Article
Convergence Results for Contractive Type Set-Valued Mappings
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Abstract: In this work, we study an iterative process induced by a contractive type set-valued mapping in a complete metric space and show its convergence, taking into account computational errors.

Keywords: complete metric space; contractive mapping; fixed point; iterate; set-valued mapping

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1. Introduction and Preliminaries

For more than sixty years, the fixed point theory has been an important area of nonlinear analysis. One of its main topics is the analysis of the existence of fixed points of contractive type maps. See, for example, [1–25]. Many existence results can be found in [6,7,21] This topic is well developed for single-valued mappings [1,3,8–10,17] as well as for set-valued mappings [14–16,20,22,26,27]. In this work, we analyze an iterative process induced by a contractive type set-valued mapping in a complete metric space. More precisely, we analyze a fixed point problem corresponding to a contractive type set-valued mapping $T$ acting in a complete metric space and study an iterative method which generates an approximate solution under the presence of computational errors which are always present in calculations.

Denote by $\mathbb{Z}$ the set of all integers, by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{R}_+$ the interval $(0, \infty)$. Assume that $(X, \rho)$ is a complete metric space. For every point $u \in X$ and every $a > 0$, set

$$B(u, a) = \{ y \in X : \rho(u, y) \leq a \}.$$  

For every point $u \in X$ and every nonempty set $A \subset X$, define

$$\rho(u, A) = \inf \{ \rho(u, y) : y \in A \}.$$  

For every $s \in \mathbb{R}_+$, set

$$\lfloor s \rfloor = \max (\mathbb{Z} \cap [s, \infty)).$$  

For each pair of nonempty sets $C, D \subset X$ define,

$$H(C, D) = \max \{ \sup \{ \rho(x, D) : x \in C \}, \sup \{ \rho(y, C) : y \in D \} \}.$$  

Assume that $\phi : [0, \infty) \to [0, 1]$ is a decreasing function such that

$$\phi(t) < 1 \text{ for each } t \in (0, \infty),$$  

$$\phi(0) = 1, \quad (1)$$  

In $T : X \to 2^X \setminus \{ \emptyset \}$, $T(x)$ is closed for each $x \in X$, and $X_0 \subset X$ is a nonempty compact set such that

$$T(X_0) \subset X_0 \quad (2)$$
and where, for every $x \in X_0$ and each $y \in X$,

$$H(T(x), T(y)) \leq \phi(\rho(x, y))\rho(x, y). \quad (3)$$

Clearly, $T$ is a contractive type set-valued mapping $[17,21]$. We are interested in solving the problem

$$\text{Find } z \in X \text{ satisfying } z \in T(z).$$

In practice, we can only obtain an approximate solution of this problem $z \in X$ such that $\rho(z, T(z))$ is small. In order to meet this goal, we use the following algorithm:

 Initialization: select an arbitrary point $x_0 \in X$ and a small positive constant $\delta$.

 Iterative step: given a current iteration point $x_n$, calculate the next iteration point $x_{n+1}$ such that

$$B(x_{n+1}, \delta) \cap \{\xi \in T(x_n) : \rho(\xi, x_n) \leq \delta + \rho(x_n, T(x_n))\} \neq \emptyset.$$ 

Clearly, at each iterative step $x_{n+1}$ is an approximate solution of the problem

$$\rho(\xi, x_n) \to \min, \, \xi \in T(x_n).$$

Since the space $X$ is not compact, a solution to the problem above does not exist in general. Note that we only assume the existence of the compact set $X_0$ but do not assume that it is given.

In this paper, we show that if $\delta$ is small enough, then our algorithm generates approximate solutions of the fixed point problem.

**Proposition 1.** There exists $x_T \in X_0$ such that

$$x_T \in T(x_T).$$

**Proof.** We prove that

$$\inf\{\rho(\xi, T(\xi)) : \xi \in X_0\} = 0.$$ 

Let us assume the contrary. Then,

$$\Delta := \inf\{\rho(x, T(x)) : x \in X_0\} > 0. \quad (4)$$

In view of (1), there is a positive number $\epsilon$ satisfying

$$\epsilon < \Delta(1 - \phi(\Delta))/2. \quad (5)$$

In view of (4), there exists

$$x_0 \in X_0 \quad (6)$$

such that

$$\rho(x_0, T(x_0)) < \Delta + \epsilon. \quad (7)$$

By (7), there exists

$$x_1 \in T(x_0) \quad (8)$$

such that

$$\rho(x_0, x_1) < \Delta + \epsilon. \quad (9)$$

It follows from (1), (3), (4), (6), (8) and (9) that

$$\Delta \leq \rho(x_1, T(x_1)) \leq H(T(x_0), T(x_1)) \leq \phi(\rho(x_0, x_1))\rho(x_0, x_1) \leq \phi(\Delta)\rho(x_0, x_1) \leq \phi(\Delta)(\Delta + \epsilon)$$

and

$$\Delta(1 - \phi(\Delta)) \leq \epsilon.$$
The inequality above contradicts relation (5). Therefore, 
\[ \Delta = 0. \]

