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The Fourier–Legendre Series of Bessel Functions of the First Kind and the Summed Series Involving ${}_1F_2$ Hypergeometric Functions That Arise from Them

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Abstract: The Bessel function of the first kind $J_N(kx)$ is expanded in a Fourier–Legendre series, as is the modified Bessel function of the first kind $I_N(kx)$. The purpose of these expansions in Legendre polynomials was not an attempt to rival established *numerical* methods for calculating Bessel functions but to provide a form for $J_N(kx)$ useful for *analytical* work in the area of strong laser fields, where analytical integration over scattering angles is essential. Despite their primary purpose, one can easily truncate the series at 21 terms to provide 33-digit accuracy that matches the IEEE extended precision in some compilers. The analytical theme is furthered by showing that infinite series of like-powered contributors (involving ${}_1F_2$ hypergeometric functions) extracted from the Fourier–Legendre series may be summed, having values that are inverse powers of the eight primes $1/\left(2^i 3^j 5^k 7^l 11^m 13^n 17^o 19^p\right)$ multiplying powers of the coefficient k.

Keywords: Bessel functions; Fourier–Legendre series; Laplace series; generalized hypergeometric functions; polynomial approximations; computational methods

MSC: 33C10; 42C10; 41A10; 33F10; 65D20; 68W30; 33D50; 33C05

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1. Introduction

Bessel functions find application in countless fields, so there has naturally been a great deal of research into how best to calculate them. Just a few of these approaches range from polynomial approximations [1,2] to solving integral equations [3] to rational approximations [4,5] to expansions in Jacobi or Chebyshev polynomials [6]. The latter are particularly prized by many workers [7–9].

However, the present paper has an entirely different goal from *numerical* approximations to Bessel functions, a representation of the Bessel function $J_N(kx)$ in a form that will be useful for *analytical* integration over angular variables. The work most closely related to the present approach is trigonometric approximations [10].

This need arises from the use of the Strong Field Approximation (SFA) [11–16] to calculate atomic transition amplitudes in intense laser fields. Unlike perturbation expansions that will not converge if an applied laser field is large, the SFA is an analytical approximation that is non-perturbative. Keating [17] applied it to the production of the positive antihydrogen ion and found that the reduction of the transition amplitudes to an analytic form required the expansion of the resultant Bessel functions $J_N(kx)$ in a series of spherical harmonics. I have been unable to discover such a Laplace series [18] in the literature, nor does there seem to be an expansion of the Bessel function in a series of Legendre polynomials, to which the Laplace series reduces in the common case where the function is independent of the azimuthal angle. I recreate herein Keating's derivation, stripped of the specialized SFA terminology, and find that I can recast his coefficients from another layer of infinite series to a ${}_2F_3$ generalized hypergeometric function. I also extend the method to the modified Bessel function of the first kind $I_N(kx)$.

Though designed as an analytical tool, this Fourier–Legendre series is easily converted to a series in powers; I do so as a check on its validity. Computer code is given in Appendix A for these series that provide 33-digit accuracy, matching the IEEE extended precision in some compilers.

Of further analytical significance, like-powered contributors (involving ${}_1F_2$ hypergeometric functions) extracted from the Fourier–Legendre series may be summed, having values that are inverse powers of the eight primes $1/\left(2^i 3^j 5^k 7^l 11^m 13^n 17^o 19^p\right)$ multiplying even powers of the coefficient k.

2. The Fourier–Legendre Series of a Bessel Function of the First Kind

I begin with the assumption [19] that the series

$$J_N(kx) = \sum_{L=0}^{\infty} a_{LN}(k) P_L(x) \tag{1}$$

converges uniformly. (Let D be a region in which the above series converges for each value of x. Then, the series can be said to converge uniformly in D if, for every $\varepsilon > 0$, there exists a number $N'(\varepsilon)$ such that, for $n > N'$, it follows that

$$\left| J_N(kx) - \sum_{L=0}^{n} a_{LN}(k) P_L(x) \right| = \left| \sum_{L=n+1}^{\infty} a_{LN}(k) P_L(x) \right| < \varepsilon$$

for all x in D. The coefficients are given by the orthogonality of the Legendre polynomials,

$$a_{LN}(k) = \frac{2L+1}{2} \int_{-1}^{1} J_N(kx) P_L(x) dx \;. \tag{2}$$

To find these coefficients, I follow Keating's approach, first using Heine's integral representation of the Bessel function [20] for integer indices

$$J_N(kx) = \frac{i^{-N}}{\pi} \int_0^{\pi} e^{ikx\cos\theta} \cos(N\theta) d\theta \;, \tag{3}$$

so that

$$a_{LN}(k) = \frac{2L+1}{2} \int_{-1}^{1} \left[\frac{i^{-N}}{\pi} \int_0^{\pi} e^{ikx\cos\theta} \cos(N\theta) d\theta \right] P_L(x) dx \;. \tag{4}$$

By switching the order of integration,

$$a_{LN}(k) = \frac{2L+1}{2} \frac{i^{-N}}{\pi} \int_0^{\pi} \left[\int_{-1}^{1} e^{ikx\cos\theta} P_L(x) dx \right] \cos(N\theta) d\theta \;, \tag{5}$$

and expanding the plane wave in a series of products of spherical Bessel functions $j_m(y)$ and Legendre polynomials [21,22],

$$e^{ikx\cos\theta} = \sum_{l'} (2l'+1) i^{l'} j_{l'}(k\cos\theta) P_{l'}(x) \;, \tag{6}$$

one has,

$$\begin{aligned} a_{LN}(k) &= \tfrac{2L+1}{2} \tfrac{i^{-N}}{\pi} \int_0^{\pi} \left[\int_{-1}^{1} \left(\sum_{l'=0}^{\infty} (2l'+1) i^{l'} j_{l'}(k\cos\theta) P_{l'}(x) \right) P_L(x) dx \right] \cos(N\theta) d\theta \\ &= \tfrac{2L+1}{2} \tfrac{i^{-N}}{\pi} \int_0^{\pi} \left[\left(\sum_{l'=0}^{\infty} (2l'+1) i^{l'} j_{l'}(k\cos\theta) \tfrac{2}{2l'+1} \delta_{l'L} \right) \right] \cos(N\theta) d\theta \\ &= (2L+1) \tfrac{i^{L-N}}{\pi} \int_0^{\pi} j_L(k\cos\theta) \cos(N\theta) d\theta \;. \end{aligned} \tag{7}$$

Using the series expansion [23]

$$j_l(x) = \frac{1}{2}\sqrt{\pi}\left(\frac{x}{2}\right)^l \sum_{M=0}^{\infty} \frac{\left(-\frac{1}{4}\right)^M x^{2M}}{M!\Gamma\left(l+M+\frac{3}{2}\right)} \tag{8}$$

this becomes

$$a_{LN}(k) = \frac{(2L+1)}{\sqrt{\pi}} i^{L-\mathrm{N}}\mathrm{N}2^{-L-1}\left(\sum_{M=0}^{\infty} \frac{\left(-\frac{1}{4}\right)^M k^{L+2M}}{M!\Gamma\left(L+M+\frac{3}{2}\right)}\right)\int_0^{\pi} \cos^{L+2M}(\theta)\cos(\mathrm{N}\theta)\,d\theta. \tag{9}$$

Gröbner and Hofreiter [24] extended an integral over the interval $\left[0, \frac{\pi}{2}\right]$ that has three branches, to the interval $[0, \pi]$ with a prefactor $(1+(-1)^{m+n})$ that renders the central one of the three possibilities nonzero only for even values for $m+n$.

$$\int_0^{\pi} cos^m\theta\, cos(n\theta)d\theta = \left(1+(-1)^{m+n}\right)\frac{\pi}{2^{m+1}}\binom{m}{\frac{m-n}{2}} \quad [\text{where } m \geq n > -1,\ m-n = 2K]\,. \tag{10}$$

The other two branches, being for odd $m+n$ on $\left[0, \frac{\pi}{2}\right]$, are zero on $[0, \pi]$ when this prefactor is included. (Numerical integration also confirms that the contributions from the $\left[\frac{\pi}{2}, \pi\right]$ interval cancel the contributions from the $\left[0, \frac{\pi}{2}\right]$ on these branches). The fifth edition of Gradshteyn and Ryzhik [21] (p. 417 No. 3.631.17) (in which m and n are reversed in meaning) nevertheless included all three branches and this prefactor on the interval $[0, \pi]$. By their seventh edition, Gradshteyn and Ryzhik removed this integral entirely despite the correctness of the central branch on the interval $[0, \pi]$. Neither source noted the lower limit on m that I found, that $m \geq n > -1$.

The final form for the coefficient set of the Fourier–Legendre series for the Bessel function $J_N(kx)$ is then

$$\begin{aligned} a_{LN}(k) &= \sqrt{\pi}(2L+1)2^{-L-1}i^{L-\mathrm{N}}\textstyle\sum_{M=0}^{\infty} \frac{\left(\left(-\frac{1}{4}\right)^M k^{L+2M}\right)}{2^{L+2M+1}\left(M!\Gamma\left(L+M+\frac{3}{2}\right)\right)} \\ &\times \left(1+(-1)^{L+2M+\mathrm{N}}\right)\binom{L+2M}{\frac{1}{2}(L+2M-\mathrm{N})} \\ &= \frac{\sqrt{\pi}2^{-2L-2}(2L+1)k^L i^{L-N}}{\Gamma\left(\frac{1}{2}(2L+3)\right)}\left(1+(-1)^{L+N}\right)\binom{L}{\frac{L-N}{2}} \\ &\times\ {}_2F_3\left(\frac{L}{2}+\frac{1}{2}, \frac{L}{2}+1; L+\frac{3}{2}, \frac{L}{2}-\frac{N}{2}+1, \frac{L}{2}+\frac{N}{2}+1; -\frac{k^2}{4}\right) \\ &= \sqrt{\pi}\,2^{-2L-2}(2L+1)k^L i^{L-N}\left(1+(-1)^{L+N}\right)\Gamma(L+1) \\ &\times\ {}_2\tilde{F}_3\left(\frac{L}{2}+\frac{1}{2}, \frac{L}{2}+1; L+\frac{3}{2}, \frac{L}{2}-\frac{N}{2}+1, \frac{L}{2}+\frac{N}{2}+1; -\frac{k^2}{4}\right) \end{aligned}, \tag{11}$$

where the final two steps are new with the present work. I have included the final form using regularized hypergeometric functions [25]

$${}_2F_3(a_1, a_2; b_1, b_2, b_3; z) = \Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\,{}_2\tilde{F}_3(a_1, a_2; b_1, b_2, b_3; z) \tag{12}$$

and cancelled the $\Gamma(b_i)$ with gamma functions in the denominators of the prefactors. Each give infinities that arise whenever $N > 1$ is an integer larger than L and of the same parity, resulting in indeterminacies in a computation when one tries to use the conventional form of the hypergeometric function.

