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Elasticity Problem with a Cusp between Thin Inclusion and Boundary

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Abstract: This paper concerns an equilibrium problem for an an elastic body with a thin rigid inclusion crossing an external boundary of the body at zero angle. The inclusion is assumed to be exfoliated from the surrounding elastic material that provides an interfacial crack. To avoid nonphysical interpenetration of the opposite crack faces, we impose inequality type constraints. Moreover, boundary conditions at the crack faces depend on a positive parameter describing a cohesion. A solution existence of the problem with different conditions on the external boundary is proved. Passages to the limit are analyzed as the damage parameter tends to infinity and to zero. Finally, an optimal control problem with a suitable cost functional is investigated. In this case, a part of the rigid inclusion is located outside of the elastic body, and a control function is a shape of the inclusion.

Keywords: elastic body; thin inclusion; cusp; non-penetration boundary condition; damage parameter; optimal control

MSC: 35J88; 74G22

1. Introduction

The engineering practice demonstrates a big variety of composite structures having thin inclusions. In particular, thin inclusions may cross an external boundary of the elastic body at zero angle. It is well known that, in general cases, Korn's inequalities are not valid in domains with non-smooth boundaries. This implies difficulties with a solution existence of such problems. This paper focuses on the equilibrium problem for a 2D elastic body with a thin rigid exfoliated inclusion crossing the external boundary of the body at zero angle. The exfoliation implies a presence of the crack between the inclusion and the elastic body. Moreover, the zero angle provides a non-smooth boundary in the mathematical formulation of the problem. Consequently, a proof of the solution existence requires additional arguments. We impose inequality type boundary conditions at the crack faces. In addition, these boundary conditions depend on a positive damage parameter characterizing a cohesion between the crack faces. Over the last decades, many papers have been published concerning problems for elastic bodies with exfoliated thin inclusions and inequality type boundary conditions. We refer the reader to [1–9]. Boundary value problems with inequality type boundary conditions for cohesive cracks can be found in [10–13]. The results for problems with a zero angle between the inclusion and the external boundaries are presented in [14], where a fictitious domain method was used to prove a solution existence. As for derivation of suitable models for thin inclusions in elastic structures, see [15–19] and the references therein. There are a number of applied papers related to thin inclusions in elastic bodies [20–26], as well as books with general approaches to the description of nonhomogeneous bodies [27,28].

The model considered in this paper belongs to the class of problems with a free boundary. The results obtained may be useful for other models with free boundaries, see for example [29], as well as for models of functionally graded materials [30,31].

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Citation: Khludnev, A. Elasticity Problem with a Cusp between Thin Inclusion and Boundary. *Axioms* **2022**, *12*, 1081. https://doi.org/10.3390/axioms12121081

Academic Editor: Irina Meghea

Received: 12 October 2023
Revised: 7 November 2023
Accepted: 21 November 2023
Published: 27 November 2023

From the practical point of view, the obtained results can be used for the modeling and analysis of different elastic and nonelastic structures having non-smooth boundaries.

This paper is structured as follows: In Section 2, we prove a solution existence of the problem. Variational and differential formulations of the problem are discussed. Passages to limits, as the damage parameter tends to infinity and to zero, are investigated in Sections 3 and 4. Mixed boundary conditions on the external boundary are considered in Section 5. A case of the inclusion partially extending beyond the elastic body is analyzed in Section 6. A solution existence of the optimal control problem with a suitable cost functional is proved.

2. Problem Formulation

Bounded domain in $\mathbb{R}^2$ with a smooth boundary Γ is denoted by Ω, and $\gamma \subset \Omega$ is a smooth curve such that $\bar{\gamma} \cap \Gamma = (0,0)$, $\Omega_\gamma = \Omega \setminus \bar{\gamma}$. The domain Ω_γ fits to an elastic body, and γ corresponds to a thin rigid inclusion. We assume that the angle between Γ and γ at the point $(0,0)$ is equal to zero for the undeformed state of the elastic body, see Figure 1. Denote $\Gamma_0 = \Gamma \setminus \{(0,0)\}$ and introduce Sobolev space

$$H^1_{\Gamma_0}(\Omega_\gamma) = \{v \in H^1(\Omega_\gamma) \mid v = 0 \text{ on } \Gamma_0\}.$$

Assume that $C = \{c_{ijkl}\}$ is a given elasticity tensor with the usual properties of symmetry and positive definiteness, $c_{ijkl} \in L^\infty_{loc}(\mathbb{R}^2)$, $i,j,k,l = 1,2$. Despite the cusp of the domain Ω_γ, Korn's inequality is fulfilled in the space $H^1_{\Gamma_0}(\Omega_\gamma)^2$. Indeed, consider a fictitious domain Ω_0 with a smooth boundary $\partial\Omega_0$ as depicted in Figure 2 assuming that angles between $\partial\Omega_0$ and Γ at the points x^1, x^2 are nonzero. Extended domain $\Omega_e = \Omega_\gamma \cup \Omega_0 \cup \gamma_1 \cup \gamma_2$ has the smooth cut γ. Let $v \in H^1_{\Gamma_0}(\Omega_\gamma)^2$. Since $v = 0$ on Γ_0, this function can be extended by zero to Ω_0. Denote by $\tilde{v}$ the extended function defined in Ω_e. It is clear that Korn's inequality holds for all such extended functions $\tilde{v}$. Hence, it is valid for all $v \in H^1_{\Gamma_0}(\Omega_\gamma)^2$, since $\tilde{v}$ is zero in Ω_0, i.e., there exists a constant $c_0 > 0$ such that

$$\int_{\Omega_\gamma} \sigma(v)\varepsilon(v) \geq c_0 \|v\|^2_{H^1_{\Gamma_0}(\Omega_\gamma)^2} \quad \forall v \in H^1_{\Gamma_0}(\Omega_\gamma)^2. \tag{1}$$

Here $\sigma(v)$ is defined according to Hooke's law $\sigma(v) = C\varepsilon(v)$, $\varepsilon(v) = \{\varepsilon_{ij}(v)\}$, $\varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i})$, $i,j = 1,2$. To simplify the formulae, we write $\sigma(v)\varepsilon(v)$ instead of $\sigma_{ij}(v)\varepsilon_{ij}(v)$.