Together with (4), this implies that the existence of \( \{x_n\}^{\infty}_{n=0} \subset X_0 \) for which 
\[ \lim_{n \to \infty} \rho(x_n, T(x_n)) = 0. \] (10)

Since \( X_0 \) is a compact, it has a convergent subsequence. As usual, we may assume without the loss of generality that there is a limit 
\[ x_T = \lim_{n \to \infty} x_n. \] (11)

By (3), (10) and (11), for every \( n \in \mathbb{N} \cup \{0\}, \)
\[ \rho(x_T, T(x_T)) \leq \rho(x_T, x_n) + \rho(x_n, T(x_n)) + H(T(x_n), T(x_T)) \leq 2\rho(x_T, x_n) + \rho(x_n, T(x_n)) \to 0 \text{ as } n \to \infty. \]

This implies that \( x_T \in T(x_T) \). Proposition 1 is proved. \( \square \)

**Proposition 2.** Let \( \epsilon \in \mathbb{R}_+ \). Then, there is \( \delta \in \mathbb{R}_+ \) such that for every \( x \in X_0 \) satisfying 
\[ \rho(x, T(x)) \leq \delta \]
there exists \( y \in X_0 \), for which 
\[ y \in T(y) \cap B(x, \epsilon). \]

**Proof.** Assume that the proposition does not hold. Then, for every \( n \in \mathbb{N} \), there is 
\[ x_n \in X_0 \]
for which 
\[ \rho(x_n, T(x_n)) \leq n^{-1} \] (12)
and 
\[ \{ y \in B(x_n, \epsilon) \cap X_0 : y \in T(y) \} = \emptyset. \] (13)

Since \( X_0 \) is compact, \( \{x_n\}^{\infty}_{n=1} \) has a convergent subsequence. Because, instead of the sequence, one can consider its convergent subsequence, we may assume without the loss of generality that there is a limit 
\[ x_\ast = \lim_{n \to \infty} x_n. \] (14)

In view of (3) and (12), for every \( n \in \mathbb{N} \),
\[ \rho(x_\ast, T(x_\ast)) \leq \rho(x_\ast, x_n) + \rho(x_n, T(x_n)) + H(T(x_n), T(x_\ast)) \leq 2\rho(x_\ast, x_n) + 1/n \to 0 \text{ as } n \to +\infty. \]

This implies that 
\[ x_\ast \in T(x_\ast). \]

By (14), for all sufficiently large \( n \in \mathbb{N} \),
\[ \rho(x_n, x_\ast) < \epsilon. \]

This contradicts (13) and completes the proof of Proposition 2. \( \square \)
2. The First Main Result

We begin with the following result, which shows that the approximate fixed point of $T$ is closed to its fixed points.

**Theorem 1.** Assume that $\epsilon \in \mathbb{R}_+$. Then, there is $\delta \in \mathbb{R}_+$ such that for every $x \in X$ which satisfies

$$\rho(x, T(x)) < \delta$$

there exists

$$y \in X_0 \cap B(x, \epsilon) \cap T(y).$$

**Proof.** Proposition 2 implies the existence of

$$\epsilon_1 \in (0, \epsilon/4)$$

such that the following is true:

(i) for each $z \in X_0$ satisfying

$$\rho(z, T(z)) \leq 3\epsilon_1$$

there exists

$$y \in X_0 \cap B(z, 2^{-1}\epsilon) \cap T(y).$$

Choose a positive number

$$\delta < 2^{-1}\epsilon_1(1 - \phi(\epsilon_1)).$$

(15)

Let $x \in X$ satisfy

$$\rho(x, T(x)) < \delta.$$  

(16)

By (16), there is

$$x_0 \in T(x)$$

(17)

for which

$$\rho(x, x_0) < \delta.$$  

(18)

