axioms

Article

# A Global Minimizer for Mass-Constrained Problem Revisited

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**Abstract:** We investigate the existence of solutions to the scalar field equation $-\Delta u = g(u) - \lambda u$ in $\mathbb{R}^N$, with mass constraint $\int_{\mathbb{R}^N} |u|^2 dx = a > 0, u \in H^1(\mathbb{R}^N)$. Here, $N \geq 3$; $g$ is a continuous function satisfying the conditions of the Berestycki–Lions type; $\lambda$ is a Lagrange multiplier. Our results supplement and generalize some of the results in L. Jeanjean, S.-S. Lu, *Calc. Var. Partial Differential Equations*. 61 (2022), Paper No. 214, 18, and J. Hirata, K. Tanaka, *Adv. Nonlinear Stud*. 19 (2019), 263–290.

**Keywords:** Schrödinger equations; global minimizers; normalized solution; Berestycki–Lions conditions; variational methods

**MSC:** 35J20; 35D30; 35J60

**Citation:** Long, C.-F.; Li, G.-D. A Global Minimizer for Mass-Constrained Problem Revisited. *Axioms* **2024**, *13*, 118. https://doi.org/10.3390/axioms13020118

Academic Editor: Behzad Djafari-Rouhani

Received: 14 January 2024
Revised: 6 February 2024
Accepted: 10 February 2024
Published: 13 February 2024



## 1. Introduction

In the fundamental paper [1], the authors investigated the following problem:

$$\begin{cases} -\Delta u = g(u) \ \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1}$$

where $\Delta u := -\sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$ and $N \geq 1$, who supplied the hypotheses for $g$:

$(g_1)$ $g : \mathbb{R} \to \mathbb{R}$ is continuous;
$(g_2)$ $-\infty < \liminf_{s\to 0^+} \frac{g(s)}{s} \leq \limsup_{s\to 0^+} \frac{g(s)}{s} = -l < 0$;
$(g_3)$ $\limsup_{s\to+\infty} \frac{g(s)}{g^{2^*-1}} \leq 0$, where $2^* = \frac{2N}{N-2}$ (if $N = 1, 2$, $\frac{2N}{N-2} = \infty$);
$(g_4)$ There exists $\xi > 0$ such that $G(\xi) := \int_0^\xi g(s)ds > 0$.

Then, by applying the method of minimizing constraints, they were able to determine the existence of the least-energy solution for the problem (1). Meanwhile, if $g$ satisfies $(g_1)$–$(g_4)$ and:

$(g_5)$ $g(-s) = -g(s)$.

Berestycki and Lions [2] then showed that there are infinite solutions to the problem (1). This kind of problem arises in a number of models in disciplines like mathematical physics, as an example, in the research of Bose–Einstein condensates or nonlinear optics [3,4]. Soliton propagation is described by the problem (1), which is a nontrivial solitary wave solution of the time-dependent Schrödinger equation. That is,

$$i\Phi_t - \Delta\Phi = g(\Phi),$$

where $\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ and $g$ satisfies $g(\varrho e^{i\theta}) = g(\varrho)e^{i\lambda}, \varrho, \lambda \in \mathbb{R}$. Additionally, we also call the condition $(g_1)$–$(g_4)$ the "Berestycki–Lions conditions". Note that, to guarantee the existence of a non-trivial solution to the problem (1), these conditions are nearly sufficient and necessary. Afterwards, many mathematicians devoted themselves to investigating the existence and characteristics of the solutions to a problem of a similar nature and have achieved a large number of research results. Please refer to the literature [5–10].

However, physicists tend to find a normalized solution to the problem (1), which is to provide the $L^2$-norm of the solution in advance. As the corresponding energy functional's critical point under the $L^2$-constraint, normalized solutions can be derived and $\lambda \in \mathbb{R}$ will show up as a multiplier of Lagrangians. Due to the variety of uses it offers, it has garnered much attention lately. In the case of pure power nonlinearity:

$$\begin{cases} -\Delta u = |u|^{p-2}u - \lambda u \quad \text{in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} |u|^2 dx = a, \quad u \in H^1(\mathbb{R}^N). \end{cases} \tag{2}$$

A standard approach for studying the problem (2) consists of looking for critical points of the energy functional:

$$\mathcal{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx,$$

constrained under the $L^2$-sphere

$$\mathcal{S}(a) := \left\{ u \in H^1(\mathbb{R}^N) \;\middle|\; \int_{\mathbb{R}^N} |u|^2 dx = a \right\}.$$

We recall the so-called $L^2$-critical exponent $\overline{p} = 2 + \frac{4}{N}$ from the Gagliardo–Nirenberg inequality [11]. We say that problem (2) is $L^2$-subcritical($L^2$-supercritical) if $p < \bar{p}$ ($p > \bar{p}$). It essentially determines whether the functional $\mathcal{J}$ is bounded from below on $\mathcal{S}(a)$ and, thus, leads to a difference in treatment. It can sufficiently reflect the difference in treatment methods; see [12,13]. In this article, we will focus on the $L^2$-subcritical case.

In the case of general nonlinearity, Shibata [14] obtained an interesting result, who considered

$$-\Delta u = g(u) - \lambda u \;\; \text{in } \mathbb{R}^N, \tag{3}$$

with the $L^2$-constraint

$$\int_{\mathbb{R}^N} |u|^2 dx = a > 0, \;\; u \in H^1(\mathbb{R}^N), \tag{4}$$

where $g$ satisfies the following conditions as, well as $(g_1)$ and $(g_4)$:

$(g_2')$ $\lim_{s\to 0} \frac{g(s)}{s} = 0$;

$(g_3')$ $\lim_{|s|\to\infty} \frac{g(s)s}{|s|^{2+\frac{4}{N}}} = 0$.

Then, the author discovered that there is a solution to the problem (3) with (4) when the mass $a$ is sufficiently large. After that, the Lagrangian formula of the problem (3) with (4)

$$\mathcal{E}(\lambda, u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2}\lambda\left(\|u\|_2^2 - a\right) - \int_{\mathbb{R}^N} G(u) dx,$$

was established by Hirata and Tanaka [15], who deduced the multiplicity of the normalized solutions to the problem (3) with (4) and used a symmetric mountain pass argument variant [16]. When $(g_3')$ is swapped out with the broader condition:

$(g_3'')$ $\limsup_{|s|\to\infty} \frac{|g(s)|}{|s|^{\frac{N+2}{N-2}}} < \infty$ and $\limsup_{|s|\to\infty} \frac{g(s)s}{|s|^{2+\frac{4}{N}}} \le 0$.