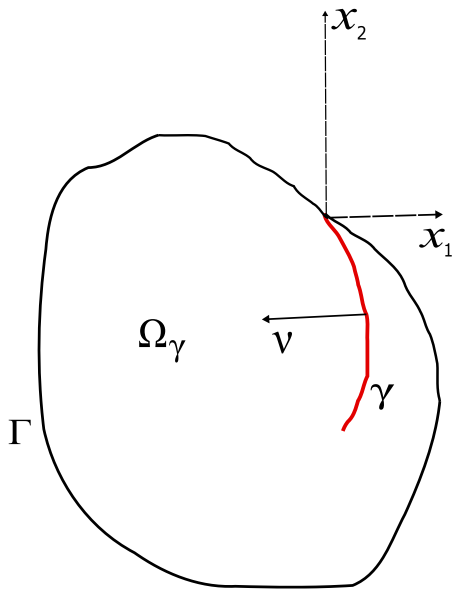

Figure 1. Geometry of the problem.

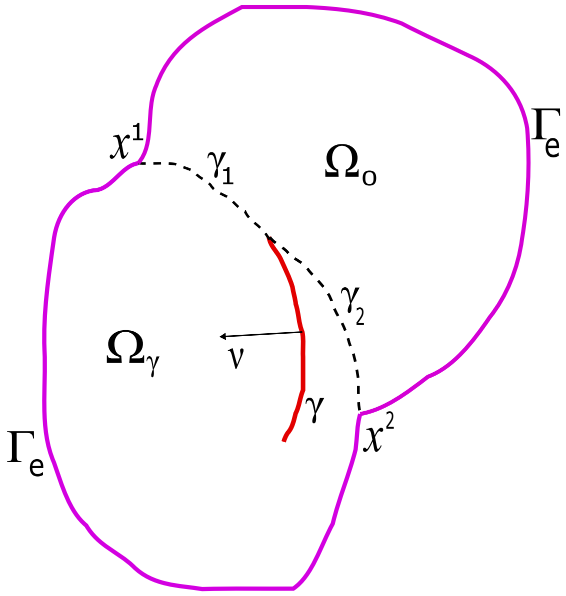

Figure 2. Fictitious domain Ω_0.

Denote by $R_0(\gamma)$ the space of infinitesimal rigid rotations,

$$R_0(\gamma) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = b(-x_2, x_1),\ b \in \mathbb{R}, x = (x_1, x_2) \in \gamma\}.$$

Let $f \in L^2(\Omega)^2$ be a given external force acting on the elastic body. Introduce a set of admissible displacements

$$S = \{v \in H^1_{\Gamma_0}(\Omega_\gamma)^2 \mid\ [v_\nu] \geq 0 \text{ on } \gamma;\ v|_{\gamma^-} \in R_0(\gamma)\},$$

where $v_\nu = v\nu,\ \ [h] = h^+ - h^-;\ h^\pm$ are traces of the function h on the crack faces $\gamma^\pm$, respectively. The signs $\pm$ fit to positive and negative crack faces with respect to the outward unit normal vector ν to γ.

An equilibrium problem for the body Ω_γ and the inclusion γ is formulated as follows: unknown functions are the displacement field $u^\alpha = (u_1^\alpha, u_2^\alpha)$, the stress tensor $\sigma = \{\sigma_{ij}\},\ i,j = 1,2$, defined in Ω_γ, as well as $\rho_0^\alpha \in R_0(\gamma)$, satisfying the following equations and boundary conditions

$$-\mathrm{div}\,\sigma = f,\ \sigma = C\varepsilon(u^\alpha) \text{ in } \Omega_\gamma, \tag{2}$$

$$u^\alpha = 0 \text{ on } \Gamma_0;\ u^\alpha = \rho_0^\alpha \text{ on } \gamma^-, \tag{3}$$

$$[u_\nu^\alpha] \geq 0, -\sigma_\nu^+ + \frac{1}{\alpha}[u_\nu^\alpha] \geq 0,\ -\sigma_\tau^+ + \frac{1}{\alpha}[u_\tau^\alpha] = 0 \text{ on } \gamma, \tag{4}$$

$$[u_\nu^\alpha](-\sigma_\nu^+ + \frac{1}{\alpha}[u_\nu^\alpha]) = 0 \text{ on } \gamma, \tag{5}$$

$$\int_\gamma [\sigma\nu]\rho = 0\ \ \forall \rho \in R_0(\gamma). \tag{6}$$

Here $\alpha > 0$ is a damage parameter characterizing a cohesion between the crack faces; $\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \sigma_\tau = \sigma_{ij}\nu_j\tau_i,\ \nu = (\nu_1, \nu_2),\ \tau = (-\nu_2, \nu_1)$.

Relations (2) are the equilibrium equation and constitutive law (Hooke's law). The first inequality in (4) provides a mutual non-penetration between the crack faces $\gamma^\pm$. The identity (6) provides a zero moment acting on the inclusion γ. The contact set between the crack faces is unknown a priori, and the model (2)–(6) corresponds to a free boundary approach.

The problem (2)–(6) has a unique solution. Indeed, consider the energy functional $G_\alpha : H^1_{\Gamma_0}(\Omega_\gamma)^2 \to \mathbb{R}$,

$$G_\alpha(v) = \frac{1}{2}\int\limits_{\Omega_\gamma} \sigma(v)\varepsilon(v) - \int\limits_{\Omega_\gamma} fv + \frac{1}{2\alpha}\int\limits_{\gamma} [v]^2.$$

Then the minimization problem

$$\inf_{v \in S} G_\alpha(v) \tag{7}$$

has a solution. To prove this statement, it suffices to use Korn's inequality (1) and note that the set S is weakly closed. Indeed, we have $[v_\nu] \in H^{1/2}(\gamma)$. Hence, the set S is closed and consequently weakly closed. The solution of the problem (7) satisfies the variational inequality

$$u^\alpha \in S, \tag{8}$$

$$\int\limits_{\Omega_\gamma} \sigma(u^\alpha)\varepsilon(\bar{u} - u^\alpha) - \int\limits_{\Omega_\gamma} f(\bar{u} - u^\alpha) + \tag{9}$$

$$+\frac{1}{\alpha}\int\limits_{\gamma} [u^\alpha][\bar{u} - u^\alpha] \geq 0 \ \ \forall\, \bar{u} \in S.$$

The following statement takes place.

