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New Conditions for Testing the Oscillation of Solutions of Second-Order Nonlinear Differential Equations with Damped Term

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Abstract: This paper deals with the oscillatory behavior of solutions of a new class of second-order nonlinear differential equations. In contrast to most of the previous results in the literature, we establish some new criteria that guarantee the oscillation of all solutions of the studied equation without additional restrictions. Our approach improves the standard integral averaging technique to obtain simpler oscillation theorems for new classes of nonlinear differential equations. Two examples are presented to illustrate the importance of our findings.

Keywords: second order; damping term; oscillation; differential equations

MSC: 34C10; 34K11

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1. Introduction

In this work, we consider the asymptotic and oscillatory properties of solutions to a class of differential equations of the form

$$\left(r(\top)\left(y'(\top)\right)^{\xi}\right)' + p(\top)\left(y'(\top)\right)^{\xi} + q(\top)f(y(\top)) = 0, \ \top \geq \top_0 > 0, \tag{1}$$

where $\xi \geq 1$ is a ratio of two odd positive integers, and

(H_1) $r(\top) \in C^1\left(I_{\top_0}, \mathbb{R}^+\right)$,

$$p(\top) \text{ and } q(\top) \in C\left(I_{\top_0}, \mathbb{R}\right), \tag{2}$$

where $I_{\top_0} = [\top_0, \infty)$;

(H_2) $f(\top) \in C(\mathbb{R}, \mathbb{R})$, $yf(y) > 0$ for $y \neq 0$, and $f(y)/y^{\xi} \geq \mu$, for $y \neq 0$ and for some $\mu > 0$.

Definition 1 ([1])**.** *By a solution y of Equation (1), we mean a function $y \in C([\top_y^*, \infty))$, $\top_y^* \in I_{\top_0}$, which satisfies (1) on $[\top_y^*, \infty)$ for every $\top \geq \top_y^* \geq \top_0$, and $r(\top)(y'(\top))^{\xi} \in C^1([\top_y^*, \infty))$. Our attention is restricted to those solutions y of (1) that exist on some half line I and satisfy*

$$\sup\left\{|y(\top)| : \top_y \leq \top < \infty\right\} > 0, \ \top_y \geq \top_y^*.$$

Definition 2 ([2])**.** *A solution of Equation (1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is called non-oscillatory. Moreover, we say that Equation (1) oscillates if all its solutions oscillate, otherwise we say that it does not oscillate.*

Oscillation theory is considered one of the most important theories in all fields of science, including physics and engineering. It also plays a crucial role in the advancement of science and technology and in developing solutions to contemporary challenges. Through continued research and development in this field, significant progress can be achieved in various scientific and applied fields. It attempts to explain how objects or phenomena can change over time and is used to understand advanced natural and technological phenomena in the modern world. Currently, oscillation theory is of particular importance because of its practical applications in various fields such as communications, astronomy, and others. Its importance is not diminished even in the field of medicine, as it contributes to examining and analysing vital signals such as heartbeats and brain waves, which enables doctors to better evaluate health conditions and diagnose diseases.

Both the concept of symmetry and the oscillation theory play pivotal roles in understanding nature at different levels, and there can be overlaps and interactions between them in certain contexts of physical research. Oscillations can be viewed as a kind of temporal symmetry in many dynamical systems. Likewise, a physical system that oscillates regularly can be described as a type of temporal symmetry in that phenomenon, where conditions repeat periodically. On the other hand, the concept of symmetry plays an important role in describing fundamental interactions and particle interactions. It is worth noting that in some cases, symmetries lead to phenomena such as oscillation between different types of particles, such as neutrino oscillation. Theories that combine symmetry and oscillation may reveal new and unexpected phenomena. For example, broken symmetry could be responsible for generating particle masses in the Standard Model of particle physics; see, for instance [3–9].

Their approximations lead to very large linear systems and many properties can be understood using the approximated solutions [10–12]. Damping is crucial in control systems to prevent and minimise feedback-induced oscillations, so the damping differential equation is used in the models of mechanical systems, electrical circuits, acoustics, civil engineering, and control theory (economic cycles). Second-order differential equations with a damping term play a central role in many scientific and engineering fields, helping to understand and analyse dynamical systems and develop new technologies. They are used in many different fields, and the analysis of solutions to these equations often includes the term "damping" to describe resistance to movement or the gradual degradation of energy, which may be the result of movement in an elastic medium or under the presence of friction. Second-order differential equations can also be used to study damping vibrations in structures and machines. Damping is also used to describe the effect that resistors, capacitors, and coils have on the current and voltage in the system. The term damping describes the interaction between particles and their surrounding environment, and its importance extends to planetary science and astronomy, as these equations are used in studying the motion of planets and other astronomical bodies and the damping effects resulting from gravity and resistance [13–18].

