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On Behavior of Solutions for Nonlinear Klein–Gordon Wave Type Models with a Logarithmic Nonlinearity and Multiple Time-Varying Delays

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Abstract: In this paper, we study the existence and exponential stability of solutions to a class of nonlinear delay Klein–Gordon wave type models on a bounded domain. Such models include multiple time-varying delays, frictional damping, and nonlinear logarithmic source terms. After showing the local existence result of the solutions using Faedo–Galerkin's method and logarithmic Sobolev inequality, the global existence is analyzed. Then, under some appropriate conditions, energy decay estimates and exponential stability results of the global solutions are investigated.

Keywords: nonlinear Klein–Gordon equation; multiple time-varying delays; nonlocal equation; logarithmic source term; asymptotic behavior; energy decay

MSC: 35A01; 35B40; 35L05; 34K20; 35Q40

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1. Introduction and Mathematical Setting of the Problem

In this paper, we consider the following nonlocal initial-boundary value problem for a class of nonlinear Klein–Gordon wave type equations with frictional damping, logarithmic nonlinearity, and multiple time-varying delays in velocity:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \mathrm{div}(\mathcal{K}(\mathbf{x})\nabla u) + d_0(t)\frac{\partial u}{\partial t} + \Psi(\mathbf{x}, t; \frac{\partial u}{\partial t}) \\
&\qquad\qquad + \mathcal{M}(\| u \|^2_{L^2(\Omega)})u = u \ln | u |^\theta, \ \mathbf{x} \in \Omega, \ t > 0, \\
&\frac{\partial u}{\partial t}(\mathbf{x}, t') = h_0(\mathbf{x}, t'), \ (\mathbf{x}, t') \in \mathcal{Q}_0 = \Omega \times [-\delta(0), 0), \\
&u(\mathbf{x}, 0) = \phi_0(\mathbf{x}), \ \ \frac{\partial u}{\partial t}(\mathbf{x}, 0) = \phi_1(\mathbf{x}), \ \ \mathbf{x} \in \Omega, \\
&u(\mathbf{x}, t) = 0, \ \mathbf{x} \in \Gamma, \ t > 0,
\end{aligned} \tag{1}$$

where $u(\mathbf{x}, t) \in \mathbb{R}$ is the variable state, the operator $\mathcal{M}$ is a continuous function on $[0, +\infty)$, the domain Ω is a boundary subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\Gamma = \partial\Omega$, and the term $\mathcal{K}$ is the conductivity tensor function on $\overline{\Omega}$, with $\overline{\Omega} = \Omega \cup \partial\Omega$ the closure of Ω. The functions ϕ_0, ϕ_1, and h_0 are the initial/initial history data to be specified later. The operator Ψ, which describes multiple time-varying delays related to velocity $\frac{\partial u}{\partial t}$, is defined as in [1]

$$\Psi(\mathbf{x}, t; \frac{\partial u}{\partial t}) = \sum_{i=1}^{n} d_i(t)\frac{\partial u}{\partial t}(\mathbf{x}, t - e_i(t)), \tag{2}$$

where the non constant weights $(d_i, i = 1, n) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ and $d_0 : \mathbb{R}^+ \longrightarrow]0, +\infty[$ are bounded and sufficiently regular functions, $(e_i, i = 1, n) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are sufficiently regular functions representing multiple time-varying delays, $\delta(0) = \max_{i=1,n}(e_i(0))$, and the parameter θ (which measures the force of nonlinear interactions) is a real number that will be specified later.

1.1. Motivation and Outline of the Paper

A strong motivation for analyzing the behavior of the nonlinear problem with multiple time-varying delays (1) comes from its application in many branches of physics and other applied sciences.

The nonlinear Klein–Gordon equation, also known as the relativistic version of the Schrodinger equation, is an important class of partial differential equations and performs a significant role in mathematical physics and many other scientific applications such as relativistic quantum mechanics. They occur in various areas of physical sciences and engineering such as solid-state physics, nonlinear optics, quantum field theory, fluid dynamics, mathematical biology, chemical kinematics, propagation of dislocations in crystals, and the behavior of elementary particles. The Klein–Gordon equation, with and without damping terms, has been extensively studied, either from a theoretical (as global existence and nonexistence, exponential decay of energy, time blow-up, asymptotic behavior of solutions) or from a numerical point of view, for different nonlinear source terms $F(u)$ as $F(u) = au \mid u \mid^{\gamma}$ or logarithmic nonlinearity $F(u) = au\, ln \mid u \mid^{\gamma}$, where the parameter a measures the force of the nonlinear interactions. For the first type of nonlinearities, we can cite, e.g., [2–5] and the references therein. Problems with logarithmic nonlinearity arise naturally in many areas such as quantum optics and transport phenomena, via a logarithmic Schrodinger equation (see, e.g., [6,7]); fluid dynamics via a logarithmic Korteweg–de Vries equation or a logarithmic Kadomtsev–Petviashvili equation (see, e.g., [8]); or material sciences, with a Cahn–Hilliard equation (see, e.g., [9]). It also arises in nuclear physics, inflation cosmology, vibration, supersymmetric fields in quantum field theory, spinless particles, and viscoelastic mechanics where a logarithmic Klein–Gordon equation is considered (see, e.g., [7,10–16]). Such logarithmic Klein–Gordon problems have been the object of numerous studies either from a theoretical or from a numerical point of view (see, e.g., [17–30] and the references therein).

The introduction of retarded arguments is to reflect the different after-effects. Different time-varying delay configurations occur naturally in various areas of physics, biologics and engineering, such as in biochemical systems, in population dynamics, in quantum chaotic systems, in relativistic quantum waves, and in the area of plasma control (e.g., in the context of thermonuclear fusion with Tokamaks). Moreover, time-varying delays in signal transmission are inevitable in many applications and practical processes. A small delay can affect considerably dynamical behaviors of the system (e.g., destabilize the system which is asymptotically stable in the absence of time delays unless additional conditions, control functions, or stabilization mechanism functions have been used). Delay terms can lead to change in the stability of dynamics and give rise to highly complex behavior including instability, oscillations, and chaos (see, e.g., [1,31–46] and the references therein). Therefore, these behaviors and aspects, by taking into account different sources of delays, motivate the study of multiple time-varying delays effects on properties of dynamical systems.

In this work, in order to take into account the influence of different sources of time delays in the velocity of signal transmission, nonlinear Klein–Gordon wave type models with logarithmic nonlinearity are modified by incorporating multiple time delays and a nonlocal operator. The proposed strategy consists in controlling these instabilities by imposing some suitable conditions involving different functions and parameters representing the multiple time-varying delays. The presence of the nonlocal term $\mathcal{M}(\| u \|^2_{L^2(\Omega)})$ and operator Ψ, which describes multiple time-varying delays, makes the mathematical study of such a class of problems particularly interesting.

Remark 1.1.

1. *The Equation* (1) *is called nonlocal because of the presence of nonlocal term* $\mathcal{M}(\| u \|^2_{L^2(\Omega)})$.

2. *The functions d_k are diffusion coefficients that represent the strength of each associated time delay. A zero coefficient means the associated previous state does not impact the system.* □

The paper is organized as follows. In the next subsections, we give some necessary notations and preliminary results. In Section 2, we prove the local existence of solutions to problem (1) using Faedo–Galerkin's approximation and logarithmic Sobolev inequality. Section 3 deals with the global existence and energy decay rate of the solutions to problem (1). In Section 4, under suitable conditions on data and involved functionals, the exponential decay of solutions, for initial data in a set of stability, is investigated. Finally, we present conclusions in Section 5.

1.2. Notations

Let $\mathcal{D}$ be a Banach space and $\mathcal{D}'$ its dual Banach space. We denote the norm on $\mathcal{D}$ by $\| \,.\, \|_{\mathcal{D}}$, the norm on $\mathcal{D}'$ by $\| \,.\, \|_{\mathcal{D}'}$, and the duality pairing on $\mathcal{D}'$ and $\mathcal{D}$ by $<,>_{\mathcal{D}',\mathcal{D}}$. We denote the norm and the scalar product in $L^2(\Omega)$ by $\| \,.\, \|$ and $(.,.)$, respectively. We say that a sequence $(w_n)_n$ of $\mathcal{D}$ (respectively, of $\mathcal{D}'$) converges weakly (respectively, weakly*) to w if and only if $\forall f \in \mathcal{D}', < f, w_n - w >$ converges to 0 (respectively, $\forall v \in \mathcal{D}, < w_n - w, v >$ converges to 0) (see e.g., [33], Part I). The dual of $L^2(\Omega)$ is identified with itself, the dual of $H_0^1(\Omega)$ is $H^{-1}(\Omega)$ and we have the following injections $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ with continuous and dense embedding. Moreover we denote by C_Ω (which depends on the geometry of domain Ω) the optimal constant of embedding inequality (this smallest possible C_Ω is called the Poincaré constant)

$$\| v \|_{L^2(\Omega)} \leq C_\Omega \| \nabla v \|_{L^2(\Omega)}, \quad \forall v \in H_0^1(\Omega). \tag{3}$$

If Ω is a convex domain with diameter d_Ω, then $C_\Omega = \frac{d_\Omega}{\pi}$.

We can now introduce the following sets (for arbitrary final time $T > 0$):

$$\begin{aligned}
&V_0(\Omega; \delta(0)) = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\mathcal{Q}_0),\\
&\mathcal{H}(0,T) = L^\infty(0,T; L^2(\Omega)),\ \mathcal{V}(0,T) = L^2(0,T; H_0^1(\Omega)),\\
&\mathcal{S}(0,T) = C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \cap C^2([0,T); H^{-1}(\Omega)).
\end{aligned}$$

Finally, for a bounded function f, we denote

$$f^{min} = \inf_{t \geq 0} | f(t) |, \quad f^{max} = \sup_{t \geq 0} | f(t) | .$$

We now state some assumptions for the various functions and operators appearing in Equation (1).

1.3. Assumptions and Preliminaries

We start by assuming that $t \in \mathbb{R}^+ \longrightarrow (r_i(t) = t - e_i(t), i = 1, n)$ are strictly increasing functions (and consequently are bijective) and $(e_i, i = 1, n)$ are $\mathcal{C}^1$ non-negative and bounded functions on $\mathbb{R}^+$. So, we have the existence of inverse functions $(f_i, i = 1, n)$ of $(r_i, i = 1, n)$. For simplicity, without loss of generality, we can suppose that $e_i(0) = \delta(0)$ for all $i = 1, \ldots, n$.

Remark 1.2. *We can define the following subdivision: $s_{-1} = -\delta(0)$, $s_0 = 0$ and $\forall j \in \mathbb{N}^*$, $s_j = \min_{i=1,n}(f_i(s_{j-1}))$, and we denote $\tau_j = s_j - s_{j-1}$, and $\mathcal{Q}_j = \Omega \times (s_{j-1}, s_j)$ for $j \geq 0$. According to the hypotheses on functions $(e_i, i = 1, n)$, we prove easily that:*

(i) the sequence $(s_j)_{j \in \mathbb{N}}$ is strictly increasing $\forall j \geq 0$,

(ii) for $j \geq 2$, if $t \in (s_{j-1}, s_j)$ then $\forall i = 1, \ldots, n$, $r_i(t) \leq s_{j-1}$,

(iii) if $t \in (s_0, s_1)$, then $\forall i = 1, \ldots, n$, $r_i(t) \in (s_{-1}, s_0)$.

