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Article

S-Embedding of Lie Superalgebras and Its Implications for Fuzzy Lie Algebras

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Abstract: This paper performed an investigation into the s-embedding of the Lie superalgebra $\vec{(}S^{1|1})$, a representation of smooth vector fields on a (1,1)-dimensional super-circle. Our primary objective was to establish a precise definition of the s-embedding, effectively dissecting the Lie superalgebra into the superalgebra of super-pseudodifferential operators ($S\psi D\odot$) residing on the super-circle $S^{1|1}$. We also introduce and rigorously define the central charge within the framework of $\vec{(}S^{1|1})$, leveraging the canonical central extension of $S\psi D\odot$. Moreover, we expanded the scope of our inquiry to encompass the domain of fuzzy Lie algebras, seeking to elucidate potential connections and parallels between these ostensibly distinct mathematical constructs. Our exploration spanned various facets, including non-commutative structures, representation theory, central extensions, and central charges, as we aimed to bridge the gap between Lie superalgebras and fuzzy Lie algebras. To summarize, this paper is a pioneering work with two pivotal contributions. Initially, a meticulous definition of the s-embedding of the Lie superalgebra $\vec{(}S^{1|1})$ is provided, emphasizing the representationof smooth vector fields on the (1,1)-dimensional super-circle, thereby enriching a fundamental comprehension of the topic. Moreover, an investigation of the realm of fuzzy Lie algebras was undertaken, probing associations with conventional Lie superalgebras. Capitalizing on these discoveries, we expound upon the nexus between central extensions and provide a novel deformed representation of the central charge.

Keywords: Lie superalgebra; s-embedding; central charge; fuzzy Lie algebras; canonical central extension

MSC: 17B66; 17B68; 16S32; 81R10; 46L87; 81T60

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1. Introduction

The study of multi-parameter deformations (MLDs) within the framework of the standard embedding (s-embedding) of the Lie algebra (LA) $\vec{(}S^{1})$ has long been a subject of profound mathematical interest. This embedding represents the vector fields on the circle S^1 and finds applications in various mathematical and physical contexts. The exploration of the MLD involves examining how this standard embedding evolves when subjected to multi-parameter deformations, thus unveiling intricate structures and revealing hidden symmetries.

In recent decades, the intersection of Lie algebras and multi-parameter deformations has been the subject of considerable exploration and debate. The work of Pogudin et al. [1] laid the groundwork, introducing the foundational concepts of the standard embedding of Lie algebras. Their focus on the vector fields on the circle was later expanded upon by Bahturin [2], who found wide-ranging applications in both mathematical and physical realms. However, it was the breakthrough research by Kanel-Belov et al. [3] that began delving into the intricacies of pseudodifferential operators, emphasizing their invaluable role in mathematical physics. However, amid this flourishing area of study, the critical review in [4–6] cautioned the academic community to ensure rigorous proofs and validations. As the field continues to grow, the convergence of these diverse perspectives promises richer understandings and breakthroughs. In the realm of pseudodifferential operators, particularly within the algebra of pseudodifferential operators denoted as $\psi\mathcal{D}\odot$ on S^1, the s-embedding has been a topic of extensive research. The pseudodifferential operator algebra is an invaluable tool in mathematical physics and provides insights into a wide array of phenomena. References [7,8] have laid a solid foundation for understanding the s-embedding within this context.

One of the central objectives of this study was to delve into the cohomology space $H^1(\vec{(}S^{1|1}), \mathcal{S}\psi\mathcal{D}\odot)$. This cohomology space plays a pivotal role in understanding the deformations of the s-embedding. Remarkably, our analysis revealed that this cohomology space is four-dimensional. Furthermore, we provide explicit formulations for four generating one-cocycles. These cocycles serve as key elements in the classification and characterization of tiny deformations (def) of the standard embedding (SE) of the Lie superalgebra $\vec{(}S^{1|1})$. This Lie superalgebra represents vector fields on the super-circle $S^{1|1}$ and resides within the larger Lie superalgebra $\mathcal{S}\psi\mathcal{D}\odot$, which comprises super-pseudodifferential operators defined on $S^{1|1}$.

Some works appear to operate within the framework of Lie superalgebras, central extensions, vertex superalgebras, and superalgebras (cf. [9–11]). These are common mathematical structures in both cases. But, in our work, we had distinct research objectives and a mathematical focus. We emphasize the s-embedding of $\vec{(}S^{1|1})$ and its connections to various mathematical domains, while the cited work concentrated on minimal generating sets, root systems, and commutant vertex algebras within the context of a different Lie superalgebra.

In the course of our investigation into these deformations and cohomology spaces, we uncovered intriguing connections with fuzzy Lie algebras. Fuzzy Lie algebras are a unique and emerging area of research that introduces non-commutative structures into the realm of Lie algebras. For more detail about these structures, see [12–14]. By examining the deformations and cohomology spaces in parallel with the concepts from fuzzy Lie algebras, we aimed to establish a deeper connection between these two mathematical domains. This interdisciplinary exploration sought to bridge the gap between traditional Lie theory and non-commutative algebra, offering new insights and perspectives on both.

The second phase of our study focused on the integrability relations of the defined quantity. This analysis revealed the existence of four distinct families of non-trivial definitions, each parameterized by an even parameter. Our objective was to derive explicit formulas that meticulously describe these families. It was through a careful contraction procedure applied to these definitions that we obtained four one-parameter definitions of the superembedding of $\vec{(}S^{1|1})$ into the Poisson–Lie superalgebra $\mathcal{S}\varrho$ of super-pseudodifferential operators defined on $S^{1|1}$. Each parameter within these definitions corresponds to a fascinating algebraic curve within the parameter space, creating a rich interplay between deformation theory and the fuzzy Lie algebra framework.

The well-established non-trivial central extension of $\mathcal{S}\psi\mathcal{D}\odot$ significantly influences the central extension of the superalgebra $\vec{(}S^{1|1})$, as detailed in [15]. In particular, the two-cocycle that generates this extension plays a critical role in defining the central extension of $\vec{(}S^{1|1})$. Leveraging our findings, we not only elucidated this intricate relationship, but also derived a "deformed" representation of the central charge. This central charge arises from the deformations of the superembedding we constructed, further emphasizing the connections between central extensions and fuzzy Lie algebra structures.

In summary, the novelty in this paper is represented by two key contributions. Firstly, we provide a precise definition for the s-embedding of the Lie superalgebra $\vec{(}S^{1|1})$, focusing on its representation of smooth vector fields on a $(1,1)$-dimensional super-circle, enhancing the foundational understanding. Secondly, we explored the domain of fuzzy Lie algebras, seeking connections with traditional Lie superalgebras. Leveraging these findings, we elucidated the relationship between central extensions and derived a novel "deformed" representation of the central charge.

MLD = multi-parameter deformation.
LA = Lie algebra.
LSA = Lie superalgebra.
$\vec{(}S^1)$ = Lie algebra of vector fields on the circle (S^1).
$\vec{(}S^{1|1})$ = Lie superalgebra of vector fields on the super-circle $(S^{1|1})$.
$\psi\mathcal{D}\odot$ = algebra of pseudodifferential operators on S^1.
$\mathcal{S}\psi\mathcal{D}\odot$ = superalgebra of pseudodifferential operators on $S^{1|1}$.
(SE) = standard embedding.

2. Background

2.1. The Lie Superalgebra of (Contact) Vector Fields on $S^{1|1}$

The super-circle $S^{1|1}$ is defined by $\mathbb{C}^\infty(S^{1|1})$. Let us explicate the elements of $\mathbb{C}^\infty(S^{1|1})$. An element $\mathbb{C}^\infty(S^{1|1})$ must be written as

$$\mathsf{T}(a,\vartheta) = f_0(a) + \vartheta f_1(a),$$

where $f_0(a)$ and $f_1(a)$ are in $\mathbb{C}^\infty(S^1)$ and a and ϑ are even and odd variables, respectively. Let us remark that we have $\vartheta^2 = 0$. In fact, the even elements in $\mathbb{C}^\infty(S^{1|1})$ are the functions $\mathsf{T}(a,0) = f_0(a)$, and the functions $\mathsf{T}(a,\vartheta) - \mathsf{T}(a,0) = \vartheta f_1(a)$ are odd elements. The parity of a homogeneous function T is denoted by $p(\mathsf{T})$.

Assume that $\vec{(}S^{1|1})$ is the super-space of vector fields on $S^{1|1}$:

$$\vec{(}S^{1|1}) = \{\mathsf{T}(a,\vartheta)\partial_a + \tilde{\mathsf{T}}(a,\vartheta)\partial_\vartheta\},$$

where ∂_ϑ and ∂_a stand, respectively, for $\frac{\partial}{\partial\vartheta}$ and $\frac{\partial}{\partial a}$. The even vector fields are linear combination of the fields $f(a)\partial_a$ and $\vartheta f(a)\partial_\vartheta$, while the odd ones are a combination of the fields $\vartheta f(a)\partial_a$ and $f(a)\partial_\vartheta$. The bilinear operation of the superbracket between two vector fields is defined for two homogeneous vector fields as follows:

$$[\chi,\ \aleph] = \chi \circ \aleph - (-1)^{p(\chi)p(\aleph)}\aleph \circ \chi,$$

where $p(\chi)$ and $p(\aleph)$ are the parities of χ and $\aleph$, respectively.

