On Universality of Some Beurling Zeta-Functions

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Abstract: Let \( \mathcal{P} \) be the set of generalized prime numbers, and \( \zeta_{\mathcal{P}}(s), s = \sigma + it \), denote the Beurling zeta-function associated with \( \mathcal{P} \). In the paper, we consider the approximation of analytic functions by using shifts \( \zeta_{\mathcal{P}}(s + i\tau), \tau \in \mathbb{R} \). We assume the classical axioms for the number of generalized integers and the mean of the generalized von Mangoldt function, the linear independence of the set \( \{ \log p : p \in \mathcal{P} \} \), and the existence of a bounded mean square for \( \zeta_{\mathcal{P}}(s) \). Under the above hypotheses, we obtain the universality of the function \( \zeta_{\mathcal{P}}(s) \). This means that the set of shifts \( \zeta_{\mathcal{P}}(s + i\tau) \) approximating a given analytic function defined on a certain strip \( \bar{\sigma} < \sigma < 1 \) has a positive lower density. This result opens a new chapter in the theory of Beurling zeta functions. Moreover, it supports the Linnik–Ibragimov conjecture on the universality of Dirichlet series. For the proof, a probabilistic approach is applied.

Keywords: Beurling zeta-function; generalized integers; generalized primes; Haar measure; random element; universality; weak convergence

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1. Introduction

A positive integer \( q > 1 \) is called prime if it has only two divisors, \( q \) and 1. Thus, \( 2, 3, 5, 7, 11, \ldots \) are prime numbers. Integer numbers \( k > 1 \) that have divisors different from \( k \) and 1 are called composite. It is well known that the set of all primes is infinite, and this was first proved by Euclid. By the fundamental theorem of arithmetic, every integer \( k > 1 \) has a unique representation as a product of prime numbers. Thus,

\[
k = q_1^{a_1} \cdots q_r^{a_r}, \quad a_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},
\]

and \( q_j \) is the \( j \)th prime number, \( j = 1, \ldots, r \), with some \( r \in \mathbb{N} \).

Investigations of the number of prime numbers

\[
\pi(x) \overset{\text{def}}{=} \sum_{q \leq x} 1, \quad x \to \infty,
\]

were more complicated. We recall that \( a = O(b), a \in \mathbb{C}, b > 0 \), means that there exists a constant \( c > 0 \) such that \( |a| \leq cb \). Comparatively recently, in 1896 Hadamard [1] and de la Vallée-Poussin [2] proved independently the asymptotic formula

\[
\pi(x) = \frac{x}{\log x} + O\left(x e^{-c\sqrt{\log x}}\right), \quad c > 0.
\]
For this, they applied the Riemann idea [3] of using the function
\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{q} \left(1 - \frac{1}{q^s}\right)^{-1}, \quad s = \sigma + it, \ \sigma > 1, \]
now called the Riemann zeta-function. The distribution low of prime numbers was found.

Prime numbers have generalizations. The system \( P \) of real numbers \( 1 < p_1 < p_2 < \cdots < p_n < \cdots \) such that \( \lim_{n \to \infty} p_n = \infty \) are called generalized prime numbers. Generalized prime numbers were introduced by Beurling in [4], and are studied by many authors. The system \( P \) generates the associated system \( N_P \) of generalized integers consisting of finite products of the form
\[ p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \quad \alpha_j \in \mathbb{N}_0, \ j = 1, \ldots, r, \]
with some \( r \in \mathbb{N} \).

The main problem in the theory of generalized primes is the asymptotic behavior of the function
\[ \pi_P(x) \overset{\text{def}}{=} \sum_{p \leq x, p \in P} 1, \quad x \to \infty. \]

The function \( \pi_P(x) \) is closely connected to the number of generalized integers
\[ N_P(x) \overset{\text{def}}{=} \sum_{m \leq x, m \in N_P} 1, \quad x \to \infty. \]