Theorem 1. *For smooth solutions, problem formulations (2)–(6) and (8) and (9) are equivalent.*

Proof. To simplify notations, we omit the symbol α. We first check that the equilibrium equation follows from (8) and (9), see (2). To this end, we have to substitute in (9) test functions of the form $\bar{u} = u \pm v,\ v \in C_0^\infty(\Omega_\gamma)^2$. Next, we choose test functions $\bar{u} = u \pm v,\ [v_\nu] = 0$ on $\gamma,\ v \in S$, and substitute in (9). This implies

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(v) - \int\limits_{\Omega_\gamma} fv + \frac{1}{\alpha}\int\limits_{\gamma} [u][v] = 0,$$

and consequently, taking into account the equilibrium equation, we obtain

$$-\int\limits_{\gamma} [\sigma_\nu]v_\nu - \int\limits_{\gamma} [\sigma_\tau v_\tau] + \frac{1}{\alpha}\int\limits_{\gamma} [u_\tau][v_\tau] = 0. \tag{10}$$

Since v_τ^+ is arbitrary on γ, from (10) we derive the third relation of (4). The relation (10) can be written in the form

$$-\int\limits_{\gamma} [\sigma_\nu]v_\nu - \int\limits_{\gamma} [\sigma_\tau]v_\tau^- + \int\limits_{\gamma} [\sigma_\tau]v_\tau^- + \int\limits_{\gamma} (\sigma_\tau^- - \frac{1}{\alpha}[u_\tau])v_\tau^- = 0 \tag{11}$$

which gives

$$-\int\limits_{\gamma} [\sigma\nu]v^- + \int\limits_{\gamma} v_\tau^-(\sigma_\tau^+ - \frac{1}{\alpha}[u_\tau]) = 0 \tag{12}$$

providing, by the third relation of (4), the identity (6). $\square$

Now we can choose test functions in (9) in the form $\bar{u} = u + v, v \in H^1_\Gamma(\Omega_\gamma)^2, v^+_\nu \geq 0$ on γ, supp $v \subset \bar{Q}$, see Figure 3. It provides the following relation

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(v) - \int\limits_{\Omega_\gamma} fv + \frac{1}{\alpha}\int\limits_\gamma [u]v^+ \geq 0.$$

Consequently, integrating by parts, we have

$$\int\limits_\gamma (\sigma^+_\nu v^+_\nu + \sigma^+_\tau v^+_\tau) - \frac{1}{\alpha}\int\limits_\gamma [u_\nu]v^+_\nu - \frac{1}{\alpha}\int\limits_\gamma [u_\tau]v^+_\tau \leq 0. \tag{13}$$

In view of the choice of v and the last relation of (4), from (13) the second relation of (4) follows.

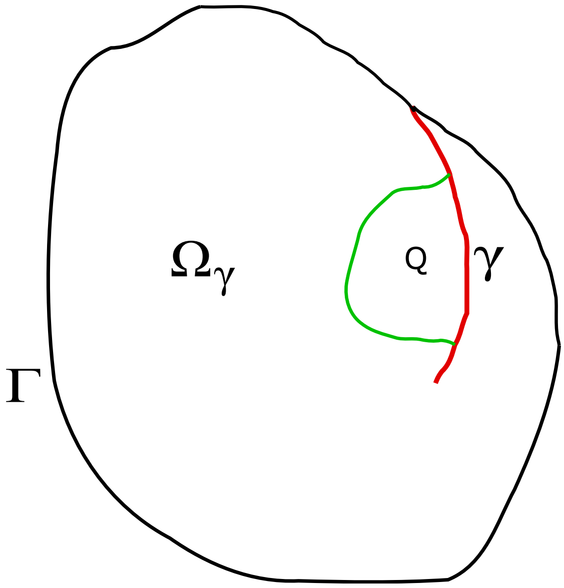

Figure 3. Neighborhood Q.

It remains to check the condition (5). To this end, suppose that at a given point $x^0 \in \gamma$ we have $[u_\nu(x^0)] > 0$. Take test functions in (9) of the following form: $\bar{u} = u \pm \beta\psi$, supp $\psi \subset \bar{Q}$, β is a small parameter and Q is a small neighborhood of the point x^0, ψ is a quite smooth function, see Figure 3. We obtain

$$\int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\psi) - \int\limits_{\Omega_\gamma} f\psi + \frac{1}{\alpha}\int\limits_\gamma [u]\psi^+ = 0,$$

consequently, by the third relation of (4),

$$\int\limits_\gamma \sigma^+_\nu \psi^+_\nu - \frac{1}{\alpha}\int\limits_\gamma [u_\nu]\psi^+_\nu = 0,$$

and thus $\sigma^+_\nu(x^0) - \frac{1}{\alpha}[u_\nu(x^0)] = 0$, i.e., $[u_\nu(x^0)](\sigma^+_\nu(x^0) - \frac{1}{\alpha}[u_\nu(x^0)]) = 0$. On the other hand, assuming that $-\sigma^+_\nu(x^0) + \frac{1}{\alpha}[u_\nu(x^0)] < 0$, we easily derive $[u_\nu(x^0)] = 0$, and (5) easily follows. Thus, from (8) and (9) we have derived all equations and boundary conditions (2)–(6).

Now we prove the converse statement. Let (2)–(6) be fulfilled. We take $\bar{u} \in S$ and multiply the first equation from (2) by $\bar{u} - u$. Integrating over Ω_γ, one obtains

$$\int_{\Omega_\gamma} (\operatorname{div} \sigma + f)(\bar{u} - u) = 0,$$

and consequently,

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \pm \frac{1}{\alpha} \int_\gamma [u][\bar{u} - u] + \\ + \int_\gamma [\sigma\nu(\bar{u} - u)] = 0. \tag{14}$$

To derive the variational inequality (9) from (14), it suffices to prove that

$$-\int_\gamma [\sigma\nu(\bar{u} - u)] + \frac{1}{\alpha} \int_\gamma [u][\bar{u} - u] \geq 0. \tag{15}$$

But the inequality (15) easily follows from boundary conditions (4)–(5). Thus, the equivalence of (8) and (9) and (2)–(6) is completely proved.