The structure of the contact on $S^{1|1}$ can be given by the one-form:

$$\alpha = da + \vartheta d\vartheta.$$

Denoting $\mathcal{K}(1)$ as the Lie-sub-superalgebra of $\vec{(}S^{1|1})$, where the Lie action on α corresponds to a function multiplication, it can be observed that any element in $\mathcal{K}(1)$ follows the form described in [15].

$$v_{\mathsf{T}} = \mathsf{T}\partial_a + \frac{(-1)^{p(\mathsf{T})+1}}{2}\gamma(\mathsf{T})\gamma,$$

where $\mathsf{T} \in \mathbb{C}^\infty(S^{1|1}),\ p(\mathsf{T})$ is the parity of T and $\gamma = \partial_\vartheta - \vartheta\partial_a$. The bracket is given by

$$[v_{\mathsf{T}},\ v_H] = v_{\{\mathsf{T},H\}},$$

where

$$\{\mathsf{T},H\} = \mathsf{T}H' - \mathsf{T}'H + \frac{(-1)^{p(\mathsf{T})+1}}{2}\gamma(\mathsf{T})\gamma(H);\ H' := \partial_a(H). \tag{1}$$

Equation (1) is taken directly from [16].

The LSA $\mathcal{K}(1)$ is called the LSA of contact vector fields. The vector field $\gamma = \partial_\theta - \vartheta\partial_a$ on $S^{1|1}$ maps $\mathsf{T} = f(a) + \vartheta g(a)$ to $\gamma(\mathsf{T}) = g(a) - f'(a)\vartheta$, so that $\gamma^2 = \frac{1}{2}[\gamma,\gamma] = -\partial_a$. The usual rule of Leibniz: $\frac{\partial}{\partial a} \circ g = g'(a) + g(a)\frac{\partial}{\partial a}$ on $\mathbb{C}^\infty(S^1)$, is replaced on $\mathbb{C}^\infty(S^{1|1})$ by:

$$\gamma \circ \mathsf{T} = \gamma(\mathsf{T}) + (-1)^{p(\mathsf{T})}\, \mathsf{T}\, \gamma.$$

2.2. Detailed Overview of the Super-Space $\mathcal{S}\psi\mathcal{DO}$

This subsection elaborates on the intricate concepts and mathematical formulations derived from the seminal work referenced as ABO. Central to this discussion is the series defined as follows:

$$\mathcal{SP} = \Big\{ X = \sum_{k=-M}^{\infty} \sum_{\epsilon} x_{k,\epsilon}(a,\vartheta) y^{-k} \bar{\vartheta}^{\epsilon} \,\Big|\, x_{k,\epsilon} \in \mathbb{C}^\infty(S^{1|1}), \epsilon \in \{0,1\}, M \in \mathbb{N} \Big\}, \tag{2}$$

where the series X represents an aggregation of terms indexed by k and ϵ, with $x_{k,\epsilon}$ as complex-valued smooth functions on the super-circle $S^{1|1}$. In this formulation, y is analogous to the derivative with respect to the coordinate a, and $\bar{\vartheta}$ corresponds to the derivative with respect to the Grassmannian variable ϑ, bearing a parity of one.

This framework gives rise to the super-space, which forms the foundational structure for the super-commutative algebra of super-pseudodifferential symbols on $S^{1|1}$. This algebra is characterized by conventional multiplication operations. The space $\mathcal{SP}$ is endowed with a Poisson LSA (Lie superalgebra) structure, which is articulated by the brackets defined as:

$$\{X,Y\} = \frac{\partial X}{\partial y}\frac{\partial Y}{\partial a} - \frac{\partial X}{\partial a}\frac{\partial Y}{\partial y} - (-1)^{p(X)}\left(\frac{\partial X}{\partial \vartheta}\frac{\partial Y}{\partial \bar{\vartheta}} + \frac{\partial X}{\partial \bar{\vartheta}}\frac{\partial Y}{\partial \vartheta}\right). \tag{3}$$

Here, the Poisson brackets incorporate both the even and odd derivatives, reflecting the supergeometry underlying the algebra.

Furthermore, the associative superalgebra of super-pseudodifferential operators, $\mathcal{S}\psi\mathcal{DO}$, is constructed on the same super-circle $S^{1|1}$. Although it shares a similar vector space structure with $\mathcal{SP}$, the multiplication in $\mathcal{S}\psi\mathcal{DO}$ adheres to a distinct rule, expanding the algebraic interactions:

$$X \circ Y = \sum_{\alpha \geq 0, \nu = 0,1} \frac{(-1)^{p(X)+1}}{\alpha!} (\partial_y^\alpha \partial_{\bar{\vartheta}}^\nu X)(\partial_a^\alpha \partial_\vartheta^\nu Y). \tag{4}$$

This rule of composition is pivotal as it induces a super-commutator, a fundamental element in the study of superalgebras, defined by:

$$[X,Y] = X \circ Y - (-1)^{p(X)p(Y)} Y \circ X.$$

The super-commutator, thus, encapsulates the non-commutative nature of the algebra, offering a nuanced view of the interactions between different elements in this superalgebraic structure.

The format of the LSA can be constructed as $\mathcal{S}\psi\mathcal{D}\odot$ to the Poisson algebra $\mathcal{S}\varrho$. Consider the super-commutative SA Λ generated by $(\vartheta;\bar{\vartheta})$. Then, $\mathcal{S}\varrho = \varrho \otimes \Lambda$, where ϱ is the space of symbols of ordinary pseudodifferential operators $\psi\mathcal{D}\odot(S^1)$. Let us define the linear isomorphisms:

$$\Phi_h : \varrho \longrightarrow \varrho \tag{5}$$

as

$$\Phi_h(x(a)y^l) = a(x)h^l y^l, \text{where } 0 < h \leq 1.$$

Now, the multiplication on ϱ is given by

$$X \circ_h Y = \Phi_h^{-1}(\Phi_h(X) \circ \Phi_h(Y)).$$

The rule of Leibniz in the odd variables can take a form by letting:

$$\bar{\vartheta} \circ_h \vartheta = h - \vartheta\bar{\vartheta}.$$

Now, the composition on $\mathcal{S}\varrho$ affected by h is:

$$(X \otimes \alpha) \circ_h (Y \otimes \beta) = (X \circ_h Y)\alpha \cdot \beta,$$

where $X,\ Y \in \varrho$ and $\alpha, \beta \in \Lambda$. We will denote by $\mathcal{S}\psi\mathcal{D}\odot_h$ the associative algebra $\mathcal{S}\varrho$ endowed with the farmer composition $\circ_h$. This compositioninduces the usual LSA structure on $\mathcal{S}\psi\mathcal{D}\odot_h$ given by the super-commutator $[X, Y]_h = X \circ_h Y - (-1)^{p(X)p(Y)} Y \circ_h X$. One has

$$[X, Y]_h = \{X, Y\} + O(h)$$

and therefore, $\lim_{h\to 0}[X, Y]_h = \{X, Y\}$. This is mean that the LSA $\mathcal{S}\psi\mathcal{D}\odot$ belongs to the Poisson superalgebra $\mathcal{S}\varrho$.

Moreover, $\mathcal{S}\psi\mathcal{D}\odot$ injects the similarity of the Adler trace expressed on the LA $\psi\mathcal{D}\odot$ of pseudodifferential operators on S^1 (cf. [15,17]). The right super-residue of the super-pseudodifferential operator $X = \sum_{k=-M}^{\infty} \sum_{\epsilon=0,1} a_{k,\epsilon}(a, \vartheta) y^{-k} \bar{\vartheta}^{\epsilon}$ (see (2)) is given by

$$\mathrm{Sres}(X) = a_{-1,\,1}(a, \vartheta)$$

and the functional $\mathrm{Str}(X)$ known from the Gelfand–Adler trace will be defined on $\mathcal{S}\psi\mathcal{D}\odot$ by

$$\mathrm{Str}(X) = \int_{S^{1|1}} \mathrm{Sres}(X) Ber(a, \vartheta).$$

while the Berezin integral is of the form

$$\int_{S^{1|1}} (g_0(a) + \vartheta g_1(a)) Ber(a, \vartheta) = \int_{S^1} g_1 da$$

2.3. Fuzzy Lie Algebras

Given the detailed mathematical context of the super-space $\mathcal{S}\psi\mathcal{D}\mathcal{O}$ and its underlying principles, we now explore the concept of fuzzy subsets within the realm of vector spaces. This exploration extends the notion of traditional vector spaces into the domain of fuzzy logic, providing a unique perspective on how subsets can be characterized in a less binary and more-gradual manner.

In the world of linear algebra, vector spaces over a field F are fundamental constructs. These spaces are typically defined with crisp, clear boundaries. However, when we introduce the concept of fuzzy subsets, these boundaries become more nuanced. A fuzzy subset μ of a vector space V can be thought of as a "soft" or "blurred" version of a traditional subset. Unlike conventional subsets, where an element either belongs or does not belong to a set, a fuzzy subset assigns a degree of membership to each element, ranging from 0 (completely outside the set) to 1 (completely inside the set).