In order to derive the solution of (1), we can use the following process: we solve the problem on $\mathcal{Q}_1$ by using the initial/initial history data and obtain the solution. Then, the solution on $\mathcal{Q}_2$ is obtained by using the solution on $\mathcal{Q}_1$ to generate the initial data at s_1. This advancing process is repeated for $\mathcal{Q}_3$, $\mathcal{Q}_4$, ..., until the final set is reached. □

For the tensor function $\mathcal{K}$, we suppose that the following assumptions hold.
(H1) We assume that the conductivity tensor function $\mathcal{K} \in W^{1,\infty}(\overline{\Omega})$ is a symmetric, positive definite matrix function and is uniformly elliptic, i.e., there exist constants $0 < \nu \leq \mu$ such that $(\forall v \in \mathbb{R}^N)$

$$\nu \|v\|_2^2 \leq v^T \mathcal{K} v \leq \mu \|v\|_2^2 \text{ in } \overline{\Omega}, \tag{4}$$

where $\|.\|_2$ is the Euclidean norm.

For the nonlinear operator $\mathcal{M}$, we set the following hypothesis.
(H2) $\mathcal{M} \in C([0,\infty), \mathbb{R})$ such that $\mathcal{M}(\zeta) \geq \vartheta_0 \geq 0$, $\forall \zeta \in [0,\infty)$, with ϑ_0 a constant. We denote $\mathcal{M}_p(\zeta) = \int_0^\zeta \mathcal{M}(\eta) d\eta$.

Finally, we impose the following assumption for the parameter θ: there exists $\alpha > 0$ (depending on θ) such that

(AAA) (i) $\nu - \frac{\theta\alpha^2}{2\pi} > 0$ and (ii) $\frac{\theta(1 + N(1 + \ln\alpha)) + \vartheta_0}{2} > 0$.

Remark 1.3.

- *If $\theta < 0$, the relation (i) is always true for any $\alpha > 0$ and (ii) is true for all α such that $0 < \alpha < \exp(\frac{-N-1-\frac{\vartheta_0}{\theta}}{N})$. Consequently, it is always possible to find $\alpha > 0$ such that the relations (i) and (ii) hold.*
- *If $\theta > 0$, the relation (i) is true for all α such that $0 < \alpha < \sqrt{\frac{2\pi\nu}{\theta}}$ and (ii) is true for all α such that $\alpha > \exp(\frac{-N-1-\frac{\vartheta_0}{\theta}}{N})$. Consequently it is always possible to find α such that the relations (i) and (ii) hold, provided that $\sqrt{\frac{2\pi\nu}{\theta}} > \exp(\frac{-N-1-\frac{\vartheta_0}{\theta}}{N})$, i.e., that θ satisfies the following inequality*

$$\theta \leq \Theta_N(\nu, \vartheta_0), \tag{5}$$

with $\Theta_N(\nu,\vartheta_0)$ satisfying $\sqrt{\frac{2\pi\nu}{\Theta_N(\nu,\vartheta_0)}} = \exp(\frac{-N-1-\frac{\vartheta_0}{\Theta_N(\nu,\vartheta_0)}}{N})$. We prove easily that the sequence $\Theta_N(\nu,\vartheta_0)$ is decreasing with $\Theta_\infty(\nu,\vartheta_0) = \lim_{N\to\infty} \Theta_N(\nu,\vartheta_0) = 2\pi e^2\nu$ (the limit $\Theta_\infty(\nu,\vartheta_0) \approx 46.4268\nu$ is independent of ϑ_0). In this case, α satisfies

$$\alpha_0 = \exp(\frac{-N-1-\frac{\vartheta_0}{\theta}}{N}) < \alpha < \alpha_1 = \sqrt{\frac{2\pi\nu}{\theta}}. \tag{6}$$

Lemma 1.1 (Logarithmic Sobolev inequality, see, e.g., [47])**.** *Let $v \in H_0^1(\Omega)$, then for any positive constant α, the following estimates hold:*

$$\begin{aligned}\int_\Omega v^2 \ln | v | dx \leq \frac{1}{2} \| v \|^2_{L^2(\Omega)} \ln \| v \|^2_{L^2(\Omega)} \\ + \frac{\alpha^2}{2\pi} \| \nabla v \|^2_{L^2(\Omega)} - \frac{N(1+\ln\alpha)}{2} \| v \|^2_{L^2(\Omega)}.\end{aligned} \tag{7}$$

Lemma 1.2 (Logarithmic Gronwall inequality, see [19] or, e.g., [25])**.** *Given a positive time T and positive constants K and A_0 with $A_0 \geq 1$, let l be a nonnegative function such that $l \in L^1((0,T))$. If a function $A : (0,T) \longrightarrow [1,+\infty[$ satisfies, for any $t \in (0,T)$,*

$$A(t) \leq A_0 + K \int_0^t l(s) A(s) ln A(s) ds,$$

then $A(t) \leq A_0^{\exp(K \int_0^t l(s)ds)}$, for any $t \in (0,T)$.

Lemma 1.3. *Let $\epsilon \in (0,1)$, then there exists $d_\epsilon > 0$ such that $r \mid \ln r \mid \leq r^2 + d_\epsilon r^{1-\epsilon}$, for any $r > 0$.*

Proof. We prove easily that d_ϵ is the maximum of the following function $g : r \in]0,+\infty[\longrightarrow g(r) = r^\epsilon(|\ln r| - r)$. □

In the sequel, we will always denote C (or C_i) as some positive constant, which may be different at each occurrence.

2. Existence of Local Solution

In this section, we shall study the existence of local solutions for problem (1). For this, let $T > 0$ be a fixed and but arbitrary real number, and we denote $\mathcal{Q} = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$.

Introducing now the following new functions (for $i = 1, \ldots, n$)

$$w^i(\mathbf{x},t;\eta) = \frac{\partial u}{\partial t}(\mathbf{x}, t - \eta e_i(t)), \;\; for \; (\mathbf{x},t;\eta) \in \mathcal{Q} \times (0,1). \tag{8}$$

Then, we have

$$\begin{aligned} & w^i(\mathbf{x},t;0) = \frac{\partial u}{\partial t}(\mathbf{x},t), \;\; w^i(\mathbf{x},t;1) = \frac{\partial u}{\partial t}(\mathbf{x}, t - e_i(t)), \\ & \text{and} \\ & e_i(t)\frac{\partial w^i}{\partial t}(\mathbf{x},t;\eta) + (1 - \eta e_i'(t))\frac{\partial w^i}{\partial \eta}(\mathbf{x},t;\eta) = 0, \text{ for } (\mathbf{x},t;\eta) \in \mathcal{Q} \times (0,1). \end{aligned} \tag{9}$$

Consequently, problem (1) becomes (for $i = 1, \ldots, n$)

$$\begin{aligned} & \frac{\partial^2 u}{\partial t^2} - \operatorname{div}(\mathcal{K}(\mathbf{x})\nabla u) + d_0(t)\frac{\partial u}{\partial t} + \sum_{i=1}^n d_i(t) w^i(\mathbf{x},t;1) \\ & \qquad\qquad + \mathcal{M}(\| u \|^2_{L^2(\Omega)})u = u \ln | u |^\theta, \quad for \; (\mathbf{x},t) \in \mathcal{Q}, \\ & e_i(t)\frac{\partial w^i}{\partial t}(\mathbf{x},t;\eta) + (1 - \eta e_i'(t))\frac{\partial w^i}{\partial \eta}(\mathbf{x},t;\eta) = 0, \; for \; (\mathbf{x},t;\eta) \in \mathcal{Q} \times (0,1), \\ & w^i(\mathbf{x},t;0) = \tfrac{\partial u}{\partial t}(\mathbf{x},t) \; for \; (\mathbf{x},t) \in \mathcal{Q}, \\ & w^i(\mathbf{x},0;\eta) = h_0(\mathbf{x}, -\eta\delta(0)) := w_0^i(\mathbf{x};\eta), \;\; for \; (\mathbf{x},\eta) \in \Omega \times (0,1), \\ & u(\mathbf{x},0) = \phi_0(\mathbf{x}), \;\; \frac{\partial u}{\partial t}(\mathbf{x},0) = \phi_1(\mathbf{x}), \quad for \; \mathbf{x} \in \Omega, \\ & u(\mathbf{x},t) = 0, \;\; for \; (\mathbf{x},t) \in \Sigma. \end{aligned} \tag{10}$$

For this solution u, the corresponding energy function $E_u : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is defined by (for all $t \geq 0$)

$$\begin{aligned} E_u(t) = & \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} + (\mathcal{K}\nabla u, \nabla u) + \mathcal{M}_p(\| u \|^2_{L^2(\Omega)}) + \frac{\theta}{2} \| u \|^2_{L^2(\Omega)} \\ & - \int_\Omega u^2 \ln | u |^\theta \, d\mathbf{x} + \lambda \sum_{i=1}^n e_i(t) \| w^i(.,t;.) \|^2_{L^2(\Omega \times (0,1))}, \end{aligned} \tag{11}$$

where $\lambda > 0$ is a fixed parameter that satisfies the following condition **(H3)**

$$D_1 = \frac{1}{n}\inf_{t\geq 0}(d_0(t) - \frac{1}{2}\sum_{i=1}^{n} \mid d_i(t) \mid) > \lambda > D_2 = \max_{i=1,n}(\sup_{t\geq 0}\frac{\mid d_i(t) \mid}{2(1-e_i'(t))}),$$
$$\text{under the following additional assumption} \tag{12}$$
$$\frac{1}{n}\inf_{t\geq 0}(d_0(t) - \frac{1}{2}\sum_{i=1}^{n} \mid d_i(t) \mid) > \max_{i=1,n}(\sup_{t\geq 0}\frac{\mid d_i(t) \mid}{2(1-e_i'(t))}).$$

For typographical convenience, we will denote the energy at time t by $E(t)$ in place of $E_u(t)$ if no confusion arises.

Remark 2.1. *If the weight function d_0 is a non-increasing function, we can replace the energy function E by*

$$\begin{aligned} E(t) = \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} + (\mathcal{K}\nabla u, \nabla u) + \mathcal{M}_p(\| u \|^2_{L^2(\Omega)}) + \frac{\theta}{2} \| u \|^2_{L^2(\Omega)} \\ - \int_\Omega u^2 \ln \mid u \mid^\theta d\mathbf{x} + \lambda \sum_{i=1}^{n} d_0(t) e_i(t) \| w^i(.,t;.) \|^2_{L^2(\Omega\times(0,1))} . \end{aligned} \tag{13}$$

By adapting easily the condition (12) and relation (15), and with very minimal modifications, the results of the paper remain valid.

