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Fixed Point Theorems for Set-Valued Contractions in Metric Spaces

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Abstract: In this paper, the concepts of Wardowski-type set-valued contractions and Işik-type set-valued contractions are introduced and fixed point theorems for such contractions are established. A positive answer to the open Question is given. Examples to support main theorems and an application to integral inclusion are given.

Keywords: fixed point; contraction; generalized contraction; set-valued contraction; metric space

MSC: 47H10; 54H25

1. Introduction and Preliminaries

Wardowski [1] introduced the notion of F-contraction mappings and the generalized Banach contraction principle by proving that every F-contractions on complete metric spaces have only one fixed point, where $F : (0, \infty) \to (-\infty, \infty)$ is a function such that

(F1) $F$ is strictly increasing;
(F2) for all sequence $\{s_n\} \subset (0, \infty)$,

$$\lim_{n \to \infty} s_n = 0 \iff \lim_{n \to \infty} F(s_n) = -\infty;$$
(F3) there exists a point $q \in (0, 1)$ : $\lim_{t \to 0^+} t^q F(t) = 0$.

Among several results ([2–18]) generalizing Wardowski’s result, Piri and Kumam [19] introduced the concept of Suzuki-type F-contractions and obtained related fixed point results in complete metric spaces, where $F : (0, \infty) \to (-\infty, \infty)$ is a strictly increasing function such that

(F4) $\inf F = -\infty$;
(F5) $F$ is continuous on $(0, \infty)$.

Nazam [20] generalized Wardowski’s result to four maps defined on b-metric spaces and proved the existence of a common fixed point by using conditions (F2), (F3) and

(F6) $\tau + F(rs_n) \leq F(s_n) \iff \tau + F(r^n s_n) \leq F(r^n-1 s_{n-1})$ for each $r > 0, n \in \mathbb{N}$, where $\tau > 0$.

Younis et al. [18] generalized Nazam’s result in b-metric spaces using only condition (F1). That is, they only used the strictly growth of $F : (0, \infty) \to (-\infty, \infty)$ and distinguished two cases: $s = 1$ and $s > 1$, where $s$ is the coefficient of b-metric spaces. Younis et al. [21] introduced the notion of Suzuki–Geraghty-type generalized $(F, \psi)$-contractions and generalized the result of [14] in partial b-metric spaces along with Geraghty-type contraction with conditions (F1), (F4) and (F5), and they gave applications to graph the theory and solution of some integral equations. Younis and Singh [22] extended Wardowski’s result to b-metric-like spaces and obtained the sufficient conditions for the existence of solutions of some class of Hammerstein integral equations and fractional differential equations.
On the other hand, Abbas et al. [23] and Abbas et al. [24] extended and generalized Wadorski’s result to two self mappings on partially ordered metric space and fuzzy mappings on metric spaces, respectively, and proved the existence of a fixed point using conditions (F1), (F2) and (F3).

Note that for a function $F : (0, \infty) \rightarrow (-\infty, \infty)$, the following are equivalent:

1. (F2) is satisfied;
2. (F4) is satisfied;
3. $\lim_{t \to 0^+} F(t) = -\infty$.

Hence, we have that

$$\lim_{n \to \infty} s_n = 0 \Rightarrow \lim_{n \to \infty} F(s_n) = -\infty$$

whenever (F4) holds.

Very recently, Fabiano et al. [25] gave a generalization of Wardowski’s result [1] by reducing the condition on function $F : (0, \infty) \rightarrow (-\infty, \infty)$ and by using the right limit of function $F : (0, \infty) \rightarrow (-\infty, \infty)$. They proved the following Theorem 1.

**Theorem 1** ([25]). Let $(E, \rho)$ be a complete metric space. Suppose that $T : E \rightarrow E$ is a map such that for all $x, y \in E$ with $\rho(Tx, Ty) > 0$,

$$\tau + F(\rho(Tx, Ty)) \leq F(\rho(x, y))$$

where $\tau > 0$ and $F : (0, \infty) \rightarrow (-\infty, \infty)$ is a function. If (F1) is satisfied, then $T$ possesses only one fixed point.

In [25], Fabiano et al. asked the following question:

**Question** ([25]). Can conditions for the function $F$ be reduced to (F1) and (F2), and can the proof be made simpler in some results for multivalued mappings in the same way as it was presented in [25] for single-valued mappings?

In this paper, we give a positive answer to the above question by extending the above theorem to set-valued maps and obtain a fixed point result for Işik-type set-valued contractions. We give examples to interpret main results and an application to integral inclusion.

Let $(E, \rho)$ be a metric space. We denote by $CL(E)$ the family of all nonempty closed subsets of $E$, and by $CB(E)$ the set of all nonempty closed and bounded subsets of $E$.

Let $H(\cdot, \cdot)$ be the generalized Pompeiu–Hausdorff distance [26] on $CL(E)$, i.e., for all $A, B \in CL(E)$,

$$H(A, B) = \begin{cases} \max\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A)\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise,} \end{cases}$$

where $\rho(a, B) = \inf\{\rho(a, b) : b \in B\}$ is the distance from the point $a$ to the subset $B$.

Let $\delta(A, B) = \sup\{\rho(a, b) : a \in A, b \in B\}$. When $A = \{x\}$, we denote $\delta(A, B)$ by $\delta(x, B)$.