In these definitions, the sums are taking counting multiplicities of \( p \) and \( m \). Distribution results for generalized numbers were obtained by Beurling [4], Borel [5], Diamond [6–8], Malvin [9], Nyman [10], Ryavec [11], Hilberdink and Lapidus [12], Stankus [13], Zhang [14], and others. The important place in generalized number theory is devoted to making relations between \( N_P(x) \) and \( \pi_P(x) \). We mention some of them. From a general Landau’s theorem for prime ideals [15], we have the estimate
\[ N_P(x) = ax + O\left(x^\beta\right), \quad a > 0, \ 0 \leq \beta < 1, \quad (1) \]
that implies
\[ \pi_P(x) = \frac{x}{2} \frac{du}{\log u} + O\left(x e^{-c \sqrt{\log x}}\right), \quad c > 0. \]

Nyman proved [10] that the estimates
\[ N_P(x) = ax + O\left(\frac{x}{(\log x)^a}\right), \quad a > 0, \quad (2) \]
and
\[ \pi_P(x) = \frac{x}{2} \frac{du}{\log u} + O\left(\frac{x}{(\log x)^{a_1}}\right), \quad a_1 > 0, \]
with arbitrary \( a > 0 \) and \( a_1 > 0 \) are equivalent. Beurling observed [4] that the relation
\[ \pi_P(x) \sim \frac{x}{\log x}, \quad x \to \infty, \]
is implied by (2) with \( a > 3/2 \).
It is important to stress that Beurling began to use zeta-functions for investigations of the function \( \pi_P(x) \). These zeta-functions \( \zeta_P(s) \), now called Beurling zeta-functions, are defined in some half-plane \( \sigma > \sigma_0 \), by the Euler product

\[
\zeta_P(s) = \prod_{p \in P} \left(1 - \frac{1}{p^s}\right)^{-1},
\]
or by the Dirichlet series

\[
\zeta_P(s) = \sum_{m \in \mathbb{N}_P} \frac{1}{m^s},
\]
where \( \sigma_0 \) depends on the system \( P \).

Suppose that (1) is true. Then, the partial summation shows that the series for \( \zeta_P(s) \) is absolutely convergent for \( \sigma > 1 \),

\[
\zeta_P(s) = s \int_1^\infty \frac{N_P(x)}{x^{s+1}} \, dx,
\]
the function \( \zeta_P(s) \) is analytic for \( \sigma > 1 \), and the equality

\[
\sum_{m \in \mathbb{N}_P} \frac{1}{m^s} = \prod_{p \in P} \left(1 - \frac{1}{p^s}\right)^{-1}
\]
is valid.

Analytic continuation for the function \( \zeta_P(s) \) is not an easy problem. If (1) is true, then (3) implies

\[
\zeta_P(s) = \frac{as}{s-1} + s \int_1^\infty \frac{R(x)}{x^{s+1}} \, dx, \quad R(x) = O \left( x^\beta \right), \quad 0 \leq \beta < 1.
\]

This gives analytic continuation for \( \zeta_P(s) \) to the half-plane \( \sigma > \beta \), except for the point \( s = 1 \) which is a simple pole with residue \( a \).

Beurling zeta-functions are attractive analytic objects; investigations of their properties lead to interesting results, and require new methods. Various authors put much effort into showing that the Beurling zeta-functions have similar properties to classical ones. We mention a recent paper [16] containing deep zero-distribution results for \( \zeta_P(s) \).

In this paper, we investigate the analytic properties of the function \( \zeta_P(s) \). The approximation of analytic functions is one of the most important chapters of function theory. It is well known that the Riemann zeta-function \( \zeta(s) \) is universal in the sense of approximation of analytic functions. More precisely, this means that every non-vanishing analytic function defined on the strip \( \{s \in \mathbb{C} : 1/2 < \sigma < 1\} \) can be approximated with desired accuracy by using shifts \( \zeta(s + i\tau), \tau \in \mathbb{R} \). Universality of \( \zeta(s) \) and other zeta-functions has deep theoretical (zero-distribution, functional independence, set denseness, moment problem, ...) and practical (approximation problem, quantum mechanics) applications. On the other hand, the universality theory of zeta-functions has some interior problems (effectivization, description of a class of universal functions, Linnik–Ibragimov conjecture, see Section 1.6 of [17], ...); therefore, investigations of universality are continued, see [17–23].