A number of authors, such as [19–23], have paid attention to the oscillation of equations of the form

$$\big(r(\top)\varphi(y(\top))y'(\top)\big)' + p(\top)y'(\top) + q(\top)f(y(\top)) = 0.$$

On the other hand, the authors in [24–27] examined the oscillation of the following differential equation:

$$\big(r(\top)y'(\top)\big)' + p(\top)y'(\top) + q(\top)f(y(\top)) = 0 \tag{3}$$

and some special cases under conditions

$$p(\top) \geq 0 \text{ and } q(\top) > 0. \tag{4}$$

Moreover, Rogovchenko [28] showed a sufficient condition for the oscillation of Equation (3) assuming that (2) is satisfied and

$$f'(x) \text{ exists, } f'(x) \geq k \text{ for some } k > 0.$$

Also, Grace [29] shed light on finding criteria that guarantee the oscillatory behavior of all solutions of Equation (3).

Motivation

Adding a damping term to a differential equation may change the character of the solutions, for example, giving rise to oscillations. For example, if we look at equation

$$y'''(\top) + \left(4\top^3\right)^{-1} y(\top) = 0, \tag{5}$$

we find that all of its solutions are non-oscillatory as follows: $y_1 = 1.2696$, $y_2 = 1.83763$ and $y_3 = -0.10716$. While if we introduce the damping term $(4\top^2)^{-1} y'(\top)$ into Equation (5) it becomes as follows:

$$y'''(\top) + \left(4\top^2\right)^{-1} y'(\top) + \left(4\top^3\right)^{-1} y(\top) = 0.$$

We find that the behavior of its solutions is different, as we obtain two conjugate solutions (oscillatory solutions) and another solution (nonoscillatory solution) as follows: $y_{1,2} = 1.5490 \pm 0.3925i$ and $y_3 = -0.097912$. Thus, the study of this type of equations can be considered related to the oscillation theory. Through this work, we aim to establish new relationships that can be used to obtain new oscillation criteria for the solutions of the Equation (1). Our results are an extension of the findings presented in the previous literature, for example [19–22,28,30–32], which studied the Equation (1) with $\xi = 1$. On the other hand, our results work to develop and improve some of the previous findings; for instance, contrary to [24–27], we do not need additional constraints, including the constraints in (4). Therefore, the scope of application of our results extends to include more models that previous studies did not cover.

This paper is organized as follows. In Section 1, we present the significance of studying oscillations in many areas of life, which is the primary motivation for our study. In Section 2, we present previous findings and the abbreviations that will be used throughout the paper. Then, in Section 3, we provide some results of oscillation for the solutions of the studied equation. We also present and discuss some examples to illustrate the importance of our results in Section 4. Finally, in Section 5, we offer a brief overview of the main conclusions and present some suggestions and open problems for future work.

2. Preliminaries

Now, assume that $U \in C(D, \mathbb{R}^+)$ and

$$D = \{(\top, \varsigma) : \top_0 \leq \varsigma \leq \top < \infty\} \text{ and } D_0 = \{(\top, \varsigma) : \top_0 \leq \varsigma < \top < \infty\}.$$

If $U(\top, \top) = 0$, $U(\top, \varsigma) > 0$ and there is $(\top, \varsigma) \in D_0$ for U has a nonpositive continuous partial derivative with respect to ς, and there is there is a function $\widetilde{u} \in l_{\{loc\}}(D, \mathbb{R})$ such that

$$\frac{\partial U(\top, \varsigma)}{\partial \varsigma} = -\widetilde{u}(\top, \varsigma)(U(\top, \varsigma)))^{\xi/(\xi+1)}, \tag{6}$$

then we say that $U \in W_\xi$.

Now we present some results obtained previously, in order to compare them with the main results we reached in this work.