For $A, B \in CL(E)$, let $D(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$. Then, we have that for all $A, B \in CL(E)$

$$D(A, B) \leq H(A, B) \leq \delta(A, B).$$
Note that the following Lemma 1 can be obtained by applying the assumptions of Lemma 1 to Theorem 4.29 of [27]. In fact, let $F : (0, \infty) \rightarrow (-\infty, \infty)$ be monotonically increasing ($x < y$ implies $F(x) \leq F(y)$) and $\{p_n\}$ be a given sequence of $(0, \infty)$ such that

$$\lim_{n \to \infty} p_n = l, \quad \text{where } l > 0.$$ 

Then, it follows from Theorem 4.28 of [27] that we obtain the conclusion of Lemma 1. Here, we give another proof of Lemma 1.

**Lemma 1.** Let $l > 0$, and let $\{t_n\}, \{s_n\} \subset (l, \infty)$ be non-increasing sequences such that

$$t_n < s_n, \forall n = 1, 2, 3, \ldots \quad \text{and} \quad \lim_{n \to \infty} t_n = \lim_{n \to \infty} s_n = l.$$ 

If $F : (0, \infty) \rightarrow (-\infty, \infty)$ is strictly increasing, then we have

$$\lim_{n \to \infty} F(t_n) = \lim_{n \to \infty} F(s_n) = F(l^+) \geq F(l).$$ 

where $F(l^+)$ denotes $\lim_{t \to l^+} F(t)$.

**Proof.** As $F$ is strictly increasing, the function $F_\epsilon : (0, \infty) \rightarrow F((0, \infty))$ defined by $F_\epsilon(t) = F(t)$ $\forall t \in (0, \infty)$, is bijective and continuous on $(0, \infty)$. We infer that

$$\lim_{t \to l^+} F_\epsilon(t) \geq F_\epsilon(l), \quad \lim_{n \to \infty} F_\epsilon(t_n) = \lim_{t \to l^+} F_\epsilon(t) \quad \text{and} \quad \lim_{n \to \infty} F_\epsilon(s_n) = \lim_{t \to l^+} F_\epsilon(t).$$

Since $\{t_n\}$ and $\{s_n\}$ are non-increasing, it follows from the strict increasingness of $F$ that

$$F_\epsilon(t_{n+1}) < F_\epsilon(t_n) < F_\epsilon(s_n) \leq F_\epsilon(s_{n-1}).$$

Hence, we obtain that

$$\lim_{t \to l^+} F_\epsilon(t) = \lim_{n \to \infty} F_\epsilon(t_{n+1}) \leq \lim_{n \to \infty} F_\epsilon(t_n) \leq \lim_{n \to \infty} F_\epsilon(s_n) \leq \lim_{n \to \infty} F_\epsilon(s_{n-1}) \leq \lim_{t \to l^+} F_\epsilon(t),$$

which implies

$$\lim_{n \to \infty} F_\epsilon(t_n) = \lim_{n \to \infty} F_\epsilon(s_n) = F_\epsilon(l^+).$$

Since $F_\epsilon(t) = F(t)$ $\forall t \in (0, \infty)$, we have the desired result. \Box

**Lemma 2** ([28]). Let $(E, \rho)$ be a metric space. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k)$ is the smallest index for which

$$m(k) > n(k) > k, \quad \rho(x_{m(k)}, x_{n(k)}) \geq \varepsilon \quad \text{and} \quad \rho(x_{m(k)-1}, x_{n(k)}) < \varepsilon. \quad (1)$$

Further, if

$$\lim_{n \to \infty} \rho(x_n, x_{n+1}) = 0,$$

then we have that

$$\lim_{k \to \infty} \rho(x_{n(k)}, x_{m(k)}) = \lim_{k \to \infty} \rho(x_{n(k)+1}, x_{m(k)}) = \lim_{k \to \infty} \rho(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \quad (2)$$

**Lemma 3.** Let $(E, \rho)$ be a metric space, and let $A, B \in \text{CL}(E)$. If $a \in A$ and $\rho(a, B) < c$, then there exists $b \in B$ such that $\rho(a, b) < c$.

**Proof.** Let $\varepsilon = c - \rho(a, B)$. It follows from the definition of infimum that there exists $b \in B$ such that $\rho(a, b) < \rho(a, B) + \varepsilon$. Hence, $\rho(a, b) < c$. \Box
Theorem 2. Let \( T : E \to CL(E) \) be a set-valued map with compact subsets of \( E \).

Proof. Assume that \( \tau \) is a constant such that \( \rho(a, b) + \phi(\rho(a, b)) < c \) for all \( a, b \in E \), where \( \phi : [0, \infty) \to [0, \infty) \) is a strictly increasing function. We infer that \( \rho(a, b) \leq d(a, b) \). Then, we have that

\[
\rho(a, b) + \phi(\rho(a, b)) < c.
\]

Lemma 5. If \( (E, \rho) \) is a metric space, then \( K(E) \subset CL(E) \), where \( K(E) \) is the family of nonempty compact subsets of \( E \).

2. Fixed Point Results

Let \( (E, \rho) \) be a metric space, and let \( F : (0, \infty) \to (-\infty, \infty) \) be a strictly increasing function. A set-valued map \( T : E \to CL(E) \) is called a Wardowski-type contraction if the following condition holds:

There exists a constant \( \tau > 0 \) such that for all \( x, y \in E \) with \( H(Tx, Ty) > 0 \),

\[
\tau + F(H(Tx, Ty)) \leq F(m(x, y)),
\]

where \( m(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2}[\rho(x, Ty) + \rho(y, Tx)]\} \).

We now prove our main result.