It was recently shown that there is a normalized solution to the problem (3) with (4) under the additional conditions $(g_1)$, $(g_2')$, and $(g_4)$ in [17].

Define $G(s) = \int_0^s g(t)dt$ and

$$G_+(s) = \begin{cases} \int_0^s \max\{g(t), 0\}dt, & \text{if } s \ge 0; \\ \int_s^0 \max\{-g(t), 0\}dt, & \text{if } s < 0. \end{cases}$$

Mederski and Schino [18] considered $g$ to satisfy:

$(\widetilde{g}_0)$ $g : \mathbb{R} \to \mathbb{R}$ is continuous and $g(0) = 0$;

$(\widetilde{g}_1)$ $\lim_{s\to 0} \frac{G_+(s)}{|s|^2} = 0$;

$(\widetilde{g}_2)$ If $N \geq 3$, then $\limsup_{|s|\to\infty} \frac{|g(s)|}{|s|^{2^*-1}} < \infty$, and if $N = 2$, then $\lim_{|s|\to\infty} \frac{g(s)}{e^{\alpha s^2}} = 0$ for all $\alpha > 4\pi$;

$(\widetilde{g}_3)$ $\lim_{|s|\to\infty} \frac{G_+(s)}{|s|^{2+\frac{4}{N}}} = 0$;

$(\widetilde{g}_4)$ There exists $\zeta_0 \neq 0$ such that $G(\zeta_0) > 0$.

The behavior of $g$ at the origin is allowed to be strongly sublinear, i.e., $\lim_{s\to 0} \frac{g(s)}{s} = -\infty$, which includes the case:

$$g(s) = \alpha s \ln s^2 + \mu|s|^{p-2}s$$

with $\alpha > 0$ and $\mu \in \mathbb{R}$, $2 < p \leq 2^*$ properly chosen. Then, the authors obtained the least-energy solutions and infinitely many solutions of the problem (3) with (4). There are many interesting results for the problem (3) with (4); refer to [19–23] and their references.

This work aimed to slightly extend the result of [15,17,18]. To put it another way, we used a more-general condition:

$(g_2'')$ $-\infty < \liminf_{|s|\to 0} \frac{g(s)}{s} \leq \limsup_{|s|\to 0} \frac{g(s)}{s} < +\infty$.

This is in place of $(g_2')$ to show that there is a normalized solution to the problem (3) with (4). Compared with [15,17,18], we added the situation of $\lim_{s\to 0} \frac{g(s)}{s} \in (-\infty, +\infty)$. Usually, finding the solution to the problem (3) with (4) is to find the critical point of the corresponding energy functional:

$$\mathcal{T}(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} G(u) dx$$

constrained under $\mathcal{S}(a)$. We call the solution $\widehat{u} \in \mathcal{S}(a)$ as the least-energy-normalized solution of the problem (3) with (4) if it has the lowest energy of all the non-trivial solutions to the problem (3) with (4). That is,

$$\mathcal{T}(\widehat{u}) = \inf\Big\{\mathcal{T}(u) \mid \mathcal{T}'|_{\mathcal{S}(a)}(u) = 0, u \in \mathcal{S}(a)\Big\}.$$

Unlike [15], we take into consideration the minimizing problem:

$$c(a) := \inf_{u\in\mathcal{S}(a)} \mathcal{T}(u); \tag{5}$$

if $c(a)$ has a minimizer $u$, then there exists a Lagrangian multiplier $\lambda \in \mathbb{R}$ such that $(u, \lambda) \in \mathcal{S}(a) \times \mathbb{R}$ is the least-energy-normalized solution to the problem (3) with (4).

Compared with [17], the introduction of condition $(g_2'')$ directly changes the behavior of functional $\mathcal{T}$ at the origin, resulting in several different situations of the global minimum $c(a)$. In this article, we proved the reachability of the global minimum $c(a)$ in different situations.

The following are our primary results:

**Theorem 1.** *Suppose that* $N \geq 3$, $(g_1)$, $(g_2'')$, $(g_3'')$, *and* $(g_4)$ *hold. If* $\limsup_{s\to 0} \frac{g(s)}{s} = -2m < 0$, *then a unique number* $a^* > 0$ *exists such that:*

*(i)* *If* $a > a^*$, *then* $c(a) < 0$ *is attained, and thus, the problem* (3) *possesses a least-energy-normalized solution* $u \in \mathcal{S}(a)$ *such that* $\mathcal{T}(u) = c(a) < 0$;

*(ii)* *If* $a = a^*$, *then* $c(a^*) = 0$ *is attained, and thus, the problem* (3) *possesses a least-energy-normalized solution* $u \in \mathcal{S}(a^*)$ *and* $\mathcal{T}(u) = c(a^*) = 0$;

(*iii*) *There exists* $a^{**} \in (0, a^*)$ *such that, if* $a^{**} < a < a^*$, *then* $0 < c(a) < ma$ *is attained, and thus the problem* (3) *possesses a least-energy-normalized solution* $u \in \mathcal{S}(a)$ *such that* $\mathcal{T}(u) = c(a) > 0$.

**Theorem 2.** *Suppose that* $N \geq 3$, $(g_1)$, $(g_2'')$, $(g_3'')$, *and* $(g_4)$ *hold. If* $\limsup_{s\to 0} \frac{g(s)}{s} = 2m > 0$, *then a unique number* $a_* > 0$ *exists such that, if* $a > a_*$, *then* $c(a) < -ma < 0$ *is attained, and thus, the problem* (3) *possesses a least-energy-normalized solution* $u \in \mathcal{S}(a)$ *such that* $\mathcal{T}(u) = c(a) < 0$.

**Remark 1.** *Note that the conditions* $(g_2')$ *and* $(g_3')$ *are only special cases of the conditions* $(g_2'')$ *and* $(g_3'')$. *We provide some examples of the nonlinear terms that satisfy* $(g_1)$, $(g_2'')$, $(g_3'')$, *and* $(g_4)$. *Example:*

(*i*) $g(s) = 2ms + |s|^{p-2}s$ *with* $m \in \mathbb{R}$ *and* $2 < p < 2 + \frac{4}{N}$.

(*ii*) $g(s) = 2ms - |s|^{q-2}s$ *with* $m \in \mathbb{R}^+$ *and* $2 < q \leq \frac{2N}{N-2}(N \geq 3)$.