Theorem 1 ([29], Theorem 6). *Assume that there exist functions $\rho \in C^1(I_{\top_0}, \mathbb{R}^+)$, $\phi \in C(I_{\top_0}, \mathbb{R})$, and $U \in W_\xi$, $\xi = 1$ and $f'(x) \geq K$. If*

$$0 < \inf_{\varsigma \geq \top_0} \Phi(\top) \leq \infty, \tag{7}$$

$$\limsup_{\top \to \infty} U^{-1}(\top, \top^*) \int_{\top^*}^{\top} r(\varsigma)\rho(\varsigma)\left[\widetilde{u}(\top,\varsigma) - \pi(\varsigma)\sqrt{U(\top,\varsigma)}\right]^2 d\varsigma < \infty,$$

$$\limsup_{\top \to \infty} U^{-1}(\top, \top_0) \int_{\top^*}^{\top} (U(\top,\varsigma)\rho(\varsigma)q(\varsigma) - F(\top,\varsigma))d\varsigma \geq \phi(\top^*),$$

and

$$\lim_{\top \to \infty} \int_{\top_0}^{\infty} \phi_+^2(\varsigma)(v(\varsigma)r(\varsigma))^{-1/\xi} d\varsigma = \infty,$$

for $\top > \top_0$, and y $\top^* \geq \top_0$, where

$$\begin{aligned} \Phi(\top) &= \liminf_{\top \to \infty} \frac{U(\top,\varsigma)}{U(\top,\top_0)}, \\ \phi_+(\top) &= \max(\phi(\top), 0) \end{aligned}$$

and

$$F(\top,\varsigma) = (4K)^{-1}\rho(\varsigma)r(\varsigma)\left(\widetilde{u}(\top,\varsigma) - \pi(\varsigma)\sqrt{U(\top,\varsigma)}\right)^2,$$

where $\pi(\varsigma) = (r(\varsigma)\rho'(\varsigma) - p(\varsigma)\rho(\varsigma))/(r(\varsigma)\rho(\varsigma))$, then (3) is oscillatory.

Theorem 2 ([28], Theorem 2). *Assume that there exist functions $g \in C^1(I_{\top_0}, \mathbb{R}^+)$, $\phi \in C(I_{\top_0}, \mathbb{R})$. Assume further that $U \in W_\xi$, $\xi = 1$ and $f'(x) \geq K$. Set*

$$\widetilde{\rho}(\top) = \exp\left(-2\int^{\top} g(\varsigma)d\varsigma\right)$$

and

$$\Psi(\top) = K^{-1}\widetilde{\rho}(\top)\left(Kq(\top) - p(\top)g(\top) - [r(\top)g(\top)]' + r(\top)g^2(\top)\right).$$

If

$$\limsup_{\top \to \infty} U^{-1}(\top, \top_0) \int_{\top_0}^{\top} r(\varsigma)\rho(\varsigma)\left(\widetilde{u}(\top,\varsigma) + \frac{p(\varsigma)}{r(\varsigma)}\sqrt{U(\top,\varsigma)}\right)^2 d\varsigma < \infty,$$

$$\limsup_{\top \to \infty} U^{-1}(\top, \top^*) \int_{\top^*}^{\top} U(\top,\varsigma)\Psi(\varsigma) - \frac{r(\varsigma)\widetilde{\rho}(\varsigma)}{4K}\left(\widetilde{u}(\top,\varsigma) + \frac{p(\varsigma)}{r(\varsigma)}\sqrt{U(\top,\varsigma)}\right)^2 d\varsigma \geq \phi(\top^*)$$

and

$$\limsup_{\top \to \infty} \int_{\top_0}^{\top} \frac{\phi_+^2(\varsigma)}{\widetilde{\rho}(\varsigma)r(\varsigma)} d\varsigma = \infty,$$

then (3) is oscillatory.

Now, define the functions

$$v(\top) = \exp\left[-(\xi+1)\int_{\top_0}^{\top}\left(\rho^{1/\xi}(\varsigma) - \frac{p(\varsigma)}{(\xi+1)r(\varsigma)}\right)d\varsigma\right], \tag{8}$$

$$\psi(\top) = v(\top)\left[\mu q(\top) + r(\top)\rho^{(\xi+1)/\xi}(\top) - (r(\top)\rho(\top))' - p(\top)\rho(\top)\right] \tag{9}$$

and

$$\widetilde{F}(\top,\varsigma) = U(\top,\varsigma)\psi(\varsigma) - \beta^{\xi}(\xi+1)^{-(\xi+1)}v(\varsigma)r(\varsigma)\widetilde{u}^{\xi+1}(\top,\varsigma), \text{ for some } \beta \in [1,\infty)$$

3. Main Results

In this section we present two main results in the form of theorems. In the first theorem, using the Riccati technique, we obtained criteria that ensure the oscillation of all solutions of the studied equation. In the second theorem, using other analytical techniques, we present another criterion that reaches the same conclusion as in the first theorem.