Theorem 2. Let \( (E, \rho) \) be a complete metric space. If \( T : E \to CL(E) \) is a Wardowski-type set-valued contraction, then \( T \) possesses a fixed point.

Proof. Let \( x_0 \in E \) be a point, and let \( x_1 \in Tx_0 \).

If \( x_1 \in Tx_1 \), then the proof is completed.

Assume that \( x_1 \notin Tx_1 \). Then, \( \rho(x_1, Tx_1) > 0 \), because \( Tx_1 \in CL(X) \). Hence, \( H(Tx_0, Tx_1) \geq d(x_1, Tx_1) > 0 \). From (3) we have that

\[
\tau + F(H(Tx_0, Tx_1)) \leq F(m(x_0, x_1)).
\]

We infer that

\[
m(x_0, x_1) = \max\{\rho(x_0, x_1), \rho(x_0, Tx_0), \rho(x_1, Tx_1), \frac{1}{2}[\rho(x_0, Tx_1) + \rho(x_1, Tx_0)]\}
\]

= \max\{\rho(x_0, x_1), \rho(x_1, Tx_1)\}, \text{ because that } \rho(x_0, Tx_0) \leq \rho(x_0, x_1) \text{ and }
\]

\[
\frac{1}{2}[\rho(x_0, Tx_1) + \rho(x_1, Tx_0)] \leq \frac{1}{2}[\rho(x_0, x_1) + \rho(x_1, Tx_1)]
\]
If \( m(x_0, x_1) = \rho(x_1, Tx_1) \), then from (4) we obtain that
\[
F(\rho(x_1, Tx_1)) < \tau + F(H(Tx_0, Tx_1)) \leq F(\rho(x_1, Tx_1)),
\]
which is a contradiction. Thus, \( m(x_0, x_1) = \rho(x_0, x_1) \). It follows from (4) that
\[
\frac{1}{2} \tau + F(\rho(x_1,Tx_1)) < \tau + F(H(Tx_0,Tx_1)) \leq F(\rho(x_0,x_1)). \tag{5}
\]
Since (F1) is satisfied, we obtain that
\[
\rho(x_1, Tx_1) < F^{-1}(\frac{1}{2} \tau + F(H(Tx_0,Tx_1))).
\]
Applying Lemma 3, there exists \( x_2 \in Tx_1 \) such that
\[
\rho(x_1, x_2) < F^{-1}(\frac{1}{2} \tau + F(H(Tx_0,Tx_1))),
\]
which implies
\[
F(\rho(x_1, x_2)) < \frac{1}{2} \tau + F(H(Tx_0,Tx_1)) \leq F(\rho(x_0,x_1)) - \frac{1}{2} \tau. \tag{6}
\]
Again from (3) we have that
\[
\frac{1}{2} \tau + F(\rho(x_2, Tx_2)) < \tau + F(H(Tx_1, Tx_2)) \leq F(\rho(x_1, x_2)) \tag{7}
\]
which implies
\[
\rho(x_2, Tx_2) < F^{-1}(\frac{1}{2} \tau + F(H(Tx_1, Tx_2))).
\]
By Lemma 3, there exists \( x_3 \in Tx_2 \) such that
\[
\rho(x_2, x_3) < F^{-1}(\frac{1}{2} \tau + F(H(Tx_1, Tx_2))).
\]
Hence, we obtain that
\[
F(\rho(x_2, x_3)) < \frac{1}{2} \tau + F(H(Tx_1, Tx_2)) \leq F(\rho(x_1, x_2)) - \frac{1}{2} \tau. \tag{8}
\]
Inductively, we have that for all \( n \in \mathbb{N} \),
\[
x_n \in Tx_{n-1}
\]
and
\[
F(\rho(x_n, x_{n+1})) < \frac{1}{2} \tau + F(H(Tx_{n-1}, x_n)) \leq F(\rho(x_{n-1}, x_n)) - \frac{1}{2} \tau. \tag{9}
\]
Because \( F \) is a strictly increasing function,
\[
\rho(x_n, x_{n+1}) < \rho(x_{n-1}, x_n), \ \forall n \in \mathbb{N}.
\]
Hence, there exists \( r \geq 0 \) such that
\[
\lim_{n \to \infty} \rho(x_n, x_{n+1}) = r.
\]
Assume that \( r > 0 \). By Lemma 1, we have that
\[
\lim_{n \to \infty} F(\rho(x_n, x_{n+1})) = \lim_{n \to \infty} F(\rho(x_{n-1}, x_n)) = \lim_{t \to r^+} F(t) = F(r^+) \geq F(r). \tag{10}
\]
Taking limit \( n \to \infty \) in (9) and using (10), we obtain that
\[
F(r^+) \leq F(r^+) - \frac{1}{2} \tau,
\]
which is a contradiction, because \( \tau > 0 \). Thus, we obtain that
\[
\lim_{n \to \infty} \rho(x_n, x_{n+1}) = 0. \tag{11}
\]

Now, we show that \( \{x_n\} \) is a Cauchy sequence. Assume that \( \{x_n\} \) is not a Cauchy sequence. Then, there exists \( \epsilon > 0 \) for which we can find subsequences \( \{x_{m(k)}\} \) and \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( m(k) \) is the smallest index for which (1) holds. That is, the following are satisfied:
\[
m(k) > n(k) > k, \ \rho(x_{m(k)}, x_{n(k)}) \geq \epsilon \quad \text{and} \quad \rho(x_{m(k)-1}, x_{n(k)}) < \epsilon.
\]