For the special cases of $N = 0, 1$, the order of the hypergeometric functions is reduced since the parameters $a_2 = b_3$ and $a_1 = b_2$, resp., giving

$$\begin{aligned} a_{L0}(k) &= \frac{\sqrt{\pi}i^L 2^{-2L-2}(2L+1)k^L}{\Gamma\left(\frac{1}{2}(2L+3)\right)}\left(1+(-1)^L\right)\binom{L}{\frac{L}{2}} \\ &\times\ {}_1F_2\left(\frac{L}{2}+\frac{1}{2}; \frac{L}{2}+1, L+\frac{3}{2}; -\frac{k^2}{4}\right) \\ &= \sqrt{\pi}i^L 2^{-2L-2}(2L+1)k^L\Gamma\left(\frac{L}{2}+1\right)\left(1+(-1)^L\right)\binom{L}{\frac{L}{2}} \\ &\times\ {}_1\tilde{F}_2\left(\frac{L}{2}+\frac{1}{2}; \frac{L}{2}+1, L+\frac{3}{2}; -\frac{k^2}{4}\right) \end{aligned}, \tag{13}$$

and

$$\begin{aligned} a_{L1}(k) &= \frac{\sqrt{\pi} i^{L-1} 2^{-2L-2}(2L+1)k^L}{\Gamma\left(\frac{1}{2}(2L+3)\right)} \left(1+(-1)^{L+1}\right)\binom{L}{\frac{L-1}{2}} \\ &\times \; {}_1F_2\left(\frac{L}{2}+1; \frac{L}{2}+\frac{3}{2}, L+\frac{3}{2}; -\frac{k^2}{4}\right) \\ &= i^{L-1} 2^{-L-2}(2L+1)k^L \Gamma\left(\frac{L}{2}+1\right)\left(1+(-1)^{L+1}\right) \\ &\times \; {}_1\tilde{F}_2\left(\frac{L}{2}+1; \frac{L}{2}+\frac{3}{2}, L+\frac{3}{2}; -\frac{k^2}{4}\right) \end{aligned}, \tag{14}$$

In each special case, the first form involving a hypergeometric function has no indeterminacies, but I include the regularized hypergeometric function version for completeness.

3. A Numerical Check

Numerical checks are an essential element of any analytical work, particularly in checking convergence. Much of my research in physics has revolved around writing my own programs to diagonalize very large Hamiltonian matrices, and I have found that I routinely need to calculate in quadruple precision (33 digits) to gain consistent double-precision results. Since this research often contains Bessel functions, it made sense to craft a quadruple precision approximation for $J_0(kx)$ that would both allow for numerical checks and be useful for some future project of mine, or of other researchers, calculated in quadruple precision. Though Castellanos and Rosenthal obtain fifteen-decimal accuracy with their Rational Chebyshev Approximations [26] and there is a ZBESJ Bessel function double-precision package that has 18-digit accuracy (UNIVAC double precision) [27], I found no prepackaged quadruple precision routines in a search of software libraries. For this reason, I have displayed all coefficients in what follows in 33 digits, accuracy that matches the IEEE selected_real_kind (33, 4931) extended precision available on some compilers (the second argument, 4931, indicates a range of 10^{-4931} to $10^{4931} - 1$ using 128 bits [28]). These results are given in Appendix A so that that readers may simply copy and paste them into a calculation routine.

The first 22 terms in the sum (1) are then, with $k = 1$

$$\begin{aligned} J_0(x) \cong\; & 0.919730410089760239314421194080620 0 P_0(x) \\ -\; & 0.157942058625851887573713967144363 7 P_2(x) \\ +\; & 0.003438400944601109232996887872072915 P_4(x) \\ -\; & 0.0000291972184882872969366059098612 5663 P_6(x) \\ +\; & 1.31735695244778097765561656314328 0 \times 10^{-7}\; P_8(x) \\ -\; & 3.684500844208203027173771096058866 \times 10^{-10}\; P_{10}(x) \\ +\; & 7.011830032993845928208803328211457 \times 10^{-13}\; P_{12}(x) \\ -\; & 9.665964369858912263671995372753346 \times 10^{-16}\; P_{14}(x) \\ +\; & 1.009636276824546446525342170924936 \times 10^{-18}\; P_{16}(x) \\ -\; & 8.266656955927637858991972584174117 \times 10^{-22}\; P_{18}(x) \\ +\; & 5.448244867762758725890082837839430 \times 10^{-25}\; P_{20}(x) \\ -\; & 2.952527182137354751675774606663400 \times 10^{-28}\; P_{22}(x) \\ +\; & 1.338856158858534469080898670096200 \times 10^{-31}\; P_{24}(x) \\ -\; & 5.154913186088512926193234837816582 \times 10^{-35}\; P_{26}(x) \\ +\; & 1.706231577038503450138564028467634 \times 10^{-38}\; P_{28}(x) \\ -\; & 4.906893556427796857473097979568289 \times 10^{-42}\; P_{30}(x) \\ +\; & 1.237489200717479383020539576221293 \times 10^{-45}\; P_{32}(x) \\ -\; & 2.759056237537871868604555688548364 \times 10^{-49}\; P_{34}(x) \\ +\; & 5.477382207172712629199714648396409 \times 10^{-53}\; P_{36}(x) \\ -\; & 9.744200345578852550688946057050674 \times 10^{-57}\; P_{38}(x) \\ +\; & 1.562280711659504489828025148995770 \times 10^{-60}\; P_{40}(x) \\ -\; & 2.269056283827394368836057470594599 \times 10^{-64}\; P_{42}(x)\;. \end{aligned} \tag{15}$$

If one wishes to check the convergence of this series at, say, the $|\varepsilon| < 5 \times 10^{-8}$ level of the polynomial approximation given by E. E. Allen (and reproduced in Abramowitz and Stegun) [1],

$$\begin{aligned} J_0(x) &\cong 1 - 2.25\left(\frac{x}{3}\right)^2 + 1.26562\left(\frac{x}{3}\right)^4 - 0.316387\left(\frac{x}{3}\right)^6 + 0.0444479\left(\frac{x}{3}\right)^8 \\ &- 0.0039444\left(\frac{x}{3}\right)^{10} + 0.0002100\left(\frac{x}{3}\right)^{12}, \end{aligned} \tag{16}$$

at the latter's limiting range of $x = 3$ (with a value of -0.260052), one finds that truncating the series after the $P_{12}(x)$ term (to obtain contributions through x^{12}) is insufficient, giving two fewer digits of accuracy (-0.260045). This shows that the optimization Allen must have performed to obtain his approximation has a significant effect.

Testing accuracy at the 15-digit level shows that the series can be truncated after the $P_{24}(x)$ term in the result, -0.260051954901933. This $|\varepsilon| < 5 \times 10^{-16}$ is much better than one would expect from doubling the number of powers in the unoptimized approximation in $P_{12}(x)$. In addition, the *range* of applicability of the new Legendre approximation truncated after $P_{24}(x) \sim x^{24}$, if one is satisfied with $|\varepsilon| < 5 \times 10^{-8}$, roughly doubles to $J_0(6) \cong 0.15064531$.

Finally, in the quality checking step, I found that if a 33-digit version of (15) is imported back into the software I had been using to generate it, *Mathematica 7*, I only obtained a result accurate to $|\varepsilon| < 1 \times 10^{-32}$, but if I imported the 34-digit version actually displayed in (15), I obtained a result accurate to $|\varepsilon| < 1 \times 10^{-34}$ over the range $-3 \le x \le 3$, with $J_0(3) \cong -0.2600519549019334376241546959773\,31$. Accuracy exceeds $|\varepsilon| < 7 \times 10^{-15}$ over the range $-8 \le x \le 8$, with $J_0(8) \cong 0.171650807137560$.

4. Summing a Set of Infinite Series

If Allen's polynomial approximation [1] has its powers of three folded into the coefficients, it becomes

$$\begin{aligned} J_0(x) \cong 1 - 0.25\,x^2 + 0.0156249\,x^4 - 0.000434001\,x^6 + 6.77456 \times 10^{-6}\,x^8 - 6.6799 \times 10^{-8}\,x^{10} \\ +3.952 \times 10^{-10}\,x^{12}\,. \end{aligned} \tag{17}$$

Noting that the third term is $1/64.0004$, one is led to wonder if each of these coefficients is made up of inverse powers of primes if one were to rederive a similar approximation using a process with greater accuracy.