Let V be a vector space over a field F. A fuzzy subset μ of V is considered a fuzzy subspace of V if it satisfies the following conditions:

$$\mu(x + y) \geq \min\{\mu(x), \mu(y)\} \text{ for all } x, y \in V,$$
$$\mu(\alpha x) \geq \mu(x) \text{ for all } x \in V, \alpha \in F.$$

Note: Condition (2) implies that $\mu(-x) \geq \mu(x)$ and $\mu(0) \geq \mu(x)$ for all x in V.

If μ is a fuzzy subspace of a vector space V, then the following properties hold:

$$\mu(x) = \mu(-x),$$
$$\mu(x - y) = \mu(0) \text{ implies } \mu(x) = \mu(y),$$
$$\mu(x) < \mu(y) \text{ implies } \mu(x - y) = \mu(x) = \mu(y - x),$$

for all $x, y \in V$.

For a fuzzy subset μ of a vector space V, the following statements are equivalent:

1. μ is a fuzzy subspace of V;
2. Each nonempty $U(\mu, t) = \{x \in L | \mu(x) \geq t\}$ is a subspace of V.

Note: This theorem establishes the equivalence between fuzzy subspaces and conventional subspaces within the context of fuzzy sets.

A fuzzy set μ, defined as a map $\mu : L \to [0,1]$, is termed a fuzzy Lie subalgebra of L over a field F if it is a fuzzy subspace of L and satisfies the additional condition:

$$\mu([x,y]) \geq \min\{\mu(x), \mu(y)\},$$

for all $x, y \in L$ and $\alpha \in F$.

Example 1. *Consider the real vector space $\mathbb{R}^3$ with the Lie bracket operation $[x,y] = x \times y$, where $x, y \in \mathbb{R}^3$. Define a fuzzy set μ on $\mathbb{R}^3$ such that:*

$$\mu(x) = \begin{cases} 0.9, & \text{if } x = (0,0,0), \\ 0.6, & \text{if } x = (c,0,0), c = 0, \\ 0.2, & \text{otherwise.} \end{cases}$$

By direct calculations, it is observed that μ satisfies the conditions of a fuzzy Lie subalgebra, making it a fuzzy Lie algebra (cf. [18–20]).

A fuzzy set $\mu : L \to [0,1]$ is considered a fuzzy Lie ideal of a Lie algebra L if it satisfies the following conditions:

$$\begin{aligned} \mu(x+y) &\geq \min\{\mu(x), \mu(y)\}, \\ \mu(\alpha x) &\geq \mu(x) \text{ for all } x \in L \text{ and } \alpha \in F, \\ \mu([x,y]) &\geq \mu(x) \text{ for all } x, y \in L \text{ and } \alpha \in F. \end{aligned}$$

A fuzzy set $\mu : L \to [0,1]$ is considered a fuzzy Lie ideal of a Lie algebra L if it satisfies the following conditions:

$$\begin{aligned} \mu(x+y) &\geq \min\{\mu(x), \mu(y)\} \text{ for all } x, y \in L, \\ \mu(\alpha x) &\geq \mu(x) \text{ for all } x \in L \text{ and } \alpha \in F, \\ \mu([x,y]) &\geq \min\{\mu(x), \mu(y)\} \text{ for all } x, y \in L. \end{aligned}$$

The definition of a fuzzy Lie ideal is equivalent to the condition that, for each nonempty set, $U(\mu, t) = \{x \in L | \mu(x) \geq t\}$, $U(\mu, t)$ is a Lie ideal of L.

2.4. The Structure of $\mathcal{S}\varrho$ as a $\vec{(}S^{1|1})$-Module

In this section, we will follow the same method used in [8,16]. We define the usual embedding of $\vec{(}S^{1|1})$ into $\mathcal{S}\varrho$ as follows:

$$\pi(\mathsf{T}\partial_a + H\partial_\vartheta) = \mathsf{T}y + H\bar{\vartheta}, \tag{6}$$

which induces a $\vec{(}S^{1|1})$-module structure on $\mathcal{S}\varrho$.

We assign the following $\mathbb{Z}$-grading to the PSA $\mathcal{S}\varrho$:

$$\mathcal{S}\varrho = \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathcal{S}\varrho_n, \tag{7}$$

where $\widetilde{\bigoplus}_{n\in\mathbb{Z}} = (\bigoplus_{n<0}) \bigoplus \prod_{n\geq 0}$, and

$$\mathcal{S}\varrho_n = \left\{ \left(\mathsf{T} y^{-n} + H y^{-n-1}\bar{\vartheta} \right) \,\middle|\, \mathsf{T},\, H \in \mathbb{C}^\infty(S^{1|1}) \right\}$$

is the homogeneous degree $-n$ subspace.

Each element A of $\mathcal{S}\psi\mathcal{D}\odot$ can be defined as

$$A = \sum_{k\in\mathbb{Z}} (\mathsf{T}_k + H_k y^{-1}\bar{\vartheta}) y^{-n}, \quad \mathsf{T}_k,\, H_k \in \mathbb{C}^\infty(S^{1|})$$

The *order* of A is defined as

$$\mathrm{o}(A) = \sup\{k \mid \mathsf{T}_k \neq 0 \text{ or } H_k \neq 0\}.$$

This defines a non-decreasing filtration on $\mathcal{S}\psi\mathcal{D}\odot$ as follows:

$$\mathsf{T}_n = \{A \in \mathcal{S}\psi\mathcal{D}\odot,\, \mathrm{o}(A) \leq -n\},$$

where n is an integer. Consequently, we have the filtration:

$$\ldots \subset \mathsf{T}_{n+1} \subset \mathsf{T}_n \subset \ldots \tag{8}$$

This filtration is compatible with the multiplication and the Poisson bracket. Specifically, for $X \in \mathsf{T}_n$ and $Y \in \mathsf{T}_m$, we have $X \circ Y \in \mathsf{T}_{n+m}$ and $\{X, Y\} \in \mathsf{T}_{n+m-1}$. This filtration endows $\mathcal{S}\psi\mathcal{D}\odot$ with an associative filtered superalgebra structure.

Let $Hr(\mathcal{S}\psi\mathcal{D}\odot) = \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathsf{T}_n/\mathsf{T}_{n+1}$ denote the associated graded space. The filtration (8) is also compatible with the usual action of $\vec{(}S^{1|1})$ on $\mathcal{S}\psi\mathcal{D}\odot$. In fact, if $v \in \vec{(}S^{1|1})$ and $X \in \mathsf{T}_n$, then

$$v.X = [\pi(v), X] \in \mathsf{T}_n.$$

The induced $\vec{(}S^{1|1})$-module structure on the quotient $\mathsf{T}_n/\mathsf{T}_{n+1}$ is isomorphic to the $\vec{(}S^{1|1})$-module $\mathcal{S}\varrho_n$. Therefore, the $\vec{(}S^{1|1})$-module $Hr(\mathcal{S}\psi\mathcal{D}\odot)$ is isomorphic to the graded $\vec{(}S^{1|1})$-module $\mathcal{S}\varrho$, i.e.,

$$\mathcal{S}\varrho \simeq \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathsf{T}_n/\mathsf{T}_{n+1}.$$

3. Computations of the Space $H^1(\vec{(}S^{1|1}), \mathcal{S}\psi\mathcal{D}\odot))$

In the current section, we will adopt the same policy as in [8,16,21] to establish the first CHS of $\vec{(}S^{1|1})$ with coefficients in $\mathcal{S}\psi\mathcal{D}\odot$.

Let an LSA $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ be acting on a super-space $\mathrm{Y} = \mathrm{Y}_0 \oplus \mathrm{Y}_1$. For $p \geq 1$, let the space $\mathbb{C}^p(\mathfrak{h}, \mathrm{Y})$ of p-cochains be the $\mathbb{Z}_2$ graded space of skew symmetric P-linear functions on $\mathfrak{h}$ with the range in V, and let $\mathbb{C}^0(\mathfrak{h}, \mathrm{Y}) = V$. The Chevalley–Eilenberg operator δ (see [22]) transforms $\mathbb{C}^p(\mathfrak{h}, \mathrm{Y})$ into $\mathbb{C}^{p+1}(\mathfrak{h}, V)$. In particular, δ converts a linear function $c : \mathfrak{h} \to \mathrm{Y}$ $(\in \mathbb{C}^1(\mathfrak{h}, V))$ into a bilinear function $\delta c : \mathfrak{h} \times \mathfrak{h} \to \mathrm{Y}$ defined by

$$\delta c(a, b) = c([a, b]) - (-1)^{p(a)p(c)} a.c(b) + (-1)^{p(b)(p(a)+p(c))} b.c(a),\ \forall\, b, a \in \mathfrak{h} \tag{9}$$

where $p(a) \in \mathbb{Z}_2$ represents the parity of a.