It follows from (3) that
\[
F(\rho(x_{n(k)+1}, Tx_{m(k)})) < \tau + F(\rho(x_{n(k)+1}, Tx_{m(k)})) \leq \tau + F(H(Tx_{n(k)}, Tx_{m(k)})) \leq F(m(x_{n(k)}, x_{m(k)})). \tag{12}
\]

We infer that
\[
\epsilon \leq \rho(x_{n(k)}, x_{m(k)}) \leq \max\{\rho(x_{n(k)}, x_{m(k)}), \rho(x_{n(k)}, Tx_{n(k)}), \rho(x_{m(k)}, Tx_{m(k)}),
\]
\[
\frac{1}{2} \left[ \rho(x_{n(k)}, Tx_{m(k)}) + \rho(x_{m(k)}, Tx_{n(k)}) \right]\}
\leq \max\{\rho(x_{n(k)}, x_{n(k)+1}), \rho(x_{m(k)}, x_{m(k)+1})\}
\leq \frac{1}{2} \left[ \rho(x_{n(k)}, x_{m(k)+1}) + \rho(x_{m(k)}, x_{n(k)+1}) \right]. \tag{13}
\]

Taking limit as \( k \to \infty \) on both sides of (13) and using (2), we obtain that
\[
\lim_{k \to \infty} m(x_{n(k)}, x_{m(k)}) = \epsilon. \tag{14}
\]

Since \( F \) is strictly increasing, from (12) we have that
\[
\rho(x_{n(k)+1}, Tx_{m(k)}) < F^{-1}(\tau + F(\rho(x_{n(k)+1}, Tx_{m(k)})).
\]

By applying Lemma 3, there exists \( y_{m(k)} \in Tx_{m(k)} \) such that
\[
\rho(x_{n(k)+1}, y_{m(k)}) < F^{-1}(\tau + F(\rho(x_{n(k)+1}, Tx_{m(k)})).
\]

Hence,
\[
F(\rho(x_{n(k)+1}, y_{m(k)})) < \tau + F(\rho(x_{n(k)+1}, Tx_{m(k)}). \tag{15}
\]

Thus, it follows from (12) that
\[
F(\rho(x_{n(k)+1}, y_{m(k)})) < \tau + F(\rho(x_{n(k)+1}, y_{m(k)})) < \tau + F(\rho(x_{n(k)+1}, Tx_{m(k)}) \leq \tau + F(H(Tx_{n(k)}, Tx_{m(k)}) \leq F(m(x_{n(k)}, x_{m(k))})
\]
which leads to
\[
\rho(x_{n(k)+1}, y_{m(k)}) < m(x_{n(k)}, x_{m(k)}), \ \forall k = 1, 2, 3, \ldots \tag{16}
\]
By taking \( \lim \sup \) as \( k \to \infty \) in (16) and using (14), we have that
\[
\lim_{k \to \infty} \sup \rho(x_{n(k)+1}, y_{m(k)}) \leq \epsilon. \tag{17}
\]
Since
\[
\rho(x_{n(k)+1}, Tx_{m(k)}) \leq \rho(x_{n(k)+1}, y_{m(k)}), \\
\rho(x_{n(k)+1}, x_{m(k)}) \\
\leq \rho(x_{n(k)+1}, Tx_{m(k)}) + \rho(Tx_{m(k)}, y_{m(k)}) \\
\leq \rho(x_{n(k)+1}, y_{m(k)}) + \rho(x_{m(k)+1}, y_{m(k)}).
\tag{18}
\]
Taking \( \lim \inf \) as \( k \to \infty \) in (18) and using (2), we obtain that
\[
\epsilon \leq \lim_{k \to \infty} \inf \rho(x_{n(k)+1}, y_{m(k)}). \tag{19}
\]
It follows from (17) and (19) that
\[
\lim_{k \to \infty} \rho(x_{n(k)+1}, y_{m(k)}) = \epsilon. \tag{20}
\]
By applying Lemma 1 to (15) with (14), (16) and (20), we obtain that
\[
F(e^+) \leq \tau + F(e^+) \leq F(e^+)
\]
which leads to a contradiction. Hence, \( \{x_n\} \) is a Cauchy sequence. From the completeness of \( E \), there exists
\[
x_* = \lim_{n \to \infty} x_n \in E.
\]
It follows from (3) that
\[
F(\rho(x_{n+1}, Tx_*)) < \tau + F(\rho(x_{n+1}, Tx_*)) \\
\leq \tau + F(H(Tx_n, Tx_*)) \leq F(m(x_n, x_*)), \tag{21}
\]
where \( m(x_n, x_*) = \max\{\rho(x_n, x_*), \rho(x_n, x_{n+1}), \rho(x_n, Tx_*), \frac{1}{2}[\rho(x_n, x_{n+1}) + \rho(x_n, Tx_*)]\} \).
Since \( F \) is strictly increasing, from (21) we have that
\[
\rho(x_{n+1}, Tx_*) < m(x_n, x_*), \tag{22}
\]
and thus
\[
\lim_{n \to \infty} \rho(x_{n+1}, Tx_*) = \lim_{n \to \infty} m(x_n, x_*) = \rho(x_*, Tx_*). \tag{23}
\]
Assume that \( \rho(x_*, Tx_*) > 0 \). By Lemma 1, we have that
\[
\lim_{n \to \infty} F(\rho(x_{n+1}, Tx_*)) = \lim_{n \to \infty} F(m(x_n, x_*)) \\
= \lim_{t \to \rho(x_*, Tx_*)} F(t) = F(\rho(x_*, Tx_*)^+). \tag{24}
\]
Applying (24) to (21), we obtain that
\[
F(\rho(x_*, Tx_*)^+) \leq \tau + F(\rho(x_*, Tx_*)^+) \leq F(\rho(x_*, Tx_*)^+)
\]
which leads to a contradiction. Hence, \( \rho(x_*, Tx_*) = 0 \), and \( x_* \in Tx_* \). \( \square \)