In addition to (2)–(6), we can provide one more differential formulation of the problem (8) and (9). It reads as follows (again omitting the symbol α): it is necessary to find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, defined in Ω_γ, and a thin inclusion displacement $\rho_0 \in R_0(\gamma)$ defined on γ such that

$$-\operatorname{div} \sigma = f, \ \sigma = C\varepsilon(u) \ \text{ in } \ \Omega_\gamma, \tag{16}$$

$$u = 0 \ \text{ on } \ \Gamma_0, \tag{17}$$

$$[u_\nu] \geq 0, \ u^- = \rho_0 \ \text{ on } \ \gamma, \tag{18}$$

$$-\int_\gamma [\sigma\nu \cdot u] + \frac{1}{\alpha} \int_\gamma [u]^2 = 0, \tag{19}$$

$$-\int_\gamma [\sigma\nu \cdot \bar{u}] + \frac{1}{\alpha} \int_\gamma [u][\bar{u}] \geq 0 \ \ \forall \bar{u} \in S. \tag{20}$$

Let us check that formulations (16)–(20) and (8) and (9) are equivalent for smooth solutions. Assume that (8) and (9) take place. This provides the equilibrium equation, see (2). From (8) and (9) we easily derive

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(u) - \int_{\Omega_\gamma} fu + \frac{1}{\alpha} \int_\gamma [u]^2 = 0. \tag{21}$$

Making the integration by parts in (21), we obtain (19). By (21), the variational inequality (9) can be rewritten as

$$\int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u}) - \int_{\Omega_\gamma} f\bar{u} + \frac{1}{\alpha} \int_\gamma [u][\bar{u}] \geq 0 \ \ \forall \bar{u} \in S,$$

thus, integrating by parts, the inequality (20) is obtained. Consequently, all relations (16)–(20) are derived from (8) and (9).

Conversely, let (16)–(20) be fulfilled. We take $\bar{u} \in S$ and multiply the first equation from (16) by $\bar{u} - u$. Integrating over Ω_γ, we obtain

$$\int_{\Omega_\gamma} (\operatorname{div} \sigma + f)(\bar{u} - u) = 0.$$

Thus,

$$\int_\gamma [\sigma\nu(\bar{u} - u)] + \int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \pm \frac{1}{\alpha}\int_\gamma [u][\bar{u} - u] = 0. \tag{22}$$

We see that for obtaining the variational inequality (9) from (22), it is enough to prove the inequality

$$\int_\gamma [\sigma\nu(\bar{u} - u)] - \frac{1}{\alpha}\int_\gamma [u][\bar{u} - u] \le 0. \tag{23}$$

But the inequality (23) easily follows from (19)–(20). Thus, the equivalence of (8) and (9) and (16)–(20) is proved for smooth solutions.

3. Passage to Limit in (8) and (9) as $\alpha \to \infty$

In this section, we justify a passage to limit in the model (8) and (9) as $\alpha \to \infty$. It turns out that the limit model fits to zero friction at the positive crack face γ^+.

From (9), it follows

$$\int_{\Omega_\gamma} \sigma(u^\alpha)\varepsilon(u^\alpha) - \int_{\Omega_\gamma} fu^\alpha + \frac{1}{\alpha}\int_\gamma [u^\alpha]^2 = 0. \tag{24}$$

By Korn's inequality valid in the space $H^1_{\Gamma_0}(\Omega_\gamma)^2$, the relation (24) implies a uniform in α estimate

$$\|u^\alpha\|_{H^1_{\Gamma_0}(\Omega_\gamma)^2} \le c.$$

Choosing a subsequence, if necessary, we assume that as $\alpha \to \infty$

$$u^\alpha \to u \text{ weakly in } \; H^1_{\Gamma_0}(\Omega_\gamma)^2, \tag{25}$$

$$[u^\alpha] \to [u] \text{ weakly in } \; L^2(\gamma)^2, \tag{26}$$

$$u|_{\gamma^-} = \rho^\infty \in R_0(\gamma). \tag{27}$$

By (25)–(27), a passage to the limit as $\alpha \to \infty$ can be fulfilled in (8) and (9) which implies

$$u \in S, \; \int_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int_{\Omega_\gamma} f(\bar{u} - u) \ge 0 \;\; \forall\, \bar{u} \in S. \tag{28}$$

Thus, we arrive at the following assertion.

Theorem 2. *Solutions of the problems* (8) *and* (9) *converge in the sense* (25)–(27) *to the solution of* (28) *as* $\alpha \to \infty$.

The problem (28) admits a differential formulation. It is necessary to find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\},\ i, j = 1, 2,$ defined in Ω_γ, and $\rho^\infty \in R_0(\gamma)$ such that

$$-\text{div}\, \sigma = f,\ \sigma = C\varepsilon(u) \text{ in } \Omega_\gamma, \tag{29}$$

$$u = 0 \text{ on } \Gamma_0;\ u = \rho^\infty \text{ on } \gamma^-, \tag{30}$$

$$[u_\nu] \geq 0,\ \sigma_\nu^+ \leq 0,\ \sigma_\tau^+ = 0,\ [u_\nu]\sigma_\nu^+ = 0 \text{ on } \gamma, \tag{31}$$

$$\int\limits_\gamma [\sigma\nu]\rho = 0 \ \ \forall \rho \in R_0(\gamma). \tag{32}$$

We can prove the following assertion.

Theorem 3. *For smooth solutions, problem formulations* (28) *and* (29)–(32) *are equivalent.*

We do not prove this theorem since the arguments used are simpler as compared to those of Theorem 1.

The model (29)–(32) is characterized by zero friction at the positive crack face γ^+.

4. Passage to Limit in (8) and (9) as $\alpha \to 0$

This section concerns a passage to limit as $\alpha \to 0$ in the model (8) and (9). We will prove that the limit model corresponds to the case without exfoliation of the rigid inclusion from the surrounding elastic body, i.e., to the case without a crack between γ and the elastic body.

Since the relation (24) takes place, we have the uniform in α estimate

$$\|u^\alpha\|_{H^1_{\Gamma_0}(\Omega_\gamma)^2} \leq c. \tag{33}$$

Moreover, (24), (33) imply

$$\int\limits_\gamma [u^\alpha]^2 \leq c\alpha. \tag{34}$$

Introduce a set of admissible displacements for the limit problem

$$S_0 = \{v \in H^1_{\Gamma_0}(\Omega_\gamma)^2 \mid \ v|_\gamma \in R_0(\gamma)\}.$$

Note that we have $[v] = 0$ on γ for $v \in S_0$. By (33), (34), we can assume that as $\alpha \to 0$,

$$u^\alpha \to u \text{ weakly in } \ H^1_{\Gamma_0}(\Omega_\gamma)^2, \tag{35}$$

$$[u^\alpha] \to [u] \text{ weakly in } \ L^2(\gamma)^2, \tag{36}$$

$$u|_\gamma = \rho^0 \in R_0(\gamma). \tag{37}$$

In particular, we have $[u] = 0$ on γ.