Theorem 3. *Assume that there exists a function $\rho \in C^1([\top_0,\infty),\mathbb{R})$ such that, for some $\beta \in [1,\infty)$ and $U \in W_\xi$,*

$$\limsup_{\top\to\infty} U^{-1}(\top,\top_0)\int_{\top_0}^{\top}\Big[U(\top,\varsigma)\psi(\varsigma)-\beta^{\xi}(\xi+1)^{-(\xi+1)}v(\varsigma)r(\varsigma)\widetilde{u}^{\xi+1}(\top,\varsigma)\Big]\mathrm{d}\varsigma=\infty \quad (10)$$

with $v(\top)$ and $\psi(\top)$ given in (8) and (9), respectively. Then all solutions of Equation (1) are oscillatory.

Proof. Let $x(\top)>0$ be a solution of (1) for $\top\geq\top_0^*\geq\top_0$. Setting

$$u(\top)=v(\top)r(\top)\left[\frac{(x'(\top))^{\xi}}{x^{\xi}(\top)}+\rho(\top)\right],\quad \top\geq\top_0^*, \quad (11)$$

we have

$$\begin{aligned} u'(\top) &= \frac{v'(\top)}{v(\top)}u(\top)+v(\top)\frac{\left(r(\top)(x'(\top))^{\xi}\right)'}{x^{\xi}(\top)} \\ &\quad -\xi v(\top)r(\top)\left[\frac{u(\top)}{v(\top)r(\top)}-\rho(\top)\right]^{(\xi+1)/\xi}+v(\top)(r(\top)\rho(\top))'. \end{aligned} \quad (12)$$

According to [33], it is

$$N_1^{1+1/\xi}-(N_1-N_2)^{1+1/\xi}\leq\frac{N_2^{1/\xi}}{\xi}[(\xi+1)N_1-N_2]. \quad (13)$$

Taking

$$N_1=u(\top)/(v(\top)r(\top)) \text{ and } N_2=\rho(\top),$$

in (13), we obtain

$$\begin{aligned} (N_1-N_2)^{(\xi+1)/\xi} &= \left[\frac{u(\top)}{v(\top)r(\top)}-\rho(\top)\right]^{(\xi+1)/\xi}\geq\left(\frac{u(\top)}{v(\top)r(\top)}\right)^{(\xi+1)/\xi} \\ &\quad -\frac{\rho^{1/\xi}(\top)}{\xi}\left[(\xi+1)\frac{u(\top)}{v(\top)r(\top)}-\rho(\top)\right]. \end{aligned}$$

It follows from (1), (8), and (12) that

$$u'(\top)\leq-\psi(\top)-\xi\left(\frac{u^{\xi+1}(\top)}{v(\top)r(\top)}\right)^{1/\xi}, \quad (14)$$

which we rewrite as

$$U(\top,\varsigma)u'(\top)\leq-U(\top,\varsigma)\psi(\top)-U(\top,\varsigma)\xi\left(\frac{u^{\xi+1}(\top)}{v(\top)r(\top)}\right)^{1/\xi}. \quad (15)$$

By integrating (15) from $\top_1^*$ to $\top$, for all $\top \geq \top_1^* \geq \top_0^*$ and all $\beta \geq 1$, we have

$$\begin{aligned} & \int_{\top_1^*}^{\top} U(\top,\varsigma)\psi(\varsigma)\mathrm{d}\varsigma + \int_{\top_1^*}^{\top} \widetilde{u}(\top,\varsigma)(U(\top,\varsigma))^{\xi/(\xi+1)}u(\top)\mathrm{d}\varsigma \\ & +\frac{\xi}{\beta}\int_{\top_1^*}^{\top} U(\top,\varsigma)u^{\xi+1/\xi}(\varsigma)\frac{1}{(v(\varsigma)r(\varsigma))^{1/\xi}}\mathrm{d}\varsigma \\ \leq \quad & U(\top,\top_1^*)u(\top_1^*) - \xi\beta^{-1}(\beta-1)\int_{\top_1^*}^{\top} U(\top,\varsigma)u^{\xi+1/\xi}(\varsigma)\frac{1}{(v(\varsigma)r(\varsigma))^{1/\xi}}\mathrm{d}\varsigma. \end{aligned} \tag{16}$$

Also, according to [34], it is

$$M_1^{(\xi+1)/\xi} - \frac{\xi+1}{\xi}M_1^{1/\xi}M_2 \geq -\frac{1}{\xi}M_2^{(\xi+1)/\xi}. \tag{17}$$