The following example interprets Theorem 2.
Example 1. Let $E = [0, 1]$ and $\rho(x, y) = |x - y|$, $\forall x, y \in E$. Then $(E, \rho)$ is a complete metric space. Define a set-valued map $T : E \rightarrow CL(E)$ by

$$Tx = \begin{cases} \{1\}, & (x = 0) \\ \{\frac{1}{2}, \frac{1}{2}\}, & (0 < x \leq 1). \end{cases}$$

Let $\tau = \ln \frac{2.1}{2}$ and $F(t) = \ln t$, $\forall t > 0$. We show that $T$ is a Wardowski-type set-valued contraction. We now consider the following two cases.

First, let $x = 0$ and $0 < y \leq 1$.

Then $H(Tx, Ty) = \frac{3}{5}$. We obtain that

$$\tau + F(H(Tx, Ty)) - F(\rho(x, Tx)) = \ln \frac{2.1}{2} + \ln \frac{3}{5} - \ln 1 = \ln 6.3 - \ln 10 \approx -0.46 < 0.$$

Thus,

$$\tau + F(H(Tx, Ty)) < F(\rho(x, Tx)),$$

which implies

$$\tau + F(H(Tx, Ty)) < F(m(x, y)).$$

Second, let $0 \leq x < 1$ and $y = 1$.

Then $H(Tx, Ty) = \frac{4}{5}$. We infer that

$$\tau + F(H(Tx, Ty)) - F(\rho(y, Ty)) = \ln \frac{2.1}{2} + \ln \frac{4}{5} - \ln 1 = \ln 8.4 - \ln 10 \approx -0.17 < 0.$$

Thus,

$$\tau + F(H(Tx, Ty)) < F(\rho(y, Ty))$$

which leads to

$$\tau + F(H(Tx, Ty)) < F(m(x, y)).$$

Hence, $T$ is a Wardowski-type set-valued contraction. The assumptions of Theorem 2 are satisfied. By Theorem 2, $T$ possesses two fixed points, $\frac{2}{5}$ and $\frac{1}{2}$.

Remark 1. Theorem 2 is a positive answer to Question 4.3 of [25].

Remark 2. Theorem 2 is an extension of Theorem 2.2 [13] to set-valued maps without conditions (F2) and (F3).

By Theorem 2, we have the following results.

Corollary 1. Let $(E, \rho)$ be a complete metric space. Suppose that $T : E \rightarrow CL(E)$ is a set-valued map such that for all $x, y \in E$ with $H(Tx, Ty) > 0$,

$$\tau + F(H(Tx, Ty)) \leq F(l(x, y))$$

(25)
where \( \tau > 0 \) and \( F : (0, \infty) \to (-\infty, \infty) \) is a function, and

\[
l(x, y) = \max \{ \rho(x, y), \frac{1}{2} [\rho(x, Tx) + \rho(y, Ty)], \frac{1}{2} [\rho(x, Ty) + \rho(y, Tx)] \}.
\]

If (F1) is satisfied, then \( T \) possesses a fixed point.

**Proof.** Since \( l(x, y) \leq m(x, y) \), \( F(l(x, y)) \leq F(m(x, y)) \). Thus, (25) implies (2). By Theorem 2, \( T \) possesses a fixed point. \( \square \)

**Corollary 2.** Let \((E, \rho)\) be a complete metric space. Suppose that \( T : E \to \text{CL}(E) \) is a set-valued map such that for all \( x, y \in E \) with \( H(Tx, Ty) > 0 \),

\[
\tau + F(H(Tx, Ty)) \leq F(\rho(x, y))
\]

(26)

where \( \tau > 0 \) and \( F : (0, \infty) \to (-\infty, \infty) \) is a function. If (F1) is satisfied, then \( T \) possesses a fixed point.

**Proof.** Since \( \rho(x, y) \leq m(x, y) \) and (F1) holds, (26) implies (2). By Theorem 2, \( T \) possesses a fixed point. \( \square \)

**Corollary 3.** Let \((E, \rho)\) be a complete metric space. Suppose that \( T : E \to \text{CL}(E) \) is a set-valued map such that for all \( x, y \in E \) with \( H(Tx, Ty) > 0 \),

\[
\tau + F(H(Tx, Ty)) \leq F(a \rho(x, y) + b \rho(x, Tx) + \rho(y, Ty)) + e \rho(x, Ty) + \rho(y, Tx))
\]

(27)

where \( \tau > 0 \) and \( F : (0, \infty) \to (-\infty, \infty) \) is a function, and \( a, b, c, e \geq 0 \) and \( a + b + c + 2e = 1 \).

If (F1) is satisfied, then \( T \) possesses a fixed point.