Set

$$\begin{array}{l} Z^1(\mathfrak{h}, \mathrm{Y}) = \{c \in \mathrm{Hom}(\mathfrak{h},\ V) | \delta c = 0\} \\ B^1(\mathfrak{h}, \mathrm{Y}) = \{c \in \mathrm{Hom}(\mathfrak{h},\ V) | \exists v \in \mathrm{Y} \text{ and } c(a) = a \cdot v, \forall a \in \mathfrak{h}\} \\ H^1(\mathfrak{h}, \mathrm{Y}) = Z^1(\mathfrak{h}, \mathrm{Y}) / B^1(\mathfrak{h}, \mathrm{Y}). \end{array} \tag{10}$$

The spaces $Z^1(\mathfrak{h}, \mathrm{Y}),\ B^1(\mathfrak{h}, \mathrm{Y})$ and $H^1(\mathfrak{h}, \mathrm{Y})$ are the space of one-cocycles, the space of one coboundaries, and the first CHS.

The space $\mathrm{Hom}(\mathfrak{h}, \mathrm{Y})$ is $\mathbb{Z}_2$-graded via

$$\mathrm{Hom}(\mathfrak{h}, \mathrm{Y})_b = \oplus_{a \in \mathbb{Z}_2} \mathrm{Hom}(\mathfrak{h}_x, \mathrm{Y}_{x+y});\ y \in \mathbb{Z}_2. \tag{11}$$

According to the $\mathbb{Z}_2$-grading (11), each c in $\mathbb{Z}^1(\mathfrak{h}, \mathrm{Y})$ is converted into $(c', c'') \in \mathrm{Hom}(\mathfrak{h}_0, \mathrm{Y}) \oplus \mathrm{Hom}(\mathfrak{h}_1, \mathrm{Y})$ with respect to the equations below:

$$\begin{array}{llcll} (\mathfrak{E}_1) & c'([h_1, h_2]) + h_2 \times c'(h_1) - h_1 \times c'(h_2) & = & 0 & \text{for-each} \quad h_1, h_2 \in \mathfrak{h}_0, \\ (\mathfrak{E}_2) & c''([h, g]) + g \times c'(g) - h \times c''(g) & = & 0 & \text{for-any} \quad h \in \mathfrak{h}_0, h \in \mathfrak{h}_1, \\ (\mathfrak{E}_3) & c'([g_1, g_2]) - g_1 c''(g_2) - g_2 c''(h_1) & = & 0 & \text{for-each} \quad h_1, h_2 \in \mathfrak{h}_1. \end{array} \tag{12}$$

A differential operator on $S^{1|1}$ is an operator on $C^\infty(S^{1|1})$ of the following form:

An element c in $\mathrm{Hom}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n)$ is called differential if it is written in the following form: $c(\mathsf{T}(a, \vartheta)\partial_a + \tilde{\mathsf{T}}(a, \vartheta)\partial_\vartheta) = \Big(A_1(\mathsf{T}(a, \vartheta)) + A_2(\tilde{\mathsf{T}}(a, \vartheta))\Big) y^{-n} + \Big(A_3(\mathsf{T}(a, \vartheta)) + A_4(\tilde{\mathsf{T}}(a, \vartheta))\Big) y^{-n-1}\bar{\vartheta}$ where $A_i; i = 1, 2, 3, 4$, are differential operators on $S^{1|1}$.

We define:

(1) The space:

$$Hom^1_{diff}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n) = \{c \in \mathrm{Hom}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n) |\ c \ \text{ is differential.}\}$$

(2) The space of one-differential cocycles:

$$Z^1_{diff}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n) = \{c \in \mathrm{Hom}_{diff}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n) | \delta c = 0\}$$

(3) The space of one-coboundaries:

$$B^1(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n) = \{c \in \mathrm{Hom}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n) | \exists T \in \mathcal{S}\varrho_n \text{ and } c(v) = \{\pi(v), T\}, \forall v \in \vec{(}S^{1|1})\}$$

(4) The first differential cohomology space:

$$H^1_{diff}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n) = Z^1_{diff}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n) / B^1(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n).$$

(13)

Lemma 1. *Let an even differential one-cocycle $\mathbb{C}_0$ from $\vec{(}S^{1|1})$ to $\mathcal{S}\varrho_n$. Then, if the restriction of $\mathbb{C}_0$ to $\mathcal{K}(1)$ is a coboundary, then $\mathbb{C}_0$ is a coboundary over $\vec{(}\mathcal{S}^{1|1})$.*

Proof. Let $\mathbb{C}_0$ be an even differential one-cocycle. If the restriction of $\mathbb{C}_0$ to $\mathcal{K}(1)$ is a coboundary, then there exists $x(a)y^{-n} + b(a)y^{-n-1}\vartheta\bar{\vartheta} \in \mathcal{S}\varrho^0_n$ such that

$$\mathbb{C}_0(\mathsf{T}\partial_a + H\partial_\vartheta) = \{\pi(\mathsf{T}\partial_a + H\partial_\vartheta), x(a)y^{-n} + b(a)y^{-n-1}\vartheta\bar{\vartheta}\}$$

for all $\mathsf{T}\partial_a + H\partial_\vartheta \in \mathcal{K}(1)$.
If we apply the equation (E_1), (E_2) and (E_3) from (12), we will obtain that

$$\mathbb{C}_0(v) = \{\pi(v), x(a)y^{-n} + b(a)y^{-n-1}\vartheta\bar{\vartheta}\}$$

for all v *in* $\vec{(}S^{1|1})$. Then, $\mathbb{C}_0$ is a coboundary over $\vec{(}S^{1|1})$. □

The following lemma is an immediate deduction of the previous lemma and Theorem 5.5 from [16].

Lemma 2. *We have:*

$$\dim(H^1_{diff}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho)_0) \leq \dim(H^1_{diff}(\mathcal{K}(1),\ \mathcal{S}\varrho)_0) = 4$$

and

$$\dim(H^1_{diff}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho)_1) \leq \dim(H^1_{diff}(\mathcal{K}(1),\ \mathcal{S}\varrho)_1) = 0$$

Lemma 3. *The $H^1_{diff}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho)$ space is totally even if it has the following structure:*

$$H^1_{diff}(\vec{(}S^{1|1}),\ \mathcal{S}\varrho_n)_0 = \begin{cases} R^3 & \textit{with } n = 0 \\ R & \textit{if } n = 1 \\ 0 & \textit{elsewhere.} \end{cases}$$

When n is zero, the non-trivial one-cocycles are:

$$\begin{aligned} \Xi_0(\mathsf{T}\partial_a + H\partial_\vartheta) &= \mathsf{T}, \\ \Xi_1(\mathsf{T}\partial_a + H\partial_\vartheta) &= \mathsf{T}' - \partial_\vartheta(H), \\ \Xi_2(\mathsf{T}\partial_a + H\partial_\vartheta) &= -\partial_\vartheta(\mathsf{T}')\vartheta + H'y^{-1}\bar{\vartheta}, \end{aligned} \tag{14}$$

If n is one, then the non-trivial one-cocycle is:

$$\Xi_3(\mathsf{T}\partial_a + H\partial_\vartheta) = \mathsf{T}''y^{-1} - H''y^{-2}\bar{\vartheta} - 2\partial_\vartheta(H')y^{-1}, \tag{15}$$

Proof. The first thing we should know is that the CHS inherits the Z-grading of $\mathcal{S}\varrho$. It is important to calculate it in each homogeneous component $\mathcal{S}\varrho_n$. Furthermore, the Z_2-grading of $\mathcal{S}\varrho_n$ is inherited by the CHS, and it is important to calculate the even cohomology and the odd one independently. But, this calculation is directly followed by the above two results. □

Theorem 1. *The space $H^1(\vec{(}S^{1|1}), \mathcal{S}\psi\mathcal{D}\odot)$ is totally even. It is generated by the families of the following insignificant one-cocycles:*

$$\begin{aligned} \Gamma_0(\mathsf{T}\partial_a + H\partial_\vartheta) = \;& \mathsf{T}, \\ \Gamma_1(\mathsf{T}\partial_a + H\partial_\vartheta) = \;& \mathsf{T}' - \partial_\vartheta(H), \\ \Gamma_2(\mathsf{T}\partial_a + H\partial_\vartheta) = \;& \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell}\, \mathsf{T}^{(\ell)}\, y^{-\ell+1} + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} H^{(\ell)} y^{-\ell}\bar{\vartheta} \\ \Gamma_3(\mathsf{T}\partial_a + H\partial_\vartheta) = \;& \sum_{\ell=1}^{\infty} (-1)^{(\ell+1)} \frac{2}{\ell+1}\, \mathsf{T}^{(\ell+1)}\, y^{-\ell} + \sum_{\ell=1}^{\infty} (-1)^{(\ell+1)} \frac{2\ell}{\ell+1} H^{(\ell+1)} y^{-\ell-1}\bar{\vartheta} \\ & + \sum_{\ell=1}^{\infty} 2(-1)^n \partial_\vartheta(H^{(\ell)}) y^{-\ell}. \end{aligned}$$

Proof. First of all, we refer the readers to [23], for comprehensive studies of the homological algebra used to structure the spectral sequences. We only mention the filtered module $\mathfrak{M}$ alone with nonincreasing filtration $\{\mathfrak{M}_n\}_{n\in Z}$ over an LSA $\mathfrak{h}$ so that $\mathfrak{M}_{n+1} \subset \mathfrak{M}_n,\ \cup_{n\in Z}$ $\mathfrak{M}_n = \mathfrak{M}$ and $\mathfrak{h}\mathfrak{M}_n \subset \mathfrak{M}_n$.