Theorem 2.1. *Assume that the hypotheses* **(AAA)** *and* **(H1)–(H3)** *hold. Then, for the initial/initial history conditions $(\phi_0, \phi_1, h_0) \in V_0(\Omega, \delta(0))$, there exists a local solution u of problem (1), with $u \in \mathcal{S}(0,T)$ and $w^i \in C^0([0,T]\times[0,1]; L^2(\Omega)) \cap C^1([0,T)\times[0,1]; H^{-1}(\Omega))$, for $i = 1,\ldots,n$, such that E is a nonincreasing function and satisfies (for $t \in (0,T)$)*

$$\frac{dE}{dt}(t) \leq -2R_1(\lambda) \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} - 2R_2(\lambda)\sum_{i=1}^{n} \| w^i(.,t;1) \|^2_{L^2(\Omega)} \leq 0 \tag{14}$$

where $R_1(\lambda)$ and $R_2(\lambda)$ are given by

$$R_1(\lambda) = n(D_1 - \lambda) \text{ and } R_2(\lambda) = n(\lambda - D_2)\min_{i=1,n}(\inf_{t\geq 0}(1 - e_i'(t))). \tag{15}$$

Proof. To establish the existence result of a weak solution to problem (1), we apply the Faedo–Galerkin method, derive a priori estimates, and then pass to the limit in the approximate solutions using compactness arguments. We approximate Equation (1) by projecting them onto finite m dimensional subspaces, and then we take the limit in m. For this, let $(v_k)_{k\in\mathbb{N}}$ be the orthogonal basis of $H_0^1(\Omega)$, which is orthogonal in $L^2(\Omega)$, and define $V_m = span\{v_1, ..., v_m\}$, for $m \geq 1$. From $(v_k)_{k=1,m}$, we can find a sequence $(\psi_k)_{k=1,m}$ with $\psi_k(\mathbf{x}, 0) = v_k(\mathbf{x})$, for $k = 1, m$, such that $(\psi_k)_{k=1,m}$ is orthogonal in $L^2(\Omega\times(0,1))$ and then define $\Psi_m = span\{\psi_1, ..., \psi_m\}$, for $m \geq 1$ (as, e.g., in [38]). Let u_{m0}, u_{m1} be sequences of V_m and w_{m0} be a sequence of Ψ_m such that $u_{m0} \longrightarrow \phi_0$ in $H_0^1(\Omega)$, $u_{m1} \longrightarrow \phi_1$ in $L^2(\Omega)$ and $w^i_{m0} \longrightarrow w^i_0$ in $L^2(\Omega\times(0,1))$ as $m \to \infty$. For each $m \in \mathbb{N}^*$, we would like to define the Faedo–Galerkin approximation solution $(u_m, (w^i_m)_{i=1,n})$ of the problem (1). Setting

$$u_m(\mathbf{x},t) = \sum_{k=1}^{m} h_{km}(t) v_k(\mathbf{x}) \text{ and } w^i_m(\mathbf{x},t;\eta) = \sum_{k=1}^{m} g^i_{km}(t)\psi_k(\mathbf{x},\eta),$$

where $(h_{km})_{k=1,m}$ and $(g^i_{km})_{k=1,m}$ are unknown functions and replacing $(u,(w^i)_{i=1,n})$ by $(u_m,(w^i_m)_{i=1,n})$ in (1), we obtain a.e. $t \in (0,T)$, the system of Galerkin equations (for all $(v,\psi) \in V_m \times \Psi_m$ and $i = 1,\ldots,n$)

$$\begin{aligned}
&(\frac{\partial^2 u_m}{\partial t^2}, v) + (\mathcal{K}\nabla u_m, \nabla v) + (d_0(t)\frac{\partial u_m}{\partial t}, v) + \sum_{i=1}^{n}(d_i(t) w^i_m(\mathbf{x},t;1), v) \\
&\qquad\qquad + (\mathcal{M}(\| u_m \|^2_{L^2(\Omega)}) u_m, v) = (u_m \ln | u_m |^\theta, v), \\
&\int_0^1 (e_i(t)\frac{\partial w^i_m}{\partial t}(.,t;\eta), \psi(\eta)) d\eta + \int_0^1 ((1-\eta e_i'(t))\frac{\partial w^i_m}{\partial \eta}(.,t;\eta), \psi(\eta)) d\eta = 0, \\
&u_m(\mathbf{x},0) = u_{m0}(\mathbf{x}),\ \ \frac{\partial u_m}{\partial t}(\mathbf{x},0) = u_{m1}(\mathbf{x}),\ \text{ for } \mathbf{x} \in \Omega \\
&w^i_m(\mathbf{x},0;\eta) = w^i_{m0}(\mathbf{x};\eta)\ \text{ for } (\mathbf{x},\eta) \in \Omega \times (0,1).
\end{aligned} \tag{16}$$

By virtue of the standard theory of ordinary differential equations, the problem (16) has a local solution that is extended to a maximal interval $[0,T_m)$ (with $0 < T_m < T$, for any given $T > 0$). The following estimate will give the local solution being extended to the whole interval $[0,T)$.

Replacing (v,ψ) by $(\frac{\partial u_m}{\partial t}, \lambda w^i_m)$ in (16) and using the relation

$$\begin{aligned}
(u_m \ln | u_m |^\theta, \frac{\partial u_m}{\partial t}) &= \int_\Omega \frac{\theta}{2} u_m \ln | u_m |^2 \frac{\partial u_m}{\partial t} d\mathbf{x} \\
&= \frac{d}{dt}(\frac{\theta}{4}\int_\Omega (u_m)^2 \ln | u_m |^2\, d\mathbf{x} - \frac{\theta}{4} \| u_m \|^2_{L^2(\Omega)}), \\
&= \frac{d}{dt}(\frac{1}{2}\int_\Omega (u_m)^2 \ln | u_m |^\theta\, d\mathbf{x} - \frac{\theta}{4} \| u_m \|^2_{L^2(\Omega)}),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\| \frac{\partial u_m}{\partial t} \|^2_{L^2(\Omega)} + (\mathcal{K}\nabla u_m, \nabla u_m) + \mathcal{M}_p(\| u_m \|^2_{L^2(\Omega)}) \\
&\qquad\qquad + \frac{\theta}{2} \| u_m \|^2_{L^2(\Omega)} - \int_\Omega (u_m)^2 \ln | u_m |^\theta\, d\mathbf{x}\Big) \\
&\qquad = -d_0(t) \| \frac{\partial u_m}{\partial t} \|^2_{L^2(\Omega)} - \sum_{i=1}^{n} d_i(t)(w^i_m(.,t;1), \frac{\partial u_m}{\partial t}), \\
&\frac{1}{2}\frac{d}{dt}(\lambda e_i(t) \| w^i_m(.,t;.) \|^2_{L^2(\Omega\times(0,1))}) \\
&\qquad = \frac{1}{2}\lambda e_i'(t) \| w^i_m(.,t;.) \|^2_{L^2(\Omega\times(0,1))} - \frac{1}{2}\lambda e_i'(t) \| w^i_m(.,t;.) \|^2_{L^2(\Omega\times(0,1))} \\
&\qquad - \lambda((1-e_i'(t)) \| w^i_m(.,t;1) \|^2_{L^2(\Omega)} - \| w^i_m(.,t;0) \|^2_{L^2(\Omega)}), \\
&\qquad = -\lambda\Big((1-e_i'(t)) \| w^i_m(.,t;1) \|^2_{L^2(\Omega)} - \| w^i_m(.,t;0) \|^2_{L^2(\Omega)}\Big), \\
&u_m(\mathbf{x},0) = u_{m0}(\mathbf{x}),\ \ \frac{\partial u_m}{\partial t}(\mathbf{x},0) = u_{m1}(\mathbf{x}),\ \text{ for } \mathbf{x} \in \Omega \\
&w^i_m(\mathbf{x},0;\eta) = w^i_{m0}(\mathbf{x};\eta)\ \text{ for } (\mathbf{x},\eta) \in \Omega \times (0,1).
\end{aligned} \tag{17}$$

Let us introduce the energy E_m of the solution u_m of problem (16) (as (11))

$$\begin{aligned}
E_m(t) = &\| \frac{\partial u_m}{\partial t} \|^2_{L^2(\Omega)} + (\mathcal{K}\nabla u_m, \nabla u_m) + \mathcal{M}_p(\| u_m \|^2_{L^2(\Omega)}) + \frac{\theta}{2} \| u_m \|^2_{L^2(\Omega)} \\
&- \int_\Omega (u_m)^2 \ln | u_m |^\theta\, d\mathbf{x} + \lambda \sum_{i=1}^{n} e_i(t) \| w^i_m(.,t;.) \|^2_{L^2(\Omega\times(0,1))}.
\end{aligned} \tag{18}$$

Then, (since $w_m^i(.,.;0) = \frac{\partial u_m}{\partial t}$)

$$\frac{1}{2}\frac{dE_m}{dt}(t) = -\left(d_0(t) \parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)} - \lambda \sum_{i=1}^{n} \parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)}\right) - \sum_{i=1}^{n} d_i(t)(w_m^i(.,t;1), \frac{\partial u_m}{\partial t}) - \lambda \sum_{i=1}^{n}(1 - e_i'(t)) \parallel w_m^i(.,t;1) \parallel^2_{L^2(\Omega)}. \tag{19}$$

By Young's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{dE_m}{dt}(t) \leq& -\left(d_0(t) \parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)} - \lambda \sum_{i=1}^{n} \parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)}\right) + \frac{1}{2}\sum_{i=1}^{n} \mid d_i(t) \mid \parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)} \\ &+ \frac{1}{2}\sum_{i=1}^{n} \mid d_i(t) \mid \parallel w_m^i(.,t;1) \parallel^2_{L^2(\Omega)} - \lambda \sum_{i=1}^{n}(1 - e_i'(t)) \parallel w_m^i(.,t;1) \parallel^2_{L^2(\Omega)}\end{aligned} \tag{20}$$

and then, according to the result (12)

$$\begin{aligned}\frac{1}{2}\frac{dE_m}{dt}(t) \leq& -(d_0(t) - \sum_{i=1}^{n}(\lambda + \frac{\mid d_i(t) \mid}{2})) \parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)} \\ &- \sum_{i=1}^{n}(\lambda(1 - e_i'(t)) - \frac{\mid d_i(t) \mid}{2}) \parallel w_m^i(.,t;1) \parallel^2_{L^2(\Omega)} \\ \leq& -R_1(\lambda) \parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)} - R_2(\lambda) \sum_{i=1}^{n} \parallel w_m^i(.,t;1) \parallel^2_{L^2(\Omega)} \leq 0,\end{aligned} \tag{21}$$

where $R_1(\lambda)$ and $R_2(\lambda)$ are given by (15).

From **(H2)**, (4), (7) and (21), we can deduce that

$$\begin{aligned}&\parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)} + (\nu - \frac{\theta\alpha^2}{2\pi}) \parallel \nabla u_m \parallel^2_{L^2(\Omega)} + \frac{\theta(1 + N(1 + \ln\alpha)) + \vartheta_0}{2} \parallel u_m \parallel^2_{L^2(\Omega)} \\ &+ 2R_1(\lambda)\int_0^t \parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)} ds + 2R_2(\lambda)\sum_{i=1}^{n}\int_0^t \parallel w_m^i(.,s;1) \parallel^2_{L^2(\Omega)} ds \\ &+ \sum_{i=1}^{n} \parallel w_m^i(.,t;.) \parallel^2_{L^2(\Omega\times(0,1))} \leq E_m(0) + \frac{\theta}{2} \parallel u_m \parallel^2_{L^2(\Omega)} \ln \parallel u_m \parallel^2_{L^2(\Omega)}.\end{aligned} \tag{22}$$

According to assumption **(AAA)**, we have that

$$\frac{1}{K_1} = \nu - \frac{\theta\alpha^2}{2\pi} > 0 \text{ and } \frac{1}{K_2} = \frac{\theta(1 + N(1 + \ln\alpha)) + \vartheta_0}{2} > 0$$

and then

$$\begin{aligned}&\parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)} + \frac{1}{K_1} \parallel \nabla u_m \parallel^2_{L^2(\Omega)} + \frac{1}{K_2} \parallel u_m \parallel^2_{L^2(\Omega)} \\ &+ 2R_1(\lambda)\int_0^t \parallel \frac{\partial u_m}{\partial t} \parallel^2_{L^2(\Omega)} ds + 2R_2(\lambda)\sum_{i=1}^{n}\int_0^t \parallel w_m^i(.,s;1) \parallel^2_{L^2(\Omega)} ds \\ &+ \sum_{i=1}^{n} \parallel w_m^i(.,t;.) \parallel^2_{L^2(\Omega\times(0,1))} \leq E_m(0) + \frac{\theta}{2} \parallel u_m \parallel^2_{L^2(\Omega)} \ln \parallel u_m \parallel^2_{L^2(\Omega)}.\end{aligned} \tag{23}$$