One can easily expand the Legendre polynomials into their constituent terms and gather like powers in (15) to give an updated polynomial approximation,

$$\begin{aligned} J_0(x) &\cong 1.000000000000000000000000000000000000\,x^0 \\ &- 0.2500000000000000000000000000000000000\,x^2 \\ &+ 0.01562500000000000000000000000000000000\,x^4 \\ &- 0.0004340277777777777777777777777777777778\,x^6 \\ &+ 6.781684027777777777777777777777777778 \times 10^{-6}\,x^8 \\ &- 6.781684027777777777777777777777777778 \times 10^{-8}\,x^{10} \\ &+ 4.709502797067901234567901234567901235 \times 10^{-10}\,x^{12} \\ &- 2.402807549524439405391786344167296548 \times 10^{-12}\,x^{14} \\ &+ 9.385966990329841427311665406903502142 \times 10^{-15}\,x^{16} \\ &- 2.896903392077111551639402903365278439 \times 10^{-17}\,x^{18} \\ &+ 7.242258480192778879098507258413196097 \times 10^{-20}\,x^{20} \end{aligned}$$

$$
\begin{aligned}
& - \quad 1.496334396734045222954237036862230599 \times 10^{-22}\, x^{22} \\
& + \quad 2.597802772107717400962217077885817011 \times 10^{-25}\, x^{24} \\
& - \quad 3.842903509035084912666001594505646466 \times 10^{-28}\, x^{26} \\
& + \quad 4.901662639075363409012757135849038860 \times 10^{-31}\, x^{28} \\
& - \quad 5.446291821194848232236396817610043178 \times 10^{-34}\, x^{30} \\
& + \quad 5.318644356635593976793356267197307791 \times 10^{-37}\, x^{32} \\
& - \quad 4.600903422695150498956190542558224733 \times 10^{-40}\, x^{34} \\
& + \quad 3.550079801462307483762492702591222788 \times 10^{-43}\, x^{36} \\
& - \quad 2.458504017633176927813360597362342651 \times 10^{-46}\, x^{38} \\
& + \quad 1.5365650110207355798833503733514641567 \times 10^{-49}\, x^{40} \\
& - \quad 8.7106860035189091830121903251216788929 \times 10^{-53}\, x^{42} \\
& \cong \quad 1 - \frac{x^2}{2^2} + \frac{x^4}{2^6} - \frac{x^6}{2^8 3^2} + \frac{x^8}{2^{14} 3^2} - \frac{x^{10}}{2^{16} 3^2 5^2} + \frac{x^{12}}{2^{20} 3^4 5^2} - \frac{x^{14}}{2^{22} 3^4 5^2 7^2} + \frac{x^{16}}{2^{30} 3^4 5^2 7^2} \\
& - \quad \frac{x^{18}}{2^{32} 3^8 5^2 7^2} + \frac{x^{20}}{2^{36} 3^8 5^4 7^2} - \frac{x^{22}}{2^{38} 3^8 5^4 7^2 11^2} + \frac{x^{24}}{2^{44} 3^{10} 5^4 7^2 11^2} - \frac{x^{26}}{2^{46} 3^{10} 5^4 7^2 11^2 13^2} \\
& + \quad \frac{x^{28}}{2^{50} 3^{10} 5^4 7^4 11^2 13^2} - \frac{x^{30}}{2^{52} 3^{12} 5^6 7^4 11^2 13^2} + \frac{x^{32}}{2^{62} 3^{12} 5^6 7^4 11^2 13^2} \\
& - \quad \frac{x^{34}}{2^{64} 3^{12} 5^6 7^4 11^2 13^2 17^2} + \frac{x^{36}}{2^{68} 3^{16} 5^6 7^4 11^2 13^2 17^2} - \frac{x^{38}}{2^{70} 3^{16} 5^6 7^4 11^2 13^2 17^2 19^2} \\
& + \quad \frac{x^{40}}{2^{76} 3^{16} 5^8 7^4 11^2 13^2 17^2 19^2} - \frac{x^{42}}{2^{78} 3^{18} 5^8 7^6 11^2 13^2 17^2 19^2}\,. \qquad (18)
\end{aligned}
$$

One can see from the repeating digits that the fourth through sixth lines in (18) have inverses that are powers of primes, and one can even see that the fifth term is $2^2 5^2$ times the sixth. Subsequent terms are not at all obviously inverse powers of primes. Indeed, revealing those powers as analytic entities required many terms in excess of those required to achieve quadruple precision in the numerical results and significantly higher precision.

One might suppose that that including series terms through $P_{24}(x)$ would be sufficient for revealing the inverse powers in the coefficient of, say, the x^{16} term, but that is not the case. Increasing the precision from 33 to 48 digits did not improve the situation enough. At that series truncation and with 48-digit precision, the inverse of the coefficient of the x^{16} term is **106542032486495**.616348409991752462411671619456197, whose integer part is in bold face. (I used the algebra and calculus computer software *Mathematica* 7 for this work). This is not a product of low-level primes. Adding one more term, $P_{26}(x)$, is sufficient to bring it to **106542032486400**.113376300998684305400416345779209, whose integer part is $2^{30}3^4 5^2 7^2$. One additional term makes this **106542032486400**.000104784167278249059923631013442, and with every additional term added, the integer part remains the same, while the fractional part diminishes by several decimal places. The coefficient of the x^{24} term required 40 terms in the series, with 48-digit precision, to establish convergence. The coefficient of the x^{42} term required 74 terms in the series, with 50-digit precision, to establish convergence.

All of the coefficients in (18) include contributions from all 74 terms in the series, calculated with 50-digit precision. These were then truncated to the 37-digit precision displayed therein, except for the last two that required 38 digits. Taking their reciprocals after truncation did not forestall revealing their integer parts as powers of primes. The truncated power series gives $J_0(3) \cong -0.2600519549019334376241546959773314809$, which matches *Mathematica*'s BesselJ[0,3]$\cong -0.2600519549019334376241546959773314368$ (when set to 37-digit precision) within an error of $|\varepsilon| < 4 \times 10^{-35}$, as does the inverse prime version, $J_0(3) \cong -\frac{906580249291695598055948762576794956 62633}{348615048725002045174179287542005760000000} = -0.2600519549019334376241546959773314809$.

Close examination of the inverse prime version shows that I have managed to translate into integer powers the denominators of the first 22 terms of the well-known series representation [21] (p. 970 No. 8.440)

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(k+\nu+1)}. \tag{19}$$

(There is a fascinating analog to this result arising from studies of the Bessel difference equation [29]). But the outcome worth the trouble of this investigation is that this process yields a set of infinite sums whose values are inverse powers of primes. The question of why so many Legendre polynomials are required to obtain sufficiently accurate coefficients in (18) led to an examination of precisely how a given x^n contributor from each Legendre polynomial contributes to the respective coefficient.

Looking back at the coefficients in (15) when multiplied by the constant terms in the Legendre polynomials that multiply them, whose first few are

$$\Big\{ P_0(x) = 1,\ P_2(x) = \frac{1}{2}\left(3x^2 - 1\right),\ P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \\ P_6(x) = \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right) \Big\}, \tag{20}$$

there is no reason to suspect that

$$\begin{aligned} &0.91973041008976023931442119408062 0\\ &-\frac{1}{2}(-0.15794205862585188757371396714436 4)\\ &+\frac{3}{8}(0.0034384009446011092329968878720729 2)\\ &-\frac{5}{16}(-0.000029197218488287296936605909861256 6)\\ &+\cdots = 1 \end{aligned} \tag{21}$$

rather than some other number close to *1*, but one sees uniform convergence up through the accuracy of the calculation as one adds additional terms, as seen in Table 1.

One may more formally conclude that

$$\sum_{L=0}^{\infty}{}^{(2)} \frac{\sqrt{\pi} i^L (2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L}\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}}\left(-\frac{L}{2}\right)_{\frac{L}{2}} 2^{-3L-2} {}_1F_2\left(\frac{L}{2}+\frac{1}{2}; \frac{L}{2}+1, L+\frac{3}{2}; -\frac{1}{4}\right)}{\frac{L}{2}!\left(\frac{1}{2}-L\right)_{\frac{L}{2}}\Gamma\left(\frac{1}{2}(2L+3)\right)} = 1 \tag{22}$$

where the superscript "(2)" on the sum indicates one is summing even values only (or one may retain the factor $\left(1+(-1)^L\right)$ in the sum as I do for similar equations below), which is a result I have not seen in the literature. The Pochhammer symbols $\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}}$ and so on derive from a shift to [30] (p. 468 No. 7.3.1.206)

$$P_n(x) = 2^{-n}\binom{2n}{n} x^n {}_2F_1\left(\frac{1}{2}-\frac{n}{2}, -\frac{n}{2}; \frac{1}{2}-n; \frac{1}{x^2}\right) \tag{23}$$

in the explicit sum [30] (p. 430 No. 7.2.1.1).

One may check that the series (22) converges under the Cauchy criterion. That is, if one calls a given term in (22) u_L and define the partial sums

$$S_n = \sum_{L=0}^{n} u_L\ , \tag{24}$$

then the numerical series (22) is convergent if for each $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that

$$|S_m - S_n| < \varepsilon \tag{25}$$

for all $m > n > N$. If I take, say, $\varepsilon = 10^{-21}$, it can be seen that the difference of the ninth and eighth lines $S_{10} - S_9$ in Table 1, above, fulfills this bound (with $m = 10$ and $n = 9$). If one instead takes $\varepsilon = 2 \times 10^{-46}$, it can be seen that the difference of the last two lines $S_{16} - S_{15}$ fulfills this tighter bound.

Table 1. The constant term of the Legendre series approximation as increasing numbers of terms are added from (15), to 48-digit accuracy.

