$, respectively. Just as Problem (2) is decomposed into simpler sub-problems, we obtain the minimal distance:

$$
d^{\text{min}}(N_{0.4}) = 2(d^{\text{min}}_{12}(N_{0.4}) + d^{\text{min}}_{13}(N_{0.4}) + d^{\text{min}}_{23}(N_{0.4})),
$$

where the equalities:

$$
\begin{align*}
d^{\text{min}}_{12}(N_{0.4}) &= \min_{p_{12} \in [0, 1]} \sum_{j=1}^{3} |p_{12} - p_{12}^j| = \min_{p_{12} \in [0, 1]} (|p_{12} - 0.2| + |p_{12} - 0.3| + |p_{12} - 0.4|), \\
d^{\text{min}}_{13}(N_{0.4}) &= \min_{p_{13} \in [0, 1]} \sum_{j=1}^{3} |p_{13} - p_{13}^j| = \min_{p_{13} \in [0, 1]} (|p_{13} - 0.6| + |p_{13} - 0.4| + |p_{13} - 0.1|), \\
d^{\text{min}}_{23}(N_{0.4}) &= \min_{p_{23} \in [0, 1]} \sum_{j=1}^{3} |p_{23} - p_{23}^j| = \min_{p_{23} \in [0, 1]} (|p_{23} - 0.7| + |p_{23} - 0.5| + |p_{23} - 0.8|)
\end{align*}
$$

hold. As we noted in Section 3.1, in the case of a different FPR model, for instance those having additive transitivity constraints, such a decomposition is not possible. Then, one needs to use any suitable software designed to solve mathematical programming problems. Solving the resulting elementary problems with one variable leads to the minimum distance $d^{\text{min}}(N_{0.4}) = 2$ and the matrix:

$$
P(N_{0.4}) = \begin{pmatrix} 0.5 & 0.3 & 0.4 \\ 0.7 & 0.5 & 0.7 \\ 0.6 & 0.3 & 0.5 \end{pmatrix}
$$

(13)

of preference degrees, which is the embedded T1FR, that is $P^{\alpha}_{0.4} = P(N_{0.4})$.

Step 2. For $a = a_2 = 1$, we construct the $\alpha$-cut $N_1 = \{2, 3\}$ of the FS $\tilde{N}$ of DMs according to (9). We solve Problem (3) with the set $f = N_1 = \{2, 3\}$ of DMs in a manner similar to what we used in Step 1. The matrix:

$$
P(N_1) = \begin{pmatrix} 0.5 & 0.35 & 0.25 \\ 0.65 & 0.5 & 0.65 \\ 0.75 & 0.35 & 0.5 \end{pmatrix}
$$

(14)

is a solution to Problem (3) with the minimum distance $d^{\text{min}}(N_1) = 1.4$ for the crisp set $N_1 = \{2, 3\}$ of DMs. This matrix is the embedded T1FR, that is $P^{\alpha}_{1} = P(N_1)$.

The final step. The resulting T2FR is given by

$$
\hat{P} = \left\{ \begin{pmatrix} 0.5 & 0.35 & 0.25 \\ 0.65 & 0.5 & 0.65 \\ 0.75 & 0.35 & 0.5 \end{pmatrix}, 1 \right\}, \quad \begin{pmatrix} 0.5 & 0.3 & 0.4 \\ 0.7 & 0.5 & 0.7 \\ 0.6 & 0.3 & 0.5 \end{pmatrix}, \quad \begin{pmatrix} 0.5 & 0.3 & 0.4 \end{pmatrix}
$$
We checked the compliance of the obtained result with Definition 2 using Example 2.

In Section 3.2, using a decomposition approach, we justify the representation of a group T2FR in the form:

\[ \tilde{\mu} = \frac{1}{N} \sum_{i=1}^{N} \mu_i \]

We intended to construct the elements \( \tilde{\mu}_{ik} = \{(r_{ik}, \mu_{\tilde{p}_{ik}}(r_{ik})) : r_{ik} \in U_{ik}\} \), \( i,k = 1,2,3 \) of the group T2FR \( \tilde{P} = (\tilde{p}_{ik})_{i,k=1,2,3} \). For \( i,k = 1 \), the set of primary membership degrees has the form \( U_{11} = \{p_{11}(N_{0,4}), p_{11}(N_1)\} = \{0.5\} \) according to (8), where \( p_{11}(N_{0,4}) = 0.5 \) and \( p_{11}(N_1) = 0.5 \) are the corresponding elements of the matrices \( P(N_{0,4}) \) and \( P(N_1) \) according to (13) and (14), respectively. Next, we calculated the secondary grade \( \tilde{p}_{11} = (0.5) = \max\{0.4,1\} \) according to (7). Thus, we obtained \( \tilde{p}_{11} = \{(0.5,1)\} \).