Take in (9) test functions of the form $\bar{u} = u^\alpha \pm v$, $v \in S_0$, and pass to the limit as $\alpha \to 0$ taking into account (35)–(37). It gives

$$u \in S_0,\ \int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(v) - \int\limits_{\Omega_\gamma} fv = 0 \ \ \forall\, v \in S_0. \tag{38}$$

Hence, the following assertion has been proved.

Theorem 4. *Solutions of the problems* (8) *and* (9) *converge in the sense* (35)–(37) *to the solution of* (38) *as* $\alpha \to 0$.

Notice that the problem (38) corresponds to minimization of the energy functional over the set S_0,

$$\inf_{v\in S_0}\{\frac{1}{2}\int\limits_{\Omega_\gamma}\sigma(v)\varepsilon(v)-\int\limits_{\Omega_\gamma}fv\}. \tag{39}$$

The minimization problem (39) has a unique solution satisfying (38).

To conclude this section, we provide a differential formulation of the problem (38): find a displacement field $u=(u_1,u_2)$, a stress tensor $\sigma=\{\sigma_{ij}\},\ i,j=1,2,$ defined in Ω_γ, and $\rho^0\in R_0(\gamma)$ such that

$$-\mathrm{div}\,\sigma=f,\ \sigma=C\varepsilon(u)\ \text{in}\ \Omega_\gamma, \tag{40}$$

$$u=0\ \text{on}\ \Gamma_0;\ u=\rho^0\ \text{on}\ \gamma, \tag{41}$$

$$\int\limits_\gamma[\sigma\nu]\rho=0\quad\forall\rho\in R_0(\gamma). \tag{42}$$

We see that the limit problem (40)–(42) fits to the rigid inclusion without exfoliation from the surrounding elastic body.

Problems (38) and (40)–(42) are equivalent for smooth solutions. The proof of this statement is simpler compared with that of Theorem 1.

5. Mixed Boundary Conditions on External Boundary Γ_0

Along with the problem (2)–(6), it is possible to consider mixed boundary conditions on Γ_0. Denote by n a unit normal vector to Γ. Like in the previous sections, the angle between Γ and γ at the point $(0,0)$ is zero. Suppose that γ_1 is a smooth part of Γ as depicted in Figure 4, meas $\Gamma\setminus\gamma_1>0$. Problem formulation is as follows: unknown functions are the displacement field $u^\alpha=(u_1^\alpha,u_2^\alpha)$, the stress tensor $\sigma=\{\sigma_{ij}\},\ i,j=1,2,$ defined in Ω_γ, as well as $\rho^\alpha\in R_0(\gamma)$, satisfying the following equations and boundary conditions

$$-\mathrm{div}\,\sigma=f,\ \sigma=C\varepsilon(u^\alpha)\ \text{in}\ \Omega_\gamma, \tag{43}$$

$$u^\alpha=0\ \text{on}\ \Gamma\setminus\gamma_1;\ \sigma n=0\ \text{on}\ \gamma_1;\ u^\alpha=\rho^\alpha\ \text{on}\ \gamma^-, \tag{44}$$

$$[u_\nu^\alpha]\geq 0,-\sigma_\nu^++\frac{1}{\alpha}[u_\nu^\alpha]\geq 0,\ -\sigma_\tau^++\frac{1}{\alpha}[u_\tau^\alpha]=0\ \text{on}\ \gamma, \tag{45}$$

$$[u_\nu^\alpha](-\sigma_\nu^++\frac{1}{\alpha}[u_\nu^\alpha])=0\ \text{ on }\gamma, \tag{46}$$

$$\int\limits_\gamma[\sigma\nu]\rho=0\quad\forall\rho\in R_0(\gamma). \tag{47}$$

We are planning to solve the problem (43)–(47) by minimizing a suitable functional. To this end, preliminary arguments are needed. Introduce the space

$$H^1_{\Gamma\setminus\gamma_1}(\Omega_\gamma)=\{v\in H^1(\Omega_\gamma)\mid v=0\ \text{on}\ \Gamma\setminus\gamma_1\}.$$

By adding a fictitious domain Ω_0, consider an extended domain $\Omega^e=\Omega_\gamma\cup\Omega_0\cup\gamma_2$ with a smooth cut $\gamma\cup\gamma_1\cup\{(0,0)\}$ and Lipschitz external boundary Γ_e, see Figure 2. Taking $v\in H^1_{\Gamma\setminus\gamma_1}(\Omega_\gamma)^2$, we can extend this function by zero to Ω_0. The extended function is equal to zero on Γ_e and belongs to the space $H^1_{\Gamma_e}(\Omega^e)^2$, where

$$H^1_{\Gamma_e}(\Omega^e)=\{v\in H^1(\Omega^e)\mid v=0\ \text{on}\ \Gamma_e\}.$$

Consequently, Korn's inequality holds for such extended functions. This implies that Korn's inequality is valid in the space $H^1_{\Gamma\setminus\gamma_1}(\Omega_\gamma)^2$.

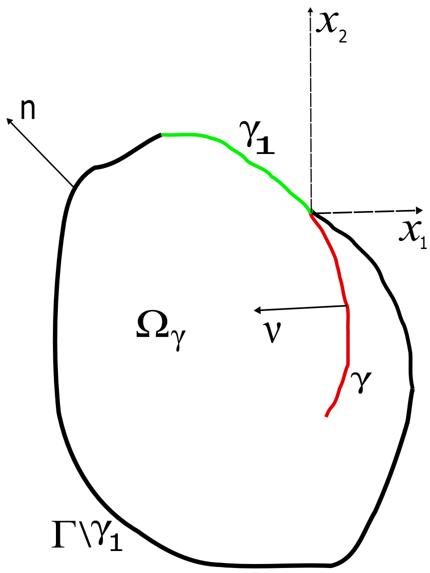

Figure 4. Partition of the external boundary.

To proceed, we introduce a set of admissible displacements

$$S_N = \{v \in H^1_{\Gamma\backslash\gamma_1}(\Omega_\gamma)^2 \mid [v_\nu] \geq 0 \text{ on } \gamma;\ v|_{\gamma^-} \in R_0(\gamma)\}.$$

The set S_N is closed and convex; hence, it is weakly closed. The above arguments imply that the problem

$$\inf_{v\in S_N} \{\frac{1}{2}\int\limits_{\Omega_\gamma} \sigma(v)\varepsilon(v) - \int\limits_{\Omega_\gamma} fv + \frac{1}{2\alpha}\int\limits_{\gamma} [v]^2\}$$

has (a unique) solution satisfying the following variational inequality

$$u^\alpha \in S_N, \tag{48}$$

$$\int\limits_{\Omega_\gamma} \sigma(u^\alpha)\varepsilon(v-u^\alpha) - \int\limits_{\Omega_\gamma} f(v-u^\alpha) + \tag{49}$$

$$+\frac{1}{\alpha}\int\limits_{\gamma} [u^\alpha][v-u^\alpha] \geq 0 \quad \forall v \in S_N.$$

The following statement holds.