Taking,

$$M_1^{(\xi+1)/\xi} = \frac{\xi}{\beta}\frac{U(\top,\varsigma)u^{(\xi+1)/\xi}(\varsigma)}{(v(\varsigma)r(\varsigma))^{1/\xi}}$$

and

$$M_2^{(\xi+1)/\xi} = -\frac{\xi\beta^{\xi}}{(\xi+1)^{\xi+1}}v(\varsigma)r(\varsigma)\widetilde{u}^{(\xi+1)}(\top,\varsigma),$$

in (17), we get

$$\begin{aligned} & u(\varsigma)\widetilde{u}(\top,\varsigma)(U(\top,\varsigma))^{\xi/(\xi+1)} + \frac{\xi}{\beta}U(\top,\varsigma)\left(\frac{u^{\xi+1}(\varsigma)}{v(\varsigma)r(\varsigma)}\right)^{1/\xi} \\ & +\frac{\beta^{\xi}}{(\xi+1)^{\xi+1}}v(\varsigma)r(\varsigma)\widetilde{u}^{\xi+1}(\top,\varsigma) \leq 0. \end{aligned} \tag{18}$$

Thus, it follows from (16) and (18) that

$$\begin{aligned} U(\top,\top_1^*)u(\top_1^*) \quad \geq \quad & \int_{\top_1^*}^{\top}\Big(U(\top,\varsigma)\psi(\varsigma) - \beta^{\xi}(\xi+1)^{-(\xi+1)}v(\varsigma)r(\varsigma)\widetilde{u}^{\xi+1}(\top,\varsigma)\Big)\mathrm{d}\varsigma \\ & +\xi\beta^{-1}(\beta-1)\int_{\top_1^*}^{\top} U(\top,\varsigma)\left(\frac{u^{\xi+1}(\varsigma)}{v(\varsigma)r(\varsigma)}\right)^{1/\xi}\mathrm{d}\varsigma. \end{aligned} \tag{19}$$

From property (6), we find

$$\begin{aligned} & \int_{\top_1^*}^{\top}\left[U(\top,\varsigma)\psi(\varsigma) - \frac{\beta^{\xi}}{(\xi+1)^{\xi+1}}v(\varsigma)r(\varsigma)\widetilde{u}^{\xi+1}(\top,\varsigma)\right]\mathrm{d}\varsigma \\ \leq \quad & U(\top,\top_1^*)|u(\top_1^*)| \leq U(\top,\top_0)|u(\top_1^*)|, \text{ for all } \top \geq \top_1^*. \end{aligned}$$

Therefore, it is

$$\begin{aligned} & \int_{\top_0}^{\top}\psi(\varsigma)U(\top,\varsigma) - \beta^{\xi}(\xi+1)^{-(\xi+1)}v(\varsigma)r(\varsigma)\widetilde{u}^{\xi+1}(\top,\varsigma)\mathrm{d}\varsigma \\ \leq \quad & U(\top,\top_0)\left(|u(\top_1^*)| + \int_{\top_0}^{\top_1^*}|\psi(\varsigma)|\mathrm{d}\varsigma\right). \end{aligned}$$

Hence,

$$\begin{aligned}
& \limsup_{\top\to\infty} U^{-1}(\top,\top_0)\int_{\top_0}^{\top}\Big[U(\top,\varsigma)\psi(\varsigma)-\beta^{\xi}(\xi+1)^{-(\xi+1)}v(\varsigma)r(\varsigma)\widetilde{u}^{\xi+1}(\top,\varsigma)\Big]\mathrm{d}\varsigma \\
\leq\ & |u(\top_1^*)|+\int_{\top_0}^{\top_1^*}|\psi(\varsigma)|\mathrm{d}\varsigma \\
<\ & \infty,
\end{aligned}$$

which contradicts (10). □

Theorem 4. *Assume that there exist functions $\rho\in C^1\big(I_{\top_0},\mathbb{R}\big)$, $\phi\in C\big(I_{\top_0},\mathbb{R}\big)$ and U belonging to the class W_ξ such that, for some $\beta>1$ and for $\top^*\geq\top_0$,*

$$\limsup_{\top\to\infty} U^{-1}(\top,\top^*)\int_{\top^*}^{\top}\widetilde{F}(\top,\varsigma)\mathrm{d}\varsigma\geq\phi(\top^*) \tag{20}$$

and

$$0<\inf_{\top\geq\top_0}\widetilde{U}(\top)\leq\infty. \tag{21}$$

If

$$\int_{\top_0}^{\infty}\phi_+^{\xi+1/\xi}(\varsigma)(v(\varsigma)r(\varsigma))^{-1/\xi}\mathrm{d}\varsigma=\infty, \tag{22}$$

then all solutions of Equation (1) are oscillatory.