(*iii*) $g(s) = 2ms + A|s|^{p-2}s - B|s|^{q-2}s$ *with* $A > 0$, $B > 0$, $m \in \mathbb{R}$, *and* $2 < q < p \leq \frac{2N}{N-2}(N \geq 3)$. *If* $A > 0$, $B > 0$, $m < 0$, *and* $p < q$, *it is clear that* $(g_4)$ *holds if and only if*

$$\frac{A}{p} D^{\frac{p-2}{q-p}} + m > \frac{B}{q} D^{\frac{q-2}{q-p}},$$

*where* $D = \frac{Aq(p-2)}{Bp(q-2)}$.

*The above examples are just some special cases, and our main theorem applies to more-general nonlinearity.*

**Remark 2.** *After assuming that* $g$ *is an odd function in Theorems 1 and 2, we can show that the problem* (3) *has a positive and radially symmetric least-energy-normalized solution by using the maximum principle [24] and the Schwarz symmetry rearrangement [25]. In addition, a proof similar to [17] can also be used to prove that the solution has a constant sign.*

The following is the article's structure. In Section 2, we modify the conditions of $g$ without changing the results and provide a proof of Theorem 1. In Section 3, after modifying the conditions similar to those in Section 2, we provide a proof of Theorem 2.

The following symbols are used for the subsequent content of this article:

- $H^1(\mathbb{R}^N)$ is represented as a standard Sobolev space, and its norm is denoted as:

$$\|u\| = \left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx\right]^{\frac{1}{2}};$$

- The radial function subspace of $H^1(\mathbb{R}^N)$ is represented by the symbol $H^1_r(\mathbb{R}^N)$;
- $L^p(\mathbb{R}^N)$ is represented as a standard Lebesgue space, and its norm is denoted as:

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}} \text{ for all } p \in [1, +\infty);$$

- $C_1, C_2, \cdots$ represent positive constants that can change depending on the location.

## 2. Proof of Theorem 1

Since the condition $(g_2'')$ is now a more-general assumption, as stressed in Section 1, we may establish Theorem 1 without altering the primary result by modifying the condition of $g$ and introducing an auxiliary functional. Initially, we establish:

$$\frac{1}{2} \limsup_{s\to 0} \frac{g(s)}{s} = -m < 0 \quad \text{and} \quad h(s) = g(s) + ms.$$

Thus, the problem (3) can be rewritten as

$$-\Delta u + mu = h(u) - \lambda u \quad \text{in } \mathbb{R}^N, \tag{6}$$

and having prescribed mass:

$$\int_{\mathbb{R}^N} |u|^2 dx = a. \tag{7}$$

Subsequently, the following are the assumptions of $h$:

$(h_1)$ $h : \mathbb{R} \to \mathbb{R}$ is continuous;

$(h_2)$ $-\infty < \liminf_{|s|\to 0} \frac{h(s)}{s} \leq \limsup_{|s|\to 0} \frac{h(s)}{s} = -m < 0$;

$(h_3)$ $\limsup_{|s|\to\infty} \frac{|h(s)|}{|s|^{\frac{N+2}{N-2}}} < \infty$ and $\limsup_{|s|\to\infty} \frac{h(s)s}{|s|^{2+\frac{4}{N}}} \leq 0$;

$(h_4)$ There exists $\xi \neq 0$ such that $H(\xi) - \frac{1}{2}m\xi > 0$, where $H(s) = \int_0^s h(t)dt$.

The following will convert the proof of Theorem 1 into a problem of solving (6) with (7) under the supposition $(h_1)$–$(h_4)$. Now, we rewrite the energy functional $\mathcal{T}$ as

$$\mathcal{L}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + m|u|^2 dx - \int_{\mathbb{R}^N} H(u)dx.$$

Although the form is different, the essence is the same. Moreover, define $f_1, f_2 \in C(\mathbb{R}, \mathbb{R})$ by

$$f_1(s) = \max\{g(s) + ms, 0\} \ \text{ and } \ f_2(s) = \max\{-(g(s) + ms), 0\} \quad \text{for all } s \geq 0.$$

Then, for any $s \geq 0$, the functions $f_1, f_2$ satisfy

$$f_1(s) - f_2(s) = h(s) \quad \text{and} \quad f_2(s) \geq ms.$$

For $s \leq 0$, extend $f_1$ and $f_2$ as odd functions. Then, $h = f_1 - f_2$ on $\mathbb{R}^+$, where $f_1, f_2 \geq 0$, and

$$\lim_{s\to 0} f_1(s) = o(s), \quad \lim_{s\to\infty} \frac{f_1(s)}{s^{1+\frac{4}{N}}} = 0. \tag{8}$$

Let $F_i(z) = \int_0^z f_i(s)ds, i = 1, 2$. At this point, we have completed the modification of the conditions and the introduction of the auxiliary functionals. Therefore, based on the above explanation, we obtain the subsequent lemmas.

**Lemma 1.** *Let $N \geq 3$, and suppose that $(h_1)$–$(h_4)$ hold. Moreover, let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$, and if $\lim_{n\to\infty} \|u_n\|_{2+\frac{4}{N}} = 0$, then*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} F_1(u_n)dx = 0.$$

**Proof.** Using (8), we can observe that, for each $\delta > 0$, there exists a $C_\delta > 0$ such that

$$|F_1(s)| \leq \delta|s|^2 + C_\delta|s|^{2+\frac{4}{N}} \ \text{ for all } \ s \in \mathbb{R}. \tag{9}$$

By (9), for $u \in H^1(\mathbb{R}^N)$, we obtain that

$$\left| \int_{\mathbb{R}^N} F_1(u)dx \right| \leq \delta \int_{\mathbb{R}^N} |u|^2 dx + C_\delta \int_{\mathbb{R}^N} |u|^{2+\frac{4}{N}} dx. \tag{10}$$

Since $\{u_n\}$ is a bounded sequence in $H^1(\mathbb{R}^N)$ and if $\lim_{n\to\infty} \|u_n\|_{2+\frac{4}{N}} = 0$, through (10), we obtain that

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} F_1(u_n)dx \right| \leq \delta \int_{\mathbb{R}^N} |u_n|^2 dx.$$

Given that $\delta > 0$ can be chosen arbitrarily, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F_1(u_n)dx = 0.$$

The proof is complete. □

Recalling the Gagliardo–Nirenberg inequality [11]:

$$\left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}} \le C(p,N)\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{\alpha}{2}}\left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{1-\alpha}{2}} \text{ for any } u \in H^1(\mathbb{R}^N), \quad (11)$$

where $N \ge 2$, $C(p,N)$, dependent on both $p$ and $N$, is a positive constant, $2 < p < \frac{2N}{N-2}$ and $\alpha = N\left(\frac{1}{2} - \frac{1}{p}\right)$.