We show that

$$\rho(x, x_0) < \epsilon_1.$$  

Assume the contrary. Then,

$$\rho(x, x_0) \geq \epsilon_1.$$  

(19)

There exists

$$x_1 \in X_0$$

(20)

such that

$$\rho(x, x_1) \leq \rho(x, x_0) + \delta.$$  

(21)

It follows from (2), (3) and (17)–(21) that

$$\rho(x, X_0) - \delta \leq \rho(x_0, X_0) \leq \rho(x_0, T(x_1))$$

$$\leq H(T(x), T(x_1)) < \phi(\rho(x, x_1))\rho(x, x_1)$$

$$\leq \phi(\epsilon_1)\rho(x, x_1) \leq \phi(\epsilon_1)\rho(x, X_0) + \delta$$

and

$$\rho(x, X_0) \leq \phi(\epsilon_1)\rho(x, X_0) + 2\delta.$$  

Together with (19), the inequality above implies that

$$\epsilon_1(1 - \phi(\epsilon_1)) \leq \rho(x, X_0)(1 - \phi(\epsilon_1)) \leq 2\delta.$$
This contradicts Equation (15) and proves that
\[ \rho(x, X_0) < \epsilon_1 \]
and that there is
\[ z \in X_0 \]  \hspace{1cm} (22)
for which
\[ \rho(x, z) < \epsilon_1. \]  \hspace{1cm} (23)
By (3), (15), (16) and (23),
\[
\rho(z, T(z)) \leq \rho(z, x) + \rho(x, T(x)) + H(T(x), T(z)) \\
\leq 2\rho(z, x) + \delta \leq 2\epsilon_1 + \delta \leq 3\epsilon_1. \]  \hspace{1cm} (24)
Property (i) and (22) and (24) imply that there is
\[ y \in X_0 \cap T(y) \cap B(z, 2^{-1}\epsilon). \]
In view of the inclusion above and (23),
\[ \rho(x, y) \leq \rho(x, z) + \rho(z, y) \leq \epsilon_1 + \epsilon/2 \leq \epsilon. \]
Theorem 1 is proved. \[ \square \]

3. The Second Main Result

Fix
\[ \theta \in X. \]
Set
\[ \text{diam}(X_0) = \sup \{\rho(z_1, z_2) : z_1, z_2 \in X_0\} < \infty. \]

Theorem 2. Let \( \epsilon \in (0, 1), M > 0, \)
\[ M_0 \geq 8(1 - \phi(1))^{-1}(2M + 3) + \rho(\theta, T(\theta)) + 4M + 1, \]  \hspace{1cm} (25)
\[ B(\theta, M) \cap X_0 \neq \emptyset, \]  \hspace{1cm} (26)
a positive number \( \delta_1 \) satisfy
\[ \delta_1 \leq 72^{-1}(1 - \phi(\epsilon/4))\epsilon, \]  \hspace{1cm} (27)
\[ n_1 = \lfloor 8(2M + 1)\delta_1^{-1}(1 - \phi(4^{-1}\delta_1))^{-1} \rfloor + 2, \]  \hspace{1cm} (28)
\[ n_2 = \lfloor 8M_0\epsilon_1^{-1}(1 - \phi(4^{-1}\epsilon))^{-1} \rfloor + 1, \]  \hspace{1cm} (29)
\[ n_0 = n_1 + n_2 \]  \hspace{1cm} (30)
and let \( \delta \in R_+ \) satisfy
\[ \delta \leq 16^{-1}(1 - \phi(\delta_1/4))\delta_1. \]  \hspace{1cm} (31)
Then, for each sequence \( \{x_i\}_{i=0}^{\infty} \subset X \) such that
\[ \rho(x_0, \theta) \leq M \]
and for each \( n \in N \cup \{0\}, \)
\[ B(x_{n+1}, \delta) \cap \{\xi \in T(x_n) : \rho(\xi, x_n) \leq \delta + \rho(x_n, T(x_n))\} \neq \emptyset \]  \hspace{1cm} (32)
the relations
\[ \rho(x_n, x_{n+1}) < \epsilon, \ B(x_n, \epsilon) \cap T(x_n) \neq \emptyset \]
hold for every integer \( n \geq n_0 \).