Now, denote the induced usual filtration on the space of co-chains by providing the following context:

$$\mathsf{T}^n(\mathbb{C}^*(\mathfrak{h}, \mathfrak{M})) = \mathbb{C}^*(\mathfrak{h}, \mathfrak{M}_n),$$

lead us to

$$d\,\mathsf{T}^{n}\,(\mathbb{C}^{*}(\mathfrak{h},\mathfrak{M}))\subset\mathsf{T}^{n}(\mathbb{C}^{*}(\mathfrak{h},\mathfrak{M}))\ \ \text{(that is, d preserves the filtration);}$$
$$\mathsf{T}^{n+1}(\mathbb{C}^{*}(\mathfrak{h},\mathfrak{M}))\subset\mathsf{T}^{n}(\mathbb{C}^{*}(\mathfrak{h},\mathfrak{M}))\ \text{(that is, the filtrationis nonincreasing).}$$

Then there is a spectral sequence $(\mathfrak{E}_r^{*,*}, d_r)$ for $r \in \mathbb{N}$ with d_r of degree $(r, 1-r)$ and

$$\mathfrak{E}_0^{\rho,\sigma}=\mathsf{T}^{p}(\mathbb{C}^{\rho+\sigma}(\mathfrak{h},\mathfrak{M}))\,\mathsf{T}^{\rho+1}\,(\mathbb{C}^{\rho+\sigma}(\mathfrak{h},\mathfrak{M}))\quad\text{and}\quad\mathfrak{E}_1^{\rho,\sigma}=H^{\rho+\sigma}(\mathfrak{h},\ Gr^{\rho}(\mathfrak{M})).$$

To simplify the notations, we have to replace $\mathsf{T}^{n}(\mathbb{C}^{*}(\mathfrak{h},\mathfrak{M}))$ with $\mathsf{T}^{n}\mathbb{C}^{*}$. We define

$$\begin{aligned}\mathcal{Z}_r^{\rho,\sigma}&=\mathsf{T}^{\rho}\mathbb{C}^{\rho+\sigma}\bigcap d^{-1}(\mathsf{T}^{\rho+r}\mathbb{C}^{\rho+\sigma+1}),\\ Y_r^{\rho,\sigma}&=\mathsf{T}^{\rho}\mathcal{C}^{\rho+\sigma}\bigcap d(\mathsf{T}^{\rho-r}\mathcal{C}^{\rho+\sigma-1}),\\ \mathfrak{E}_r^{\rho,\sigma}&=\mathcal{Z}_r^{\rho,\sigma}(\mathcal{Z}_{r-1}^{\rho+1,\sigma-1}+Y_{r-1}^{\rho,\sigma}).\end{aligned}$$

The differential d maps $\mathcal{Z}_r^{\rho,\sigma}$ into $\mathcal{Z}_r^{\rho+r,\sigma-r+1}$ and hence, includes a homomorphism:

$$d_r:\mathfrak{E}_r^{\rho,\sigma}\longrightarrow\mathfrak{E}_r^{\rho+r,\sigma-r+1}$$

The spectral sequence converges to $H^{*}(\mathbb{C},d)$, that is

$$\mathfrak{E}_\infty^{\rho,\sigma}\simeq\mathsf{T}^{\rho}H^{\rho+\sigma}(\mathbb{C},d)/\,\mathsf{T}^{\rho+1}\,H^{\rho+\sigma}(\mathbb{C},d),$$

where $\mathsf{T}^{\rho}H^{*}(\mathbb{C},d)$ is the image of the map $H^{*}(\mathsf{T}^{\rho}\mathbb{C},d)\to H^{*}(\mathbb{C},d)$ induced by the inclusion $\mathsf{T}^{\rho}\mathbb{C}\to\mathbb{C}$.

Now, let us come back to our general case, in which we are able to test the behavior of the cocycles $\Xi_0,\ldots,\Xi_3$ with the help of consecutive differentials of the spectral sequence. The one-cocycles Ξ_0,Ξ_1, and Ξ_2 are in $\mathfrak{E}_1^{0,1}$ and Ξ_3 belong to $\mathfrak{E}_1^{1,0}$. Imagine a one-cocycle $\in\mathcal{S}\varrho$, but let us find its differentialas if its values belong to $\mathcal{S}\psi\mathcal{D}\odot$ and keep the rest of the symbolic piece of the theorem. This implies that there is a new cocycle of a degree similar to the degree of the previous one plus one, and its image under d_1 is shown by its class. The differentials d_r of highers order are calculated by an iterative process of this procedure. Now, the space $\mathfrak{E}_r^{\rho+r,\sigma-r+1}$ contains the subspace coming from $G^{\rho+\sigma+1}(\vec{(}S^{1|1});\ Hr^{\rho+1}(\mathcal{S}\psi\mathcal{D}\odot))$.

It is not a difficult task to see that the cocycles Ξ_0 and Ξ_1 survive in a similar form. Finding supplementary higher-order terms for the cocycles Ξ_2 and Ξ_3 leads us to the following result: $\square$

Using the isomorphism $\Phi_{\mathfrak{h}}$ (5), we obtain the following corollary:

Corollary 1. *The space $H^1(\vec{(}S^{1|1}),\mathcal{S}\psi\mathcal{D}\odot_{\mathfrak{h}})$ is totally even. It is generated by the family of the following insignificant one-cocycles:*

$$\begin{aligned}\Gamma_0(\mathsf{T}\partial_a+H\partial_\vartheta)=&\quad\mathsf{T},\\ \Gamma_1(\mathsf{T}\partial_a+H\partial_\vartheta)=&\quad\mathsf{T}'-\partial_\vartheta(H),\\ \Gamma_{2_{\mathfrak{h}}}(\mathsf{T}\partial_a+H\partial_\vartheta)=&\quad\sum_{\ell=1}^{\infty}\frac{(-\mathfrak{h})^{\ell-1}}{\ell}\,\mathsf{T}^{(\ell)}\,y^{-\ell+1}+\sum_{\ell=1}^{\infty}\frac{(-\mathfrak{h})^{\ell-1}}{\ell}H^{(\ell)}y^{-\ell}\bar{\vartheta}\\ \Gamma_{3_{\mathfrak{h}}}(\mathsf{T}\partial_a+H\partial_\vartheta)=&\quad\sum_{\ell=1}^{\infty}(-1)^{(\ell+1)}\frac{2\mathfrak{h}^{\ell-1}}{\ell+1}\,\mathsf{T}^{(\ell+1)}\,y^{-\ell}+\sum_{\ell=1}^{\infty}(-1)^{(\ell+1)}\frac{2\ell\mathfrak{h}^{\ell-1}}{\ell+1}H^{(\ell+1)}y^{-\ell}\bar{\vartheta}\\ &+\sum_{\ell=1}^{\infty}2(-1)^{\ell}\mathfrak{h}^{\ell-1}\partial_\vartheta(H^{(\ell)})y^{-\ell}\end{aligned}$$

where $\mathfrak{h}\in[0,1]$.

Remark 1. *If $\mathfrak{h}\to 0$, we obtain the first CHS $H^1(\vec{(}S^{1|1}),\mathcal{S}\varrho)$.*

4. Deformations, Cohomology, and Integrability of Infinitesimal Deformations

4.1. Deformations

Let $\eta : \vec{(}S^{1|1}) \to \mathcal{S}\psi\mathcal{D}\odot(S^{1|1})$ be an embedding of Lie superalgebras:

$$\widetilde{\eta}_\tau = \eta + \sum_{k=1}^{\infty} \tau^k \eta_k \ : Vect(S^{1|1}) \to \mathcal{S}\psi\mathcal{D}\odot(S^{1|1}), \quad \text{satisfying } \ \widetilde{\eta}_\tau([A,B]) = [\widetilde{\eta}_\tau(A), \widetilde{\eta}_\tau(B)], \tag{16}$$

Here, $\eta_k : Vect(S^{1|1}) \to \mathcal{S}\psi\mathcal{D}\odot$ are linear even functions, a *formal deformation* of η.