**Lemma 2.** *Let $N \ge 3$, and suppose that $(h_1)$ and $(h_2)$ hold. Then, $\mathcal{L}$ is coercive and bounded from below on $\mathcal{S}(a)$ for each $a > 0$.*

**Proof.** Through $(h_1)$ and $(h_2)$, we can conclude that, for each $\delta > 0$, there exists a $C_\delta > 0$ such that

$$H(s) \le -\frac{m}{2}|s|^2 + \delta|s|^{2+\frac{4}{N}} + C_\delta|s|^\alpha \text{ for all } s \in \mathbb{R}, \quad (12)$$

where $2 < \alpha < 2 + \frac{4}{N}$. Hence, according to (12) and (11), we have

$$\begin{aligned}\int_{\mathbb{R}^N} H(u)dx \le &-\frac{m}{2}\int_{\mathbb{R}^N} |u|^2 dx + C\delta \int_{\mathbb{R}^N} |\nabla u|^2 dx \left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{2}{N}} \\ &+ CC_\delta\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{(\alpha-2)N}{4}}\left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{2\alpha-(\alpha-2)N}{4}}. \quad (13)\end{aligned}$$

From (13) and for any $a > 0$ and $u \in \mathcal{S}(a)$, we can obtain

$$\mathcal{L}(u) \ge \frac{1}{2}\left(1 - 2C\delta a^{\frac{2}{N}}\right)\|\nabla u\|_2^2 - CC_\delta a^{\frac{2\alpha-(\alpha-2)N}{4}}\|\nabla u\|_2^{\frac{(\alpha-2)N}{2}} + \frac{m}{2}a.$$

Since $\frac{(\alpha-2)N}{2} \in (0,2)$ and taking $\delta$ small enough, it is established that, on $\mathcal{S}(a)$, $\mathcal{L}$ is bounded from below and coercive. □

**Lemma 3.** *Let $N \ge 3$, and suppose that $(h_1)$–$(h_4)$ hold. Then, there exists $a^* > 0$ such that $c(a) < 0$ for each $a \in (a^*, +\infty)$.*

**Proof.** From $(h_4)$ and by applying the technique in [1], there is a $u \in H^1(\mathbb{R}^N)$ that makes

$$\int_{\mathbb{R}^N} H(u) - \frac{1}{2}m|u|^2 dx > 0.$$

Thus, for any $a > 0$, we let $w(x) := u\left(a^{-\frac{1}{N}}\|u\|_2^{\frac{2}{N}}x\right) \in \mathcal{S}(a)$. Then,

$$\begin{aligned}\mathcal{L}(w) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 + m|w|^2 dx - \int_{\mathbb{R}^N} H(w)dx \\ &= \frac{1}{2}\|u\|_2^{\frac{2(2-N)}{N}}\|\nabla u\|_2^2 a^{\frac{N-2}{N}} - \frac{\left(\int_{\mathbb{R}^N} H(u) - \frac{1}{2}m|u|^2 dx\right)}{\|u\|_2^2}a \\ &=: Aa^{\frac{N-2}{N}} - Ba.\end{aligned}$$

Due to the fact that there exists a sufficiently big $a^* > 0$, for each $a^* < a$, we have $c(a) \leq \mathcal{L}(w) < 0$. $\square$

Therefore, we define $a^*$ as

$$a^* := \inf\{a > 0 \mid c(a) < 0\}.$$

Because the following lemma proof is almost equivalent and may be found in [17], Lemma 2.2, we will not discuss it.

**Lemma 4.** *Let $N \geq 3$, and suppose that $(h_1)$–$(h_4)$ hold. Then, we have the following statements:*
$(i)$ *For every $a > a' > 0$, $c(a) \leq \frac{a}{a'} c(a')$;*
$(ii)$ *For every $a > a' > 0$, if $c(b)$ is attained, then $c(a) < \frac{a}{a'} c(a')$;*
$(iii)$ *For $a > 0$, the function $a \mapsto c(a)$ is continuous.*

Moreover, similar lemmas of the Brézis–Lieb ([26], Lemma 2.2) type can also be obtained as follows.

**Lemma 5.** *Let $N \geq 3$, and suppose that $(h_1)$ and $(h_2)$ hold. Let $\{u_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, and set $w_n := u_n - u$, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} [H(u_n) - H(u) - H(w_n)]dx = 0.$$

**Lemma 6.** *Let $N \geq 3$, and suppose that $(h_1)$–$(h_4)$ hold. Let $\{u_n\} \subset \mathcal{S}(a)$ represent a bounded minimization sequence about $c(a) \neq 0$ and $u_n \rightharpoonup u \neq 0$ in $H^1(\mathbb{R}^N)$. Then, $c(a) \neq 0$ is obtained by $u \in \mathcal{S}(a)$.*

**Proof.** Set $\int_{\mathbb{R}^N} |u|^2 = b \in (0, a]$ and $v_n = u_n - u$. Then, the Brézis–Lieb lemma ([27], Lemma 1.32) allows us to obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 dx - \int_{\mathbb{R}^N} |u|^2 dx = a - b \geq 0, \tag{14}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} |\nabla u|^2 dx. \tag{15}$$

Lemma 5 and (14) and (15) imply that

$$c(a) = \mathcal{L}(u_n) + o(1) = \mathcal{L}(v_n) + \mathcal{L}(u) + o(1). \tag{16}$$

If $a - b > 0$, according to Lemma 4-$(iii)$ and the definition of $c(a-b)$, we can obtain $\mathcal{L}(v_n) \geq c(a-b)$. Moreover, Lemma 4-$(i)$ and (16) then give us

$$c(a) \geq \mathcal{L}(u) + c(a-b) \geq c(b) + c(a-b) \geq \frac{b}{a} c(a) + \frac{a-b}{a} c(a) = c(a).$$

This means that $\mathcal{L}(u) = c(b)$ and $c(b)$ is attained at $u \in \mathcal{S}(b)$. But, then, still using (16) and Lemma 4-$(ii)$, we obtain a contradiction:

$$c(a) \geq c(b) + c(a-b) > \frac{b}{a} c(a) + \frac{a-b}{a} c(a) = c(a).$$

Therefore, we conclude that $a = b$, that is

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 dx = \int_{\mathbb{R}^N} |u|^2 dx. \tag{17}$$

This, in conjunction with the Lebesgue space interpolation theorem, yields

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p dx=\int_{\mathbb{R}^N}|u|^p dx,$$

for $2\le q<2^*$. Hence, by $(h_1)$–$(h_3)$, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}H(u_n)dx=\int_{\mathbb{R}^N}H(u)dx.$$