**Proof.** Assume that a sequence \( \{x_n\}_{n=0}^{\infty} \subset X \) satisfies
\[ \rho(x_0, \theta) \leq M \] (33)
and that, for every \( n \in \mathbb{N} \cup \{0\} \), (32) holds. In view of (26), there is
\[ \theta_1 \in B(\theta, M) \cap X_0. \] (34)
By (33) and (34),
\[ \rho(x_0, \theta_1) \leq \rho(x_0, \theta) + \rho(\theta, \theta_1) \leq 2M. \] (35)
Equations (3) and (34) imply that
\[
\rho(x_0, T(x_0)) \leq \rho(x_0, \theta_1) + \rho(\theta_1, T(\theta_1)) + H(T(\theta_1), T(x_0)) \\
\leq 2\rho(x_0, \theta_1) + \text{diam}(X_0) \leq 4M + \text{diam}(X_0). \] (36)
It follows from (32) and (36) that
\[ \rho(x_0, x_1) \leq \rho(x_0, T(x_0)) + 2\delta \leq 4M + \text{diam}(X_0) + 1. \] (37)
By (32), for each integer \( n \geq 0 \) there exists
\[ \xi_n \in B(x_{n+1}, \delta) \cap T(x_n) \] (38)
such that
\[ \rho(x_n, \xi_n) \leq \delta + \rho(x_n, T(x_n)). \] (39)
By (37) and (38),
\[ \rho(x_0, \xi_0) \leq \rho(x_0, x_1) + \rho(x_1, \xi_0) \leq 4M + \text{diam}(X_0) + 2. \]
For every \( n \in \mathbb{N} \cup \{0\} \), there is
\[ u_n \in X_0 \] (40)
for which
\[ \rho(u_n, x_n) \leq \rho(x_n, X_0) + \delta. \] (41)
Let \( n \geq 0 \) be an integer. By (3), (38), (40) and (41),
\[
\rho(u_{n+1}, x_{n+1}) \leq \rho(x_{n+1}, X_0) + \delta \\
\leq \rho(x_{n+1}, \xi_n) + \rho(\xi_n, X_0) + \delta \\
\leq 2\delta + \rho(\xi_n, T(u_n)) \leq 2\delta + H(T(x_n), T(u_n)) \\
\leq 2\delta + \phi(\rho(x_n, y_n))\rho(x_n, y_n). \] (42)
It follows from (3), (38) and (40) that
\[ \xi_n \in T(x_n), \ H(T(x_n), T(u_n)) \leq \phi(\rho(x_n, y_n))\rho(x_n, y_n). \] (43)
In view of (43), there exists
\[ v_n \in T(u_n) \] (44)
such that
\[ \rho(v_n, \xi_n) \leq \phi(\rho(x_n, u_n))\rho(x_n, u_n) + \delta. \] (45)
Equations (38), (42) and (45) imply that

\[
\rho(v_n, u_{n+1}) \leq \rho(v_n, \xi_n) + \rho(\xi_n, u_{n+1}) \\
\leq \phi(\rho(x_n, u_n))\rho(x_n, u_n) + \delta + \rho(x_{n+1}, \xi_n) + \rho(x_{n+1}, u_{n+1}) \\
\leq \phi(\rho(x_n, u_n))\rho(x_n, u_n) + 2\delta + 2\delta + \phi(\rho(x_n, u_n))\rho(x_n, u_n) \\
\leq 2\phi(\rho(x_n, u_n))\rho(x_n, u_n) + 4\delta.
\]

(46)

By (39) and (45),

\[
\rho(u_n, v_n) \leq \rho(u_n, x_n) + \rho(x_n, \xi_n) + \rho(\xi_n, v_n) \\
\leq \rho(u_n, x_n) + \rho(x_n, T(x_n)) + \delta + \phi(\rho(u_n, x_n))\rho(u_n, x_n) + \delta \\
= (1 + \phi(\rho(u_n, x_n)))\rho(u_n, x_n) + 2\delta + \rho(x_n, T(x_n)).
\]

(47)

Equations (3) and (40) imply that

\[
\rho(x_n, T(x_n)) \leq \rho(x_n, u_n) + \rho(u_n, T(u_n)) + H(T(u_n), T(x_n)) \\
\leq \rho(x_n, u_n) + \rho(u_n, T(u_n)) + \phi(\rho(u_n, x_n))\rho(u_n, x_n) \\
= (1 + \phi(\rho(u_n, x_n)))\rho(u_n, x_n) + \rho(u_n, T(u_n)).
\]

(48)

In view of (47) and (48),

\[
\rho(u_n, v_n) \leq (1 + \phi(\rho(u_n, x_n)))\rho(u_n, x_n) + 2\delta + \rho(x_n, T(x_n)) \\
\leq 2(1 + \phi(\rho(u_n, x_n)))\rho(u_n, x_n) + 2\delta + \rho(u_n, T(u_n)).
\]