For \( i = 1, k = 2 \), the set of primary membership degrees has the form:

\[ U_{12} = \{p_{12}(N_{0,4}), p_{12}(N_1)\} = \{(0.3,0.4)\} \]

according to (8), where \( p_{12}(N_{0,4}) = 0.35 \) and \( p_{12}(N_1) = 0.5 \) are the corresponding elements of the matrices \( P(N_{0,4}) \) and \( P(N_1) \) according to (13) and (14), respectively. Next, we took the secondary grades \( \mu_{\tilde{p}_{12}}(0.35) = 1 \) and \( \mu_{\tilde{p}_{12}}(0.3) = 0.4 \) according to (7). Thus, we obtained \( \tilde{p}_{12} = \{(0.35,1), (0.3,0.4)\} \). We calculated the remaining elements similarly and obtained the group T2FR in the form:

\[ \tilde{P} = \begin{pmatrix}
\{(0.5,1)\} & \{(0.35,1), (0.3,0.4)\} & \{(0.25,1), (0.4,0.4)\} \\
\{(0.65,1), (0.7,0.4)\} & \{(0.5,1)\} & \{(0.65,1), (0.7,0.4)\} \\
\{(0.75,1), (0.6,0.4)\} & \{(0.35,1), (0.5,0.4)\} & \{(0.5,1)\}
\end{pmatrix}. \]

Thus, the group T2FR constructed according to Definition 2 is in line with (15).

### 6. Results and Discussion

Since the goal of this article was to demonstrate that an FS of DMs generates a group T2FR, we focused on the results that contribute to achieving this goal. In Section 3.2, we justify the statement that the group FPR for an FS of DMs is a T2FR with constant secondary grades. This allowed us to formulate the corresponding definition in Section 4.1. In Section 4.2, using a decomposition approach, we justify the representation of a group...
T2FR for an FS of DMs in the form of a set of embedded T2FRs with constant secondary grades and propose the algorithm for constructing a group T2FR in Section 4.4. These study showed that, although in general, a T2FR is a rather complicated mathematical object, T2FRs with constant secondary grades are simple enough for practical use. We demonstrate this by examples in Section 5. A positive feature of the proposed decomposition approach is its independence from a chosen method for calculating group FPRs for subsets of DMs. The study of the properties of the group T2FR in Section 4.4 shows that the individual FPRs of DMs with lower degrees of membership to the FS of DMs lead to a greater degree of inconsistency between the group FPR and the individual FPRs of DMs. In addition, the guaranteed degree of truth of any FPR embedded in a group T2FR is determined by the lowest degree of membership of those DMs for which this FPR is their group preference.

Comparing the GDM methods [25] based on weights with the developed approach, we can conclude the following. A significant limitation of the proposed approach is an increase in computational complexity compared to aggregation methods that use weights. This is due to the need to solve the optimization problem (3) for each $\alpha$-cut $N_\alpha = \{ j \in N : \tilde{\mu}_N(j) \geq \alpha \}$ of the FS $\tilde{N}$ of DMs. This drawback limits the scope of application of the developed approach to the number of alternatives and DMs, which can be large enough to solve the optimization problems (3). Nevertheless, this drawback can be avoided if we do not set the problem of calculating the group T2FR in full and limit ourselves to obtaining the group T2FR $\tilde{P}e_2^\alpha = \{(P(N_\alpha), \alpha)\}$ with the constant secondary grade $\alpha$ corresponding to an acceptable fixed value $\alpha \in A$ of the degree of truth.

7. Conclusions

The conducted research showed that, in addition to aggregation methods that use weights, a moderator can use the proposed approach based on representing a group of DMs in the form of an FS. This FS can describe some property of the DM group, for example the competence of the DMs. The application of the FS theory to solve a GDM problem in such a formulation seems quite logical. Since we used the Kemeny optimization scheme, the proposed approach inherits the advantages, disadvantages, and scope of application of the Kemeny scheme in practice. Thus, this approach ensures consistency in a group judgment when eliminating alternatives and makes it possible to construct a group FPR with specified properties. In addition, the proposed method constructs a group FPR with dependence on the subset of DMs (generally fuzzy) that take part in decision-making. Finally, we note that our method not only extends the area of applications of group decision-making to the case of a fuzzy set of DMs, but can also provide new approaches to solving other formulations of fuzzy decision-making problems.

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Abbreviations

The following abbreviations are used in this manuscript:

DM decision-maker
MF membership function
FS fuzzy set
T2FR Type-2 fuzzy relation
T1FR Type-1 fuzzy relation
FPR fuzzy preference relation

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