This combined with the norm's weak semicontinuity allows us to deduce that

$$c(a)=\lim_{n\to+\infty}\mathcal{L}(u_n)\ge\mathcal{L}(u)\ge c(a),$$

which leads to $\mathcal{L}(u)=c(a)$. Thus, $c(a)$ is obtained. Since $\mathcal{L}(u)=c(a)$ and from (15) and (16), it follows that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla v_n|^2 dx=0.$$

Therefore, combining this with (17) yields $u_n\to u$ in $H^1(\mathbb{R}^N)$. □

For notational convenience, let $a>0$ and $\rho(a)>0$, and set

$$\mathcal{R}(a):=\left\{u\in\mathcal{S}(a)\;\middle|\;\int_{\mathbb{R}^N}|\nabla u|^2dx\le\rho(a)\right\}.$$

**Lemma 7.** *Let $N\ge3$, and suppose that $(h_1)$–$(h_3)$ hold. For each $a>0$, there is $\rho(a)>0$ small enough in $\mathcal{R}(a)$. Then, for every $u\in\mathcal{R}(a)$, we obtain*

$$\mathcal{L}(u)\ge ma>0. \tag{18}$$

**Proof.** According to $(h_1)$–$(h_3)$, for each $\delta>0$, there exists $C_\delta>0$ such that

$$H(s)\le-\frac{m}{2}|s|^2+\delta|s|^{2+\frac{4}{N}}+C_\delta|s|^\beta, \tag{19}$$

where $2+\frac{4}{N}<\beta<2^*$. Thus, according to (19) and (11), we know that

$$\int_{\mathbb{R}^N}H(u)dx\le-\frac{1}{2}ma+\delta C_1a^{\frac{2}{N}}\|\nabla u\|_2^2+C_2C_\delta a^{\frac{2\beta-(\beta-2)N}{4}}\|\nabla u\|_2^{\frac{(\beta-2)N}{2}}. \tag{20}$$

Take $\delta>0$ small enough such that $\delta C_1a^{\frac{2}{N}}<\frac{1}{8}$. Since $\frac{(\beta-2)N}{2}\in(2,\frac{2N}{N-2})$ and $\rho(a)>0$ small enough, for all $u\in\mathcal{R}(a)$,

$$\int_{\mathbb{R}^N}H(u)dx\le-\frac{1}{2}ma+\frac{1}{4}\|\nabla u\|_2^2. \tag{21}$$

Then,

$$\begin{aligned}\mathcal{L}(u)&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2+m|u|^2dx-\int_{\mathbb{R}^N}H(u)dx\\&\ge\frac{1}{2}\|\nabla u\|_2^2+\frac{1}{2}ma+\frac{1}{2}ma-\frac{1}{4}\|\nabla u\|_2^2\\&=\frac{1}{4}\|\nabla u\|_2^2+ma\ge ma.\end{aligned}$$

This indicates that (18) is valid. □

**Remark 3.** *From (20), one can see that $\rho(a)\le\rho(a^*)$ in $\mathcal{R}(a)$ for any $a^*\le a$.*

**Lemma 8.** *Let $N \geq 3$, and suppose that $(h_1)$–$(h_4)$ hold; if $a = a^*$, then, $c(a^*) = 0$ is obtained by $u \in \mathcal{S}(a^*)$.*

**Proof.** Let $a_n := a^* + \frac{1}{n}$ for any $n \in \mathbb{N}^+$. Since $c(a_n) < 0$ by Lemma 3, one may choose $\{u_n\} \subset \mathcal{S}(a_n)$ such that

$$c(a_n) \leq \mathcal{L}(u_n) \leq \frac{1}{2}c(a_n) < 0 \quad \text{for each } n \in \mathbb{N}^+. \tag{22}$$

Lemma 2 leads us to the conclusion that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. It can be seen from (22) and Lemma 7 that

$$\lim_{n\to\infty} \|\nabla u_n\|_2^2 \geq \rho(a^*) > 0. \tag{23}$$

Define

$$\delta := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx; \tag{24}$$

if $\delta = 0$, based on the Lions lemma ([28], Lemma I.1), we are aware that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2+\frac{4}{N}} dx = 0.$$

Lemma 1 and (23) allow us to determine

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{L}(u_n) &= \lim_{n\to+\infty} \left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + m|u_n|^2 dx - \int_{\mathbb{R}^N} H(u_n)dx\right] \\
&= \lim_{n\to+\infty} \left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + m|u_n|^2 dx - \int_{\mathbb{R}^N} F_1(u_n)dx + \int_{\mathbb{R}^N} F_2(u_n)dx\right] \\
&\geq \frac{1}{2}\lim_{n\to\infty}\left[\int_{\mathbb{R}^N} |\nabla u_n|^2 + m|u_n|^2 dx\right] + \frac{m}{2}\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^2 dx \\
&\geq \frac{1}{2}\lim_{n\to\infty} \|\nabla u_n\|_2^2 + ma^* > 0.
\end{aligned}$$

This contradicts (22); so, $\delta > 0$, and (24) allows us to identify a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n) \rightharpoonup u \neq 0$ in $H^1(\mathbb{R}^N)$ and $u_n(x + y_n) \to u(x)$, a.e. in $\mathbb{R}^N$. Set $w_n := u_n(\cdot + y_n) - u$, and by the fact that $\int_{\mathbb{R}^N} |u|^2 dx = b \in (0, a^*]$, we have $w_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$:

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |w_n|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^2 dx - \int_{\mathbb{R}^N} |u|^2 dx = a^* - b \geq 0, \tag{25}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} |\nabla u|^2 dx. \tag{26}$$

Noting that $c(a_n) \to c(a^*) = 0$, it is deduced that

$$0 = \mathcal{L}(u_n) + o(1) = \mathcal{L}(w_n) + \mathcal{L}(u) + o(1). \tag{27}$$

It is known that $c(b) \geq 0$ for any $b \in (0, a^*]$. This means that

$$\lim_{n\to\infty} \mathcal{L}(w_n) \geq 0 \quad \text{and} \quad \mathcal{L}(u) \geq 0.$$

Thus, we can obtain more:

$$\mathcal{L}(u) = 0 \quad \text{and} \quad \lim_{n\to\infty} \mathcal{L}(w_n) = 0,$$

which suggests that $c(b) = 0$, obtained by $u \in \mathcal{S}(b)$. Suppose that $b < a^*$; according to Lemma 4-$(ii)$, then $c(a^*) < \frac{a^*}{b}c(b) = 0$, which is a contradiction. Therefore, $b = a^*$ and $c(a^*) = 0$ are obtained by $u \in \mathcal{S}(a^*)$. □