Theorem 5. *For smooth solutions, problem formulations (43)–(47) and (48)–(49) are equivalent.*

The arguments are omitted since they are similar to those of Theorem 1.

6. Optimal Control of the Inclusion Shape

We devote this section to the case of the rigid inclusion crossing the boundary Γ and partially located outside of the elastic body. Denote by γ, γ_e internal and external parts of the rigid inclusion $\gamma_0 = \gamma \cup \gamma_e \cup \{(0,0)\}$, $\Gamma \cap \gamma_0 = (0,0)$, see Figure 5. The part γ of the inclusion γ_0 is assumed to be a Lipschitz curve and γ_e is a continuous one. The angle between γ_0 and Γ at the point $(0,0)$ is zero. Tip points of the inclusion γ_0 are $x^0 = (-1,-1)$ and $z^0 = (1,1)$. Moreover, $y^0 \in \gamma$, $y^0 = (-1/2,-1)$. We suppose that the inclusion shape located between the points x^0 and y^0 may change. The aim of this section is to analyze an optimal control problem assuming that a control function is a shape of the inclusion located between x^0, y^0, and a cost functional is a displacement of the inclusion at the tip point z^0. To be more precise, consider a bounded and weakly closed set $\Xi \subset H_0^2(-1/2, -1)$ assuming that the inclusion shape located between the points x^0 and y^0 is described as a graph of the function

$$x_2 = \xi(x_1) - 1, \;\; x_1 \in (-1/2, -1), \; \xi \in \Xi.$$

In this case, the curve γ corresponding to ξ is denoted by γ_ξ, and Ω_{γ_ξ} is denoted by Ω_ξ. An equilibrium problem for the body Ω_ξ with a given function ξ and a fixed damage parameter α is formulated as follows: unknown functions are the displacement field $u^\xi = (u_1^\xi, u_2^\xi)$, the stress tensor $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, defined in Ω_ξ, as well as $\rho_0^\xi \in R_0(\gamma_\xi)$, satisfying the following equations and boundary conditions

$$-\mathrm{div}\, \sigma = f, \; \sigma = C\varepsilon(u^\xi) \text{ in } \Omega_\xi, \tag{50}$$

$$u^\xi = 0 \text{ on } \Gamma_0; \; u^\xi = \rho_0^\xi \text{ on } \gamma_\xi^-, \tag{51}$$

$$[u_\nu^\xi] \geq 0, -\sigma_\nu^+ + \frac{1}{\alpha}[u_\nu^\xi] \geq 0, \; -\sigma_\tau^+ + \frac{1}{\alpha}[u_\tau^\xi] = 0 \text{ on } \gamma_\xi, \tag{52}$$

$$[u_\nu^\xi](-\sigma_\nu^+ + \frac{1}{\alpha}[u_\nu^\xi]) = 0 \;\text{ on } \gamma_\xi, \tag{53}$$

$$\int\limits_{\gamma_\xi} [\sigma\nu]\rho = 0 \;\; \forall \rho \in R_0(\gamma_\xi). \tag{54}$$

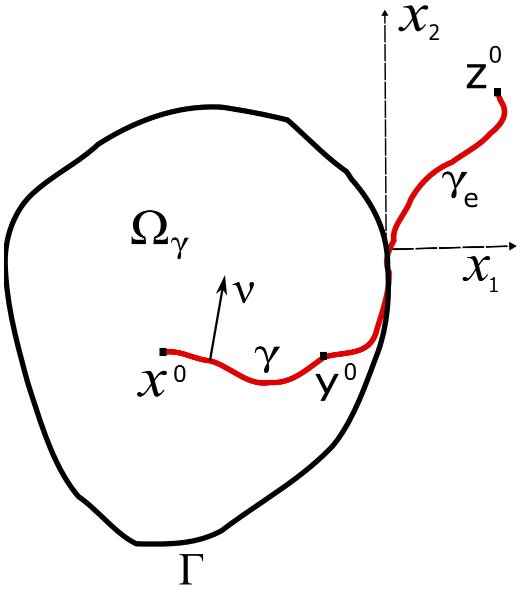

Figure 5. Inclusion $\gamma_0 = \gamma \cup \gamma_e \cup \{(0,0)\}$.

The problem (50)–(54) can be written in the variational form. To this end, introduce a set of admissible displacements

$$S_\xi = \{v \in H^1_{\Gamma_0}(\Omega_\xi)^2 \mid [v_\nu] \geq 0 \text{ on } \gamma_\xi; \; v|_{\gamma_\xi^-} \in R_0(\gamma_\xi)\}.$$

Then the problem (50)–(54) is equivalent to the following variational inequality

$$u^\xi \in S_\xi, \tag{55}$$

$$\int\limits_{\Omega_\xi} \sigma(u^\xi)\varepsilon(v-u^\xi) - \int\limits_{\Omega_\xi} f(v-u^\xi)+ \qquad (56)$$
$$+\frac{1}{\alpha}\int\limits_{\gamma_\xi}[u^\xi][v-u^\xi] \geq 0 \;\; \forall\, v \in S_\xi.$$

Moreover, we know that the problem (55) and (56) corresponds to minimization of the energy functional over the set S_ξ,

$$\inf_{v\in S_\xi}\{\frac{1}{2}\int\limits_{\Omega_\xi}\sigma(v)\varepsilon(v) - \int\limits_{\Omega\xi} fv + \frac{1}{2\alpha}\int\limits_{\gamma_\xi}[v]^2\}. \qquad (57)$$

According to Section 2, the set S_ξ is weakly closed, and the energy functional is coercive on S_ξ for any fixed $\xi \in \Xi$. Hence, the problem (57) indeed has a solution u^ξ satisfying (55) and (56).