0.919730410089760239314421194080619970661964806513
0.998701439402686183101278177652801821334364120020
0.999990839756911599063652010604829164640891568430
0.999999963887689188843944699951660807338623340119
0.999999999909168357337955807722426205787682516277
0.999999999999841620300892054094280728854756176645
0.99999999999999979732605041136082528291421094660
0.999999999999999999801573588581204243621535893434
0.999999999999999999999846581786952424267000275450
0.999999999999999999999999903953851991585994488442
0.999999999999999999999999999503204200428971030880
0.999999999999999999999999999999978412290228685090
0.999999999999999999999999999999999992008312509362
0.999999999999999999999999999999999999997449396440
0.999999999999999999999999999999999999999999290955
0.999999999999999999999999999999999999999999999827
1.00000000000000000000000000000000000000000000000

The summed series (22) was a consequence of my expansion of $J_N(kx)$ in a Fourier–Legendre series after setting $k = 1$. Including k poses no problem despite its appearance as the argumemt of the ${}_1F_2\left(\frac{L}{2}+\frac{1}{2}; \frac{L}{2}+1, L+\frac{3}{2}; -\frac{k^2}{4}\right)$ function as well as there being a k^L factor in the argument of the sum. It ends up contributing a very clean factor of k^{2h} to the right-hand side below. The 43 verified summed series given by the present approach (with $0 \le h \le 42$) are

$$\sum_{L=0}^{\infty} \frac{\sqrt{\pi} i^L 2^{-3L-2}(1+(-1)^L)(2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L}\, {}_1F_2\left(\frac{L}{2}+\frac{1}{2}; \frac{L}{2}+1, L+\frac{3}{2}; -\frac{k^2}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!} k^L$$
$$\times \left[\frac{\left(\frac{1}{2}-\frac{L}{2}\right)_{\frac{L}{2}-h}\left(-\frac{L}{2}\right)_{\frac{L}{2}-h}}{\left(\frac{1}{2}-L\right)_{\frac{L}{2}-h}}\right] = \frac{(-1)^h 2^{-2h}}{h!\Gamma(h+1)} k^{2h}\,. \tag{26}$$

To verify the final $h = 42$ summed series, I had to take the upper limit on the number of terms in the series $\ge h + 74$ in order to obtain a percent difference between left- and right-hand sides that was $\le 10^{-33}$ because the first h terms in the series do not contribute. For $h = 0$, an upper limit on the number of terms in the series $\ge h + 44$ was sufficient.

I wish to provide two alternative forms for readers seeking to sum a series involving ${}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};-\frac{k^2}{4}\right)$ hypergeometric functions, who may have somewhat different coefficients than the above: One may use the relation [31]

$$(-a)_n = (-1)^n (a-n+1)_n \tag{27}$$

and the primary definition of the Pochhammer symbol [32]

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (-a \in \mathbb{Z} \wedge -a \geq 0 \wedge n \in \mathbb{Z} \wedge n \leq -a) \tag{28}$$

to rewrite the term in square brackets in infinite sum (26) to give two alternative forms:

$$\begin{aligned}
&\sum_{L=0}^{\infty} \frac{\sqrt{\pi} i^L 2^{-3L-2}(1+(-1)^L)(2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L}\,{}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};-\frac{k^2}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!} k^L \\
&\times \left[\frac{(-1)^{\frac{L}{2}-h}\left(h+\frac{1}{2}\right)_{\frac{L}{2}-h}(h+1)_{\frac{L}{2}-h}}{\left(h+\frac{L}{2}+\frac{1}{2}\right)_{\frac{L}{2}-h}}\right] \\
&= \sum_{L=0}^{\infty} \frac{\sqrt{\pi} i^L 2^{-3L-2}(1+(-1)^L)(2L+1)\binom{L}{\frac{L}{2}}\binom{2L}{L}\,{}_1F_2\left(\frac{L}{2}+\frac{1}{2};\frac{L}{2}+1,L+\frac{3}{2};-\frac{k^2}{4}\right)}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(\frac{L}{2}-h\right)!} k^L \\
&\times \left[\frac{(-1)^{\frac{L}{2}-h}2^{2h-L}\Gamma(L+1)\Gamma\left(h+\frac{L}{2}+\frac{1}{2}\right)}{\Gamma(2h+1)\Gamma\left(L+\frac{1}{2}\right)}\right] = \frac{(-1)^h 2^{-2h}}{h!\Gamma(h+1)} k^{2h}\,.
\end{aligned} \tag{29}$$

5. Series Arising from the $J_1(x)$ Fourier–Legendre Series

The first 22 terms in the $J_1(x)$ Fourier–Legendre series (1) are

$$\begin{aligned}
J_1(x) \cong\ & 0.4635981705953810635941110039338702 P_1(x) \\
-\ & 0.0238653456584073979630720941648486 6P_3(x) \\
+\ & 0.00031972435597200476385247576232560 28P_5(x) \\
-\ & 1.9705191806665942502580629293911 12 \times 10^{-6}\ P_7(x) \\
+\ & 6.9872474730978072187917594101570 14 \times 10^{-9}\ P_9(x) \\
-\ & 1.6105000560468750278070024429533 27 \times 10^{-11}\ P_{11}(x) \\
+\ & 2.6070865924416288429392481936199 09 \times 10^{-14}\ P_{13}(x) \\
-\ & 3.1273114825407968821447136195674 42 \times 10^{-17}\ P_{15}(x) \\
+\ & 2.8914240817870507398273825966160 64 \times 10^{-20}\ P_{17}(x) \\
-\ & 2.1236645347793691992144144557203 17 \times 10^{-23}\ P_{19}(x) \\
+\ & 1.2690112017586735117145537075281 86 \times 10^{-26}\ P_{21}(x) \\
-\ & 6.2902019391359257635768713587386 00 \times 10^{-30}\ P_{23}(x) \\
+\ & 2.6281357969893254525738707742672 13 \times 10^{-33}\ P_{25}(x) \\
-\ & 9.3815755627230761092832580506676 42 \times 10^{-37}\ P_{27}(x) \\
+\ & 2.8943372424159840409418590610224 19 \times 10^{-40}\ P_{29}(x) \\
-\ & 7.7944441041041716843950942611748 14 \times 10^{-44}\ P_{31}(x) \\
+\ & 1.8482007598181701348958670523067 67 \times 10^{-47}\ P_{33}(x) \\
-\ & 3.8882496397739122256945358903292 44 \times 10^{-51}\ P_{35}(x)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 7.3069787188071236330441200058516188 \times 10^{-55}\, P_{37}(x) \\
&- \quad 1.2340225304566215711275900996477796 \times 10^{-58}\, P_{39}(x) \\
&+ \quad 1.8830677992555689156494618842554 28 \times 10^{-62}\, P_{41}(x) \\
&- \quad 2.6091228845363508612681953510458 90 \times 10^{-66}\, P_{43}(x) \qquad (30)
\end{aligned}
$$

If one wishes to check the convergence of this series at, say, the $|\varepsilon| < 1.3 \times 10^{-8}$ level given by the polynomial approximation given by E. E. Allen (and reproduced in Abramowitz and Stegun) [1] (p. 370 No. 9.4.4),

$$
\begin{aligned}
J_1(x) \;\cong\;& \frac{x}{2} - 0.062499983x^3 + 0.0026041448x^5 - 0.00005424265x^7 + 6.7568816 \times 10^{-7}\, x^9 \\
-\;& 5.3788 \times 10^{-9}\, x^{11} + 2.087 \times 10^{-11}\, x^{13}\,, \qquad (31)
\end{aligned}
$$

at the latter's limiting range of $x = 3$ (with a value of 0.339059), one finds that truncating the series after the $P_{13}(x)$ term (to obtain contributions through x^{13}) is insufficient, giving one fewer digits of accuracy (0.339060), again showing that the optimization Allen must have performed to obtain his approximation had a significant effect.

Testing accuracy at the 15-digit level shows that the series can be truncated after the $P_{25}(x)$ term in the result, 0.339058958525937. This $|\varepsilon| < 4 \times 10^{-17}$ is better than one would expect for doubling the number of powers from the $P_{13}(x)$ truncation. In addition, the *range* of applicability of the new Legendre polynomial approximation truncated after $P_{24}(x) \sim x^{24}$, if one is satisfied with $|\varepsilon| < 1 \times 10^{-8}$, roughly doubles to $J_0(6) \cong -0.276684$.

Finally, I confirm that the series through $P_{43}(x)$ does give 33-digit accuracy for $-3 \leq x \leq 3$, with $J_1(3) \cong 0.339058958525936458925514597206 48$. Accuracy exceeds $|\varepsilon| < 9 \times 10^{-15}$ over the range $-8 \leq x \leq 8$, with $J_1(8) \cong 0.234636346853916$.