(49)

It follows from (44), (46) and (49) that

\[
\rho(u_n, v_n) \leq 2(1 + \phi(\rho(u_n, x_n)))\rho(u_n, x_n) + 2\delta + \rho(u_n, T(u_n)).
\]

(50)

Let \( n \geq 0 \) be an integer. Equations (3), (40) and (50) imply that

\[
\rho(u_{n+1}, u_0) \leq \rho(u_{n+1}, v_{n+1}) + \rho(v_{n+1}, u_{n+1}) \\
\leq \rho(u_{n+1}, v_{n+1}) + 2\phi(\rho(x_{n+1}, u_{n+1}))\rho(x_{n+1}, u_{n+1}) + 4\delta \\
\leq \rho(u_{n+1}, T(u_{n+1})) + 2\delta + 2(1 + \phi(\rho(x_{n+1}, u_{n+1}))\rho(x_{n+1}, u_{n+1}) + 4\delta \\
\leq \rho(u_{n+1}, x_n) + \rho(v_{n+1}, T(u_{n+1})) \\
+ (2 + 4\phi(\rho(x_{n+1}, u_{n+1})))\rho(x_{n+1}, u_{n+1}) + 6\delta \\
\leq H(T(u_n), T(u_{n+1})) + 2\phi(\rho(x_n, u_n))\rho(x_n, u_n) + 4\delta \\
+ (2 + 4\phi(\rho(x_{n+1}, u_{n+1})))\rho(x_{n+1}, u_{n+1}) + 6\delta \\
\leq \phi(\rho(u_n, u_{n+1}))\rho(u_n, u_{n+1}) + 6\rho(x_{n+1}, u_{n+1}) + 10\delta + 2\rho(x_n, u_n).
\]

(51)

By (33), (34) and (41),

\[
\rho(u_0, x_0) \leq \rho(x_0, X_0) + 1 \leq \rho(x_0, \theta_1) + 1 \leq 2M + 1.
\]

(52)

We prove that there is \( k \in \{0, \ldots, n_1\} \) such that

(a)

\[
\rho(x_i, u_i) \leq \delta_1 \text{ for each integer } i \geq k
\]

(53)

and \( \rho(x_i, u_i) > \delta_1 \) for every non-negative integer \( i < k \).

Clearly, it is sufficient to show that there is

\[
k \in \{0, \ldots, n_1\}
\]

such that (53) holds.

Assume that \( i \in \mathbb{N} \cup \{0\} \) and that

\[
\rho(x_i, u_i) > \delta_1 / 4.
\]

(54)
By (31), (42), (44) and (54),
\[ \rho(x_{i+1}, u_{i+1}) \leq 2\delta + \phi(\rho(x_i, u_i)) \rho(u_i, u_i) \leq 2\delta + \phi(4^{-1}\delta_1) \rho(x_i, u_i) \]
and
\[ \rho(x_i, u_i) - \rho(x_{i+1}, u_{i+1}) \geq (1 - \phi(4^{-1}\delta_1)) \rho(x_i, u_i) - 2\delta \]
\[ \geq (1 - \phi(4^{-1}\delta_1))\delta_1/4 - 2\delta \geq (1 - \phi(4^{-1}\delta_1))\delta_1/8. \]

Thus, the following property holds:

(b) For every \( i \in \mathbb{N} \cup \{0\} \) satisfying \( \rho(x_i, u_i) > \delta_1/4 \), we have
\[ \rho(x_i, u_i) - \rho(x_{i+1}, u_{i+1}) \geq (1 - \phi(4^{-1}\delta_1))\delta_1/8. \]

Assume that \( i \in \mathbb{N} \cup \{0\} \) and that
\[ \rho(x_i, u_i) \leq \delta_1. \] (55)

There are two cases:
\[ \rho(x_i, u_i) \leq \delta_1/4; \] (56)
\[ \rho(x_i, u_i) > \delta_1/4. \] (57)

If (57) holds, then property (b) and (55) imply that
\[ \rho(x_{i+1}, u_{i+1}) \leq \rho(x_i, u_i) \leq \delta_1. \]

Assume that (56) holds. Then, by (31) and (42),
\[ \rho(x_{i+1}, u_{i+1}) \leq \rho(x_i, u_i) + 2\delta \leq 2\delta + \delta_1/4 \leq \delta_1. \]

Thus, in both cases we obtained
\[ \rho(x_{i+1}, u_{i+1}) \leq \delta_1. \]

Therefore, the following property holds:

(c) If an integer \( i \geq 0 \) and \( \rho(x_i, u_i) \leq \delta_1 \), then \( \rho(x_{i+1}, u_{i+1}) \leq \delta_1 \).