Proof. Let $x(\top)>0$ be a solution of (1) on $[\top_0^*,\infty)$, $\top_0^*\geq\top_0$. Proceeding exactly as in the proof of Theorem 3, we get the inequality (19), it is obvious that

$$\begin{aligned}
\phi(\top_1^*)\ &\leq\ \limsup_{\top\to\infty}\frac{1}{U(\top,\top_1^*)}\int_{\top_1^*}^{\top}\left[U(\top,\varsigma)\psi(\varsigma)-\frac{\beta^{\xi}}{(\xi+1)^{\xi+1}}v(\varsigma)r(\varsigma)\widetilde{u}^{\xi+1}(\top,\varsigma)\right]\mathrm{d}\varsigma \\
&\leq\ u(\top_1^*)-\frac{\xi(\beta-1)}{\beta}\liminf\frac{1}{U(\top,\top_1^*)}\int_{\top_1^*}^{\top}U(\top,\varsigma)\left(\frac{u^{\xi+1}(\varsigma)}{v(\varsigma)r(\varsigma)}\right)^{\frac{1}{\xi}}\mathrm{d}\varsigma,
\end{aligned}$$

for all $\top>\top_1^*$ and for any $\beta\geq1$. This implies that

$$\phi(\top_1^*)+\xi\beta^{-1}(\beta-1)\liminf_{\top\to\infty}U^{-1}(\top,\top_1^*)\int_{\top_1^*}^{\top}U(\top,\varsigma)\frac{u^{\frac{\xi+1}{\xi}}(\varsigma)}{(v(\varsigma)r(\varsigma))^{\frac{1}{\xi}}}\mathrm{d}\varsigma\leq u(\top_1^*) \tag{23}$$

and

$$\begin{aligned}
& \liminf_{\top\to\infty}U^{-1}(\top,\top_1^*)\int_{\top_1^*}^{\top}U(\top,\varsigma)\frac{u^{\frac{\xi+1}{\xi}}(\varsigma)}{(v(\varsigma)r(\varsigma))^{\frac{1}{\xi}}}\mathrm{d}\varsigma \\
\leq\ & \beta(\xi\beta-\xi)^{-1}(u(\top_1^*)-\phi(\top_1^*)) \\
<\ & \infty.
\end{aligned} \tag{24}$$

It follows from (21) that there exists $v>0$ such that

$$\widetilde{U}(\top)>v(\top). \tag{25}$$

Now, we claim that

$$\int_{\top_1^*}^{\infty}\frac{u^{\frac{\xi+1}{\xi}}(\varsigma)}{(v(\varsigma)r(\varsigma))^{\frac{1}{\xi}}}\mathrm{d}\varsigma=\infty. \tag{26}$$

By (26) and for any constant $\eta > 0$, there exists a $\top_2^* > \top_1^*$ such that

$$\int_{\top_1^*}^{\top} \frac{u^{\frac{\xi+1}{\xi}}(\varsigma)}{(v(\varsigma)r(\varsigma))^{\frac{1}{\xi}}} d\varsigma > \frac{\eta}{v}, \text{ for all } \top \geq \top_2^*. \tag{27}$$

Integrating from $\top_1^*$ to $\top$ and using (27), we see that

$$\begin{aligned} & U^{-1}(\top,\top_1^*)\int_{\top_1^*}^{\top} U(\top,\varsigma)\frac{u^{\frac{\xi+1}{\xi}}(\varsigma)}{(v(\varsigma)r(\varsigma))^{\frac{1}{\xi}}} d\varsigma \\ = \; & U^{-1}(\top,\top_1^*)\int_{\top_1^*}^{\top} U(\top,\varsigma) d\left[\int_{\top_1^*}^{\varsigma} \frac{u^{\frac{\xi+1}{\xi}}(\zeta)}{(v(\zeta)r(\zeta))^{\frac{1}{\xi}}} d\zeta\right], \end{aligned}$$

for all $\top \geq \top_1^*$. Hence,

$$\begin{aligned} U^{-1}(\top,\top_1^*)\int_{\top_1^*}^{\top} U(\top,\varsigma)\left(\frac{u^{\xi+1}(\varsigma)}{v(\varsigma)r(\varsigma)}\right)^{\frac{1}{\xi}} d\varsigma &= U^{-1}(\top,\top_1^*)\int_{\top_1^*}^{\top}\left(-\frac{\partial U(\top,\varsigma)}{\partial\varsigma}\right) \\ &\quad\times\left(\int_{\top_1^*}^{\varsigma}\left(\frac{u^{\xi+1}(\zeta)}{v(\zeta)r(\zeta)}\right)^{\frac{1}{\xi}} d\zeta\right) d\varsigma \\ &\geq \frac{\eta}{v} U^{-1}(\top,\top_1^*)\int_{\top_2^*}^{\top}\left(-\frac{\partial U(\top,\varsigma)}{\partial\varsigma}\right) d\varsigma \\ &= \frac{\eta}{v} U(\top,\top_2^*)U^{-1}(\top,\top_1^*) \\ &\geq \frac{\eta}{v} U^{-1}(\top,\top_0)U(\top,\top_2^*). \end{aligned}$$