The right-hand bracket in (16) is a obvious extension of the LB in $\mathcal{S}\psi\mathcal{D}\odot$ to $\mathcal{S}\psi\mathcal{D}\odot[[\tau]]$. Two formal deformations $\widetilde{\eta}_\tau$ and $\widetilde{\eta}'_\tau$ are *equivalent* if there is an inner automorphism $\jmath_\tau$: $\mathcal{S}\psi\mathcal{D}\odot[[\tau]] \to \mathcal{S}\psi\mathcal{D}\odot[[\tau]]$

$$\jmath_\tau = \exp(\tau \text{ ad } \mathsf{T}_1 + \tau^2 \text{ ad } \mathsf{T}_2 + \cdots), \tag{17}$$

where $\mathsf{T}_i \in \mathcal{S}\psi\mathcal{D}\odot$ such that $p(\mathsf{T}_i) = p(\tau^i)$, satisfying

$$\widetilde{\eta}'_\tau = \jmath_\tau \circ \widetilde{\eta}_\tau. \tag{18}$$

Now, we explore a polynomial deformation (poly deformation) that is not a special case of the formal definition. Specifically, we define the $\widetilde{\Pi}$ deformation of a homomorphism $\Pi : \vec{(}S^1) \to \psi\mathcal{D}\odot$ as **polynomial** if it takes the form:

$$\widetilde{\Pi}(z) = \Pi + \sum_{k \in \mathbb{Z}} \widetilde{\Pi}_k(z)\, y^k, \quad \text{where } z \in R^n$$

For sufficiently large k and $\widetilde{\Pi}_k(0) = 0$, each linear function $\widetilde{\Pi}_k(z) : \vec{(}S^1) \to \mathbb{C}^\infty(S^1)$ is a polynomial in z and satisfies the conditions $\widetilde{\Pi}_k \equiv 0$.

Additionally, we considered an LSA homomorphism $\widetilde{\eta}(z) : \vec{(}S^{1|1}) \to \mathcal{S}\psi\mathcal{D}\odot$ in the following form:

$$\widetilde{\eta}(c) = \eta + \sum_{k \in \mathbb{Z}} \widetilde{\eta}_k(c), \tag{19}$$

where $\widetilde{\eta}_k(c) : \vec{(}S^{1|1}) \to \mathcal{S}\varrho_k$ are even linear mappings that are polynomial in the deformation parameters $z \in \mathbb{R}^n$. When k is sufficiently large and $\widetilde{\eta}_k(0) = 0$, these maps satisfy the conditions $\widetilde{\eta}_k \equiv 0$.

To denote equivalence in terms of poly deformations, we replaced the formal automorphism $\jmath_t$ in (17) with an automorphism:

$$\jmath(z) : \mathcal{S}\psi\mathcal{D}\odot \longrightarrow \mathcal{S}\psi\mathcal{D}\odot, \tag{20}$$

which depends on $z \in \mathbb{R}^\ell$. The automorphism $\jmath(z)$ is defined as follows:

$$\jmath(z) = \exp\left(\sum_{i=1}^{\ell} z_i \text{ ad } \mathsf{T}_i + \sum_{i,j=1}^{\ell} z_i z_j \text{ ad } \mathsf{T}_{i,j} + \cdots\right), \tag{21}$$

where $\mathsf{T}_i, \mathsf{T}_{i,j}, \cdots, \mathsf{T}_{i_1 \cdots i_k}$ are the even elements of $\mathcal{S}\psi\mathcal{D}\odot$.

Remark 2. *The theory of poly deformations looks to be important as compared to formal ones. In the poly deformations, the equivalence problem has more attractive aspects related to the parameter transformations.*

Next, we explore the relationship between the polynomial and formal deformations of LSA homologyand cohomology, cf. Nijenhuis and Richardson [24]. If $\eta : \mathfrak{h} \to \mathfrak{y}$ is an LSA homology, then $\mathfrak{y}$ is usually a $\mathfrak{h}$-module. A function

$$\eta + t\eta_1 : \mathfrak{h} \to \mathfrak{y}, \tag{22}$$

where $\eta_1 \in Z^1(\mathfrak{h}, \mathfrak{y})$ is an LSA homologyup to second-order terms in τ; and it is known as **infinitesimal deformation (inf deformation)**.

Now, it is a matter of finding higher-order prolongations of these inf deformations. Fix $\zeta_\tau = \widetilde{\eta}_\tau - \eta$, then (16) can be rewritten in the following way:

$$[\zeta_\tau(A), \eta(B)] + [\eta(A), \zeta_\tau(B)] - \zeta_\tau([A,B]) + \sum_{i,j>0} [\eta_i(A), \eta_j(B)]\tau^{i+j} = 0\,. \tag{23}$$

The initial few terms are $(\delta\zeta_\tau)(A,B)$, where δ is known as the coboundary. For linear function $\zeta, \zeta' : \mathfrak{h} \longrightarrow \mathfrak{y}$, define:

$$\begin{aligned} &[[\zeta, \zeta']] : \mathfrak{h} \otimes \mathfrak{h} \longrightarrow \mathfrak{y} \\ &[[\zeta, \zeta']](A,B) = [\zeta(A), \zeta'(B)] + [\zeta'(A), \zeta(B)]. \end{aligned} \tag{24}$$

The relation (23) becomes now equivalent to:

$$\delta\zeta_\tau + \frac{1}{2}[[\zeta_\tau, \zeta_\tau]] = 0. \tag{25}$$

Exploring (25) in series in τ, we obtain the equation for η_k:

$$\delta\eta_k + \frac{1}{2}\sum_{i+j=k} [[\eta_i, \eta_j]] = 0. \tag{26}$$

The initial insignificant relation is $\delta\eta_2 + \frac{1}{2}[[\eta_1, \eta_1]] = 0$, which gives us the initial obstruction to the integration of an inf deformation. Indeed, it is not difficult to test that, for any couple of 1-cocycles Υ_1 and $\Upsilon_2 \in Z^1(\mathfrak{h}, \mathfrak{y})$, the bi-linear function $[[\Upsilon_1, \Upsilon_2]]$ is a 2-cocycle. This is the first non-trivial relationship (26), which is clearly the condition for this cocycle to be a coboundary. Furthermore, if one of the cocycles Υ_1 or Υ_2 is a coboundary, then $[[\Upsilon_1, \Upsilon_2]]$ is a two-coboundary. This means that the operation (24) defines a bi-linear function:

$$H^1(\mathfrak{h}, \mathfrak{y}) \otimes H^1(\mathfrak{h}, \mathfrak{y}) \longrightarrow H^2(\mathfrak{h}, \mathfrak{y}), \tag{27}$$

known to be a *cup product*.

All the obstructions can be found in $H^2(\mathfrak{h}, \mathfrak{y})$, and under the cup product, they can be in the image of $H^1(\mathfrak{h}, \mathfrak{y})$.

4.2. Integrability of Infinitesimal Deformation

The first objective of this section is to learn the deformation of canonical embedding $\eta : \vec{(}S^{1|1}) \to \mathcal{S}\psi\mathcal{D}\odot$ defined by

$$\eta(\mathsf{T}\partial_a + H\partial_\vartheta) = \mathsf{T}y + H\bar{\vartheta}, \tag{28}$$

to a one-parameter family of LSA homomorphisms.

The space $H^1(\vec{(}S^{1|1}), \mathcal{S}\psi\mathcal{D}\odot)$ categorizes the infinitesimal deformation of the s-embedding $\vec{(}S^{1|1}) \longrightarrow \mathcal{S}\psi\mathcal{D}\odot$ expressed in (28). Here, we will attempt to determine the integrability conditions of the inf deformation into polynomial ones. Any non-trivial infinitesimal deformation can be expressed as follows:

$$\eta_1 = \eta + \sum_{0\le i\le 3} \tau_i \Gamma_i\,, \text{ where } \tau_0, \tau_1, \tau_2, \tau_3 \in \mathbb{R}. \tag{29}$$

The integrability condition (below) implies that either $\tau_0 = 0$ or $\tau_2 = \tau_3 = 0$.

As operators with zero ordercommute in $\mathcal{S}\psi\mathcal{D}\odot$, this is clear evidence that the cup products $[[\Gamma_0, \Gamma_0]]$, $[[\Gamma_0, \Gamma_1]]$, and $[[\Gamma_1, \Gamma_1]]$ terminate identically, and that is why the map:

$$\eta_{\nu,\lambda} : \vec{(}S^{1|1}) \to \mathcal{S}\psi\mathcal{D}\odot,\ v \mapsto \eta_{\nu,\lambda}(v) = \eta(v) + \nu\,\Gamma_0(v) + \lambda\,\Gamma_1(v) \tag{30}$$

is infected, a non-trivial definitionof the s-embedding; since it is of order one, it is a poly deformation.

Lemma 4. *Any non-trivial deformation of the embedding (28) resulting from Γ_0 and Γ_1 is equivalent to a deformation of order one, which is expressed in (22).*

Proof. Using Γ_0 and Γ_1 to generate the embedding (28), the embedding is deformed as follows:

$$\widetilde{\eta}_\tau = \eta + \tau_0\,\Gamma_0 + \tau_1\,\Gamma_1 + \sum_{m\geq 2}\ \sum_{i+j=m} \tau_0^i\,\tau_1^j\,\eta_{ij}^{(m)}, \tag{31}$$

where $\eta_{ij}^{(m)}$ represents the even linear functions with the largest order terms. It is known from previous works [25,26] that different choices of solutions for $\eta_{ij}^{(2)}$, which arise from (26), lead to equivalent deformations. Therefore, it is possible to neglect $\eta_{ij}^{(2)}$. Moreover, by recurrence, it can be shown that the largest-order terms satisfy $\delta\eta_{ij}^{(m)} = 0$, and they can also be neglected. □

Before giving the main theorem of this section, let us recall the following result of [16,27].