One can again expand the Legendre polynomials into their constituent terms and gather like powers in (30) to give an updated polynomial approximation,

$$
\begin{aligned}
J_1(x) \;\cong\;& 0.5000000000000000000000000000000000\, x \\
-\;& 0.06250000000000000000000000000000000\, x^3 \\
+\;& 0.002604166666666666666666666666666667\, x^5 \\
-\;& 0.00005425347222222222222222222222222222\, x^7 \\
+\;& 6.7816840277777777777777777777777778 \times 10^{-7}\, x^9 \\
-\;& 5.6514033564814814814814814814814815 \times 10^{-9}\, x^{11} \\
+\;& 3.3639305693342151675485008818342151 68 \times 10^{-11}\, x^{13} \\
-\;& 1.5017547184527746283698664651045603 43 \times 10^{-13}\, x^{15} \\
+\;& 5.2144261057388007929509252260575011 90 \times 10^{-16}\, x^{17} \\
-\;& 1.4484516960385557758197014516826392 19 \times 10^{-18}\, x^{19} \\
+\;& 3.2919356728148994904993214810969073 17 \times 10^{-21}\, x^{21} \\
-\;& 6.2347266530585217623093209869259608 27 \times 10^{-24}\, x^{23} \\
+\;& 9.9915491234912207729316041457146808 13 \times 10^{-27}\, x^{25} \\
-\;& 1.3724655389411017545235719980377308 81 \times 10^{-29}\, x^{27} \\
+\;& 1.6338875463584544696709190452830129 53 \times 10^{-32}\, x^{29} \\
-\;& 1.7019661941233900725738740055031384 93 \times 10^{-35}\, x^{31} \\
+\;& 1.5643071637163511696451047844697964 09 \times 10^{-38}\, x^{33} \\
-\;& 1.2780287285264306941544973729328402 04 \times 10^{-41}\, x^{35} \\
+\;& 9.3423152670060723256907702699769020 73 \times 10^{-45}\, x^{37} \\
-\;& 6.1462600440829423195334014934058566 27 \times 10^{-48}\, x^{39}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 3.658488121477941856865119936551105135 \times 10^{-51}\ x^{41} \\
&- \quad 1.979701364436115723411861437527654294 \times 10^{-54}\ x^{43} \\
&= \quad \frac{x}{2} - \frac{x^3}{2^4} + \frac{x^5}{2^7 3} - \frac{x^7}{2^{11} 3^2} + \frac{x^9}{2^{15} 3^2 5} - \frac{x^{11}}{2^{18} 3^3 5^2} + \frac{x^{13}}{2^{21} 3^4 5^2 7} - \frac{x^{15}}{2^{26} 3^4 5^2 7^2} \\
&+ \quad \frac{x^{17}}{2^{31} 3^6 5^2 7^2} - \frac{x^{19}}{2^{34} 3^8 5^3 7^2} + \frac{x^{21}}{2^{37} 3^8 5^4 7^2 11} - \frac{x^{23}}{2^{41} 3^9 5^4 7^2 11^2} \\
&+ \quad \frac{x^{25}}{2^{45} 3^{10} 5^4 7^2 11^2 13} - \frac{x^{27}}{2^{48} 3^{10} 5^4 7^3 11^2 13^2} + \frac{x^{29}}{2^{51} 3^{11} 5^5 7^4 11^2 13^2} - \frac{x^{31}}{2^{57} 3^{12} 5^6 7^4 11^2 13^2} \\
&+ \quad \frac{x^{33}}{2^{63} 3^{12} 5^6 7^4 11^2 13^2 17} - \frac{x^{35}}{2^{66} 3^{14} 5^6 7^4 11^2 13^2 17^2} + \frac{x^{37}}{2^{69} 3^{16} 5^6 7^4 11^2 13^2 17^2 19}, \\
&- \quad \frac{x^{39}}{2^{73} 3^{16} 5^7 7^4 11^2 13^2 17^2 19^2} + \frac{x^{41}}{2^{77} 3^{17} 5^8 7^5 11^2 13^2 17^2 19^2} - \frac{x^{43}}{2^{80} 3^{18} 5^8 7^6 11^3 13^2 17^2 19^2}
\end{aligned} \tag{32}
$$

with the latter form an integer-power realization of first 22 terms of the well-known series representation [21] (GR5 p. 970 No. 8.440). This time, all terms in the first half were truncated to the 37-digit precision displayed therein.

The truncated power series gives $J_1(3) \cong 0.3390589585259364589255145972064788 94$, which matches *Mathematica*'s BesselJ[1,3]$\cong$ 0.339058958525936458925514597206478897 0 (when set to 37-digit precision) within an error of $|\varepsilon| < 3 \times 10^{-36}$, as does the inverse prime version $J_1(3) \cong \frac{2326694457886355371234768432489832510 4584007}{6862212011744777099744371238984744960 0000000} = 0.339058958525936458925514597206478894 1$.

The Legendre series, via the above development, gives another set of infinite series (of which I have confirmed $0 \leq h \leq 43$),

$$
\begin{aligned}
&\sum_{L=1}^{\infty} \frac{\sqrt{\pi} i^{L-1} \left(1 + (-1)^{L+1}\right)(2L+1) 2^{-3L-2} \binom{L}{\frac{L-1}{2}} \binom{2L}{L} (-1)^{-h + \frac{L}{2} - \frac{1}{2}}}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(-h + \frac{L}{2} - \frac{1}{2}\right)!} k^L \\
&\times \quad {}_1F_2\left(\frac{L}{2} + 1; \frac{L}{2} + \frac{3}{2}, L + \frac{3}{2}; -\frac{k^2}{4}\right) \left[\frac{\left(\frac{1}{2} - \frac{L}{2}\right)_{\frac{L-1}{2} - h} \left(-\frac{L}{2}\right)_{\frac{L-1}{2} - h}}{\left(\frac{1}{2} - L\right)_{\frac{L-1}{2} - h}} \right] \\
&= \quad \sum_{L=1}^{\infty} {}^{(2)} \frac{\sqrt{\pi} i^{L-1} \left(1 + (-1)^{L+1}\right)(2L+1) 2^{-3L-2} \binom{L}{\frac{L-1}{2}} \binom{2L}{L}}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(-h + \frac{L}{2} - \frac{1}{2}\right)!} k^L \\
&\times \quad {}_1F_2\left(\frac{L}{2} + 1; \frac{L}{2} + \frac{3}{2}, L + \frac{3}{2}; -\frac{k^2}{4}\right) \left[(-1)^{-h + \frac{L}{2} - \frac{1}{2}} \frac{(h+1)_{-h + \frac{L}{2} - \frac{1}{2}} \left(h + \frac{3}{2}\right)_{-h + \frac{L}{2} - \frac{1}{2}}}{\left(h + \frac{L}{2} + 1\right)_{-h + \frac{L}{2} - \frac{1}{2}}} \right] \\
&= \quad \sum_{L=1}^{\infty} {}^{(2)} \frac{\sqrt{\pi} i^{L-1} \left(1 + (-1)^{L+1}\right)(2L+1) 2^{-3L-2} \binom{L}{\frac{L-1}{2}} \binom{2L}{L}}{\Gamma\left(\frac{1}{2}(2L+3)\right)\left(-h + \frac{L}{2} - \frac{1}{2}\right)!} k^L \\
&\times \quad {}_1F_2\left(\frac{L}{2} + 1; \frac{L}{2} + \frac{3}{2}, L + \frac{3}{2}; -\frac{k^2}{4}\right) \left[(-1)^{-h + \frac{L}{2} - \frac{1}{2}} \frac{2^{2h - L + 1} \Gamma(L+1) \Gamma\left(h + \frac{L}{2} + 1\right)}{\Gamma(2h+2) \Gamma\left(L + \frac{1}{2}\right)} \right] \\
&= \quad \frac{(-1)^h 2^{-2h-1}}{h! \Gamma(h+2)} k^{2h+1}.
\end{aligned} \tag{33}
$$

To verify the final summed series, with $h = 43$, I had to take the upper limit on the number of terms in the series $\geq h + 78$ in order to obtain a percent difference between left- and right-hand sides that was $\leq 10^{-33}$ because the first h terms in the series do not contribute. For $h = 0$, an upper limit on the number of terms in the series $\geq h + 45$ was sufficient.

6. Series Arising from the $I_n(x)$ Fourier–Legendre Series

Because the modified Bessel functions of the first kind $I_N(kx)$ are related to the ordinary Bessel functions by the relation [21] (p. 961 No. 8.406.3)

$$I_n(z) = i^{-n} J_n(iz) \;, \tag{34}$$

one merely needs to multiply by i^{-n} and set $k = i$ in (11) to obtain the $I_0(x)$ Fourier–Legendre series, the first 24 terms of which are

$$\begin{aligned} I_0(x) \;\cong\;& 1.0865210970235898158379419234925O6\, P_0(x) \\ &+ 0.17580468192152426626059513542612 50\, P_2(x) \\ &+ 0.0037090092440528825339238381655270 33\, P_4(x) \\ &+ 0.000030951052709924321986137446087 77602\, P_6(x) \\ &+ 1.3812597347197735383200523052245O6 \times 10^{-7}\, P_8(x) \\ &+ 3.8343126010863730053177889061255 73 \times 10^{-10}\, P_{10}(x) \\ &+ 7.2571724500962139367206676604119 78 \times 10^{-13}\, P_{12}(x) \\ &+ 9.9627469788360180201284331116359 75 \times 10^{-16}\, P_{14}(x) \\ &+ 1.0372513461100526309637054770467 36 \times 10^{-18}\, P_{16}(x) \\ &+ 8.4704968632404753393434993216041 16 \times 10^{-22}\, P_{18}(x) \\ &+ 5.5705413998588522192782604835236 87 \times 10^{-25}\, P_{20}(x) \\ &+ 3.0133473832345288502246890418232 01 \times 10^{-28}\, P_{22}(x) \\ &+ 1.3643380053535272726384790931752 49 \times 10^{-31}\, P_{24}(x) \\ &+ 5.2460884671622816489446609893595 65 \times 10^{-35}\, P_{26}(x) \\ &+ 1.7344170522365465259795626101693 36 \times 10^{-38}\, P_{28}(x) \\ &+ 4.9829298896312035606869677624018 21 \times 10^{-42}\, P_{30}(x) \\ &+ 1.2555464305598776215872017907003 57 \times 10^{-45}\, P_{32}(x) \\ &+ 2.7970965964017064134440688215081 93 \times 10^{-49}\, P_{34}(x) \\ &+ 5.5489556770499634839096738450714 89 \times 10^{-53}\, P_{36}(x) \\ &+ 9.8652062252050832472129850965315 73 \times 10^{-57}\, P_{38}(x) \\ &+ 1.5807636916523069830994437619446 73 \times 10^{-60}\, P_{40}(x) \\ &+ 2.2946883314792052816008147199140 93 \times 10^{-64}\, P_{42}(x) \\ &+ 3.0317714955807038951271099333866 07 \times 10^{-68}\, P_{44}(x) \\ &+ 3.6612007726805987529908521860251 67 \times 10^{-72}\, P_{46}(x) \;. \end{aligned} \tag{35}$$

If one wishes to check the convergence of this series at, say, the $|\varepsilon| < 1.6 \times 10^{-7}$ level of the polynomial approximation given by E. E. Allen (and reproduced in Abramowitz and Stegun) [1] (p. 378 No. 9.8.1),

$$\begin{aligned} I_0(x) \;\cong\;& 1 + 3.5156229\left(\frac{x}{3.75}\right)^2 + 3.0899424\left(\frac{x}{3.75}\right)^4 + 1.2067492\left(\frac{x}{3.75}\right)^6 \\ +\;& 0.2659732\left(\frac{x}{3.75}\right)^8 + 0.0360768\left(\frac{x}{3.75}\right)^{10} + 0.0045813\left(\frac{x}{3.75}\right)^{12} \\ =\;& 1 + 0.25\, x^2 + 0.0156252\, x^4 + 0.00043394\, x^6 + 6.801234 \times 10^{-6}\, x^8 + 6.56017 \times 10^{-8} x^{10} \\ +\;& 5.9240 \times 10^{-10}\, x^{12} \end{aligned} \tag{36}$$

at the latter's limiting range of $x = 3.75$ (with a value of 9.11895), one finds that truncating the series after the $P_{12}(x)$ term (to obtain contributions through x^{12}) is insufficient, giving two fewer digits of accuracy (9.1187). This again shows that the optimization Allen must have performed to obtain his approximation has a significant effect.