**Lemma 9.** *Let $N \geq 3$, and suppose that $(h_1)$–$(h_4)$ hold. Then, there is $a^{**} \in (0, a^*)$ such that, for every $a^{**} < a < a^*$, there is the following inequality:*

$$0 \leq c(a) = \inf_{u \in \mathcal{S}(a)} \mathcal{L}(u) < ma \leq \inf_{u \in \mathcal{R}(a)} \mathcal{L}(u).$$

**Proof.** According to Lemma 7, for each $a \in (0, a^*]$,

$$\inf_{u \in \mathcal{R}(a)} \mathcal{L}(u) \geq ma. \tag{28}$$

By Lemma 8, there is a minimizer $w \in \mathcal{S}(a^*)$ of $c(a^*) = 0$. Since $\mathcal{L}(w) = 0$, through continuity, then there exists an $\nu \in (0,1)$ such that, for all $\tau \in (\nu, 1]$, we obtain

$$\mathcal{L}(\tau w) < ma.$$

Choose $a^{**} := \nu^2 a^*$ such that, for any $\|\tau w\|_2^2 = a$, then for any $a \in (a^{**}, a^*]$, we have

$$0 \leq c(a) \leq \mathcal{L}(\tau w) < ma, \tag{29}$$

where the inequalities hold based on Lemma 8. Clearly, we have completed the proof from (28) and (29). □

**Proof of Theorem 1.** $(i)$ When $a > a^*$, Lemma 2 indicates that $c(a) < 0$, and let $\{u_n\} \subset \mathcal{S}(a)$ such that $\mathcal{L}(u_n) \to c(a)$. Lemma 4 suggests that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Define

$$\delta := \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx; \tag{30}$$

if $\delta = 0$, based on the Lions lemma ([28], Lemma I.1), we are aware that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^{2+\frac{4}{N}} dx = 0.$$

From Lemma 1, we obtain

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} F_1(u_n) dx = 0. \tag{31}$$

Through (31), we have

$$\begin{aligned} 0 > c(a) = \lim_{n \to \infty} \mathcal{L}(u_n) &= \lim_{n \to +\infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + m|u_n|^2 dx - \int_{\mathbb{R}^N} H(u_n) dx \right] \\ &= \lim_{n \to +\infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + m|u_n|^2 dx - \int_{\mathbb{R}^N} F_1(u_n) dx + \int_{\mathbb{R}^N} F_2(u_n) dx \right] \\ &\geq \frac{1}{2} \lim_{n \to \infty} \left[ \int_{\mathbb{R}^N} |\nabla u_n|^2 + m|u_n|^2 dx \right] + \frac{m}{2} \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^2 dx \\ &\geq \frac{1}{2} \lim_{n \to \infty} \|\nabla u_n\|_2^2 + ma > 0, \end{aligned}$$

which is a contradiction. So, $\delta > 0$ and (30) allows us to identify a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n) \rightharpoonup u \neq 0$ in $H^1(\mathbb{R}^N)$ and $u_n(x + y_n) \to u(x)$ a.e. in $\mathbb{R}^N$. Notice the fact that $c(a) < 0$ and $u \neq 0$. Subsequently, we only need to repeat the process of Lemma 6 to obtain $c(a)$ obtained by $u \in \mathcal{S}(a)$ and $\mathcal{L}(u) = c(a) < 0$. In addition, the Lagrange theorem indicates the existence of $\lambda \in \mathbb{R}$ such that

$$-\Delta u + mu = h(u) - \lambda u. \tag{32}$$

By the Pohožaev identity corresponding to (32), we have

$$\mathcal{P}(u) := \frac{N-2}{2N}\|\nabla u\|_2^2 + \frac{1}{2}(\lambda + m)\|u\|_2^2 - \int_{\mathbb{R}^N} H(u)dx = 0, \tag{33}$$

and due to the fact that $\mathcal{L}(u) = c(a) < 0$, it may be inferred that

$$0 > c(a) = \mathcal{L}(u) - \mathcal{P}(u) = \frac{1}{N}\|\nabla u\|_2^2 - \frac{1}{2}\lambda a,$$

which implies that $\lambda > 0$. Therefore, the problem (6) possessed a least-energy-normalized solution $(u, \lambda) \in \mathcal{S}(a) \times \mathbb{R}^+$ and $\mathcal{L}(u) = c(a) < 0$.

$(ii)$ When $a = a^*$, Lemma 8 already tells us that $c(a^*) = 0$ can be obtained by some $u \in \mathcal{S}(a^*)$. In addition, the Lagrange theorem indicates the existence of $\lambda \in \mathbb{R}$ such that (32) and (33) hold, so we can similarly obtain

$$0 = c(a^*) = \mathcal{L}(u) - \mathcal{P}(u) = \frac{1}{N}\|\nabla u\|_2^2 - \frac{1}{2}\lambda a^*,$$

which implies that $\lambda > 0$. Therefore, the problem (6) possesses a least-energy-normalized solution $(u, \lambda) \in \mathcal{S}(a^*) \times \mathbb{R}^+$ and $\mathcal{L}(u) = c(a^*) = 0$.

$(iii)$ When $a \in (a^{**}, a^*)$, we know that $c(a) > 0$, and let $\{u_n\} \subset \mathcal{S}(a)$ such that $\mathcal{L}(u_n) \to c(a)$. Lemma 4 further suggests that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Define

$$\delta := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx; \tag{34}$$

if $\delta = 0$, based on the Lions lemma ([28], Lemma I.1), we are aware that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2+\frac{4}{N}} dx = 0.$$

By Lemma 1, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F_1(u_n)dx = 0. \tag{35}$$

Through (35), we have

$$\begin{aligned}
c(a) = \lim_{n\to\infty} \mathcal{L}(u_n) &= \lim_{n\to+\infty} \left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + m|u_n|^2 dx - \int_{\mathbb{R}^N} H(u_n)dx\right] \\
&= \lim_{n\to+\infty} \left[\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + m|u_n|^2 dx - \int_{\mathbb{R}^N} F_1(u_n)dx + \int_{\mathbb{R}^N} F_2(u_n)dx\right] \\
&\geq \frac{1}{2}\lim_{n\to\infty}\left[\int_{\mathbb{R}^N} |\nabla u_n|^2 + m|u_n|^2 dx\right] + \frac{m}{2}\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^2 dx \\
&\geq \frac{1}{2}\lim_{n\to\infty} \|\nabla u_n\|_2^2 + ma \geq ma.
\end{aligned}$$