Note that a displacement of the γ_e- points is defined according to the function ρ_0^ξ from (51). If

$$\rho_0^\xi(x) = (c^1,c^2) + b(-x_2,x_1),\; x=(x_1,x_2)\in\gamma_\xi,$$

then

$$\rho_0^\xi(x) = (c^1,c^2) + b(-x_2,x_1),\; x=(x_1,x_2)\in\gamma_e.$$

In particular, we can find a displacement of the inclusion γ_0 at the tip point z^0, i.e., $\rho_0^\xi(1,1) = (\rho_{01}^\xi(1,1), \rho_{02}^\xi(1,1))$. Thus, for any $\xi\in\Xi$ a value of a cost functional

$$J(\xi) = |\rho_{02}^\xi(1,1)|$$

can be defined. An optimal control problem to be analyzed below is formulated as follows

$$\inf_{\bar\xi\in\Xi} J(\bar\xi). \qquad (58)$$

We see that the inclusion shape is defined by the function ξ. Solving optimal control problem (58), we find the best inclusion shape that minimizes the cost functional. The following statement takes place.

Theorem 6. *There exists a solution of the optimal control problem* (58).

Proof. Consider a minimizing sequence $\xi^m \in \Xi$. By boundedness of Ξ and the imbedding theorem, we can assume that as $m\to\infty$,

$$\xi^m \to \xi \quad \text{weakly in } H_0^2(-1/2,-1),\; \xi\in\Xi, \qquad (59)$$
$$\xi^m \to \xi \text{ strongly in } C^1[-1/2,-1].$$

Introduce the notation

$$S^m = \{v\in H^1_{\Gamma_0}(\Omega_m)^2 \mid [v_\nu]\geq 0 \text{ on } \gamma_{\xi_m};$$
$$v|_{\gamma^-_{\xi^m}} \in R_0(\gamma_{\xi^m})\},$$

where $\Omega_m = \Omega \setminus \bar{\gamma}_{\xi^m}$. For any m, a solution of the problem

$$u^m \in S^m, \tag{60}$$

$$\int_{\Omega_m} \sigma(u^m)\varepsilon(v - u^m) - \int_{\Omega_m} f(v - u^m) + \tag{61}$$
$$+\frac{1}{\alpha} \int_{\gamma_{\xi^m}} [u^m][v - u^m] \geq 0 \;\; \forall\, v \in S^m$$

can be found. We can provide a transformation of the independent variables

$$\begin{cases} x_1 &= y_1 \\ x_2 &= y_2 - \varphi(y)(\xi^m(y_1) - \xi(y_1)), \end{cases} \tag{62}$$

where $x = (x_1, x_2) \in \Omega_\xi,\ y = (y_1, y_2) \in \Omega_m$. A smooth function φ with compact support in Ω_ξ is chosen in such a way that $\varphi = 1$ in a neighborhood of the union

$$\bigcup \{(y_1, y_2) \mid y_2 = \xi^m(y_1) - 1,\ y_1 \in [-1/2, -1]\},$$

where all functions ξ, ξ^m are extended by zero outside of $[-1/2, -1]$. Denote

$$u_m(x) = u^m(y), x \in \Omega_\xi,\ y \in \Omega_m.$$

Next, we can use the arguments of paper [32] where a perturbation of the inclusion shapes is analyzed, and obtain a priori estimates from (60) and (61) being uniform in m,

$$\|u_m\|_{H^1_{\Gamma_0}(\Omega_\xi)^2} \leq c. \tag{63}$$

By (63), we assume that as $m \to \infty$,

$$u_m \to u \;\text{ weakly in }\; H^1_{\Gamma_0}(\Omega_\xi)^2,$$

and moreover, it is proved

$$u_m \to u \text{ strongly in } H^1_{\Gamma_0}(\Omega_\xi)^2. \tag{64}$$

Making a transformation of the independent variables (62) in (60) and (61) and passing to the limit as $m \to \infty$, by (64), we obtain

$$u \in S_\xi, \tag{65}$$

$$\int_{\Omega_\xi} \sigma(u)\varepsilon(v - u) - \int_{\Omega_\xi} f(v - u) + \tag{66}$$
$$+\frac{1}{\alpha} \int_{\gamma_\xi} [u][v - u] \geq 0 \;\; \forall\, v \in S_\xi$$

which means $u = u^\xi$. It is turned out (see [32]) that if

$$\rho^m = u^m \;\text{ on }\; \gamma^-_{\xi^m},$$

then as $m \to \infty$,

$$\rho_m(x) \to \rho^\xi_0(x) \tag{67}$$

for all $x \in \gamma$, and consequently, for all $x \in \gamma_e$, where $\rho_0^\xi = (\rho_{01}^\xi, \rho_{02}^\xi) = u^\xi|_{\gamma_\xi^-}$. Then, by (67),

$$\inf_{\bar{\xi}\in\Xi} J(\bar{\xi}) = \lim_{m\to\infty} \inf J(\xi^m) = \lim_{m\to\infty} \inf |\rho_{m2}(1,1)| = |\rho_{02}^\xi(1,1)| =$$
$$= J(\xi) \geq \inf_{\bar{\xi}\in\Xi} J(\bar{\xi}).$$

Thus, the limit function ξ from (59) is a solution of the optimal control problem (58). Theorem 6 is proved. □

7. Conclusions

This paper provides a rigorous mathematical analysis of the problem for an elastic body with a thin rigid inclusion crossing the external boundary of the body at zero angle. It is supposed that the inclusion is exfoliated from the surrounding body which implies a presence of the interfacial crack. Inequality type constraints are imposed at the crack faces proving a mutual non-penetration between them. In addition to this factor, boundary conditions depend on a positive damage parameter describing the adhesion between the opposite crack faces. These boundary conditions imply that the problem considered refers to the problem with an unknown set of a contact. The solution existence of the problem is proved, and asymptotic analysis is fulfilled with respect to the damage parameter assuming that this parameter tends to infinity and to zero. Therefore, in the frame of a high–level mathematical model, we prove a correctness of the boundary value problem and analyze the limit models. Moreover, the existence of a solution to the optimal control problem is proved which allows us to find the optimal shape of the rigid inclusion. The approaches used in this paper can be useful for the analysis of various complex elastic and inelastic structures formulated in terms of free boundary problems, and in particular, for those having zero angles of the boundary.