Testing accuracy at the 15-digit level shows that the series can be truncated after the $P_{28}(x)$ term in the result, 9.1189458608445666, with $|\varepsilon| < 5 \times 10^{-17}$. In addition, the *range* of applicability of the new Legendre approximation truncated after $P_{24}(x) \sim x^{24}$, if one is satisfied with $|\varepsilon| < 1.6 \times 10^{-7}$, roughly doubles to $I_0(7.5) \cong 268.1613$.

Finally, in the quality checking step, I found that if I imported a 33-digit version of (35) back into the software I had been using to generate it, *Mathematica 7*, I only obtained a result accurate to$|\varepsilon| < 1 \times 10^{-32}$, but if I imported the 34-digit version version actually displayed in (35), I obtained a result accurate to $|\varepsilon| < 1 \times 10^{-34}$ over the range $-3.75 \le x \le 3.75$, with $I_0(3.75) \cong 9.118945860844566690670997606599715$. Accuracy exceeds $|\varepsilon| < 6 \times 10^{-14}$ over the range $-8 \le x \le 8$, with $I_0(8) \cong 427.5641157218$.

If one turns one's attention to the latter form of the polynomial approximation (36) and compare it with the J_0 polynomial approximation (17), one sees that the I_0 version is approximately the J_0 version with all of the negative signs reversed. That the correspondence is not exact for the higher-power terms is likely a result of the optimization scheme in the two cases having slightly different ranges of validity.

One may apply (34) to (19), giving $i^{\nu}(-1)^k i^{2k+\nu}\left(\frac{x}{2}\right)^{2k+\nu} = (-1)^{\nu}\left(\frac{x}{2}\right)^{2k+\nu}$ for each numerator, which indeed gives the series [21] (GR5 p. 971 No. 8.445)

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(k+\nu+1)} \,. \tag{37}$$

I, thus, need not display the 24-term polynomial approximation for I_0 apart from the two terms unneeded for accuracy in the J_0 version that are necessary for I_0:

$$\begin{aligned} \ldots \quad &+ \quad 4.499321282809353916845139630744668 8497 \times 10^{-56}\, x^{44} \\ &+ \quad 2.126333309456216406826625534378387 9252 \times 10^{-59}\, x^{46} \,. \end{aligned} \tag{38}$$

One obtains the other terms by simply negating the negative signs in (18). This is not true of (35) because the arguments of the Legendre polynomials do not undergo $x \to ix$ since they derive from the definition of the Fourier–Legendre series, (1). The k-dependence is entirely within the coefficients $a_{LN}(k)$.

Furthermore, the I_0 Legendre series expansion leads to no new set of summed series since these would simply be (26) with $k = i\kappa$.

Although (1) allows one to easily compute the Fourier–Legendre series for any $J_n(x)$ or $I_n(x)$, to enable readers to find these series for higher indices by recursion [21] (p. 979 No. 8.471.1, p. 981 No. 8.486.1) I give the first 24 terms in the $I_1(x)$ Fourier–Legendre series to complete the required pair:

$$\begin{aligned} I_1(x) \quad \cong \quad & 0.5386343421852555592809081051666336\, P_1(x) \\ + \quad & 0.02618069164825977449795296407260333\, P_3(x) \\ + \quad & 0.0003419851912550806236210094361507344\, P_5(x) \\ + \quad & 2.077651971699656963860267070724864 \times 10^{-6}\, P_7(x) \\ + \quad & 7.299001518662431414905576324932877 \times 10^{-9}\, P_9(x) \\ + \quad & 1.671443482954853739162527767203215 \times 10^{-11}\, P_{11}(x) \\ + \quad & 2.692744551459235232734936666452704 \times 10^{-14}\, P_{13}(x) \\ + \quad & 3.218106754771162455853759838545282 \times 10^{-17}\, P_{15}(x) \\ + \quad & 2.966624646773403074824196435937542 \times 10^{-20}\, P_{17}(x) \\ + \quad & 2.173686883720901031568436047655748 \times 10^{-23}\, P_{19}(x) \\ + \quad & 1.296326265789554875546711294998344 \times 10^{-26}\, P_{21}(x) \end{aligned}$$

$$
\begin{aligned}
&+ \quad 6.41485510215141573359658829657883 3 \times 10^{-30}\, P_{23}(x) \\
&+ \quad 2.676389925142875786863074285387196 \times 10^{-33}\, P_{25}(x) \\
&+ \quad 9.542035089444710700263714658817980 \times 10^{-37}\, P_{27}(x) \\
&+ \quad 2.940669572337884276201779460203862 \times 10^{-40}\, P_{29}(x) \\
&+ \quad 7.911705033029330504663434638574930 \times 10^{-44}\, P_{31}(x) \\
&+ \quad 1.874426565726980007813411585706840 \times 10^{-47}\, P_{33}(x) \\
&+ \quad 3.940459072597980181771454632976250 \times 10^{-51}\, P_{35}(x) \\
&+ \quad 7.400090413796917559009186360838868 \times 10^{-55}\, P_{37}(x) \\
&+ \quad 1.248984620737396858084740490332061 \times 10^{-58}\, P_{39}(x) \\
&+ \quad 1.904842982553207494042180613785837 \times 10^{-62}\, P_{41}(x) \\
&+ \quad 2.637959760920312924684635466402215 \times 10^{-66}\, P_{43}(x) \\
&+ \quad 3.332061910821697596383220274010501 \times 10^{-70}\, P_{45}(x)\,.
\end{aligned}
\tag{39}
$$

If one wishes to check the convergence of this series at, say, the $|\varepsilon| < 8 \times 10^{-9}$ level given by the polynomial approximation given by E. E. Allen (and reproduced in Abramowitz and Stegun) [1] (p. 378 No. 9.8.3),

$$
\begin{aligned}
I_1(x) \quad \cong \quad & \frac{x}{2} + 0.0625x^3 + 0.00260419x^5 + 0.0000542445x^7 + 6.79868 \times 10^{-7}x^9 + 5.48303 \times 10^{-9}\, x^{11} \\
+ \quad & 4.191 \times 10^{-11}\, x^{13}\,,
\end{aligned}
\tag{40}
$$

at the latter's limiting range of $x = 3.75$ (with a value of 7.78002), one finds that truncating the series after the $P_{13}(x)$ term (to obtain contributions through x^{13}) is insufficient, giving three fewer digits of accuracy (7.77996). Yet again this shows that the optimization Allen must have performed to obtain his approximation has a significant effect.

Testing accuracy at the 15-digit level shows that the series can be truncated after the $P_{27}(x)$ term in the result, 7.780015229824415, with $|\varepsilon| < 4 \times 10^{-16}$. In addition, the *range* of applicability of the new Legendre approximation truncated after $P_{27}(x) \sim x^{27}$, if one is satisfied with $|\varepsilon| < 5 \times 10^{-9}$, roughly doubles to $I_1(6.5) \cong 97.735011$.

Finally, in my quality checking step, I found that if I imported a 33-digit version of (39) back into the software I had been using to generate it, *Mathematica 7*, I only obtained a result accurate to$|\varepsilon| < 1 \times 10^{-32}$, but if I imported the 34-digit version actually displayed in (35), I obtained a result accurate to $|\varepsilon| < 1 \times 10^{-34}$ over the range $-3.75 \le x \le 3.75$, with $I_1(3.75) \cong 7.7800152298244158649886762775161 13$. Accuracy exceeds $|\varepsilon| < 1 \times 10^{-18}$ over the range $-8 \le x \le 8$, with $I_1(8) \cong 399.873136782560098$.

Because the above-noted correspondence between the power-series versions of $J_0(x)$ and $I_0(x)$ (reversing all of the negative signs in the former to achieve the latter) applies as well to $J_1(x)$ and $I_1(x)$, there is no need to display the power-series version of the latter except for the additional term,

$$
\ldots + 9.781133223498595471402477458140584456 \times 10^{-58}\, x^{45}\,. \tag{41}
$$

7. Conclusions

I have found the Fourier–Legendre series of modified Bessel functions of the first kind $I_N(kx)$ based on that found by Keating [17] for the Bessel functions of the first kind $J_N(kx)$ and have shown that Keating's coefficients, comprised of infinite series, can be reduced to ${}_2F_3$ functions. For $N = 0$ and 1, I gave numerical values for those coefficients up through x^{46} with 33-digit accuracy.