Our purpose is to prove the universality of the function \( \zeta_P(s) \) with a certain system \( P \). We began studying the approximation of analytic functions by shifts \( \zeta_P(s + i\tau) \) in [24]. Suppose that the estimate (1) is valid. Let

\[
M_P(\sigma, T) = \int_0^T |\zeta_P(\sigma + it)|^2 \, dt,
\]
\[ \hat{\sigma} = \inf \left\{ \sigma : M_P(\sigma, T) \ll \sigma T, \quad \sigma > \max \left( \frac{1}{2}, \beta \right) \right\}. \]

Suppose that \( \hat{\sigma} < 1 \) and define
\[ D = D_P = \{ s \in \mathbb{C} : \hat{\sigma} < \sigma < 1 \}. \]

Here, and in the sequel, the notation \( a \ll c \), \( a, c \in \mathbb{C} \), \( b > 0 \), shows that there exists a constant \( c = c(\epsilon) > 0 \) such that \( |a| \leq cb \). Denote by \( H(D) \) the space of analytic on \( D \) functions equipped with the topology of uniform convergence on compacta, and by \( \text{meas} A \) the Lebesgue measure of a measurable set \( A \subset \mathbb{R} \). The main result of [24] is the following theorem.

**Theorem 1.** Suppose that the system \( \mathcal{P} \) satisfies the axiom (1). Then there exists a closed non-empty subset \( F_P \subset H(D) \) such that, for every compact set \( K \subset D \), \( f(s) \in F_P \) and \( \epsilon > 0 \),
\[ \lim \inf \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_P(s + i\tau) - f(s)| < \epsilon \right\} > 0. \]

Moreover, the limit
\[ \lim \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_P(s + i\tau) - f(s)| < \epsilon \right\} \]
exists and is positive for all but at most countably many \( \epsilon > 0 \).

Theorem 1 demonstrates good approximation properties of the function \( \zeta_P(s) \); however, the set \( F_P \) of approximated functions is not explicitly given. The aim of this paper, using certain additional information on system \( \mathcal{P} \), is to identify the set \( F_P \).

A new approach for analytic continuation of the function \( \zeta_P(s) \) involving the generalized von Mangoldt function
\[ \Lambda_P(m) = \begin{cases} \log p & \text{if } m = p^k, p \in \mathcal{P}, k \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \]
and
\[ \psi_P(x) = \sum_{m \leq x, m \in \mathcal{P}} \Lambda_P(m) \]
was proposed in [12]. Let, for \( \alpha \in [0, 1) \) and every \( \epsilon > 0 \),
\[ \psi_P(x) = x + O(x^{\alpha + \epsilon}). \] (4)

Then, in [12], it was obtained that the function \( \zeta_P(s) \) is analytic in the half-plane \( \sigma > \alpha \), except for a simple pole at the point \( s = 1 \). It turns out that estimates of type (4) are useful for the characterization of the system \( \mathcal{P} \). It is known [12] that (1) does not imply the estimate
\[ \psi_P(x) = x + O(x^{\beta_1}) \] (5)
with \( \beta_1 < 1 \). Therefore, together with (1), we suppose that estimate (5) is valid.

Let \( \mathcal{K} \) be the class of compact subsets of strip \( D \) with the connected complement, and \( H_0(K) \) with \( K \in \mathcal{K} \) the class of continuous functions on \( K \) that are analytic in the interior of \( K \). Moreover, let
\[ L(\mathcal{P}) = \{ \log p : p \in \mathcal{P} \}. \]

Note, that the following theorem supports the Linnik–Ibragimov conjecture.
Theorem 2. Suppose that the system $\mathcal{P}$ satisfies the axioms (1) and (5), and $L(\mathcal{P})$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\epsilon > 0$,

$$\lim_{T \to \infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{\mathcal{P}}(s + it) - f(s)| < \epsilon \right\} > 0.$$ 

Moreover, the limit

$$\lim_{T \to \infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta_{\mathcal{P}}(s + it) - f(s)| < \epsilon \right\}$$

exists and is positive for all but at most countably many at $\epsilon > 0$.