Moreover, we have, according to the expression of $E_m(0)$, for large m: $E_m(0) \leq C_I(\phi_0, \phi_1, h_0)$, where C_I is a positive constant depending on the initial/initial history data (ϕ_0, ϕ_1, h_0), and then

$$\begin{aligned} &\| \frac{\partial u_m}{\partial t} \|^2_{L^2(\Omega)} + \frac{1}{K_1} \| \nabla u_m \|^2_{L^2(\Omega)} + \frac{1}{K_2} \| u_m \|^2_{L^2(\Omega)} + 2R_1(\lambda) \int_0^t \| \frac{\partial u_m}{\partial t} \|^2_{L^2(\Omega)} ds \\ &\qquad + 2R_2(\lambda) \sum_{i=1}^n \int_0^t \| w^i_m(., s; 1) \|^2_{L^2(\Omega)} ds \\ &\qquad + \sum_{i=1}^n \| w^i_m(., t; .) \|^2_{L^2(\Omega\times(0,1))} \leq C_I + \frac{\theta}{2} \| u_m \|^2_{L^2(\Omega)} \ln \| u_m \|^2_{L^2(\Omega)} . \end{aligned} \tag{24}$$

Since $u_m(\mathbf{x}, t) = u_m(\mathbf{x}, 0) + \int_0^t \frac{\partial u_m}{\partial t}(\mathbf{x}, s) ds$, then, for large m, (according to (24))

$$\begin{aligned} \| u_m(., t) \|^2_{L^2(\Omega)} &\leq 2 \| u_m(0) \|^2_{L^2(\Omega)} + C_1 \int_0^t \| \frac{\partial u_m}{\partial t} \|^2_{L^2(\Omega)} ds \\ &\leq C_0 + C_2 \int_0^t \| u_m \|^2_{L^2(\Omega)} \ln \| u_m \|^2_{L^2(\Omega)} ds \end{aligned} \tag{25}$$

According to logarithmic Gronwall inequality, we can deduce the estimate (for all t)

$$\| u_m(., t) \|^2_{L^2(\Omega)} \leq (1 + C_0)^{exp(C_2 t)} \leq C_T \tag{26}$$

and then from inequality (24) follows (for a.e. $t \in (0, T)$)

$$\begin{aligned} &\| \frac{\partial u_m}{\partial t} \|^2_{L^2(\Omega)} + \| \nabla u_m \|^2_{L^2(\Omega)} + \| u_m \|^2_{L^2(\Omega)} + \int_0^t \| \frac{\partial u_m}{\partial t} \|^2_{L^2(\Omega)} ds \\ &\qquad + \sum_{i=1}^n \int_0^t \| w^i_m(., s; 1) \|^2_{L^2(\Omega)} ds + \sum_{i=1}^n \| w^i_m(., t; .) \|^2_{L^2(\Omega\times(0,1))} \leq C. \end{aligned} \tag{27}$$

Consequently,

$$\begin{aligned} &(u_m) \text{ is bounded in } L^\infty(0, T; H^1_0(\Omega)) \times L^\infty(0, T; L^2(\Omega)), \\ &(\frac{\partial u_m}{\partial t}) \text{ is bounded in } L^\infty(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega)), \\ &(w^i_m) \text{ is bounded in } L^\infty(0, T; L^2(\Omega \times (0, 1))), \\ &(w^i_m(., .; 1)) \text{ is bounded in } L^2(0, T; L^2(\Omega)). \end{aligned} \tag{28}$$

This result makes it possible to extract from (u_m, w^i_m) a subsequence also denoted by (u_m, w^i_m) and such that (by using Aubin–Lions compactness lemma, see [48] Theorem 5.1, p. 58)

$$\begin{aligned} &u_m \overset{*}{\rightharpoonup} u \text{ weakly star in } L^\infty(0, T; H^1_0(\Omega)), \\ &\tfrac{\partial u_m}{\partial t} \overset{*}{\rightharpoonup} \tfrac{\partial u}{\partial t} \text{ weakly star in } L^\infty(0, T; L^2(\Omega)), \\ &\tfrac{\partial u_m}{\partial t} \rightharpoonup \tfrac{\partial u}{\partial t} \text{ weakly in } L^2(0, T; L^2(\Omega)), \\ &w^i_m \overset{*}{\rightharpoonup} w^i \text{ weakly star in } L^\infty(0, T; L^2(\Omega \times (0, 1))), \\ &w^i_m(., .; 1) \rightharpoonup w^i(., .; 1) \text{ weakly in } L^2(\mathcal{Q}), \\ &u_m \longrightarrow u \text{ strongly in } L^2(\mathcal{Q}) \text{ (and so } u_m \longrightarrow u \text{ a.e. in } \mathcal{Q}), \ (*), \\ &u_m \ln | u_m |^\theta \longrightarrow u \ln | u |^\theta \text{ a.e. in } \mathcal{Q} \\ &\qquad\qquad\qquad\qquad \text{(from the continuity of } y \longrightarrow yln | y |^\theta \text{ and (*))}, \\ &\mathcal{M}(\| u_m \|^2_{L^2(\Omega)}) \longrightarrow \mathcal{M}(\| u \|^2_{L^2(\Omega)}) \text{ strongly in } L^2(0, T) \\ &\qquad\qquad\qquad\qquad \text{(from the continuity of } \mathcal{M} \text{ and (*))}. \end{aligned} \tag{29}$$

Moreover, since $\sup_{0<r\leq 1} \mid r\ln r \mid = \frac{1}{e}$, we can deduce, for all $q > 2$ if $N \leq 2$ and $2 < q \leq \frac{2N}{N-2}$ if $N \geq 3$, that

$$\begin{aligned} \| u_m \ln \mid u_m \mid^{\theta} \|_{L^2(\Omega)} &\leq \frac{\theta^2}{e}\left(\frac{1}{2}mes(\Omega) + \frac{1}{q-2} \| u_m \|^q_{L^q(\Omega)}\right) \\ &\leq C(1 + \| \nabla u_m \|^q_{L^2(\Omega)}). \end{aligned} \tag{30}$$

Consequently, from (30) we obtain (according to the last result of (29) and Lebesgue dominated convergence arguments)

$$u_m \ln \mid u_m \mid^{\theta} \longrightarrow u \ln \mid u \mid^{\theta} \ \text{ weakly } \text{ in } L^2(\mathcal{Q}) \tag{31}$$

and (according to (29) and (16)) the boundedness of $\frac{\partial^2 u_m}{\partial t^2}$ in $L^\infty(0, T; H^{-1}(\Omega))$ and

$$\frac{\partial^2 u_m}{\partial t^2} \overset{*}{\rightharpoonup} \frac{\partial^2 u}{\partial t^2} \text{ weakly star in } L^\infty(0, T; H^{-1}(\Omega)). \tag{32}$$

By using (29)–(32) and density properties of spaces spanned, respectively, by $(v_k)_k$ and $(\psi_k)_k$, we can pass to the limit ($m \to +\infty$) in a standard way in (16). So we omit the details. The limit $(u; w^i)$, for $i = 1, n$, then satisfies the following system (for all $(v, \psi) \in H_0^1(\Omega) \times L^2(\Omega \times (0, 1))$)

$$\begin{cases} < \frac{\partial^2 u}{\partial t^2}, v >_{H^{-1},H_0^1} + (\mathcal{K}\nabla u, \nabla v) + (d_0(t)\frac{\partial u}{\partial t}, v) + \sum_{i=1}^{n}(d_i(t)w^i(\mathbf{x}, t; 1), v) \\ \qquad + (\mathcal{M}(\| u \|^2_{L^2(\Omega)})u, v) = (u \ln \mid u \mid^{\theta}, v), \\ \int_0^1 (e_i(t)\frac{\partial w^i}{\partial t}(., t; \eta), \psi(., \eta))d\eta + \int_0^1 ((1 - \eta e_i'(t))\frac{\partial w^i}{\partial \eta}(., t; \eta), \psi(., \eta))d\eta = 0, \\ u(\mathbf{x}, 0) = \phi_0(\mathbf{x}), \ \frac{\partial u_m}{\partial t}(\mathbf{x}, 0) = \phi_1(\mathbf{x}), \text{ for } \mathbf{x} \in \Omega \\ w^i(\mathbf{x}, 0; \eta) = w_0^i(\mathbf{x}; \eta), \text{ for } (\mathbf{x}, \eta) \in \Omega \times (0, 1). \end{cases} \tag{33}$$

Finally, by a similar argument as to show (21), we can deduce

$$\frac{dE}{dt}(t) \leq -2R_1(\lambda) \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} - 2R_2(\lambda)\sum_{i=1}^{n} \| w^i(., t; 1) \|^2_{L^2(\Omega)} \leq 0 \tag{34}$$

where $R_1(\lambda)$ and $R_2(\lambda)$ are given in (15).
Then, E is a nonincreasing function. This completes the proof. □

Remark 2.2. *If $T_{max} = \sup\{T > 0 :$ the solution u existing on $[0, T]\}$ is the maximal existence time of the weak solution to problem (1), and then if $T_{max} = +\infty$, we say that the solution u is global and if $T_{max} < +\infty$, the solution u blows up and T_{max} is the blow-up time.*

3. Global Existence and Energy Decay Estimate

In this section, we prove the global existence and energy decay rate of the solutions to problem (10).

We started by introducing the functions $\mathcal{I}$ and $\mathcal{J}$ such that (for $v \in H_0^1(\Omega)$)

$$\mathcal{I}(v) = \nu \| \nabla v \|^2_{L^2(\Omega)} + \frac{\vartheta_0}{2} \| v \|^2_{L^2(\Omega)} - \theta \int_{\Omega} v^2 \ln \mid v \mid d\mathbf{x} \text{ and } \mathcal{J}(v) = \mathcal{I}(v) + \frac{\theta}{2} \| v \|^2_{L^2(\Omega)}.$$

For $r > 0$ and $v \in H_0^1(\Omega)$, we obtain

$$\mathcal{J}(rv) = \mathcal{I}(rv) + \frac{r^2\theta}{2} \| v \|^2_{L^2(\Omega)} = r^2(\mathcal{I}(v) + \frac{\theta(1 - 2\ln(r))}{2} \| v \|^2_{L^2(\Omega)}).$$

Then

$$\frac{\partial}{\partial r}(\mathcal{J}(rv)) = 2r(\mathcal{I}(v) + \frac{\theta(1-2\ln(r))}{2} \parallel v \parallel^2_{L^2(\Omega)}) - r\theta \parallel v \parallel^2_{L^2(\Omega)}$$
$$= 2r(\mathcal{I}(v) - \theta\ln(r) \parallel v \parallel^2_{L^2(\Omega)}).$$

Therefore, for an arbitrary element $v \in H^1_0(\Omega)/\{0\}$, we have $r\frac{\partial}{\partial r}(\mathcal{J}(rv)) = 2\mathcal{I}(rv)$, $\lim_{r\to 0^+} \mathcal{J}(rv) = 0$, $\lim_{r\to+\infty} \mathcal{J}(rv) = -\infty$, and $\frac{\partial}{\partial r}J(rv) > 0$ if $0 < r < \overline{r}(v)$ and $\frac{\partial}{\partial r}\mathcal{J}(rv) < 0$ if $r > \overline{r}(v)$, where the value $\overline{r}(v) = \exp(\frac{\mathcal{I}(v)}{\theta \parallel v \parallel^2_{L^2(\Omega)}})$ is the unique solution of $\frac{\partial}{\partial r}\mathcal{J}(rv) = 0$. Furthermore, we have that $\frac{\partial^2}{\partial r^2}\mathcal{J}(rv)|_{r=\overline{r}} = -2\theta \parallel v \parallel^2_{L^2(\Omega)} < 0$, $\arg\max_{r>0} \mathcal{J}(rv) = \overline{r}(v)$ and $\mathcal{I}(\overline{r}(v)v) = 0$.