Property (c) implies that in order to show that there is \( k \in \{0, \ldots, n_1\} \) for which (a) is valid, it is sufficient to show the existence of \( k \in \{0, \ldots, n_1\} \) for which
\[ \rho(x_i, u_i) \leq \delta_1. \]

Assume the contrary. Then, for any \( i \in \{0, \ldots, n_1\} \), we have
\[ \rho(x_i, u_i) > \delta_1 \]

and in view of property (b),
\[ \rho(x_i, u_i) - \rho(x_{i+1}, u_{i+1}) \geq (1 - \phi(4^{-1}\delta_1))\delta_1/8. \]

Together with (52), this implies that
\[ 2M + 1 \geq \rho(u_0, x_0) \geq \rho(u_0, x_0) - \rho(u_{n_1}, x_{n_1}) \]
\[ = \sum_{i=0}^{n_1-1} (\rho(u_i, x_i) - \rho(u_{i+1}, x_{i+1})) \geq 8^{-1}n_1\delta_1(1 - \phi(4^{-1}\delta_1)) \]
and
\[ n_1 \leq 8(2M + 1)\delta_1^{-1}(1 - \phi(4^{-1}\delta_1))^{-1}. \]
We show that
\[ \rho(x_i, u_i) \leq \delta_1 \]
and, therefore, there is
\[ k \in \{0, \ldots, n_1\} \]
for which property (a) holds.

Properties (a) and (b) and (52) imply that for each \( i \in \{0, \ldots, k\} \),
\[ \rho(x_i, u_i) \leq \rho(x_0, u_0) \leq 2M + 1 \]
and that
\[ \rho(x_i, u_i) \leq 2M + 1 \] (58)
for each integer \( i \geq 0 \). By (51), for any \( n \in \mathbb{N} \cup \{0\} \),
\[ \rho(u_{n+1}, u_{n+2}) \leq \phi(\rho(u_n, u_{n+1}))\rho(u_n, u_{n+1}) + 6\rho(x_n, u_{n+1}) + 10\delta + 2\rho(x_n, u_n) \leq \phi(\rho(u_n, u_{n+1}))\rho(u_n, u_{n+1}) + 8(2M + 2). \] (59)

By (3), (25), (27), (31), (50) and (52),
\[ \rho(u_0, u_1) \leq \rho(u_0, v_0) + \rho(v_0, u_1) \leq \rho(u_0, v_0) + 2\phi(\rho(x_0, u_0))\rho(x_0, u_0) + 4\delta \leq \rho(u_0, T(u_0)) + 2\delta + 4\rho(x_0, u_0) + 2\rho(x_0, u_0) + 4\delta \leq \rho(u_0, T(u_0)) + 6\rho(x_0, u_0) + 1 \leq \rho(u_0, T(u_0)) + 6(2M + 1) + 1 \leq \rho(u_0, \theta) + \rho(\theta, T(\theta)) + H(T(\theta), T(\theta_0)) + 6(2M + 2) \leq 6(2M + 2) + \rho(\theta, T(\theta)) \leq \rho(\theta, T(\theta)) + 6(2M + 2) + 3M + 1 < M_0. \] (60)

We show that
\[ \rho(u_k, u_{k+1}) \leq M_0. \] (61)

If \( k = 0 \), then in view of (60), inequality (61) is true. Consider the case
\[ k > 0. \]

Assume that \( i \in \mathbb{N} \cup \{0\} \) and that
\[ \rho(u_i, u_{i+1}) \leq M_0. \]

There are two cases:
\[ \rho(u_i, u_{i+1}) \leq 1; \rho(u_i, u_{i+1}) > 1. \]

If
\[ \rho(u_i, u_{i+1}) \leq 1, \]
then, in view of (25), (51) and (58),
\[ \rho(u_{i+1}, u_{i+2}) \leq \rho(u_i, u_{i+1}) + 16M + 16 < 16M + 17 < M_0. \]

If
\[ \rho(u_i, u_{i+1}) > 1, \]
then, by (25), (51) and (58),
\[ \rho(u_{i+1}, u_{i+2}) \leq \phi(\rho(u_i, u_{i+1}))\rho(u_i, u_{i+1}) + 8(2M + 2) \leq \phi(1)\rho(u_i, u_{i+1}) + 8(2M + 2) \]
Thus, we have shown that if a non-negative integer \( i \) satisfies
\[
\rho(u_i, u_{i+1}) \leq M_0,
\]
then
\[
\rho(u_{i+1}, u_{i+2}) \leq M_0.
\]
Together with (60), this implies that \( \rho(u_i, u_{i+1}) \leq M_0 \) for all integers \( i \geq 0 \) and, in particular, (61) holds.