Notice that the requirement on the set $L(\mathcal{P})$ is sufficiently strong, it shows that the numbers of the system $\mathcal{P}$ must be different. The simplest example is the system

$$\mathcal{P} = \{ q + \alpha : q \text{ is prime} \},$$

where $\alpha$ is a transcendental number.

An example of $\mathcal{P}$ with a bounded mean square is given in [25].

For the proof of Theorem 2, we will build the probabilistic theory of the function $\zeta_{\mathcal{P}}(s)$ in the space of analytic functions $H(D)$.

The paper is organized as follows. In Section 2, we introduce a certain probability space, and define the $H(D)$ valued random element. Section 3 is devoted to the ergodicity of one group of transformations. In Section 4, we approximate the mean of the function $\zeta_{\mathcal{P}}(s)$ by an absolutely convergent Dirichlet series. Section 5 is the most important. In this section, we prove a probabilistic limit theorem for the function $\zeta_{\mathcal{P}}(s)$ on a weakly convergent probability measure in the space $H(D)$, and identify the limit measure. Section 6 gives the explicit form for the support of the limit measure of Section 5. In Section 7, the universality of the function $\zeta_{\mathcal{P}}(s)$ is proved.

2. Random Element

Define the Cartesian product

$$\Omega_{\mathcal{P}} = \prod_{p \in \mathcal{P}} \{ s \in \mathbb{C} : |s| = 1 \}.$$ 

The set $\Omega_{\mathcal{P}}$ consists of all functions $\omega : \mathcal{P} \to \{ s \in \mathbb{C} : |s| = 1 \}$. In $\Omega_{\mathcal{P}}$, the operation of pointwise multiplication and product topology can be defined, and this makes $\Omega_{\mathcal{P}}$ a topological group. Since the unit circle is a compact set, the group $\Omega_{\mathcal{P}}$ is compact. Denote by $\mathcal{B}(\mathbb{X})$, the Borel $\sigma$-field of the space $\mathbb{X}$. Then, the compactness of $\Omega_{\mathcal{P}}$ implies the existence of the probability Haar measure $m_{\mathcal{P}}$ on $(\Omega_{\mathcal{P}}, \mathcal{B}(\Omega_{\mathcal{P}}))$, and we have the probability space $(\Omega_{\mathcal{P}}, \mathcal{B}(\Omega_{\mathcal{P}}), m_{\mathcal{P}})$.

Denote the elements of $\Omega_{\mathcal{P}}$ by $\omega = (\omega(p) : p \in \mathcal{P})$. Since the Haar measure $m_{\mathcal{P}}$ is the product of Haar measures on unit circles, $\{ \omega(p) : p \in \mathcal{P} \}$ is a sequence of independent complex-valued random variables uniformly distributed on the unit circle.

Extend the functions $\omega(p), p \in \mathcal{P}$, to the generalized integers $\mathcal{N}_{\mathcal{P}}$. Let

$$m = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \in \mathcal{N}_{\mathcal{P}}.$$ 

Then we put

$$\omega(m) = \omega^{\alpha_1}(p_1) \cdots \omega^{\alpha_r}(p_r).$$

(6)

Now, for $s \in D$ and $\omega \in \Omega_{\mathcal{P}}$, define

$$\zeta_{\mathcal{P}}(s, \omega) = \sum_{m \in \mathcal{N}_{\mathcal{P}}} \frac{\omega(m)}{m^s}.$$
Lemma 1. Under the hypotheses of Theorem 2, $\zeta_P(s, \omega)$ is an $H(D)$-valued random element defined on the probability space $(\Omega_P, B(\Omega_P), m_P)$.