Associated with $\mathcal{J}$, we have the well-known Nehari Manifold (the set of all nontrivial stationary solutions to the problem (10))

$$\mathcal{N} = \{v \in H^1_0(\Omega)/\{0\} : (\frac{\partial}{\partial r}J(rv))|_{r=1} = 0\}.$$

Equivalently,

$$\mathcal{N} = \{v \in H^1_0(\Omega)/\{0\} : \mathcal{I}(v) = 0\}.$$

We define the potential well depth d (also known as mountain pass level), as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [49], by

$$d = \inf_{v\in H^1_0(\Omega)/\{0\}} \sup_{r>0} \mathcal{J}(rv). \tag{35}$$

As Payne and Sattinger noted in [50], the potential well depth d can be also characterized as

$$d = \inf_{v\in\mathcal{N}} \mathcal{J}(v). \tag{36}$$

Moreover, the weak solution of (10) blows up when $\mathcal{I}(u(t)) < 0$. If $\theta < 0$, then $\mathcal{I}(u(t)) \geq 0$, for all t and then the weak solution of (10) is global. Consequently, in the sequel, we assume that $\theta > 0$.

Now, we can introduce the following spaces

$$\begin{aligned} \mathcal{N}^+ &= \{v \in H^1_0(\Omega)/\{0\} : \mathcal{I}(v) > 0\} \cup \{0\}, \\ \mathcal{N}^- &= \{v \in H^1_0(\Omega)/\{0\} : \mathcal{I}(v) < 0\}, \\ \mathcal{W}_1 &= \{v \in \mathcal{N}^+ : \mathcal{J}(v) < d\} \text{ (stable set)}, \\ \mathcal{W}_2 &= \{v \in \mathcal{N}^- : \mathcal{J}(v) < d\} \text{ (unstable set)}. \end{aligned} \tag{37}$$

The space $\mathcal{W}_1$ is corresponding to the set of stability for the problem (10).

Theorem 3.1. *Assume that the hypotheses* **(AAA)–(H2)** *hold, and that the initial condition* $(\phi_0, \phi_1) \in H^1_0(\Omega) \times L^2(\Omega)$ *and initial history condition* $h_0 \in L^2(\mathcal{Q}_0)$. *Let u be a local weak solution to* (10) *and* $E_a \in]0, d]$. *If* $\phi_0 \in \mathcal{W}_1$ *and* $E(0) < E_a$, *then u is a global solution.*

Proof. First, from Lemma A3, we have that $u(t) \in \mathcal{W}_1$, for every $t \in [0, T_{max})$. Prove now that $T_{max} = \infty$. For this purpose, it is sufficient to prove the boundedness (in time) of $\parallel \frac{\partial u}{\partial t} \parallel^2_{L^2(\Omega)} + \parallel \nabla u \parallel^2_{L^2(\Omega)}$.

Since $0 < E(0) < E_a \leq d$, we have for all $t \in [0, T_{max})$ (from the definition of $E(t)$ and the positivity of $\mathcal{I}(u(t))$)

$$\begin{aligned}
&\| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} \leq E(t) \leq E(0) < E_a < d, \\
&\frac{\theta}{2} \| u \|^2_{L^2(\Omega)} \leq E(t) \leq E(0) < E_a < d.
\end{aligned} \tag{38}$$

Moreover, from Lemma 1.1, we can deduce

$$\begin{aligned}
&\nu \| \nabla u \|^2_{L^2(\Omega)} = \mathcal{I}(u(t)) - \frac{\vartheta_0}{2} \| u \|^2_{L^2(\Omega)} + \theta \int_\Omega u^2 \ln | u | d\mathbf{x} \\
&\leq \mathcal{I}(u(t)) + \frac{\theta}{2} \| u \|^2_{L^2(\Omega)} (\ln(\| u \|^2_{L^2(\Omega)}) - N(1 + \ln \alpha) - \frac{\vartheta_0}{\theta}) + \frac{\theta \alpha^2}{2\pi} \| \nabla u \|^2_{L^2(\Omega)} \\
&\leq E(t) + \frac{\theta}{2} \| u \|^2_{L^2(\Omega)} (\ln(\| u \|^2_{L^2(\Omega)}) - N(1 + \ln \alpha) - \frac{\vartheta_0}{\theta}) + \frac{\theta \alpha^2}{2\pi} \| \nabla u \|^2_{L^2(\Omega)}
\end{aligned} \tag{39}$$

and then

$$(\nu - \frac{\theta \alpha^2}{2\pi}) \| \nabla u \|^2_{L^2(\Omega)} \leq E(t) + \frac{\theta}{2} \| u \|^2_{L^2(\Omega)} h_{\alpha,\theta,\vartheta_0}(\| u \|_{L^2(\Omega)}), \tag{40}$$

where

$$h_{\alpha,\theta,\vartheta_0}(\| u \|_{L^2(\Omega)}) = \ln(\| u \|^2_{L^2(\Omega)}) - N(1 + \ln \alpha) - \frac{\vartheta_0}{\theta}. \tag{41}$$

From (38), (A1) in Appendix A and **(AAA)**, we can deduce that

$$\begin{aligned}
&h_{\alpha,\theta,\vartheta_0}(\| u \|_{L^2(\Omega)}) \leq \ln(\tfrac{2d}{\theta}) - N(1 + \ln \alpha) - \tfrac{\vartheta_0}{\theta}, \\
&\ln(\tfrac{2d}{\theta}) - N(1 + \ln \alpha) - \tfrac{\vartheta_0}{\theta} \geq \ln(\tfrac{2d_I}{\theta}) - N(1 + \ln \alpha) - \tfrac{\vartheta_0}{\theta} \\
&\qquad \geq \frac{2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}^2_\Omega}{\theta} + N(1 + \ln \alpha) - N(1 + \ln \alpha) - \frac{\vartheta_0}{\theta} + \frac{\vartheta_0}{\theta} \\
&\qquad = \frac{2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}^2_\Omega}{\theta} > 0
\end{aligned} \tag{42}$$

and then (since $E(t) < d$)

$$\| \nabla u \|^2_{L^2(\Omega)} < d(1 + \ln(2d/\theta) - N(1 + \ln \alpha) - \frac{\vartheta_0}{\theta})(\nu - \frac{\theta \alpha^2}{2\pi})^{-1}. \tag{43}$$

Thus, according to (38) and (43), we can conclude that the solution u is global (by the continue principle). This completes the proof. □

Remark 3.1.

1. *From* (34)*, we can deduce the following boundedness results (since* $0 < E(t) < d$ *for* $t \geq 0$*)*

$$\begin{aligned}
&\int_0^t \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} ds \leq \frac{d}{2R_1(\lambda)}, \\
&\int_0^t \| w^i(s;1) \|^2_{L^2(\Omega)} ds \leq \frac{d}{2R_2(\lambda)}, \text{ for i=1,\ldots,n.}
\end{aligned} \tag{44}$$

2. *If we replace the conditions* $\phi_0 \in \mathcal{W}_1$ *and* $E(0) < E_a$*, by the existence of a real number* $t_1 \in [0; T_{max})$ *such that* $u(t_1) \in \mathcal{W}_1$ *and* $E(t_1) < E_a$*, the result of Theorem remains valid.*
3. *From* (A1) *in Appendix A,* (38) *and* **(AAA)***, we can deduce the following relations.*

(i) *If* $E_a \geq d_I$

$$\begin{aligned} h_{\alpha,\theta,\vartheta_0}(\| u \|_{L^2(\Omega)}) &\leq \ln(\frac{2E_a}{\theta}) - N(1+\ln\alpha) - \frac{\vartheta_0}{\theta}, \\ \ln(\frac{2E_a}{\theta}) - N(1+\ln\alpha) - \frac{\vartheta_0}{\theta} &\geq \ln(\frac{2d_I}{\theta}) - N(1+\ln\alpha) - \frac{\vartheta_0}{\theta} \\ &= \frac{2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}_\Omega^2}{\theta} > 0, \end{aligned} \tag{45}$$

(ii) *if* $E_a \leq d_I$

$$\begin{aligned} h_{\alpha,\theta,\vartheta_0}(\| u \|_{L^2(\Omega)}) &\leq \ln(\frac{2E_a}{\theta}) - N(1+\ln\alpha) - \frac{\vartheta_0}{\theta} \\ &\leq \ln(\frac{2d_I}{\theta}) - N(1+\ln\alpha) - \frac{\vartheta_0}{\theta} = \frac{2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}_\Omega^2}{\theta}, \end{aligned} \tag{46}$$

(iii) *if* $E_a \leq d_A$

$$\begin{aligned} h_{\alpha,\theta,\vartheta_0}(\| u \|_{L^2(\Omega)}) &\leq \ln(\frac{2E_a}{\theta}) - N(1+\ln\alpha) - \frac{\vartheta_0}{\theta} \\ &\leq \ln(\frac{2d_A}{\theta}) - N(1+\ln\alpha) - \frac{\vartheta_0}{\theta} = 0. \end{aligned} \tag{47}$$

4. Asymptotic Behavior

In this section, we prove the exponential decay of solution of problem (34). We assume that the operator $\mathcal{M}$ satisfies the following condition (for all $v \geq 0$)

$$\mathcal{M}(v)v - \mathcal{M}_p(v) \geq 0 \tag{48}$$

and we use the Nakao's Lemma [51].

Theorem 4.1. *Let the assumptions of Theorem 3.1 hold. Assume that* $E_a \leq d_I$ *and* $\nu > \frac{\theta\alpha^2}{\pi}$*. Then, if* $E_a \leq d_A$ *or* $2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}_\Omega^2 \leq \vartheta_0$*, there exist positive constants* C_s *and* δ *such that the energy* E *associated to problem (10) satisfies*

$$0 < E(t) \leq C_s e^{-\delta t}, \quad \forall t \geq 0.$$

Proof. Let $v = u$ and $\psi = \exp(-\eta e_i)w^i$ in (33), we can deduce that

$$\begin{aligned} &-\| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} + \frac{d}{dt}(\frac{\partial u}{\partial t}, u) + (\mathcal{K}\nabla u, \nabla u) + \mathcal{M}(\| u \|^2_{L^2(\Omega)}) \| u \|^2_{L^2(\Omega)} \\ &\qquad + (d_0(t)\frac{\partial u}{\partial t}, u) + \sum_{i=1}^{n}(d_i(t)w^i(.,t;1), u) = (u \ln | u |^\theta, u), \\ &\int_0^1 e_i(t)\exp(-\eta e_i(t))\frac{\partial}{\partial t} \| w^i(.,t;\eta) \|^2_{L^2(\Omega)} \, d\eta \\ &\qquad + \int_0^1 \exp(-\eta e_i(t))(1 - \eta e_i'(t))\frac{\partial}{\partial \eta} \| w^i(.,t;\eta) \|^2_{L^2(\Omega)} \, d\eta = 0, \end{aligned} \tag{49}$$

then (since $w^i(.;\eta=0)=\frac{\partial u}{\partial t}$),

$$\begin{aligned}
\frac{d}{dt}(\frac{\partial u}{\partial t},u) =& \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} -(\mathcal{K}\nabla u,\nabla u) - \mathcal{M}(\| u \|^2_{L^2(\Omega)}) \| u \|^2_{L^2(\Omega)} \\
& -d_0(t)(\frac{\partial u}{\partial t},u) - \sum_{i=1}^{n}(d_i(t)w^i(.,t;1),u) + \int_\Omega u^2 \ln | u |^\theta \, d\mathbf{x}, \\
\frac{\partial}{\partial t}(e_i(t)\exp(-\eta e_i(t)) & \| w^i(.,t;.) \|^2_{L^2(\Omega\times(0,1))}) \\
& - \int_0^1 \exp(-\eta e_i)e_i'(1-\eta e_i)) \| w^i(.t,;\eta) \|^2_{L^2(\Omega)} (t;\eta)d\eta \\
& + \int_0^1 \exp(-\eta e_i(t))(e_i + e_i'(t)(1-\eta e_i(t))) \| w^i(.,t;\eta) \|^2_{L^2(\Omega)} d\eta \\
& + exp(-e_i(t))(1-e_i'(t)) \| w^i(.,t;1) \|^2_{L^2(\Omega)} - \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} = 0.
\end{aligned} \tag{50}$$