**Proof.** It follows from (27) that

\[
\tau + F(H(Tx, Ty)) \leq F(a \rho(x, y) + b \rho(x, Tx) + \rho(y, Ty)) + e \rho(x, Ty) + \rho(y, Tx))
\]

\[
= F(a \rho(x, y) + b \rho(x, Tx) + \rho(y, Ty)) + 2e \frac{1}{2} [\rho(x, Ty) + \rho(y, Tx)]
\]

\[
\leq F((a + b + c + 2e) \max \{ \rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2} [\rho(x, Ty) + \rho(y, Tx)] \})
\]

\[
= F(m(x, y)).
\]

By Theorem 2, \( T \) possesses a fixed point. \( \square \)

**Corollary 4.** Let \((E, \rho)\) be a complete metric space. Suppose that \( T : E \to \text{CL}(E) \) is a set-valued map such that for all \( x, y \in E \) with \( H(Tx, Ty) > 0 \),

\[
\tau + F(H(Tx, Ty)) \leq F(a \rho(x, y) + b \rho(x, Tx) + \rho(y, Ty)) + c \rho(x, Ty) + \rho(y, Tx))
\]

(28)

where \( \tau > 0 \) and \( F : (0, \infty) \to (-\infty, \infty) \) is a function, and \( a, b, c \geq 0 \) and \( a + 2b + 2c = 1 \).

If (F1) is satisfied, then \( T \) possesses a fixed point.
Theorem 3. Let 

Proof. It follows from (28) that
\[
\tau + F(H(Tx, Ty)) 
\leq F(\alpha(x, y) + b[\rho(x, Tx) + \rho(y, Ty))] + c[\rho(x, Ty) + \rho(Tx)])
\]

By Corollary 1, \( T \) possesses a fixed point. \( \square \)

Corollary 5. Let \((E, \rho)\) be a complete metric space. Suppose that \( T : E \to CL(E) \) is a set-valued map such that for all \( x, y \in E \) with \( H(Tx, Ty) > 0 \),
\[
\tau + F(H(Tx, Ty)) \leq F\left(\frac{1}{2}\rho(x, Ty) + \rho(y, Tx)\right)
\]

where \( \tau > 0 \) and \( F : (0, \infty) \to (-\infty, \infty) \) is a function. If (F1) is satisfied, then \( T \) possesses a fixed point.

Proof. Since \( \frac{1}{2}\rho(x, Ty) + \rho(y, Tx) \leq \rho(x, y) \) and (F1) holds, (29) implies (25). By Corollary 1, \( T \) possesses a fixed point. \( \square \)

Corollary 6. Let \((E, \rho)\) be a complete metric space. Suppose that \( T : E \to CL(E) \) is a set-valued map such that for all \( x, y \in E \) with \( H(Tx, Ty) > 0 \),
\[
\tau + F(H(Tx, Ty)) \leq F\left(\frac{1}{2}\rho(x, Ty) + \rho(y, Tx)\right)
\]

where \( \tau > 0 \) and \( F : (0, \infty) \to (-\infty, \infty) \) is a function. If (F1) is satisfied, then \( T \) possesses a fixed point.

Proof. Since \( \frac{1}{2}\rho(x, Ty) + \rho(y, Tx) \leq \rho(x, y) \) and (F1) holds, implies (25). By Corollary 1, \( T \) possesses a fixed point. \( \square \)

Remark 3. Corollary 4 is a generalization of the main theorem of [29]. Indeed, if \( F(t) = \ln t, \forall t > 0 \) and we take \( T \) to be the self-mapping of \( E \), then Corollary 4 becomes the main theorem of [29].

Nadler [30] extended Banach’s fixed point theorem to set-valued maps. We are calling it Nadler’s fixed point theorem. We now prove the following theorem, which is a generalization of Nadler’s fixed point theorem.

Theorem 3. Let \((E, \rho)\) be a complete metric space. Suppose that \( T : E \to CL(E) \) is an Išik-type set-valued contraction, i.e., for each \( x, y \in E \) and each \( u \in Tx \), there exists \( v \in Ty \) such that
\[
\rho(u, v) \leq \phi(\rho(x, y)) - \phi(\rho(u, v))
\]

where \( \phi : [0, \infty) \to [0, \infty) \) is a function such that
\[
\lim_{t \to 0^+} \phi(t) = 0.
\]

Then, \( T \) possesses a fixed point.

Proof. Let \( x_0 \in E \), and let \( x_1 \in Tx_0 \). Then there exits \( x_2 \in Tx_1 \) such that
\[
\rho(x_1, x_2) \leq \phi(\rho(x_0, x_1)) - \phi(\rho(x_1, x_2)).
\]
Again, there exists $x_3 \in Tx_2$ such that 
\[
\rho(x_2, x_3) \leq \phi(\rho(x_1, x_2)) - \phi(\rho(x_2, x_3)).
\]

Inductively, we have a sequence $\{x_n\} \subset E$ such that for all $n = 1, 2, 3, \ldots,$
\[
x_n \in Tx_{n-1} \text{ and } \rho(x_n, x_{n+1}) \leq \phi(\rho(x_{n-1}, x_n)) - \phi(\rho(x_n, x_{n+1})).
\]  
(33)

It follows from (33) that $\{\phi(\rho(x_{n-1}, x_n))\}$ is a non-increasing sequence and bounded below by 0. Hence, there exists $r \geq 0$ such that 
\[
\lim_{n \to \infty} \phi(\rho(x_{n-1}, x_n)) = r.
\]

We show that $\{x_n\}$ is a Cauchy sequence.
Let $m, n$ be any positive integers such that $m > n$. Then we have that 
\[
\rho(x_m, x_n) 
\leq \rho(x_m, x_{m+1}) + \rho(x_{m+1}, x_{m+2}) + \cdots + \rho(x_{n-1}, x_n) 
\leq \phi(\rho(x_{m-1}, x_m)) - \phi(\rho(x_m, x_n)) 
\leq \phi(\rho(x_{m-1}, x_m)) - r.
\]  
(34)