Each of these infinite Fourier–Legendre series may be decomposed into an infinite sum of infinite series, by gathering like powers from the Legendre polynomials in each of the terms in the Fourier–Legendre series. I showed that each of these infinite sub-series converges to values that are inverse powers of the first eight primes $1/\left(2^i 3^j 5^k 7^l 11^m 13^n 17^o 19^p\right)$

multiplying powers of the coefficient k. Given the relative paucity of infinite series whose values are known (e.g., 24 pages in Gradshteyn and Ryzhik compared to their 900 pages of known integrals), having even one such to add to the total has the potential to be of use to future researchers. Herein I have added an infinite set of infinite series of ${}_1F_2$ functions whose values are now known.

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Appendix A

The code for calculating these series is given in Fortran, below, because it can be used as such or called in C and C++ programs and one wishes to avoid duplication in this paper. In Fortran 90, quadruple precision is instituted as in the following calculation of π:

```
! -----+-----+-----+-----+-----+-----+-----+-----+-----+-----+-----
! In gcc46, and later, one compiles the program with
! $ gfortran precisiontest.f90 -o precisiontest90
! and runs it with
! $ ./precisiontest90
!
! The ouptput is
!                        1234567891123456789212345678931 23
!                               4       8               16
! s_r_k(6,37)    =    3.1415927
! s_r_k(15,307)  =    3.1415926535897931
! s_r_k(33,4931) =    3.14159265358979323846264338327950 28
!
! -----+-----+-----+-----+-----+-----+-----+-----+-----+-----+-----
module precisionkinds!   use ISO_FORTRAN_ENV
   implicit none
   private
   public isp, idp, iqp
   integer, parameter :: isp = selected_real_kind(6, 37)
   integer, parameter :: idp = selected_real_kind(15, 307)
   integer, parameter :: iqp = selected_real_kind(33, 4931)
end module precisionkinds
!
program precisiontest
   use precisionkinds
   implicit none
   real (isp), parameter :: pi1 = 4*atan (1.0_isp)
   real (idp), parameter :: pi2 = 4*atan (1.0_idp)
   real (iqp), parameter :: pi4 = 4*atan (1.0_iqp)
      write (*,*) '                    1234567891123456789212345678931 23'
      write (*,*) '                          4',                  &
      '      8              16'
         write (*,*) 's_r_k(6,37)    = ',pi1
         write (*,*) 's_r_k(15,307)  = ',pi2
         write (*,*) 's_r_k(33,4931) = ',pi4
end program precisiontest
```

For readers who are modifying legacy programs, the Fortran 77 equivalent lines are as follows (two additional compilers are referenced, xlf for PowerPC and ifort for Intel Macintosh computers):

```
C $ xlf -o precisiontest77g precisiontest77g.f
C $ ./precisiontest77g
C $ ifort precisiontest77g.f -o precisiontest77g_intel
C
C                          4       8                 16
C intel gives
Cpi16srk31    =   3.14159265358979323846264338327950
C
C gfortran gives
C             =   3.14159265358979323846264338327950280
C xlf gives
C pi16Q0      =   3.1415926535897932384626433832795059
C pi16srk20   =   3.1415926535897932384626433832795059
C pi16srk31   =   3.1415926535897932384626433832795059
C pi16iesrk31 =   3.1415926535897932384626433832795059
C
C"from Bailey's DQFUN:A thread-safe double-quad precision package
C                 3.14159265358979323846264338327950288419 7169...
C
          real*16 pi16Q0,pi16srk20,pi16iesrk31,pi16srk31
C
          pi16Q0 = 4*atan (1.0Q0)
          pi16srk20 = 4*atan (Real(1.0,SELECTED_REAL_KIND(20,140)))
          pi16srk31 = 4*atan (Real(1.0,SELECTED_REAL_KIND(31)))
          pi16iesrk31 = 4*atan (Real(1.0,IEEE_SELECTED_REAL_KIND(31)))
```

In C and C++, one specifies _Float128. Mathematica and Maple input may be generated from the following by replacing the Fortran power operator "**" with "^" and deleting the continuation indicators "-" in column 6.

For (15), the code is

```
        J0(x) = 0.91973041008976023931442119408062*P(0,x) -
     -  0.15794205862585188757371396714436 37*P(2,x) +
     -  0.0034384009446011092329968878720 72915*P(4,x) -
     -  0.00002919721848828729693660590986125663*P(6,x) +
     -  1.31735695244778097765561656314328e-7*P(8,x) -
     -  3.68450084420820302717377109605886 6e-10*P(10,x) +
     -  7.01183003299384592820880332821144 7e-13*P(12,x) -
     -  9.66596436985891226367199537275334 6e-16*P(14,x) +
     -  1.00963627682454644652534217092493 6e-18*P(16,x) -
     -  8.26665695592763785899197258417411 7e-22*P(18,x) +
     -  5.44824486776275872589008283783943 0e-25*P(20,x) -
     -  2.95252718213735475167577460666340 0e-28*P(22,x) +
     -  1.33885615885853446908089867009620 0e-31*P(24,x) -
     -  5.15491318608851292619323483781658 2e-35*P(26,x) +
     -  1.70623157703850345013856402846763 4e-38*P(28,x) -
     -  4.90689355642779685747309797956828 9e-42*P(30,x) +
     -  1.23748920071747938302053957622129 3e-45*P(32,x) -
     -  2.75905623753787186860455568854836 4e-49*P(34,x) +
     -  5.47738220717271262919971464839640 9e-53*P(36,x) -
     -  9.74420034557885255068894605705067 4e-57*P(38,x) +
     -  1.56228071165950448982802514899577 0e-60*P(40,x) -
     -  2.26905628382739436883605747059459 9e-64*P(42,x)
```

and the corresponding expressions for both forms of (18) are

```
    J0(x) = 1. - 0.25*x**2 + 0.015625*x**4 -
-   0.0004340277777777777777777777777777777778*x**6 +
-   6.781684027777777777777777777777778e-6*x**8 -
-   6.781684027777777777777777777777778e-8*x**10 +
-   4.709502797067901234567901234567901235e-10*x**12 -
-   2.402807549524439405391786344167296548e-12*x**14 +
-   9.385966990329841427311665406903502142e-15*x**16 -
-   2.896903392077111551639402903365278439e-17*x**18 +
-   7.242258480192778879098507258413196097e-20*x**20 -
-   1.496334396734045222954237036862230599e-22*x**22 +
-   2.597802772107717400962217077885817011e-25*x**24 -
-   3.842903509035084912666001594505646466e-28*x**26 +
-   4.901662639075363409012757135849038860e-31*x**28 -
-   5.446291821194848232236396817610043178e-34*x**30 +
-   5.318644356635593976793356267197307791e-37*x**32 -
-   4.600903422695150498956190542558224733e-40*x**34 +
-   3.550079801462307483762492702591222788e-43*x**36 -
-   2.458504017633176927813360597362342651e-46*x**38 +
-   1.536565011020735579883350373351464l567e-49*x**40 -
-   8.710686003518909183012190325121678892 9e-53*x**42
```

and (for integer arithmetic)

```
J0(x)=x^0*-x^2/(2^2)+x^4/(2^6)-x^6/(2^8*3^2)+x^8/(2^14*3^2)
-x^10/(2^16*3^2*5^2)+x^12/(2^20*3^4*5^2)-x^14/(2^22*3^4*5^2*7^2)
+x^16/(2^30*3^4*5^2*7^2)-x^18/(2^32*3^8*5^2*7^2)+x^20/(2^36*3^8*5^4*7^2)
-x^22/(2^38*3^8*5^4*7^2*11^2)+x^24/(2^44*3^10*5^4*7^2*11^2)
-x^26/(2^46*3^10*5^4*7^2*11^2*13^2)+x^28/(2^50*3^10*5^4*7^4*11^2*13^2)
-x^30/(2^52*3^12*5^6*7^4*11^2*13^2)+x^32/(2^62*3^12*5^6*7^4*11^2*13^2)
-x^34/(2^64*3^12*5^6*7^4*11^2*13^2*17^2)
+x^36/(2^68*3^16*5^6*7^4*11^2*13^2*17^2)
-x^38/(2^70*3^16*5^6*7^4*11^2*13^2*17^2*19^2)
+x^40/(2^76*3^16*5^8*7^4*11^2*13^2*17^2*19^2)
-x^42/(2^78*3^18*5^8*7^6*11^2*13^2*17^2*19^2)
```

For (30), the code is

```
    J1(x) = 0.4635981705953810635941110039338702*P(1,x) -
-   0.02386534565840739796307209416484866*P(3,x) +
-   0.0003197243559720047638524757623256028*P(5,x) -
-   1.970519180666594250258062929391112e-6*P(7,x) +
-   6.987247473097807218791759410157014e-9*P(9,x) -
-   1.610500056046875027807002442953327e-11*P(11,x) +
-   2.607086592441628842939248193619909e-14*P(13,x) -
-   3.127311482540796882144713619567442e-17*P(15,x) +
-   2.891424081787050739827382596616064e-20*P(17,x) -
-   2.123664534779369199214414455720317e-23*P(19,x) +
-   1.269011201758673511714553707528186e-26*P(21,x) -
-   6.290201939135925763576871358738600e-30*P(23,x) +
-   2.628135796989325452573870774267213e-33*P(25,x) -
-   9.381575562723076109283258050667642e-37*P(27,x) +
-   2.894337242415984040941859061022419e-40*P(29,x) -
-   7.794444104104171684395094261174814e-44*P(31,x) +
-   1.848200759818170134895867052306767e-47*P(33,x) -
-   3.888249639773912225694535890329244e-51*P(35,x) +
-   7.306978718807123633044120058516188e-55*P(37,x) -
```

```
-   1.2340225304566215711275900996477 96e-58*P(39,x) +
```