Introduce now the following functions:

$$\Phi(t) = (\frac{\partial u}{\partial t},u) \text{ and } \Psi(t) = \frac{\lambda}{\sqrt{\theta}}\sum_{i=1}^{n}\int_0^1 e_i(t)\exp(-\eta e_i) \| w^i \|^2_{L^2(\Omega)} (t;\eta)d\eta.$$

Then,

$$\begin{aligned}
\Phi'(t) =& \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} -(\mathcal{K}\nabla u,\nabla u) - \mathcal{M}(\| u \|^2_{L^2(\Omega)}) \| u \|^2_{L^2(\Omega)} \\
& -d_0(t)\Phi(t) - \sum_{i=1}^{n}(d_i(t)w^i(\mathbf{x},t;1),u) + \int_\Omega u^2 \ln | u |^\theta \, d\mathbf{x} \\
\le & (1+\frac{(C_\Omega d_0^{max})^2}{\nu}) \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} -\frac{1}{2}(\mathcal{K}\nabla u,\nabla u) \\
& -\mathcal{M}(\| u \|^2_{L^2(\Omega)}) \| u \|^2_{L^2(\Omega)} \\
& +\frac{(C_\Omega D_\infty)^2}{\nu}\sum_{i=1}^{n} \| w^i(.,t;1) \|^2_{L^2(\Omega)} + \int_\Omega u^2 \ln | u |^\theta \, d\mathbf{x}
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
\Psi'(t) =& -\frac{\lambda}{\sqrt{\theta}}\sum_{i=1}^{n}\int_0^1 \exp(-\eta e_i(t))e_i(t) \| w^i(.,t;\eta) \|^2_{L^2(\Omega)} (t;\eta)d\eta \\
& -\frac{\lambda}{\sqrt{\theta}}\sum_{i=1}^{n}\exp(-e_i(t))(1-e_i'(t)) \| w^i(.,t;1) \|^2_{L^2(\Omega)} + \frac{\lambda}{\sqrt{\theta}} \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} \\
\le & -\frac{\lambda}{\sqrt{\theta}}\sum_{i=1}^{n} e_i(t)\exp(-e_i(t)) \| w^i(.,t;.) \|^2_{L^2(\Omega\times(0,1))} \\
& -\frac{\lambda P_\infty}{\sqrt{\theta}}\sum_{i=1}^{n} \| w^i(.,t;1) \|^2_{L^2(\Omega)} + \frac{\lambda}{\sqrt{\theta}} \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)}
\end{aligned} \tag{52}$$

where $P_\infty = \min_i((1-p_i^{max})\exp(-e_i^{max}))$, $D_\infty = \max_i(d_i^{max})$. Consequently (according to (3) and (**H1**))

$$\begin{aligned}
(\Phi+\Psi)'(t) \le & (1+\frac{(C_\Omega d_0^{max})^2}{\nu}+\frac{\lambda}{\sqrt{\theta}}) \| \frac{\partial u}{\partial t} \|^2_{L^2(\Omega)} -\frac{1}{2}(\mathcal{K}\nabla u,\nabla u) \\
& -\mathcal{M}(\| u \|^2_{L^2(\Omega)}) \| u \|^2_{L^2(\Omega)} \\
& -(-\frac{(C_\Omega D_\infty)^2}{\nu}+\frac{\lambda P_\infty}{\sqrt{\theta}})\sum_{i=1}^{n} \| w^i(.,t;1) \|^2_{L^2(\Omega)} + \int_\Omega u^2 \ln | u |^\theta \, d\mathbf{x} \\
& -\frac{\lambda}{\sqrt{\theta}}\sum_{i=1}^{n} e_i(t)\exp(-e_i(t)) \| w^i(.,t;.) \|^2_{L^2(\Omega\times(0,1))} .
\end{aligned} \tag{53}$$

Then, for all $\epsilon, \xi > 0$, we have (according to (3) and **(H1)**)

$$
\begin{aligned}
&(E(t)+\epsilon(\Phi+\Psi))'(t)+\frac{\lambda\xi}{\sqrt{\theta}}E(t) \\
&\quad\leq -(2R_1(\lambda)-\epsilon(1+\frac{(C_\Omega d_0^{max})^2}{\nu}+\frac{\lambda}{\sqrt{\theta}})-\frac{\lambda\xi}{\sqrt{\theta}}) \parallel \frac{\partial u}{\partial t} \parallel^2_{L^2(\Omega)} \\
&\qquad +(C_\Omega^2\frac{\lambda\xi}{2\sqrt{\theta}}\theta-\nu(\frac{\epsilon}{2}-\frac{\lambda\xi}{\sqrt{\theta}})) \parallel \nabla u \parallel^2_{L^2(\Omega)} -\epsilon\mathcal{M}(\parallel u \parallel^2_{L^2(\Omega)}) \parallel u \parallel^2_{L^2(\Omega)} \\
&\qquad +\frac{\lambda\xi}{\sqrt{\theta}}\mathcal{M}_p(\parallel u \parallel^2_{L^2(\Omega)}) \\
&\qquad -(2R_2(\lambda)+\epsilon(-\frac{(C_\Omega D_\infty)^2}{\nu}+\frac{\lambda P_\infty}{\sqrt{\theta}}))\sum_{i=1}^n \parallel w^i(.,t;1) \parallel^2_{L^2(\Omega)} \\
&\qquad +(\epsilon-\frac{\lambda\xi}{\sqrt{\theta}})\int_\Omega u^2 \ln|u|^\theta \, d\mathbf{x} \\
&\qquad -\frac{\lambda}{\sqrt{\theta}}\sum_{i=1}^n e_i(t)(\epsilon\exp(-e_i(t))-\xi\lambda) \parallel w^i(.,t;.) \parallel^2_{L^2(\Omega\times(0,1))} .
\end{aligned} \tag{54}
$$

So, from the logarithmic Sobolev inequality (for $\frac{\epsilon}{2} > \rho = \frac{\lambda\xi}{\sqrt{\theta}} > 0$)

$$
\begin{aligned}
&(E(t)+\epsilon(\Phi+\Psi))'(t)+\rho E(t) \\
&\quad\leq -(2R_1(\lambda)-\epsilon(1+\frac{(C_\Omega d_0^{max})^2}{\nu}+\frac{\lambda}{\sqrt{\theta}})-\rho) \parallel \frac{\partial u}{\partial t} \parallel^2_{L^2(\Omega)} \\
&\qquad -(-C_\Omega^2\frac{\theta\rho}{2}-(\epsilon-\rho)\theta\frac{\alpha^2}{2\pi}+\nu(\frac{\epsilon}{2}-\rho)) \parallel \nabla u \parallel^2_{L^2(\Omega)} \\
&\qquad -(\epsilon-\rho)\mathcal{M}(\parallel u \parallel^2_{L^2(\Omega)}) \parallel u \parallel^2_{L^2(\Omega)} \\
&\qquad -\rho(-\mathcal{M}_p(\parallel u \parallel^2_{L^2(\Omega)})+\mathcal{M}(\parallel u \parallel^2_{L^2(\Omega)}) \parallel u \parallel^2_{L^2(\Omega)}) \\
&\qquad -(2R_2(\lambda)+\epsilon(-\frac{(C_\Omega D_\infty)^2}{\nu}+\frac{\lambda P_\infty}{\sqrt{\theta}}))\sum_{i=1}^n \parallel w^i(.,t;1) \parallel^2_{L^2(\Omega)} \\
&\qquad +(\epsilon-\rho)\theta(\ln \parallel u \parallel^2_{L^2(\Omega)} -N(1+\ln\alpha))\frac{1}{2} \parallel u \parallel^2_{L^2(\Omega)} \\
&\qquad -\frac{\lambda}{\sqrt{\theta}}\sum_{i=1}^n e_i(t)(\epsilon\exp(-e_i(t))-\rho\sqrt{\theta}) \parallel w^i(.,t;.) \parallel^2_{L^2(\Omega\times(0,1))} .
\end{aligned} \tag{55}
$$

According to **(H2)** and (48), we can deduce that (according to the expression (41))

$$
\begin{aligned}
&(E(t)+\epsilon(\Phi+\Psi))'(t)+\rho E(t) \\
&\quad\leq -(2R_1(\lambda)-\epsilon(1+\frac{(C_\Omega d_0^{max})^2}{\nu}+\frac{\lambda}{\sqrt{\theta}})-\rho) \parallel \frac{\partial u}{\partial t} \parallel^2_{L^2(\Omega)} \\
&\qquad -(-\rho(C_\Omega^2\frac{\theta}{2}+(\nu-\theta\frac{\alpha^2}{2\pi}))+\frac{\epsilon}{2}(\nu-\theta\frac{\alpha^2}{\pi})) \parallel \nabla u \parallel^2_{L^2(\Omega)} \\
&\qquad -(2R_2(\lambda)+\epsilon(-\frac{(C_\Omega D_\infty)^2}{\nu}+\frac{\lambda P_\infty}{\sqrt{\theta}}))\sum_{i=1}^n \parallel w^i(.,t;1) \parallel^2_{L^2(\Omega)} (t;1) \\
&\qquad +(\epsilon-\rho)\frac{\theta}{2} \parallel u \parallel^2_{L^2(\Omega)} (h_{\alpha,\theta,\vartheta_0}(\parallel u \parallel_{L^2(\Omega)})-\frac{\vartheta_0}{\theta}) \\
&\qquad -\frac{\lambda}{\sqrt{\theta}}\sum_{i=1}^n e_i(t)\exp(-e_i(t))(\epsilon-\rho\sqrt{\theta}e_\infty) \parallel w^i(.,t;.) \parallel^2_{L^2(\Omega\times(0,1))},
\end{aligned} \tag{56}
$$

where $e_\infty = \max_i(exp(e_i^{max}))$.