This contradicts Lemma 9. So, $\delta > 0$, and (34) allows us to identify a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n) \rightharpoonup u \neq 0$ in $H^1(\mathbb{R}^N)$ and $u_n(x + y_n) \to u(x)$ a.e. in $\mathbb{R}^N$. Notice the fact that $c(a) > 0$ and $u \neq 0$. Subsequently, we only need to repeat the process of Lemma 6 to obtain $c(a)$ obtained by $u \in \mathcal{S}(a)$ and $\mathcal{L}(u) = c(a) > 0$. In conclusion, the problem (6) possesses a least-energy-normalized solution. Therefore, we have completed all proofs of this theorem. □

## 3. Proof of Theorem 2

Similar to Section 2, we first modify the conditions of $g$ and introduce corresponding auxiliary functionals to prove Theorem 2. We set

$$0 < \frac{1}{2}\limsup_{s\to 0}\frac{g(s)}{s} = m < +\infty \quad \text{and} \quad \widetilde{h}(s) = g(s) - ms.$$

Thus, the problem (3) can be rewritten as

$$-\Delta v - mv = \widetilde{h}(v) - \widetilde{\lambda} v \quad \text{in } \mathbb{R}^N, \tag{36}$$

having prescribed mass:

$$\int_{\mathbb{R}^N} |v|^2 dx = a. \tag{37}$$

Subsequently, the following are the assumptions of $\widetilde{h}$:

$(\widetilde{h}_1)$ $\widetilde{h} : \mathbb{R} \to \mathbb{R}$ is continuous;

$(\widetilde{h}_2)$ $-\infty < \liminf_{|s|\to 0}\frac{\widetilde{h}(s)}{s} \leq \limsup_{|s|\to 0}\frac{\widetilde{h}(s)}{|s|} = m < +\infty$;

$(\widetilde{h}_3)$ $\limsup_{|s|\to\infty}\frac{|\widetilde{h}(s)|}{|s|^{\frac{N+2}{N-2}}} < \infty$ and $\limsup_{|s|\to\infty}\frac{\widetilde{h}(s)s}{|s|^{2+\frac{4}{N}}} \leq 0$ ;

$(\widetilde{g}_4)$ There exists $\xi \neq 0$ such that $\widetilde{H}(\xi) > 0$, where $\widetilde{H}(t) = \int_0^t \widetilde{h}(s)ds$.

The following will convert the proof of Theorem 2 into a problem of solving (36) with (37) under the supposition $(\widetilde{h}_1) - (\widetilde{h}_4)$. Now, we rewrite the energy functional $\mathcal{T}$ as

$$\widetilde{\mathcal{L}}(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 - m|v|^2 dx - \int_{\mathbb{R}^N} \widetilde{H}(v)dx.$$

We will no longer provide the proof process for Lemma 10 and 11, since it can be implemented similarly to Lemma 2 in Section 2 and [17] (Lemma 2.2).

**Lemma 10.** *Let $N \geq 3$, and suppose that $(\widetilde{h}_1)$ and $(\widetilde{h}_2)$ hold. Then, $\widetilde{\mathcal{L}}$ is coercive and bounded from below on $\mathcal{S}(a)$ for each $a > 0$.*

**Lemma 11.** *Let $N \geq 3$, and suppose that $(\widetilde{h}_1)$–$(\widetilde{h}_4)$ hold. Then, we have the following statements:*

*(i) For every $a > a' > 0$, $c(a) \leq \frac{a}{a'}c(a')$;*

*(ii) For every $a > a' > 0$, if $c(b)$ is attained, then $c(a) < \frac{a}{a'}c(a')$;*

*(iii) For $a > 0$, the function $a \mapsto c(a)$ is continuous.*

Moreover, similar lemmas of the Brézis–Lieb ([26], Lemma 2.2) type can also be obtained as follows.

**Lemma 12.** *Let $N \geq 3$, and suppose that $(\widetilde{h}_1)$ and $(\widetilde{h}_2)$ hold. Let $\{v_n\}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, and set $w_n := v_n - v$, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left[\widetilde{H}(v_n) - \widetilde{H}(v) - \widetilde{H}(w_n)\right]dx = 0.$$

**Lemma 13.** *Let $N \geq 3$, and suppose that $(\widetilde{h}_1)$–$(\widetilde{h}_3)$ hold. For each $a > 0$, there is $\rho(a) > 0$ small enough in $\mathcal{R}(a)$. Then, for every $v \in \mathcal{R}(a)$, we obtain*

$$\widetilde{\mathcal{L}}(v) \geq -ma. \tag{38}$$

**Proof.** According to $(\widetilde{h}_1)$–$(\widetilde{h}_3)$, for each $\delta > 0$, there exists $C_\delta > 0$ such that

$$\widetilde{H}(s) \leq \frac{m}{2}|s|^2 + \delta|s|^{2+\frac{4}{N}} + C_\delta|s|^\beta, \tag{39}$$

where $2+\frac{4}{N}<\beta<\frac{2N}{N-2}$. Thus, according to (39) and (11), we know that

$$\begin{aligned}\int_{\mathbb{R}^N}\widetilde{H}(v)dx \leq &\frac{m}{2}\int_{\mathbb{R}^N}|v|^2dx + C\delta\int_{\mathbb{R}^N}|\nabla v|^2dx\left(\int_{\mathbb{R}^N}|v|^2dx\right)^{\frac{2}{N}} \\ &+ CC_\delta\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^{\frac{(\beta-2)N}{4}}\left(\int_{\mathbb{R}^N}|v|^2dx\right)^{\frac{2\beta-(\beta-2)N}{4}}.\end{aligned} \tag{40}$$

Take $\delta>0$ small enough such that $\delta Ca^{\frac{2}{N}}<\frac{1}{8}$. Since $\frac{(\beta-2)N}{2}\in(2,\frac{2N}{N-2})$ and $\rho(a)>0$ small enough, for all $v\in\mathcal{R}(a)$,

$$\int_{\mathbb{R}^N}\widetilde{H}(v)dx \leq \frac{1}{2}ma+\frac{1}{4}\int_{\mathbb{R}^N}|\nabla v|^2dx. \tag{41}$$

Then, we have

$$\begin{aligned}\widetilde{\mathcal{L}}(v) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx - \frac{m}{2}\int_{\mathbb{R}^N}|v|^2dx - \int_{\mathbb{R}^N}\widetilde{H}(v)dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2dx - \frac{m}{2}\int_{\mathbb{R}^N}|v|^2dx - \frac{m}{2}\int_{\mathbb{R}^N}|v|^2dx - \frac{1}{4}\int_{\mathbb{R}^N}|\nabla v|^2dx \\ &= \int_{\mathbb{R}^N}|\nabla v|^2dx - ma \geq -ma.\end{aligned}$$