Assume that \( i \geq k \) is an integer for which
\[
\rho(u_i, u_{i+1}) \geq \epsilon / 4.
\]
By (27), (41), (51), the equation above, property (a) and the definition of \( k \),
\[
\rho(u_{i+1}, u_{i+2}) \leq \phi(\rho(u_i, u_{i+1}))\rho(u_i, u_{i+1}) + 6\rho(x_{i+1}, u_{i+1}) + 10\delta + 2\rho(x_i, u_i)
\]
\[
\leq \phi(\epsilon / 4)\rho(u_i, u_{i+1}) + 8\delta + 10\delta
\]
\[
\leq \phi(\epsilon / 4)\rho(u_i, u_{i+1}) + 9\delta_1
\]
and
\[
\rho(u_i, u_{i+1}) - \rho(u_{i+1}, u_{i+2})
\]
\[
\geq (1 - \phi(\epsilon / 4))\rho(u_i, u_{i+1}) - 9\delta_1
\]
\[
\geq (1 - \phi(\epsilon / 4))\epsilon / 4 - 9\delta_1 \geq (1 - \phi(\epsilon / 4))\epsilon / 8.
\]
Thus we have shown that the following property holds:
(d) If \( i \geq k_1 \) is an integer and \( \rho(u_i, u_{i+1}) \geq \epsilon / 4 \), then
\[
\rho(u_i, u_{i+1}) - \rho(u_{i+1}, u_{i+2}) \geq (1 - \phi(\epsilon / 4))\epsilon / 8.
\]
We show that there exists \( j \in \{k, \ldots, k + n_2\} \) such that
\[
\rho(u_j, u_{j+1}) \leq \epsilon / 4.
\]
Assume the contrary. Then, for each \( i \in \{k, \ldots, k + n_2\} \),
\[
\rho(u_i, u_{i+1}) > \epsilon / 4
\]
and, by property (d),
\[
\rho(u_i, u_{i+1}) - \rho(u_{i+1}, u_{i+2}) \geq (1 - \phi(\epsilon / 4))\epsilon / 8.
\]
Property (d), (61) and the relation above imply that
\[
M_0 \geq \rho(u_k, u_{k+1}) \geq \rho(u_k, u_{k+1}) - \rho(u_{k+n_2-1}, u_{k+n_2})
\]
\[
= \sum_{i=k}^{k+n_2-1} (\rho(u_i, u_{i+1}) - \rho(u_{i+1}, u_{i+2})) \geq 8^{-1}\epsilon n_2 (1 - \phi(\epsilon / 4))
\]
and
\[
n_2 \leq 8M_0\epsilon^{-1}(1 - \phi(\epsilon / 4))^{-1}.
\]
This contradicts (29) and proves that there exists
\[
p \in \{k, \ldots, k + n_2\}
\]
such that
\[ \rho(u_p, u_{p+1}) \leq \epsilon/4. \]

Assume that \( i \geq p \) is an integer and
\[ \rho(u_i, u_{i+1}) \leq \epsilon/2. \]

We show that
\[ \rho(u_{i+1}, u_{i+2}) \leq \epsilon/2. \]

There are two cases:
\[ \rho(u_i, u_{i+1}) \leq \epsilon/4; \]
\[ \rho(u_i, u_{i+1}) \geq \epsilon/4. \]

Assume that the first case holds. Then, in view of (30), (31), (51), the choice of \( p \) and property (a),
\[ \rho(u_{i+1}, u_{i+2}) \leq \rho(u_i, u_{i+1}) + 10 \delta + 8 \delta_1 \leq \epsilon/4 + 9 \delta_1 \leq \epsilon/2. \]

Assume that the second case holds. Then, by our assumption and property (b),
\[ \rho(u_{i+1}, u_{i+2}) \leq \rho(u_i, u_{i+1}) - 8^{-1} e(1 - \phi(\epsilon/4)) \leq \epsilon/2. \]