Letting $m, n \to \infty$ in (34), we obtain that 
\[
\lim_{n,m \to \infty} \rho(x_m, x_n) = 0.
\]

Thus, $\{x_n\}$ is a Cauchy sequence. It follows from the completeness of $E$ that 
\[
x_* = \lim_{n \to \infty} x_n \text{ exists.} 
\]  
(35)

Now, we show that $x_*$ is a fixed point for $T$.
It follows from (31) that for $x_n \in Tx_{n-1}$, there exists $v \in Tx_*$ such that 
\[
\rho(x_n, v) \leq \phi(\rho(x_{n-1}, x_*)) - \phi(\rho(x_n, v)) \leq \phi(\rho(x_{n-1}, x_*)).
\]  
(36)

Taking limit $n \to \infty$ in Equation (36) and using (32), we infer that 
\[
\lim_{n \to \infty} \rho(x_n, v) = 0
\]

which implies 
\[
x_* = v \in Tx_*. 
\]

\[\square\]

Example 2. Let $E = \{x_n : x_n = \sum_{k=1}^{n} \in \mathbb{N}\}$ and $\rho(x, y) = |x - y|, \forall x, y \in E$. Then $(E, \rho)$ is a complete metric space.
Define a map $T : E \to CL(E)$ by
\[
Tx = \begin{cases} 
\{x_1\}, & \text{if } x = x_1 \\
\{x_1, x_2, x_3, \ldots x_{n-1}\}, & \text{if } x = x_n.
\end{cases}
\]

Let $\phi(t) = \frac{1}{2} t, \forall t \geq 0.$
We show that condition (31) is satisfied.
Consider the following two cases.
First, let $x = x_1$ and $y = x_n, n = 2, 3, 4, \ldots$. 

Then, for \( u = x_1 \in Tx \), there exists \( v = x_1 \in Ty \) such that
\[
\rho(u, v) = 0 < \frac{1}{2}\rho(x_1, x_n) = \phi(\rho(x_1, x_n)) = \phi(\rho(x_1, x_n)) - \phi(\rho(u, v)).
\]

Second, let \( x = x_n \) and \( y = x_m, m > n, n = 2, 3, \cdots \).
For \( u = x_k \in Tx \) \((k = 1, 2, 3, \cdots, n - 1)\), there exists \( v = x_k \in Ty \) such that
\[
\rho(u, v) = 0 < \frac{1}{2}\rho(x_n, x_m) = \phi(\rho(x_n, x_m)) = \phi(\rho(x_n, x_m)) - \phi(\rho(u, v)).
\]

This show that \( T \) satisfies condition (31). Thus, all conditions of Theorem 3 hold. From Theorem 3, \( T \) possesses a fixed point, \( x_\ast = x_1 \).

**Corollary 7.** Let \((E, \rho)\) be a complete metric space. Suppose that \( T : E \to CL(E) \) is a set-valued map such that for each \( x, y \in E \),
\[
H(Tx, Ty) < \phi(\rho(x, y)) - \phi(H(Tx, Ty)),
\]
where \( \phi : [0, \infty) \to [0, \infty) \) is a strictly increasing function such that
\[
\lim_{t \to 0^+} \phi(t) = 0.
\]
Then, \( T \) possesses a fixed point.

**Proof.** Let \( x, y \in E \) and let \( u \in Tx \). As \( \phi \) is strictly increasing,
\[
\rho(u, Ty) + \phi(\rho(u, Ty)) < \phi(\rho(x, y)).
\]
Applying Lemma 4, there exists \( v \in Ty \) such that
\[
\rho(u, v) + \phi(\rho(u, v)) < \phi(\rho(x, y)).
\]
By Theorem 3, \( T \) possesses a fixed point. \( \square \)

From Theorem 3 we have the following result.

**Corollary 8 ([31]).** Let \((E, \rho)\) be a complete metric space. Suppose that \( f : E \to E \) is a map such that for each \( x, y \in E \),
\[
\rho(fx, fy) \leq \phi(\rho(x, y)) - \phi(\rho(fx, fy))
\]
where \( \phi : [0, \infty) \to [0, \infty) \) is a function such that
\[
\lim_{t \to 0^+} \phi(t) = 0.
\]
Then, \( f \) possesses a fixed point.

3. **Application**

In this section, we give an application of our result to integral inclusion. Let \([a, b] \subset (\infty, \infty)\) be a closed interval, and let \( C([a, b], (\infty, \infty)) \) be the family of continuous mapping from \([a, b] \) into \((\infty, \infty)\). Let \( E = C([a, b], (\infty, \infty)) \) and \( \rho(x, y) = \sup_{t \in [a,b]} |x(t) - y(t)| \) for all \( x, y \in E \). Then, \((E, \rho)\) is a complete metric space.

Consider the Fredholm type integral inclusion:
\[
x(t) \in \int_a^b K(t, s, x(s))ds + f(t), t \in [a, b]
\] (37)
where \( f \in E, K : [a, b] \times [a, b] \times (-\infty, \infty) \to CB((-\infty, \infty)) \), and \( x \in E \) is the unknown function.