By taking ϵ and $\dfrac{\rho}{\epsilon}$ sufficiently small, we can deduce ($\nu > \frac{\theta\alpha^2}{\pi}$)

$$
(E(t)+\epsilon(\Phi+\Psi))'(t)+\rho E(t) \leq (\epsilon-\rho)\frac{\theta}{2} \parallel u \parallel^2_{L^2(\Omega)} (h_{\alpha,\theta,\vartheta_0}(\parallel u \parallel_{L^2(\Omega)})-\frac{\vartheta_0}{\theta}). \tag{57}
$$

Since $0 < E(0) < E_a \leq d_I$, we have that from (46)–(47) that (since $E_a \leq d_A$ or $2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}^2_\Omega \leq \vartheta_0$)

$$(E(t) + \epsilon(\Phi + \Psi))'(t) \leq -\rho E(t). \tag{58}$$

Prove now the following energy equivalence (for ϵ sufficiently small)

$$\alpha_1 E(t) \leq E(t) + \epsilon(\Phi + \Psi)(t) \leq \alpha_2 E(t),$$

where $\alpha_i > 0$, for $i = 1, 2$. Since

$$\begin{aligned}
&\mid (\Phi + \Psi)(t) \mid \leq \mid (\frac{\partial u}{\partial t}, u) \mid + \mid \frac{\lambda}{\sqrt{\theta}} \sum_{i=1}^{n} e_i(t) \int_0^1 \exp(-\eta e_i) \parallel w^i(.,t;\eta) \parallel^2_{L^2(\Omega)} d\eta \mid \\
&\leq \frac{\sqrt{2}}{\theta}(\parallel \frac{\partial u}{\partial t} \parallel^2_{L^2(\Omega)} + \frac{\theta}{2} \parallel u \parallel^2_{L^2(\Omega)} + \sqrt{2} \sum_{i=1}^{n} \lambda e_i(t) \parallel w^i(.,t;.) \parallel^2_{L^2(\Omega\times(0,1))}) \\
&\leq \frac{\sqrt{2}}{\theta}(\parallel \frac{\partial u}{\partial t} \parallel^2_{L^2(\Omega)} + \mathcal{J}(u)(t) + \sum_{i=1}^{n} \lambda e_i(t) \parallel w^i(.,t;.) \parallel^2_{L^2(\Omega\times(0,1))}) \\
&\leq \frac{\sqrt{2}}{\theta} E(t)
\end{aligned}$$

then

$$\alpha_1 E(t) \leq E(t) + \epsilon(\Phi + \Psi)(t) \leq \alpha_2 E(t), \tag{59}$$

where $\alpha_1 = 1 - \epsilon\frac{\sqrt{2}}{\theta} > 0$ and $\alpha_2 = 1 + \epsilon\frac{\sqrt{2}}{\theta} > 0$. From (59) and (58), we can obtain that $E(t) + \epsilon(\Phi + \Psi)(t) \leq \beta_1 exp(-\rho t)$ (where $\beta_1 > 0$ is constant depending on $E(0) + \epsilon(\Phi + \Psi)(0)$, α_1 and α_2), and by using again (59), we can deduce the result of the theorem. □

5. Conclusions

In this work, we have studied the existence and exponential stability of global solutions to nonlinear logarithmic Klein–Gordon type equations with multiple-time varying delays and a nonlocal term in a bounded domain. The logarithmic Klein–Gordon equation is the relativistic version of the logarithmic Schrodinger equation. Such logarithmic nonlinearity effects often arise in various areas of physical sciences and engineering. The introduction of retarded arguments is to reflect the different after-effects. Various time-varying delay configurations occur naturally in various areas of physics, biologics, and engineering. This natural phenomenon is due to the fact that the instantaneous rate of change of such systems does not only depend on their current time but rather on their previous history as well. Moreover, in many realistic application fields, introducing time delays into mathematical modeling has long proven to be unavoidable for correctly representing the behavior of real-world systems. It is also known that time delay is a non-negligible constraint in the process and may induce complex behaviors in the dynamical system, e.g., instability, oscillations, chaos, and poor performances. Therefore, these behaviors and aspects, by taking into account different sources of delays, motivate the study of multiple time-varying delay effects on properties of dynamical systems.

In this respect, in order to take into account the influence of different sources of time delays in the velocity of signal transmission, nonlinear Klein–Gordon wave type models with logarithmic nonlinearity are modified by incorporating multiple time delays and a nonlocal operator. The presence of nonlocal term and multiple time delays in a system leads to a more complex analysis.

The proposed strategy consists in controlling the instabilities by imposing some suitable conditions involving different functions and parameters representing the multiple time-varying delays. After obtaining the local existence result of the solutions by using Faedo–Galerkin's method and logarithmic Sobolev inequality, the global existence is derived. To show the exponential stability result under appropriate conditions, the potential well and perturbed energy methods are applied.

A future objective is to simulate and validate numerically the developed theoretical results. These studies will be the subject of a forthcoming paper. It would also be interesting to investigate the blow-up behavior of solutions of the considered problem. Moreover, the developed analysis in this work can be further applied to the studied model but

- With dynamical boundary conditions on regular non-cylindrical domains
- With switching time delays
- Or with Kirchhoff–Carrier type operators, i.e., by replacing the term $\mathcal{K}(\mathbf{x})$ in (1) by the operator $\mathcal{K}(\mathbf{x}, \| \nabla u \|^2_{L^2(\Omega)})$.

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Appendix A

In this annex we give some lemmas used in the document.

Lemma A1. *The potential depth d satisfies*

$$\begin{aligned} d \geq d_I &= \frac{\theta}{2}\exp\Big(\frac{2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}^2_\Omega + \vartheta_0 + N\theta(1+\ln\alpha)}{\theta}\Big) \\ &= \frac{\theta}{2}\exp\Big(\frac{2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}^2_\Omega + \vartheta_0}{\theta}\Big)e^N\alpha^N, \\ d_I \geq d_A &= \frac{\theta}{2}\exp\Big(\frac{\vartheta_0}{\theta}\Big)e^N\alpha^N, \end{aligned} \tag{A1}$$

where $\tilde{C}_\Omega = C_\Omega^{-1}$, C_Ω is the Poincaré constant and $\alpha > 0$ is defined in assumption **(AAA)**. *Moreover, we have*

$$\begin{aligned} \rho_0 &= \frac{\theta}{2}\exp\Big(\frac{2(\nu - \frac{\theta\alpha_1^2}{2\pi})\tilde{C}^2_\Omega + \vartheta_0}{\theta}\Big)e^N\alpha_0^N < d_I < \rho_1 = \frac{\theta}{2}\exp\Big(\frac{2(\nu - \frac{\theta\alpha_0^2}{2\pi})\tilde{C}^2_\Omega + \vartheta_0}{\theta}\Big)e^N\alpha_1^N, \\ \rho_{0a} &= \frac{\theta}{2}\exp\Big(\frac{\vartheta_0}{\theta}\Big)e^N\alpha_0^N < d_A < \rho_{1a} = \frac{\theta}{2}\exp\Big(\frac{\vartheta_0}{\theta}\Big)e^N\alpha_1^N. \end{aligned} \tag{A2}$$

Proof. According to assumption **(AAA)**, Lemma 1.1 and relation (3), we can deduce that (for all $v \in H^1_0(\Omega)$)

$$0 = \mathcal{I}(\bar{r}(v)v) \geq \Big(\big(\nu - \frac{\theta\alpha^2}{2\pi}\big)\tilde{C}^2_\Omega + \frac{N\theta}{2}(1+\ln\alpha) + \frac{\vartheta_0}{2} - \frac{\theta}{2}\ln \| \bar{r}(v)v \|^2_{L^2(\Omega)}\Big) \| \bar{r}(v)v \|^2_{L^2(\Omega)}$$

and then

$$\| \bar{r}(v)v \|^2_{L^2(\Omega)} \geq \exp\Big(\frac{2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}^2_\Omega + N\theta(1+\ln\alpha) + \vartheta_0}{\theta}\Big).$$

Consequently,

$$\begin{aligned} d \geq \mathcal{J}(\bar{r}(v)v) = \frac{\theta}{2} \| \bar{r}(v)v \|^2_{L^2(\Omega)} \geq d_I &= \frac{\theta}{2}\exp\Big(\frac{2(\nu - \frac{\theta\alpha^2}{2\pi})\tilde{C}^2_\Omega + N\theta(1+\ln\alpha) + \vartheta_0}{\theta}\Big) \\ &\geq d_A = \frac{\theta}{2}\exp\Big(\frac{\vartheta_0}{\theta}\Big)e^N\alpha^N. \end{aligned}$$

According to (6), we obtain easily the relations (A2). This completes the proof. □

Lemma A2. *Let $v \neq 0$ be in $H^1_0(\Omega)$, we have*

1. *If* $0 < \| \nabla v \|^2_{L^2(\Omega)} \leq R_N(\lambda_{min}, \nu, \vartheta_0)$ *then* $\mathcal{I}(v) \geq 0$,
   where $R_N(\lambda_{min}, \nu, \vartheta_0) = \lambda_{min} e^{N+\vartheta_0} (2\pi\nu)^{\frac{N}{2}}$ *and* λ_{min} *is the first eigenvalue of* $-\Delta$ *with homogeneous Dirichlet boundary conditions, that is*
   $$\lambda_{min} = \inf_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{\| \nabla v \|^2_{L^2(\Omega)}}{\| v \|^2_{L^2(\Omega)}}$$
   (and then $\lambda^{-1}_{min} = C_\Omega$ *is the Poincaré constant)*
2. *If* $\mathcal{I}(v) < 0$, *then* $\| \nabla v \|^2_{L^2(\Omega)} > R_N(\lambda_{min}, \nu, \vartheta_0)$.

Proof. From Lemma 1.1, for any positive constant α, we have

$$\begin{aligned}\mathcal{I}(v) &= \nu \| \nabla v \|^2_{L^2(\Omega)} + \frac{\vartheta_0}{2} \| v \|^2_{L^2(\Omega)} - \theta \int_\Omega v^2 \ln | v | \, dx \\ &\geq (\nu - \frac{\alpha^2}{2\pi}) \| \nabla v \|^2_{L^2(\Omega)} + \frac{1}{2} \| v \|^2_{L^2(\Omega)} (\theta N(1 + \ln \alpha) + \vartheta_0) - \theta \ln \| v \|^2_{L^2(\Omega)}).\end{aligned}$$

For α such that $\nu - \frac{\theta\alpha^2}{2\pi} = 0$ (i.e., $\alpha = \sqrt{\frac{2\pi\nu}{\theta}}$), we can deduce that

$$\mathcal{I}(v) \geq \frac{1}{2} \| v \|^2_{L^2(\Omega)} (\frac{\theta N}{2}(2 + \ln(2\pi\nu)) + \vartheta_0) - \theta \ln \| v \|^2_{L^2(\Omega)}).$$

We can deduce that:

- if $0 < \| \nabla v \|^2_{L^2(\Omega)} \leq R_N(\lambda_{min}, \nu, \vartheta_0)$, then $\mathcal{I}(v) \geq 0$,

where $R_N(\lambda_{min}, \nu, \vartheta_0) = \lambda_{min} e^{N+\vartheta_0} (2\pi\nu)^{\frac{N}{2}}$,

- if $\mathcal{I}(v) < 0$ then $\| \nabla v \|^2_{L^2(\Omega)} > R_N(\lambda_{min}, \nu, \vartheta_0)$.

  This completes the proof. □

Lemma A3. *Assume that the hypotheses* **(AAA)–(H2)** *and the initial/initial history conditions* $(\phi_0, \phi_1, h_0) \in H^1_0(\Omega) \times L^2(\Omega) \times L^2(\mathcal{Q}_0)$ *hold. We have, for* $E_a \in]0, d]$,

(i) *if there exists a real number* $t_1 \in [0, T_{max})$ *such that* $u(t_1) \in \mathcal{W}_2$ *and* $E(t_1) < E_a \leq d$, *then* $u(t) \in \mathcal{W}_2$, *for every* $t \in [t_1, T_{max})$. *Moreover, (for every* $t \in [t_1, T_{max})$*)*
$$\frac{\theta}{2} \| u(t) \|^2_{L^2(\Omega)} > E_a, \tag{A3}$$

(ii) *if there exists a real number* $t_1 \in [0, T_{max})$ *such that* $u(t_1) \in \mathcal{W}_1$ *and* $E(t_1) < E_a \leq d$, *then* $u(t) \in \mathcal{W}_1$, *for every* $t \in [t-1, T_{max})$. *Moreover, (for every* $t \in [t_1, T_{max})$*)*
$$\frac{\theta}{2} \| u(t) \|^2_{L^2(\Omega)} < E_a, \tag{A4}$$

T_{max} *is the maximal existence time of the weak solution to problem* (1).

Proof. Without loss of generality, we assume that $t_1 = 0$.