Funding: This work is supported by the Mathematical Center in Akademgorodok under agreement No. 075-15-2022-282 with the Ministry of Science and Higher Education of the Russian Federation.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Lazarev, N. Shape sensitivity analysis of the energy integrals for the Timoshenko-type plate containing a crack on the boundary of a rigid inclusion. *Z. Angew. Math. Phys.* **2015**, *66*, 2025–2040. [CrossRef]
2. Lazarev, N.; Itou, H. Optimal location of a rigid inclusion in equilibrium problems for inhomogeneous Kirchhoff-Love plates with a crack. *Math. Mech. Solids* **2019**, *24*, 3743–3752. [CrossRef]
3. Khludnev, A.M. On the crossing bridge between two Kirchhoff-Love plates. *Axioms* **2023**, *12*, 120. [CrossRef]
4. Rudoy, E.; Shcherbakov, V. First-order shape derivative of the energy functional for elastic plates with rigid inclusions and interfacial cracks. *Appl. Math. Optim.* **2021**, *84*, 2775–2802. [CrossRef]
5. Rudoy, E.M. On numerical solving a rigid inclusions problem in 2D elasticity. *Z. Angew. Math. Phys.* **2017**, *68*, 19. [CrossRef]
6. Khludnev, A.M.; Fankina, I.V. Equilibrium problem for elastic plate with thin rigid inclusion crossing an external boundary. *Z. Angew. Math. Phys.* **2021**, *72*, 121. [CrossRef]
7. Shcherbakov, V.V. Shape optimization of rigid inclusions in elastic plates with cracks. *Z. Angew. Math. Phys.* **2016**, *67*, 71. [CrossRef]
8. Shcherbakov, V.V. Energy release rates for interfacial cracks in elastic bodies with thin semirigid inclusions. *Z. Angew. Math. Phys.* **2017**, *68*, 26. [CrossRef]
9. Furtsev, A.I. On contact between a thin obstacle and a plate containing a thin inclusion. *J. Math. Sci.* **2019**, *237*, 530–545. [CrossRef]
10. Khludnev, A.M. On modeling thin inclusions in elastic bodies with a damage parameter. *Math. Mech. Solids* **2019**, *24*, 2742–2753. [CrossRef]
11. Khludnev, A.M. On thin Timoshenko inclusions in elastic bodies with defects. *Arch. Appl. Mech.* **2019**, *89*, 1691–1704. [CrossRef]
12. Kovtunenko, V.A.; Kunisch, K. Shape derivative for penalty-constrained nonsmooth-nonconvex optimization: Cohesive crack problem. *J. Optim. Theory Appl.* **2022**, *194*, 597–635. [CrossRef]

13. Almi, S. Energy release rate and quasi-static evolution via vanishing viscosity in a fracture model depending on the crack opening. *ESAIM Control Optim. Calc. Var.* **2017**, *23*, 791–826. [CrossRef]
14. Khludnev, A.M.; Popova, T.S. Timoshenko inclusions in elastic bodies crossing an external boundary at zero angle. *Acta Mech. Solida Sin.* **2017**, *30*, 327–333. [CrossRef]
15. Rudoy, E. Asymptotic justification of models of plates containing inside hard thin inclusions. *Technologies* **2020**, *8*, 59. [CrossRef]
16. Furtsev, A.; Rudoy, E. Variational approach to modeling soft and stiff interfaces in the Kirchhoff-Love theory of plates. *Int. J. Solids Struct.* **2020**, *202*, 562–574. [CrossRef]
17. Serpilli, M. Asymptotic interface models in magneto-electro-thermoelastic composites. *Meccanica* **2017**, *52*, 1407–1424. [CrossRef]
18. Serpilli, M.; Rizzoni, R.; Lebon, F.; Dumont, S. An asymptotic derivation of a general imperfect interface law for linear multiphysics composites. *Int. J. Solids Struct.* **2019**, *180–181*, 97–107. [CrossRef]
19. Gaudiello, A.; Sili, A. Limit models for thin heterogeneous structures with high contrast. *J. Differ. Equ.* **2021**, *302*, 37–63. [CrossRef]
20. Mallick, P. *Fiber-Reinforced Composites. Materials, Manufacturing, and Design*; Marcel Dekker: New York, NY, USA, 1993.
21. Pasternak, I.M. Plane problem of elasticity theory for anisotropic bodies with thin elastic inclusions. *J. Math. Sci.* **2012**, *186*, 31–47. [CrossRef]
22. Caillerie, D. The effect of a thin inclusion of high rigidity in an elastic body. *Math. Meth. Appl. Sci.* **1980**, *2*, 251–270. [CrossRef]
23. Goudarzi, M.; Dal Corso, F.; Bigoni, D.; Simone, A. Dispersion of rigid line inclusions as stiffeners and shear band instability triggers. *Int. J. Solids Struct.* **2021**, *210–211*, 255–272. [CrossRef]
24. El Jarroudi, M. Homogenization of an elastic material reinforced with thin rigid von Karman ribbons. *Math. Mech. Solids* **2018**, *24*, 1–27. [CrossRef]
25. Hu, K.X.; Chandra, A.; Huang, Y. On crack, rigid-line fiber, and interface interactions. *Mech. Mater.* **1994**, *19*, 15–28. [CrossRef]
26. Pingle, P.; Sherwood, J.; Gorbatikh, L. Properties of rigid-line inclusions as building blocks of naturally occurring composites. *Compos. Sci. Technol.* **2008**, *68*, 2267–2272. [CrossRef]
27. Kozlov, V.A.; Ma'zya, V.G.; Movchan, A.B. *Asymptotic Analysis of Fields in a Multi-Structure*; Oxford Mathematical Monographs; Oxford University Press: New York, NY, USA, 1999.
28. Panasenko, G. *Multi-Scale Modelling for Structures and Composites*; Springer: New York, NY, USA, 2005.
29. Boureghda, A.; Djellab, N. Du Fort-Frankel Finite difference scheme for solving of oxygen diffusion problem inside one cell. *J. Comput. Theor. Transport.* **2023**. [CrossRef]
30. Amar, L.H.H.; Bourada, F.; Bousahla, A.A.; Tounsi, A.; Benrahou, K.H.; Albalawi, H.; Tounsi, A. Buckling analysis of FG plates via 2D and quasi-3D refined shear deformation theories. *Struct. Engin. Mech.* **2023**, *85*, 765–780.
31. Tounsi, A.; Bousahla, A.A.; Tahir, S.I.; Mostefa, A.H.; Bourada, F.; Al-Osta, M.A.; Tounsi, A. Influences of different boundary conditions and hygro-thermal environment on the free vibration responses of FGM sandwich plates resting on viscoelastic foundation. *Int. J. Struct. Stab. Dyn.* **2023** , 2450117. [CrossRef]
32. Khludnev, A.M. Shape control of thin rigid inclusions and cracks in elastic bodies. *Arch. Appl. Mech.* **2013**, *83*, 1493–1509. [CrossRef]