Suppose that the following conditions are satisfied:

1. For each \( x \in E \), \( K(\cdot, x(s)) = K_x(\cdot, \cdot) \) is continuous;
2. There exists a continuous function \( Z : [a, b] \times [a, b] \to [0, \infty) \) such that for all \( t, s \in [a, b] \) and all \( u, v \in E \),

\[
|k_u(t, s) - k_v(t, s)| \leq Z(t, s)\rho(u(s), v(s))
\]

where \( k_u(t, s) \in K_u(t, s), k_v(t, s) \in K_v(t, s) \);
3. There exists a \( \alpha > 1 \) such that

\[
\sup_{t \in [a, b]} \int_a^b Z(t, s)ds \leq \frac{1}{2 + \alpha}.
\]

We apply the following theorem, known as Michael’s selection theorem, to the proof of Theorem 5.

**Theorem 4** ([32]). Let \( X \) be a paracompact space, and let \( B \) be a Banach space. Suppose that \( F : X \to B \) is a lower semicontinuous set-valued map such that for all \( x \in X \), \( F(x) \) is a nonempty closed and convex subset of \( B \). Then \( F : X \to B \) admits a continuous single valued selection.

Note that \( (-\infty, \infty) \) with absolute value norm is a Banach space and closed intervals and singleton of real numbers are a convex subset of \( (-\infty, \infty) \).

**Theorem 5.** Let \((E, \rho)\) be a complete metric space. If conditions (1st), (2nd) and (3rd) are satisfied, then the integral inclusion (37) has a solution.

**Proof.** Define a set-valued map \( T : E \to CB(E) \) by

\[
Tx = \{ y \in E : y(t) \in \int_a^b K(t, s, x(s))ds + f(t), t \in [a, b] \}.
\]

Let \( x \in E \) be given. For the set-valued map \( K_x(t, s) : [a, b] \times [a, b] \to CB((-\infty, \infty)) \), by applying Michael’s selection theorem, there exists a continuous map \( k_x(t, s) : [a, b] \times [a, b] \to (-\infty, \infty) \) such that

\[
k_x(t, s) \in K_x(t, s), \forall t, s \in [a, b].
\]

Thus,

\[
\int_a^b k_x(t, s)ds + f(t) \in Tx,
\]

and so \( Tx \neq \emptyset \).

Since \( f \) and \( k_x \) are continuous, \( Tx \in CB(E) \) for each \( x \in E \).

Let \( y_1 \in Tx \). Then,

\[
y_1(t) \in \int_a^b K(t, s, x_1(s))ds + f(t), t \in [a, b].
\]

Hence, there exists \( k_{x_1}(t, s) \in K_{x_1}(t, s), \forall t, s \in [a, b] \) such that

\[
y_1(t) = \int_a^b k_{x_1}(t, s)ds + f(t), \forall t, s \in [a, b].
\]
It follows from (2nd) that there exists \( z(t,s) \in K_{x_2}(t,s) \) such that

\[
|k_{x_1}(t,s) - z(t,s)| \leq Z(t,s)\rho(x_1(s), x_2(s)), \forall t, s \in [a,b].
\]

Let \( U : [a, b] \times [a, b] \to CB((-\infty, \infty)) \) be defined by

\[
U(t,s) = K_{x_2}(t,s) \cap \{ u \in (-\infty, \infty) : \rho(k_{x_1}(t,s), u) \leq \rho(x_1(s), x_2(s)) \}.
\]

From (1st) \( U \) is continuous. Hence, it follows that there exists a continuous map \( k_{x_2} : [a, b] \times [a, b] \to (-\infty, \infty) \) such that

\[
k_{x_2}(t,s) \in U(t,s), \forall t, s \in [a,b].
\]

Let

\[
y_2(t) = \int_a^b k_{x_2}(t,s)ds + f(t), \forall t, s \in [a,b].
\]

Then,

\[
y_2(t) \in \int_a^b K_{x_2}(t,s)ds + f(t) = \int_a^b K(t,s, x_2(s))ds + f(t), \forall t, s \in [a,b],
\]

and so \( y_2 \in Tx_2 \).

Thus, we obtain that

\[
\rho(y_1, y_2) = \left| \int_a^b k_{x_1}(t,s) - k_{x_2}(t,s)ds \right|
\leq \sup_{t \in [a,b]} \int_a^b |k_{x_1}(t,s) - k_{x_2}(t,s)|ds
\leq \sup_{t \in [a,b]} \int_a^b Z(t,s)ds\rho(x_1(s), x_2(s))
\leq \frac{1}{2 + \alpha} \rho(x_1(s), x_2(s)).
\]

Thus, we have that

\[
(1 + \frac{1}{2^\alpha}) \delta(Tx_1, Tx_2) \leq \frac{1}{2} \rho(x_1, x_2)
\]

which implies

\[
(1 + \frac{1}{2^\alpha}) H(Tx_1, Tx_2) \leq \frac{1}{2} \rho(x_1, x_2).
\]

Hence, we obtain that

\[
H(Tx_1, Tx_2) \leq \phi(\rho(x_1, x_2)) - \phi(\alpha H(Tx_1, Tx_2))
\leq <\phi(\rho(x_1, x_2)) - \phi(H(Tx_1, Tx_2)) \text{ where } \phi(t) = \frac{1}{2} t, \forall t \geq 0.
\]

By Corollary 7, \( T \) possesses a fixed point, and hence the integral inclusion (37) has a solution.  \( \square \)

4. Conclusions

Our results are generalizations and extensions of \( F \)-contractions and Işık contractions to set-valued maps on metric spaces. We give a positive answer to Question 4.3 of [25] and an application to integral inclusion.

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