In view of (25), there exists a $\top_3^* \geq \top_2^*$ such that

$$\frac{U(\top,\top_2^*)}{U(\top,\top_0)} \geq v(\top), \text{ for all } \top \geq \top_3^*.$$

Thus, we have

$$U^{-1}(\top,\top_1^*)\int_{\top_1^*}^{\top} U(\top,\varsigma)\left(\frac{u^{\xi+1}(\varsigma)}{v(\varsigma)r(\varsigma)}\right)^{\frac{1}{\xi}} d\varsigma \geq \eta \text{ for } \top \geq \top_2^*.$$

Since $\eta > 0$, we obtain

$$\liminf_{\top\to\infty} U^{-1}(\top,\top_1^*)\int_{\top_1^*}^{\top} U(\top,\varsigma)\left(\frac{u^{\xi+1}(\varsigma)}{v(\varsigma)r(\varsigma)}\right)^{\frac{1}{\xi}} d\varsigma = \infty.$$

But according to (24), we note that

$$\int_{\top_1^*}^{\infty} u^{\frac{\xi+1}{\xi}}(\varsigma)(v(\varsigma)r(\varsigma))^{\frac{-1}{\xi}} d\varsigma = \infty,$$

and from (23), we get

$$\int_{\top_1^*}^{\infty}\left(\frac{\phi^{\xi+1}(\varsigma)}{v(\varsigma)r(\varsigma)}\right)^{\frac{1}{\xi}} d\varsigma \leq \int_{\top_1^*}^{\infty}\left(\frac{u^{\xi+1}(\varsigma)}{v(\varsigma)r(\varsigma)}\right)^{\frac{1}{\xi}} d\varsigma < \infty.$$

This completes the proof. □

Remark 1. *We observe that the restrictions imposed in Theorem 4 are more tractable than those in [29] [Theorem 6], since we do not have the complex hypotheses that appear there.*

Corollary 1. *If one of the following statements is true*
(**i**) *All conditions of Theorem 3 are satisfied;*
(**ii**) *All conditions of Theorem 4 are satisfied,*
then the equation

$$\left(r(\top)\left(y'(\top)\right)^{\xi}\right)' + p(\top)\left(y'(\top)\right)^{\xi} + \mu q(\top)y^{\xi}(\top) = 0, \tag{28}$$

where $\mu > 0$, is oscillatory.

4. Applications

Example 1. *Consider the following differential equation*

$$\left(\frac{1}{\top}\left(y'(\top)\right)^{\xi}\right)' + \cos\top\left(y'(\top)\right)^{\xi} + q(\top)y^{\xi}(\top) = 0, \tag{29}$$

for $\top \geq 1$ and $\xi \geq 1$, where

$$\begin{aligned} q(\top) &= \frac{1}{\top^3} - \frac{\top\cos\top - 2\top^{-1}}{\top(\xi+1)}\left(\frac{\top\cos\top - 2\top^{-1}}{\xi+1}\right) + \left(\frac{\top\cos\top - 2\top^{-1}}{\xi+1}\right)\cos\top \\ &\quad + \left(\frac{\top\cos\top - 2\top^{-1}}{\top^{\xi}(\xi+1)}\right)'. \end{aligned}$$

Let assume that

$$\rho(\top) = \left(\frac{\top\cos\top - 2\top^{-1}}{\xi+1}\right), \Psi(\top) = \top^{-1}, v(\top) = \top^3 \text{ and } \beta \geq 1.$$

By condition (10) in Corollary 1, we conclude that

$$\begin{aligned} &\limsup_{\top\to\infty} \frac{1}{U(\top,\top_0)} \int_{\top_0}^{\top} \left[U(\top,\varsigma)\psi(\varsigma) - \frac{\beta^{\xi}}{(\xi+1)^{\xi+1}} v(\varsigma) r(\varsigma) \widetilde{u}^{\xi+1}(\top,\varsigma)\right] d\varsigma \\ = &\limsup_{\top\to\infty} \frac{1}{\top^2} \int_1^{\top} \left[\frac{(\top-\varsigma)^2}{\varsigma} - \frac{2^{\xi+1}\beta^{\xi}}{(\xi+1)^{\xi+1}} \varsigma(\top-\varsigma)^{1-\xi}\right] d\varsigma = \infty. \end{aligned}$$

That is Equation (29) is oscillatory.