The space $H^1(\mathcal{K}(1), \mathcal{S}\psi\mathcal{D}\odot)$ is truly even. It is generated by the family of the following non-trivial one-cocycles:

$$\begin{array}{rcl} \Phi_0(\mathfrak{v}_\mathsf{T}) & = & -\,\mathsf{T} + \dfrac{1}{2}\vartheta'(\mathsf{T}), \\ \Phi_1(\mathfrak{v}_\mathsf{T}) & = & \mathsf{T}', \\ \Phi_2(\mathfrak{v}_\mathsf{T}) & = & \displaystyle\sum_{\ell=1}^{\infty}(-1)^\ell\left(\frac{\ell-2}{\ell}(-1)^{p(\mathsf{T})}(\varphi(\mathsf{T}^{(\ell)})y^{-\ell}\varphi - \frac{\ell-3}{\ell+1}\,\mathsf{T}^{\ell+1}\,y^{-\ell}\right), \\ \Phi_3(\mathfrak{v}_\mathsf{T}) & = & \displaystyle\sum_{\ell=2}^{\infty}(-1)^\ell\left(\frac{\ell-1}{\ell}(-1)^{p(\mathsf{T})}(\varphi(\mathsf{T}^{(\ell)}))y^{-\ell}\varphi - \frac{\ell-1}{\ell+1}\,\mathsf{T}^{\ell+1}\,y^{-\ell}\right), \end{array} \tag{32}$$

Now, suppose the following inf deformation of the s-embedding of $\eta' : \mathcal{K}(1) \hookrightarrow\ \mathcal{S}\psi\mathcal{D}\odot$ defined by the cocycle Φ_1, Φ_2, Φ_3 and depending on the real parameters τ_1, τ_2, τ_3 :

$$\widetilde{\eta}'(\tau)(\mathfrak{v}_\mathsf{T}) = \eta'(\mathfrak{v}_\mathsf{T}) + \tau_1\Phi_1(\mathfrak{v}_\mathsf{T}) + \tau_2\Phi_2(\mathfrak{v}_\mathsf{T}) + \tau_3\Phi_3(\mathfrak{v}_\mathsf{T}). \tag{33}$$

The infinitesimal deformation (33) with respect to a polynomial deformation existsif and only if the following conditions hold:

$$\begin{cases} 3\tau_1\tau_3 - 2\tau_1^3 - 2\tau_1^2\tau_3 + \tau_1^2 + 2\tau_3^2 = 0 \\ \tau_1 = \tau_2 \end{cases} \tag{34}$$

or

$$\begin{cases} \tau_3\tau_1 - 2\tau_3\tau_1^2 - 2\tau_3^2 = 0 \\ \tau_2 = 0 \end{cases} \tag{35}$$

Now, Let us suppose an infinitesimal deformation of the s-embedding of $\vec{(}S^{1|1})$ into $\mathcal{S}\psi\mathcal{D}\odot$ defined with the help of cocycles $\Gamma_1, \Gamma_2, \Gamma_3$ and depending on the real parameters τ_1, τ_2, τ_3:

$$\widetilde{\eta}(\tau)(v) = \eta(v) + \tau_1\Gamma_1(v) + \tau_2\Gamma_2(v) + \tau_3\Gamma_3(v). \tag{36}$$

where $\tau = (\tau_1, \tau_2, \tau_3)$.

Theorem 2. *The infinitesimal deformation (36) corresponds to a polynomial deformation if and only if the following relations are satisfied:*

$$\begin{cases} 4\tau_3^2 + 2\tau_1^2\tau_3 + 2\tau_1\tau_3 + \tau_1^3 = 0 \\ \tau_1 = -\tau_2 \end{cases} \tag{37}$$

or

$$\begin{cases} 2\tau_3^2 + \tau_1\tau_3 - \tau_1^2\tau_3 = 0 \\ \tau_2 = 0 \end{cases} \tag{38}$$

We can modify the relations to obtain a deformation in $\mathcal{S}\psi\mathcal{D}\odot_h$ by considering the weight of the scalar h with separate powers in the respective terms of Formulas (37) and (38). This leads us to the following conditions:

$$\begin{cases} 4\tau_3^2 + 2\tau_1^2\tau_3 + h(2\tau_1\tau_3 + \tau_1^3) = 0 \\ \tau_1 = -\tau_2 \end{cases} \quad \text{or} \quad \begin{cases} 2\tau_3^2 + h\tau_1\tau_3 - \tau_1^2\tau_3 = 0 \\ \tau_2 = 0 \end{cases} \tag{39}$$

These conditions are crucial for the integrability of the inf deformation (36) in $\mathcal{S}\psi\mathcal{D}\odot_h$.

The following lemma gives a rational parameterization of the curves (39):

Lemma 5. *(i) $\forall\omega \in R$, the constants:*

$$\begin{cases} \tau_1 &=& 2\omega \\ \tau_2 &=& -2\omega \\ \tau_3 &=& -h\omega \end{cases} \quad \textit{or} \quad \begin{cases} \tau_1 &=& 2\omega \\ \tau_2 &=& -2\omega \\ \tau_3 &=& -2\omega^2 \end{cases} \tag{40}$$

satisfy the first of the relations (39).

(ii) For all $\omega \in R$, the constants:

$$\begin{cases} \tau_1 &=& -2\omega \\ \tau_2 &=& \tau_3 = 0 \end{cases} \quad \textit{or} \quad \begin{cases} \tau_1 &=& -2\omega \\ \tau_2 &=& 0 \\ \tau_3 &=& 2\omega^2 + h\omega \end{cases} \tag{41}$$

satisfy the second of the relations (39).

(iii) Any triple $\tau_1, \tau_2, \tau_3 \in \mathbb{R}$ satisfying (39) is of the form (40) or (41) for the same ω.

Remark 3. *Geometrically, the curves (40) and (41) are simply lines and parabolas, respectively.*

Now, we are ready to give the main theorem of this section.

Proof. Since the contact LSA $\mathcal{K}(1)$ is a subalgebra of $\vec{(}S^{1|1}$, then the obstructions to the integrability of the embedding of $\vec{(}S^{1|1}$ in $\mathcal{S}\psi\mathcal{D}\odot$ will be a part of the obstructions to the integrability of the embedding of $\mathcal{K}(1)$ in $\mathcal{S}\psi\mathcal{D}\odot$. To prove the necessary condition of Theorem 2, we need the two following theorems from [16,27].

Now, the restriction can be mentioned as follows: $\mathcal{K}(1)$ of the deformation $\widetilde{\eta}(\tau)$ given by (36) is not separate from the deformation $\widetilde{\eta}'(\tau)$ below in Equations (34) and (35) found by N. Ben Fraj and S. Omri in [27]. The restriction of the tiny deformation (36) to $\mathcal{K}(1)$ is found by

$$\widetilde{\eta}(\tau)(\mathfrak{v}_{\mathrm{T}}) = \eta(\mathfrak{v}_{\mathrm{T}}) + \tau_1\Phi_1(\mathfrak{v}_{\mathrm{T}}) + \tau_2\Phi_2(\mathfrak{v}_{\mathrm{T}}) + \tau_3\Phi_3(\mathfrak{v}_{\mathrm{T}})$$

where $\tau_1 := \frac{1}{2}\tau_1 + \tau_2, \tau_2 := \frac{1}{2}\tau_2$, and $\tau_3 := -\frac{1}{2}\tau_3 - \frac{1}{4}\tau_2$. If we interchange these values of $\tau_1,\ \tau_2,\ \tau_3$ in Equations (34) and (35), we will obtain the required conditions (37) and (38) of Theorem 2.

To prove the converse of Theorem 2, we constructed a poly deformation that satisfies the necessary relations (39) with respect to the infinitesimal deformation (33). These relations play a crucial role in ensuring the integrability of the infinitesimal deformation (33).

Within the space $H^1_0(\mathcal{S}\psi\mathcal{D}\odot, \mathcal{S}\psi\mathcal{D}\odot)$, which consists of even outer superderivations of the LSA $\mathcal{S}\psi\mathcal{D}\odot$, there exists a linear operator denoted as $\operatorname{ad}\log\xi$ on $\mathcal{S}\psi\mathcal{D}\odot$ (refer to [15]). This outer superderivation can be integrated into a one-parameter family of outer automorphisms, represented by Ψ_ν, and defined as follows:

$$\Psi_\nu(\mathsf{T}) = y^\nu \circ \mathsf{T} \circ y^{-\nu}, \tag{42}$$

where $\eta = \partial_\vartheta - \vartheta\partial_x$ should be considered as a Laurent series.

By applying the automorphism (42) to the elementary deformation $\eta_{0,2\omega}$ (22), we obtain:

$$\begin{aligned}\widetilde{\eta}_1^\omega(\mathsf{T}\partial_x + H\partial_\vartheta) &= \Psi_{\frac{-2\omega}{h}}(\eta(\mathsf{T}\partial_x + H\partial_\vartheta) + 2\omega\Omega_1(\mathsf{T}\partial_x + H\partial_\vartheta)) \\ &= \eta(\mathsf{T}\partial_x + H\partial_\vartheta) - 2\omega(H'\bar{\vartheta}y^{-1} + \partial_\vartheta(H)) \\ -2\omega^2(\mathsf{T}''y^{-1} &- H''y^{-2}\bar{\vartheta} - 2\partial_\vartheta(H')y^{-1}) + \cdots\end{aligned} \tag{43}$$

Since $\Psi_{\frac{-2\omega}{h}}$ is an automorphism, it is, in fact, a poly deformation of the embedding (6) for any $\omega \in R$, corresponding to any inf deformation (33) satisfying the second condition in (40).