(i) If $u(0) \in \mathcal{W}_2$ and $E(0) < E_a \leq d$, then the solution u satisfies $u(t) \in \mathcal{W}_2$ and $E(t) < d$, for every $t \in [0, T_{max})$. In fact, since E is a nonincreasing function, then $E(t) \leq E(0) < d$, for every $t \in [0, T_{max})$. Assume that there exists t_0 such that $u(t_0) \in \mathcal{N}$, then from the definition of d and (11), we can deduce that $d \leq \mathcal{J}(u(t_0)) \leq E(t_0) \leq E(0) < d$, which is impossible. Consequently, $u(t) \in \mathcal{W}_2$, for every $t \in [0, T_{max})$. Moreover, from the definition of d and the fact that $\mathcal{I}(u(t)) < 0$ (since $u(t) \in \mathcal{W}_2$), we can deduce that
$$E_a \leq d \leq J(\bar{r}(u)u) = \exp(2\frac{\mathcal{I}(u)}{\theta \| u \|^2_{L^2(\Omega)}}) \frac{\theta}{2} \| u \|^2_{L^2(\Omega)} < \frac{\theta}{2} \| u \|^2_{L^2(\Omega)} .$$

(ii) If $u(0) \in \mathcal{W}_1$ and $E(0) < E_a \leq d$, then the solution u satisfies $u(t) \in \mathcal{W}_1$ and $E(t) < d$, for every $t \in [0, T_{max})$. In fact, since E is a nonincreasing function, then $E(t) \leq E(0) < E_a \leq d$, for every $t \in [0, T_{max})$. Suppose that there exists t_0 such that $u(t_0) \in \mathcal{N}$, then from the definition of d and (11), we can deduce that $d \leq \mathcal{J}(u(t_0)) \leq E(t_0) \leq E(0) < d$, which is impossible. Consequently, $u(t) \in \mathcal{W}_1$, for every $t \in [0, T_{max})$. Moreover, from the definition of $E(t)$ and the positivity of $\mathcal{I}(u(t))$, we obtain $\frac{\theta}{2} \parallel u(t) \parallel^2_{L^2(\Omega)} \leq E(t) \leq E(0) < E_a$. This completes the proof.

□

References

1. Belmiloudi, A. Robust control problems associated with time-varying delay nonlinear parabolic equations. *IMA J. Math. Control Inf.* **2003**, *20*, 305–334. [CrossRef]
2. Bentrcia, T.; Mennouni, A. On the solution behavior of a nonlinear time-fractional Klein–Gordon equation: Theoretical study and numerical validation. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *125*, 107384. [CrossRef]
3. Kolkovska, N.; Dimova, M.; Kutev, N. Nonexistence of global solutions to Klein-Gordon equations with variable coefficients power-type nonlinearities. *Open Math.* **2023**, *21*, 20220584. [CrossRef]
4. Killip, R.; Stovall, B.; Visan, M. Blow-up behaviour for the nonlinear Klein–Gordon equation. *Math. Ann.* **2014**, *358*, 289–350. [CrossRef]
5. Xu, R.Z. Global existence, blow up and asymptotic behavior of solutions for nonlinear Klein-Gordon equation with dissipative term. *Math. Methods Appl. Sci.* **2010**, *33*, 831–844.
6. De Martino, S.; Falanga, M.; Godano, C.; Lauro, G. Logarithmic Schrodinger-like equation as a model for magma transport. *Europhys. Lett.* **2003**, *63*, 472–475. [CrossRef]
7. Krolikowski, W.; Edmundson, D.; Bang, O. Unified model for partially coherent solitons in logarithmically nonlinear media. *Phys. Rev. E* **2000**, *61*, 3122–3126. [CrossRef]
8. Wazwaz, A.M. Gaussian solitary waves for the logarithmic-KdV and the logarithmic-KP equations. *Phys. Scripta* **2014**, *89*, 095206. [CrossRef]
9. Cherfils, L.; Miranville, A.; Zelik, S. The Cahn-Hilliard equation with logarithmic potentials. *Milan J. Math.* **2011**, *79*, 561–596. [CrossRef]
10. Barrow, J.D.; Parsons, P. Inflationary models with logarithmic potentials. *Phys. Rev. D* **1995**, *52*, 5576–5587. [CrossRef]
11. Bialynicki-Birula, I.; Mycielski, J. Wave equations with logarithmic nonlinearities. *Bull. Acad. Polon. Sci.* **1975**, *3*, 461–466.
12. Buljan, H.; Šiber, A.; Soljačić, M.; Schwartz, T.; Segev, M.; Christodoulides, D.N. Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media. *Phys. Rev. E* **2003**, *68*, 036607. [CrossRef]
13. Cherstvy, A.G.; Safdari, H.; Metzler, R. Anomalous diffusion, nonergodicity, and ageing for exponentially and logarithmically time-dependent diffusivity: Striking differences for massive versus massless particles. *J. Phys. D Appl. Phys.* **2021**, *54*, 195401. [CrossRef]
14. Enqvist, K.; McDonald, J. Q-balls and baryogenesis in the MSSM. *Phys. Lett. B* **1998**, *425*, 309–321. [CrossRef]
15. Rosen, G. Dilatation covariance and exact solutions in local relativistic field theories. *Phys. Rev.* **1969**, *183*, 1186. [CrossRef]
16. Scott, T.C.; Zhang, X.D.; Mann, R.B.; Fee, G.J. Canonical reduction for dilatonic gravity in 3 + 1 dimensions. *Phys. Rev. D* **2016**, *93*, 084017. [CrossRef]
17. Al-Gharabli, M.M.; Messaoudi, S. Existence and a general decay result for a plate equation with nonlinear damping and a logarithmic source term. *J. Evol. Equ.* **2018**, *18*, 105–125. [CrossRef]
18. Bartkowski, K.; Górka, P. One-dimensional Klein-Gordon equation with logarithmic nonlinearities. *J. Phys. A* **2008**, *41*, 355201. [CrossRef]
19. Cazenave, T.; Haraux, A. Équations d'évolution avec non linéarité logarithmique. *Ann. Fac. Des Sci. Toulouse Série* **1980**, *2*, 21–51. [CrossRef]
20. Chen, H.; Luo, P.; Liu, G. Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity. *J. Math. Anal. Appl.* **2015**, *422*, 84–98. [CrossRef]
21. Gomes, D.A.; Pimentel, E. Time-Dependent Mean-Field Games with Logarithmic Nonlinearities. *SIAM J. Math. Anal.* **2015**, *47*, 3798–3812. [CrossRef]
22. Gorka, P. Logarithmic quantum mechanics: Existence of the ground state. *Found. Phys. Lett.* **2006**, *19*, 591–601. [CrossRef]
23. Gorka, P. Logarithmic Klein-Gordon equation. *Acta Phys. Polon. B* **2009**, *40*, 59–66.
24. Ha, T.G.; Park, S.H. Blow-up phenomena for a viscoelastic wave equation with strong damping and logarithmic nonlinearity. *Adv. Differ. Equ.* **2020**, *235*, 17. [CrossRef]
25. Han, X.S. Global existence of weak solutions for a logarithmic wave equation arising from Q-ball dynamics. *Bull. Korean Math. Soc.* **2013**, *50*, 275–283. [CrossRef]
26. Hiramatsu, T.; Kawasaki, M.; Takahashi, F. Numerical study of Q-ball formation in gravity mediation. *J. Cos. Astr. Phys.* **2010**, *6*, 008. [CrossRef]

27. Hu, Q.; Zhang, H.; Liu, G. Global exitence and exponential growth of the solution for the logarithmic Boussinesq-type equation. *J. Math. Anal. Appl.* **2016**, *436*, 990–1001. [CrossRef]
28. Lian, W.; Ahmed, M.S.; Xu, R. Global existence and blow up of solution for semilinear hyperbolic equation with logarithmic nonlinearity. *Nonlinear Anal.* **2019**, *184*, 239–257. [CrossRef]
29. Yan, J.; Qian, X.; Zhang, H.; Song, S. Two regularized energy-preserving finite difference methods for the logarithmic Klein-Gordon equation. *J. Comput. Appl. Math.* **2021**, *393*, 113478. [CrossRef]
30. Zhang, H.; Liu, G.; Hu, Q. Exponential decay of energy for a logarithmic wave equation. *J. Partial Differ. Equ.* **2015**, *28*, 269–277.
31. Ait Benhassi, E.M.; Ammari, K.S.; Boulite, S.; Maniar, L. Feedback stabilization of a class of evolution equations with delay. *J. Evol. Equ.* **2009**, *9*, 103–121. [CrossRef]
32. Belmiloudi, A. Dynamical behavior of nonlinear impulsive abstract partial differential equations on networks with multiple time-varying delays and mixed boundary conditions involving time-varying delays. *J. Dyn. Control Syst.* **2015**, *21*, 95–146. [CrossRef]
33. Belmiloudi, A. *Stabilization, Optimal and Robust Control. Theory and Applications in Biological and Physical Sciences*; Springer: London, UK, 2008.
34. Belmiloudi, A. Nonlinear optimal control problems of degenerate parabolic equations with logistic time varying delays of convolution type. *Nonlinear Anal.* **2005**, *63*, 1126–1152. [CrossRef]
35. Chellaoua, H.; Boukhatem, Y. Blow-up result for an abstract evolution problem with infinite memory and time-varying delay. *Appl. Anal.* **2022**, *101*, 4574–4597. [CrossRef]
36. Feng, B.W. Well-posedness and exponential stability for a plate equation with time-varying delay and past history. *Z. Angew. Math. Physik.* **2017**, *68*, 1–24. [CrossRef]
37. Kafini, M.; Messaoudi, S. Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay. *Appl. Anal.* **2020**, *99*, 530–547. [CrossRef]
38. Kirane, M.; Said-Houari, B. Existence and asymptotic stability of viscoelastic wave equation with a delay. *Z. Angew. Math. Phys.* **2011**, *62*, 1065–1082. [CrossRef]
39. Liu, G.; Diao, L. Energy decay of the solution for a weak viscoelastic equation with a time-varying delay. *Acta. Appl. Math.* **2018**, *155*, 9–19. [CrossRef]
40. Liu, W. General decay rate estimate for the energy of a weak viscoealstic equation with internal time-varying delay term. *Taiwan. J. Math.* **2013**, *17*, 2101–2115. [CrossRef]
41. Nicaise, S.; Pignotti, C. Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. *SIAM J. Control Opt.* **2006**, *45*, 1561–1585. [CrossRef]
42. Nicaise, S.; Pignotti, C.; Valein, J. Exponential stability of the wave equation with boundary time-varying delay. *Discret. Contin. Dyn. Syst. Ser. S* **2011**, *4*, 693–722. [CrossRef]
43. Nieto-Guadarrama, F.; Villavicencio, J. Zitterbewegung and Klein-tunneling phenomena for transient quantum waves. *Phys. Rev. A* **2020**, *101*, 042104. [CrossRef]
44. Park, S.H. Global existence, energy decay and blow-up of solutions for wave equations with time delay and logarithmic source. *Adv. Differ. Equ.* **2020**, *631*, 17. [CrossRef]
45. Piskin, E.; Yuksekkaya, H. Local existence and blow up of solutions for a logarithmic nonlinear viscoelastic wave equation with delay. *Comput. Methods Differ. Equ.* **2021**, *9*, 623–636.
46. Yang, X.G.; Zhang, J.; Wang, S. Stability and dynamics of a weak viscoelastic system with memory and nonlinear time-varying delay. *Discret. Contin. Dyn. Syst.* **2020**, *40*, 1493–1515. [CrossRef]
47. Gross, L. Logarithmic Sobolev inequalities. *Am. J. Math.* **1975**, *97*, 1061–1083. [CrossRef]
48. Lions, J.L. *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*; Dunod; Gauthier-Villars: Paris, France, 1969.
49. Ambrosetti, A.; Rabinowitz, P.H. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **1973**, *14*, 349–381. [CrossRef]
50. Payne, L.E.; Sattinger, D.H. Saddle points and instability of nonlinear hyperbolic equations. *Isr. Math. J.* **1975**, *22*, 273–303. [CrossRef]
51. Nakao, M. Decay of solutions for some nonlinear evolution equations. *J. Math. Anal. Appl.* **1997**, *60*, 542–549. [CrossRef]