Example 2. *Consider the following differential equation*

$$0 = \left(\frac{(2\top^3+1)(2+\sin\top)}{2\top^3}\left(x'(\top)\right)^{\xi}\right)' + \left(\frac{3(2\top^3+1)(2+\sin\top)}{2\top^4}\right)\left(x'(\top)\right)^{\xi} + q(\top)x^{\xi}(\top), \tag{30}$$

for $\top \geq 1$ and $\xi \geq 1$, where

$$q(\top) = \frac{1}{\top^3}\left(\left(1 - \top^3 + 2\top^2 - 6\top\right)\sin\top + 12\top\right).$$

Now, assume that

$$\rho(\top) = 0, \Psi(\top) = \left(1 - \top^3 + 2\top^2 - 6\top\right)\sin\top + 12\top, v(\top) = \top^3, \beta = 2^{-1}(1+\xi)^{(\xi+1)/\xi}.$$

Set $U(\top,\varsigma) = (\top-\varsigma)^2$, $\widetilde{u}(\top,\varsigma) = 2(\top-\varsigma)^{(1-\xi)/(\xi+1)}$.

By condition (20) in Theorem 4, we find that

$$\begin{aligned}
&\limsup_{\top\to\infty}\frac{1}{U(\top,\top^*)}\int_{\top^*}^{\top}\left[U(\top,\varsigma)\psi(\varsigma)-\frac{\beta^{\xi}}{(\xi+1)^{\xi+1}}v(\varsigma)r(\varsigma)\widetilde{u}^{\xi+1}(\top,\varsigma)\right]d\varsigma \\
= &\limsup_{\top\to\infty}\frac{1}{\top^2}\int_{\top^*}^{\top}[(\top-\varsigma)^2\left(\left(1-\varsigma^3+2\varsigma^2-6\varsigma\right)\sin\varsigma+12\varsigma\right)-2^{\xi}\beta^{\xi}(\xi+1)^{-(\xi+1)} \\
&\times\left(2\varsigma^3+1\right)(2+\sin\varsigma)(\top-\varsigma)^{1-\xi}]d\varsigma \\
&\geq\limsup_{\top\to\infty}\frac{1}{\top^2}\int_{\top^*}^{\top}[(\top-\varsigma)^2\left(\left(1-\varsigma^3+2\varsigma^2-6\varsigma\right)\sin\varsigma+12\varsigma\right) \\
&-\left(2\varsigma^3+1\right)(2+\sin\varsigma)]d\varsigma \\
= &\ 16-\top^{*3}\cos\top^*+\top^{*2}(2\cos\top^*-6+3\sin\top^*)-4\top^*\sin\top^*-3\cos\top^*=\phi(\top^*).
\end{aligned}$$

It is easy to see that condition (22) is satisfied. Therefore, Equation (30) is oscillatory.

Remark 2. *If condition (10) in Theorem 3 fails, we can use Theorem 4.*

Remark 3. *By applying Theorems 3 and 4 when $\xi = 1$, we obtain the results presented in references [24] [Theorems 17 and 19], Our results also improve those of [25,26], which imposed more restrictions on the sign of the coefficients p and q.*

5. Conclusions

Through this paper, we focus on studying some oscillatory properties of a particular class of differential equations with damping. We note that the conditions in Theorem 2 are less restrictive and more efficient than those in Theorem 1. The improvement is due to the fact that the oscillation criteria obtained in this paper are more flexible compared to those appearing [29,35], because there are no restrictions on the damping coefficient $p(\top)$. Studying this type of equation without any restrictions imposed on the functions $p(\top)$ and $q(\top)$ is an extension and improvement of previous results. Defining the optional functions U and ρ and then using them in Theorems 3 and 4 to test the oscillatory behavior of Equation (1) (or its special cases) provide strong results for testing the oscillation of its solutions. Also, for $\xi = 1$, W_{ξ} generates the class of functions W, which was studied in [24]. On the other hand, our results do not need additional restrictions to ensure the oscillation of all solutions of Equation (1) [1,24–28]. It would be worth studying the following more general form of Equation (1):

$$\left(r(\top)(y'(\top))^{\xi}\right)'+p(\top)(y'(\top))^{\beta}+q(\top)f(y(\top))=0,$$

where ξ and β are positive. Furthermore, introducing a delay term into the function $f(y(\top))$ so that it has the form $f(y(\tau(\top)))$, where $\tau(\top) < \top$, will be a fertile field for researchers. Also, the possibility of providing different conditions without resorting to setting the restriction $\xi \geq 1$ remains an inspiring point for researchers as well.

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