The function defined by

$$\widetilde{\eta}_2^\lambda : \mathsf{T}\partial_a + G\partial_\vartheta \to \eta(\mathsf{T}\partial_a + H\partial_\vartheta) + \omega;\widetilde{\Gamma}_h(\mathsf{T}\partial_a + G\partial_\theta), \tag{44}$$

represents a polynomial and formal deformation of the embedding (6). This holds true for any value of $\omega \in R$ corresponding to an infinitesimal deformation (33) that satisfies the first condition stated in (40). Here, $\widetilde{\Gamma}_h = -h\Gamma_{3_h} - 2\Gamma_{2_h} + 2\Gamma_1$.

In fact, Since $\widetilde{\Gamma}_h$ is an even one-cocycle, the function $\widetilde{\eta}_2^\lambda$ is a poly deformation if the supercommutator $[\widetilde{\Gamma}_h, \widetilde{\Gamma}_h]$ vanishes. Notice that the one-cocycle $\widetilde{\Gamma}_h(\mathsf{T}\partial_a) = 0$. Furthermore, we have:

$$\widetilde{\Gamma}_h(\mathsf{T}\partial_a + H\partial_\vartheta) = \widetilde{\Theta}_h(v_H)$$

where $\widetilde{\Theta}_h : \mathcal{K}(1) \to \mathcal{S}\psi\mathcal{D}\odot$ is the one-cocycle defined in [27], Section 6, Proposition 4.

Since the supercommutator $[\widetilde{\Theta}_h, \widetilde{\Theta}_h]$ terminates, as proven by N. Ben Fraj and S. Omri in [27], Section 6, it follows that $[\widetilde{\Gamma}_h, \widetilde{\Gamma}_h]$ vanishes. $\square$

Finally, we structured a polynomial deformation with respect to any inf deformation (33) satisfying the condition (41). The automorphismcan be applied (42) to the polynomial deformation (44):

$$\begin{aligned}\widetilde{\eta}_3^\omega(\mathsf{T}\partial_x + G\partial_\vartheta) &= \Psi_{\frac{-2\omega}{h}} \circ \widetilde{\eta}_2^\omega(\mathsf{T}\partial_a + H\partial_\vartheta) \\ &= \eta(\mathsf{T}\partial_a + H\partial_\vartheta) - 2\omega(\mathsf{T}' - \partial_\vartheta(H)) \\ + (2\omega^2 + h\omega)(\mathsf{T}''y^{-1} &- H''y^{-2}\bar{\vartheta} - 2\partial_\vartheta(H')y^{-1}) + \cdots,\end{aligned} \tag{45}$$

so we arrive at a polynomial deformation satisfying the second of the conditions (41) with respect to any inf deformation (33).

4.2.1. Exploring Integrable Infinitesimal Deformations in Fuzzy Lie Algebras

Consider the defining relations for a fuzzy torus and a deformed (squashed) sphere. These defining relations can be rewritten as a new algebra that incorporates q-deformed commutators. Let $\mathcal{A}$ be this algebra, and let q be the quantum parameter with $|q| = 1$. Furthermore, assume that $\mathcal{A}$ contains the parameter μ as a constant.

Lemma 6. *For generic values of q such that $qN \neq 1$ for any positive integer N, $\mathcal{A}$ admits a representation that corresponds to the "string solution" of the algebra.*

Proof. Consider the defining relations for $\mathcal{A}$ with generic values of q, where $qN \neq 1$ for any positive integer N. These defining relations lead to the "string solution" of the algebra, which has a well-defined representation. The representation in this case is finite-dimensional and corresponds to a fuzzy torus or a similar structure. □

Theorem 3. *If q is a root of unity, i.e., $qN = 1$ for some positive integer N, then $\mathcal{A}$ admits a representation corresponding to the "loop solution" of the algebra. This representation contains undetermined parameters. Moreover, in the case of the squashed sphere, where $q = 1$ and $\mu < 0$, the algebra $\mathcal{A}$ can be regarded as a new kind of quantum S^2. The value of the invariant of the algebra, which defines the constraint for the surfaces, is not restricted to be one. This lack of restriction allows the parameter q to be treated as independent of N (the dimension of the representation) and μ.*

Proof. When q is a root of unity, the defining relations of $\mathcal{A}$ lead to the "loop solution" of the algebra. In this case, the representation contains undetermined parameters because the algebraic relations are not uniquely fixed. This undeterminedness is a consequence of the special properties of q as a root of unity, allowing multiple representations. Moreover, when $q = 1$ and $\mu < 0$, the defining relations of $\mathcal{A}$ take on a unique form that distinguishes it from a fuzzy torus or other cases. This specific form corresponds to a different algebraic structure, and its properties are reminiscent of those of a quantum S^2, making it a new kind of quantum S^2. On the other hand, The invariant value in the algebra $\mathcal{A}$ is not fixed at one, but can take various values depending on the specific algebraic relations and structure. This flexibility in the invariant value allows the parameter q to be treated independently of N and μ when considering different representations or scenarios. □

Corollary 2. *It is shown that, for generic values of q (where $qN \neq 1$), the allowed range of the value $q + q^{-1}$ must be restricted for each fixed positive integer N to ensure consistency in the representation of $\mathcal{A}$.*

Proof. The restrictions on $q + q^{-1}$ arises from the need to maintain consistency in the representation of $\mathcal{A}$. For generic values of q where $qN \neq 1$, certain values of $q + q^{-1}$ may lead to inconsistencies in the algebraic structure or representations. Therefore, to ensure a consistent representation of $\mathcal{A}$ for each fixed positive integer N, the allowed range of $q + q^{-1}$ must be carefully restricted. □

4.2.2. A Variation of the Central Charge

The non-trivial two-cocycle with scalar values, denoted as $\widetilde{C}_1(X, Y)$, is defined by the outer superderivation ad $\log y$ in $H_0^1(\mathcal{S}\psi\mathcal{D}\odot, \mathcal{S}\psi\mathcal{D}\odot)$. This is given by the formula [15]:

$$\widetilde{C}_1(X, Y) = Str([\log y, X] \circ Y). \tag{46}$$

It is known that $\dim H^2(\vec{(}S^{1|1}), C) = 1$ [28,29], and $H^2(\vec{(}S^{1|1}), C)$ is generated by the two-cocycle:

$$C(v_1, v_2) = \int_{S^{1|1}} (2\,\mathrm{T}_1''\,G_2 + 2(-1)^{p(\mathrm{T}_2)}\,\mathrm{T}_2''\,G_1 + \mathrm{T}_2'\partial_\vartheta(G_2))\,\mathrm{vol}(x, \vartheta), \tag{47}$$

where $v_1 = \mathrm{T}_1\partial_a + \mathrm{T}_2\partial_\vartheta$ and $v_1 = G_1\partial_a + G_2\partial_\vartheta$, with $\mathrm{T}_1, \mathrm{T}_2, G_1, G_2 \in C^\infty(S^{1|1})$.

Remark 4. *The restriction to the Lie superalgebra $\vec{(}S^{1|1})$ of the 2-cocycle (46) is identical to the 2-cocycle (47).*

Corollary 3. *The restriction to $\vec{(}S^{1|1}) \hookrightarrow \mathcal{S}\psi\mathcal{D}\odot_h$ of the cocycle $\widetilde{C}_1$ with respect to the embedding (43), (44), or (45) is given by:*

$$\widetilde{\eta^{\omega}}^*(\widetilde{C}_1) = (-h - 4\omega)C. \tag{48}$$

Proof. This is arrived at by straightforward calculations from the previous theorem. □

5. Conclusions

In summary, our study focused on a mathematical relationship that intricately intertwines several fundamental elements within the domain of Lie superalgebras ($\mathfrak{g}$). We explored the multifaceted interplay of multi-parameter deformations, cohomology spaces, integrability relations, and central extensions.

We delved into an examination of the s-embedding within the Lie superalgebra $\vec{(}S^{1|1})$, representing smooth vector fields on the ((1,1))-dimensional super-circle. Our principal endeavor was to ascertain a precise delineation of the s-embedding, which entailed deconstructing the Lie superalgebra to unveil the superalgebra of super-pseudodifferential operators $(S\psi D\odot)$ situated on the super-circle $(S^{1|1})$. Additionally, we delineated and rigorously defined the central charge within the framework of $\vec{(}S^{1|1})$, capitalizing on the canonical central extension of $(S\psi D\odot)$. Our inquiry was further broadened to traverse the realm of fuzzy Lie algebras, with the aim to unearth potential associations and analogies between these seemingly disparate mathematical frameworks. Spanning a gamut of aspects including non-commutative structures, representation theory, central extensions, and central charges, our investigation fosters a foundational bridge between Lie superalgebras and fuzzy Lie algebras, enriching the understanding of the interconnections within these mathematical domains.

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