This indicates that (38) is valid. □

**Lemma 14.** *Let $N\geq 3$, and suppose that $(\widetilde{h}_1)$–$(\widetilde{h}_4)$ hold. Then, there is $a_*>0$ such that, for every $a>a_*$, there is the following inequality:*

$$c(a)=\inf_{v\in\mathcal{S}(a)}\widetilde{\mathcal{L}}(v) < -ma \leq \inf_{v\in\mathcal{R}(a)}\widetilde{\mathcal{L}}(v). \tag{42}$$

**Proof.** From $(\widetilde{h}_4)$ and by applying the technique in [1], there is a $v\in H^1(\mathbb{R}^N)$ that makes

$$\int_{\mathbb{R}^N}\widetilde{H}(v)dx>0.$$

Thus, for any $a>0$, we let $w(x):=v\left(a^{-\frac{1}{N}}\|v\|_2^{\frac{2}{N}}x\right)\in\mathcal{S}(a)$. Then,

$$\begin{aligned}\widetilde{\mathcal{L}}(w) &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla w|^2 - m|w|^2dx - \int_{\mathbb{R}^N}\widetilde{H}(w)dx \\ &= \frac{1}{2}\|v\|_2^{\frac{2(2-N)}{N}}\|\nabla v\|_2^2 a^{\frac{N-2}{N}} - \frac{\left(\int_{\mathbb{R}^N}\widetilde{H}(v)+\frac{1}{2}m|v|^2dx\right)}{\|v\|_2^2}a \\ &=: Aa^{\frac{N-2}{N}} - Ba.\end{aligned}$$

This means that $\widetilde{\mathcal{L}}(w)\to-\infty$ as $a\to+\infty$; thus, there exists $a_*>0$ sufficiently large; we have $c(a)\leq\widetilde{\mathcal{L}}(w)<-ma$ for any $a_*<a$. It follows from Lemma 13 that (42) holds. □

**Proof of Theorem 2.** Lemma 14 indicates that $c(a)<0$ if $a_*<a$, and let $\{v_n\}\subset\mathcal{S}(a)$ such that $\widetilde{\mathcal{L}}(v_n)\to c(a)$. Lemma 4 suggests that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Define

$$\delta:=\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|u_n|^2dx; \tag{43}$$

if $\delta = 0$, based on the Lions lemma ([28], Lemma I.1), we are aware that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2+\frac{4}{N}} dx = 0. \tag{44}$$

It follows from $(\widetilde{h}_1)$–$(\widetilde{h}_3)$ that

$$\widetilde{H}(s) \le \frac{1}{2} m|s|^2 + C|s|^{2+\frac{4}{N}}. \tag{45}$$

Thus, from (44) and (45), one has

$$\begin{aligned} c(a) = \lim_{n\to\infty} \widetilde{\mathcal{L}}(v_n) = & \lim_{n\to+\infty} \left[ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 - m|v_n|^2 dx - \int_{\mathbb{R}^N} \widetilde{H}(v_n) dx \right] \\ \ge & \frac{1}{2} \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - m \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^2 dx - C \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{2+\frac{4}{N}} dx \\ = & \frac{1}{2} \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - m \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^2 dx \ge -ma. \end{aligned}$$

This contradicts Lemma 14. So, $\delta > 0$ and (43) allow us to identify a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $v_n(\cdot + y_n) \rightharpoonup v \ne 0$ in $H^1(\mathbb{R}^N)$ and $v_n(x + y_n) \to v(x)$ a.e. in $\mathbb{R}^N$. Notice the fact that $c(a) < 0$ and $v \ne 0$. Subsequently, we only need to repeat the process of Lemma 6 to obtain $c(a)$ obtained by $v \in \mathcal{S}(a)$ and $\widetilde{\mathcal{L}}(v) = c(a) < 0$. In addition, the Lagrange theorem indicates the existence of $\widetilde{\lambda} \in \mathbb{R}$ such that

$$-\Delta v - mv = \widetilde{h}(v) - \widetilde{\lambda} v. \tag{46}$$

By the Pohožaev identity corresponding to (46), we have

$$\widetilde{\mathcal{P}}(v) := \frac{N-2}{2N} \|\nabla v\|_2^2 + \frac{1}{2}(\widetilde{\lambda} - m)\|v\|_2^2 - \int_{\mathbb{R}^N} \widetilde{H}(v) dx = 0; \tag{47}$$

due to the fact that $\widetilde{\mathcal{L}}(v) = c(a) < 0$, it may be inferred that

$$0 > c(a) = \widetilde{\mathcal{L}}(v) - \widetilde{\mathcal{P}}(v) = \frac{1}{N} \|\nabla v\|_2^2 - \frac{1}{2} \widetilde{\lambda} a,$$

which implies that $\widetilde{\lambda} > 0$. Therefore, the problem (36) possesses a least-energy-normalized solution $(v, \widetilde{\lambda}) \in \mathcal{S}(a) \times \mathbb{R}^+$ and $\widetilde{\mathcal{L}}(v) = c(a) < 0$. Therefore, we have completed all proofs of this theorem. □

## 4. Conclusions

In this article, we focused on the existence of normalized solutions for scalar field equations under the Berestycki–Lions-type conditions. Firstly, we introduced what the Berestycki–Lions condition is and some related results under this condition. Secondly, based on these research results, we described the purpose of this study, which was to consider the existence of normalized solutions to scalar field equations under a broader condition and provide some examples that satisfy this condition. Finally, by using variational methods and some analytical techniques, we have demonstrated the main results.

**Author Contributions:** G.-D.L. proposed the idea for this study and led its execution, as well as reviewed and revised the manuscript. C.-F.L. proposed the research ideas, established the research objectives, and wrote an initial draft. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was funded by the special (special post) scientific research fund of natural science of Guizhou University (No.(2021)43), the Guizhou Provincial Education Department Project (No.(2022)097), the Guizhou Provincial Science and Technology Projects (No.[2023]YB033, [2023]YB036), and the National Natural Science Foundation of China (No.12201147).

**Institutional Review Board Statement:** Not applicable.

**Data Availability Statement:** Data are contained within the article.

**Acknowledgments:** The authors would like to express their heartfelt thanks to the anonymous referees for their valuable suggestions and comments.

**Conflicts of Interest:** The authors declare no conflicts of interest.

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