and the corresponding expressions for both forms of (32) are

```
    J1(x)= 0.5*x - 0.0625*x**3 +
-   0.0026041666666666666666666666666666667*x**5 -
-   0.000054253472222222222222222222222222222*x**7 +
-   6.7816840277777777777777777777777778e-7*x**9 -
-   5.6514033564814814814814814814814814881e-9*x**11 +
-   3.3639305693342151675485008818342151 68e-11*x**13 -
-   1.5017547184527746283698664651045603 43e-13*x**15 +
-   5.2144261057388007929509252260575011 90e-16*x**17 -
-   1.4484516960385557758197014516826392 19e-18*x**19 +
-   3.2919356728148994904993214810969073 17e-21*x**21 -
-   6.2347266530585217623093209869259608 27e-24*x**23 +
-   9.9915491234912207729316041457146808 13e-27*x**25 -
-   1.3724655389411017545235719980377308 81e-29*x**27 +
-   1.6338875463584544696709190452830129 53e-32*x**29 -
-   1.7019661941233900725738740055031384 93e-35*x**31 +
-   1.5643071637163511696451047844697964 09e-38*x**33 -
-   1.2780287285264306941544973729328402 04e-41*x**35 +
-   9.3423152670060723256907702699769020 73e-45*x**37 -
-   6.1462600440829423195334014934058566 27e-48*x**39 +
-   3.6584881214779418568651199365511051 35e-51*x**41 -
-   1.9797013644361157234118614375276542 94e-54*x**43
```

and (for integer arithmetic)

```
J1(x)=x^2/2-x^3/(2^4)+x^5/(2^7*3*-x^7/(2^11*3^2)+x^9/(2^15*3^2*5)
-x^11/(2^18*3^3*5^2)+x^13/(2^21*3^4*5^2*7)-x^15/(2^26*3^4*5^2*7^2)
+x^17/(2^31*3^6*5^2*7^2)-x^19/(2^34*3^8*5^3*7^2)+x^21/(2^37*3^8*5^4*7^2*11)
-x^23/(2^41*3^9*5^4*7^2*11^2)+x^25/(2^45*3^10*5^4*7^2*11^2*13)
-x^27/(2^48*3^10*5^4*7^3*11^2*13^2)+x^29/(2^51*3^11*5^5*7^4*11^2*13^2)
-x^31/(2^57*3^12*5^6*7^4*11^2*13^2)+x^33/(2^63*3^12*5^6*7^4*11^2*13^2*17)
-x^35/(2^66*3^14*5^6*7^4*11^2*13^2*17^2)
+x^37/(2^69*3^16*5^6*7^4*11^2*13^2*17^2*19)
-x^39/(2^73*3^16*5^7*7^4*11^2*13^2*17^2*19^2)
+x^41/(2^77*3^17*5^8*7^5*11^2*13^2*17^2*19^2)
-x^43/(2^80*3^18*5^8*7^6*11^3*13^2*17^2*19^2)
```

For (35), the code is

```
     I(0,x) = 1.0865210970235898158379419234925 06*P(0,x) +
-   0.17580468192152426626059513542612 50*P(2,x) +
-   0.0037090092440528825339238381655270 33*P(4,x) +
-   0.0000309510527099243219861374460877 7602*P(6,x) +
-   1.3812597347197735383200523052245 06e-7*P(8,x) +
-   3.8343126010863730053177889061255 73e-10*P(10,x) +
-   7.2571724500962139367206676604119 78e-13*P(12,x) +
-   9.9627469788360180201284331116359 75e-16*P(14,x) +
-   1.0372513461100526309637054770467 36e-18*P(16,x) +
-   8.4704968632404753393434993216041 16e-22*P(18,x) +
-   5.5705413998588522192782604835236 869e-25*P(20,x) +
-   3.0133473832345288502246890418232 01e-28*P(22,x) +
-   1.3643380053535272726384790931752 49e-31*P(24,x) +
```

```
-  5.2460884671622816489446609893595 65e-35*P(26,x) +
```

and for (39), the code is

```
    I(1,x) = 0.53863434218525555928090810516663 36*P(1,x) +
```

References

1. Allen, E.E. Analytical approximations. *Math. Tables Aids Comp.* **1954**, *8*, 240–241. reproduced in *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, ed M. Abramowitz and I. A. Stegun (National Bureau of Standards, Applied mathematics series, 55, Washington, 1970), p. 369 No. 9.4.1. [CrossRef]
2. Millane, R.P.; Eads, J.L. Polynomial Approximations to Bessel Functions. *IEEE Trans. Antennas Propag.* **1995**, *51*, 1398. [CrossRef]
3. Gross, F.B. New Approximations to J_0 and J_1 Bessel Functions. *IEEE Trans. Antennas Propag.* **2003**, *43*, 904. [CrossRef]
4. Maass, F.; Martin, P. Precise analytic approximations for the Bessel function J_1. *Results Phys.* **2018**, *8*, 1234–1238. [CrossRef]
5. Martín, P.; Guerrero, A.L. Fractional approximations to the Bessel function $J_0(x)$. *J. Math. Phys.* **1985**, *26*, 705–707. [CrossRef]
6. Wimp, J. Polynomial Expansions of Bessel Functions and Some Associated Functions. *Math. Comp.* **1962**, *16*, 446–458. [CrossRef]
7. Clenshaw, C.W. Mathematical Tables. In *Chebyshev Series for Mathematical Functions*; Department of Scientific and Industrial Research, Her Majesty's Stationery Office: London, UK, 1962; Volume 5, pp. 30–35.
8. Zhang, X. Comparisons of best approximations with Chebyshev expansions for functions with logarithmic endpoint singularities. *Numer. Algorithms* **2023**, *94*, 1–25. [CrossRef]
9. Coleman, J.P.; Monaghan, A.J. Chebyshev Expansions for the Bessel Function $J_n(z)$ in the Complex Plane. *Math. Comput.* **1983**, *40*, 343–366.
10. Molinari, L.G. A Note on Trigonometric Approximations of Bessel Functions of the First Kind, and Trigonometric Power Sums. *Fundam. J. Math. Appl.* **2022**, *5*, 266–272. [CrossRef]
11. Reiss, H.R. Effect of an intense electromagnetic filed on a weakly bound system. *Phys. Rev. A* **1980**, *22*, 1786. [CrossRef]

12. Reiss, H.R. Theoretical methods in quantum optics: S-matrix and Keldysh techniques for stong-field process. *Progress Quantum Electron.* **1992**, *16*, 1. [CrossRef]
13. Reiss, H.R. Relativistic strong-field photoionization. *J. Opt. Soc. Am. B* **1990**, *7*, 574–586. [CrossRef]
14. Faisal, F.H.M. Multiple absorbtion of laser phtons by atoms. *J. Phys. B* **1973**, *6*, 4. [CrossRef]
15. Reiss, H.R. Foundations of Strong-Field Physics. *Lect. Ultrafast Intense Laser Sci.* **2010**, *1*, 41–84.
16. Reiss, H.R. Limits on Tunneling Theories of Strong-Field Ionization. *Phys. Rev. Lett.* **2008**, *101*, 043002. [CrossRef] [PubMed]
17. Keating, C.M. Using Strong Laser Fields to Produce Antihydrogen Ions. Ph.D. Dissertation, Appendix D. Available online: https://pdxscholar.library.pdx.edu/open_access_etds/4519/ (accessed on 6 February 2024).
18. Haber, H.E. The Spherical Harmonics. Available online: http://scipp.ucsc.edu/~{}haber/ph116C/SphericalHarmonics_12.pdf (accessed on 6 February 2024).
19. Kellogg, O.D. *Foundations of Potential Theory*; Springer: Berlin, Germany, 1929; p. 132.
20. Magnus, W.; Oberhettinger, F.; Soni, R.P. *Formulas and Theorems for the Special Functions of Mathematical Physics*; Springer: New York, NY, USA, 1966; p. 81. Available online: https://functions.wolfram.com/03.01.07.0005.01 (accessed on 6 February 2024).
21. Gradshteyn, I.S.; Ryzhik, I.M. *Table of Integrals, Series, and Products*, 5th ed.; Academic: New York, NY, USA, 1994; p. 987, No. 8.511.4.
22. Joachain, C.J. *Quantum Collision Theory*; North-Holland: New York, NY, USA, 1983; p. 671, (B.46).
23. Available online: http://functions.wolfram.com/03.21.06.0019.01 (accessed on 6 February 2024).
24. Gröbner, W.; Hofreiter, N. *Integraltafel Zweiter Teil Bestimmte Integrale*; Springer: Berlin, Germany, 1961; p. 110, No. 332.14a.
25. Available online: http://functions.wolfram.com/07.26.26.0001.01 (accessed on 6 February 2024).
26. Castellanos, D.; Rosenthal, W.E. Rational Chebyshev Approximations of Analytic Functions. Available online: https://www.fq.math.ca/Scanned/31-3/castellanos.pdf (accessed on 6 February 2024).
27. Amos, D.E. ALGORITHM 644: A Portable Package for Bessel Functions of a Complex Argument and Nonnegative Order. *ACM Trans. Math. Softw.* **1966**, *12*, 265–273. [CrossRef]
28. Available online: https://fortran-lang.org/en/learn/best_practices/floating_point (accessed on 6 February 2024).
29. Bohner, M.; Cuchta, T. The Bessel Difference Equation. *Proc. Am. Math. Soc.* **2017**, *145*, 1567–1580. [CrossRef]
30. Prudnikov, A.P.; Brychkov, Y.A.; Marichev, O.I. *Integrals and Series*; Gorden and Breach: New York, NY, USA, 1986; Volume 3.
31. Relation 5.2.6. Available online: https://dlmf.nist.gov/search/search?q=pochhammer&p=0&r=0 (accessed on 6 February 2024).
32. Available online: http://functions.wolfram.com/06.10.02.0001.01 (accessed on 6 February 2024).