Thus, in both cases
\[ \rho(u_{i+1}, u_{i+2}) \leq \epsilon/2. \]

Thus, we have shown that if \( i \geq p \) is an integer and \( \rho(u_i, u_{i+1}) \leq \epsilon/2 \) holds, then the relation above is true. Combined with the choice of \( p \), this implies that for each integer \( i \geq p \),
\[ \rho(u_i, u_{i+1}) \leq \epsilon/2. \]

Property (a), (27), the choice of \( p \) and the equation above imply that for each integer \( i \geq p \),
\[ \rho(x_i, x_{i+1}) \leq \rho(x_i, u_i) + \rho(u_i, u_{i+1}) + \rho(u_{i+1}, x_{i+1}) \leq \epsilon/2 + 2 \delta_1 \leq 3 \epsilon/4. \]

Let \( i \geq p \) be an integer. By (27), (31), (38) and the equation above,
\[ \xi_i \in T(x_i) \cap B(x_{i+1}, \delta), \]
\[ \rho(x_i, \xi_i) \leq \delta + 3 \epsilon/4 \leq \epsilon, \]
\[ \xi_i \in B(x_i, \epsilon) \cap T(x_i). \]

Theorem 2 is proved. \( \square \)

4. Extensions

Theorems 1 and 2 easily imply the following result.

**Theorem 3.** Let \( \epsilon \in (0, 1) \), \( M > 0 \) and
\[ B(\theta, M) \cap X_0 \neq \emptyset. \]

Then, there exist a number \( \delta \in (0, \epsilon) \) and \( n_0 \in \mathbb{N} \) such that, for each sequence, \( \{x_i\}_{i=0}^\infty \subset X \). Then,
\[ \rho(x_0, \theta) \leq M \]
and, for every \( n \in \mathbb{N} \cup \{0\}, \)
\[ B(x_{n+1}, \delta) \cap \{\xi \in T(x_n) : \rho(\xi, x_n) \leq \delta + \rho(x_n, T(x_n))\} \neq \emptyset \]
there exists
\[ y_n \in X_0 \cap B(x_n, \epsilon) \cap T(y_n) \]
for every integer \( n \geq n_0 \).

Theorem 3 easily implies the following result.

**Theorem 4.** Let \( M > 0 \) and
\[ B(\theta, M) \cap X_0 \neq \emptyset. \]
Then, there exists \( \delta \in \mathbb{R}_+ \) such that, for every sequence \( \{\delta_i\}_{i=0}^{\infty} \subset (0, \delta) \) and every sequence \( \{x_i\}_{i=0}^{\infty} \subset X \),
\[ \rho(x_0, \theta) \leq M \]
and, for every \( n \in \mathbb{N} \cup \{0\} \),
\[ B(x_{n+1}, \delta_n) \cap \{ \xi \in T(x_n) : \rho(\xi, x_n) \leq \delta_n + \rho(x_n, T(x_n)) \} \neq \emptyset \]
the following relation is valid:
\[ \lim_{n \to \infty} \rho(x_n, \{ y \in X_0 : y \in T(y_0) \}) = 0. \]

5. An Example

Assume that \( X \) is a nonempty closed set in a Banach space \((E, \| \cdot \|)\), \( \rho(x, y) = \|x - y\| \), \( x, y \in X \), \( \theta \in X \) and that, for each \( x \in X \),
\[ tx + (1 - t)\theta \in X, \quad t \in [0, 1]. \]
Assume that \( \phi : [0, \infty) \to [0, 1] \) is a decreasing function such that
\[ \phi(0) = 1, \]
\[ \phi(t) < 1, \quad t \in (0, 1), \]
\( H : X \to 2^X \setminus \{\emptyset\}, T(x) \) is compact for each \( x \in X \),
\[ H(T(x), T(y)) \leq \phi(\rho(x, y)), \quad x, y \in X \]
and that there exists a nonempty bounded set \( K \subset X \) such that
\[ T(K) \subset K. \]
Then, there exists a unique compact set \( X_T \subset X \) such that
\[ T(A_T) = A_T. \]

6. Conclusions

In the present paper, we analyze a fixed point problem corresponding to a contractive type set-valued mapping \( T \) acting in a complete metric space. We discuss a simple algorithm which generates an approximate solution under the presence of computational errors which are always present in calculations. Note that at any iterative step only a current iteration point \( x_n \) and \( T(x_n) \) are known.

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