Proof. Fix $\sigma_0 > \sigma$, and consider

$$a_m(\omega) = \frac{\omega(m)}{m^{\sigma_0}}, \quad m \in \mathcal{N}_P.$$  

Then $\{a_m : m \in \mathcal{N}_P\}$ is a sequence of complex-valued random variables on $(\Omega_P, B(\Omega_P), m_P)$. Denote by $z$ the complex conjugate of $z \in \mathbb{C}$. Suppose that $m_1 \neq m_2, m_1, m_2 \in \mathcal{N}_P$. Since the set $L(P)$ is linearly independent over $\mathbb{Q}$, in the product $\omega(m_1)\omega(m_2)$, there exists at least one factor $\omega^\alpha(p), p \in P$, with integer $\alpha \neq 0$. Therefore, denoting by $E_\xi$ the expectation of the random variable $\xi$, we have

$$E|a_m(\omega)|^2 = \frac{1}{m^{2\sigma_0}}, \quad m \in \mathcal{N}_P,$$  

(7)

$$Ea_m(\omega)a_n(\omega) = \frac{1}{m_1^{\sigma_0}m_2^{\sigma_0}} \int_{\Omega_P} \omega(m_1)\omega(m_2) dm_P = 0, \quad m_1 \neq m_2,$$

because the integral includes the factor

$$\int_{\gamma} \omega^\alpha(p) dm_\gamma = \int_{0}^{1} e^{2\pi i u} du = 0,$$

where $\gamma$ is the unit circle on $\mathbb{C}$, and $m_\gamma$ the Haar measure on $\gamma$. This and (7) show that $\{a_m\}$ is a sequence of pairwise orthogonal complex-valued random variables and the series

$$\sum_{m \in \mathcal{N}_P} E|a_m|^2 \log^2 m$$

is convergent. Hence, by the classical Rademacher theorem, see [26], the series

$$\sum_{m \in \mathcal{N}_P} \frac{\omega(m)}{m^{\sigma_0}}$$

converges for almost all $\omega$ with respect to the measure $m_P$. Therefore, by a property of the Dirichlet series, see [22], the series

$$\sum_{m \in \mathcal{N}_P} \frac{\omega(m)}{m^s}$$

(8)

converges uniformly on compact sets of the half-plane $\sigma > \sigma_0$ for almost all $\omega \in \Omega_P$.

Now, let

$$\sigma_k = \sigma + \frac{1}{k}, \quad k \in \mathbb{N},$$

and $D_k = \{s \in \mathbb{C} : \sigma > \sigma_k\}$. Denote by the set $\Omega_k \subset \Omega_P$ such that the series (8) converges uniformly on compact sets of $D_k$ for almost all $\omega \in \Omega_k$. Then, by the above remark,

$$m_P(\Omega_k) = 1.$$  

(9)

On the other hand, taking

$$\hat{\Omega} = \bigcap_k \Omega_k,$$

we obtain from (9) that $m_P(\hat{\Omega}) = 1$, and the series (8) converges uniformly on compact sets of the half-plane $\sigma > \hat{\sigma}$ of the strip $D$. Hence, $\zeta_P(s, \omega)$ is the $H(D)$-valued random element on $(\Omega_P, B(\Omega_P), m_P)$. □
Lemma 2. For almost all $\omega$, the product
\[ \prod_{p \in P} \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1} \]
converges uniformly on compact subsets of the half-plane $\sigma > \sigma_0$, and the equality
\[ \zeta_P(s, \omega) = \prod_{p \in P} \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1} \]
holds.

Proof. The series $\zeta_P(s, \omega)$ is absolutely convergent for $\sigma > 1$. Therefore, the equality of the lemma, in view of (6), is valid for $\sigma > 1$. By proof of Lemma 1, the function $\zeta_P(s, \omega)$, for almost all $\omega \in \Omega_P$, is analytic in the half-plane $\sigma > \sigma_0$. Therefore, by analytic continuation, it suffices to show that the product of the lemma, for almost all $\omega \in \Omega_P$, converges uniformly on compact subsets of the strip $D$. Write
\[ \prod_{p \in P} \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1} = \prod_{p \in P} (1 + a_p(s, \omega)) \quad (10) \]
with
\[ a_p(s, \omega) = \sum_{a=1}^{\infty} \frac{\omega^a(p)}{pa^s}. \]
We observe that the convergence of product (10) follows from that of the series
\[ \sum_{p \in P} a_p(s, \omega) \quad \text{and} \quad \sum_{p \in P} |a_p(s, \omega)|^2. \]
Set
\[ b_p(s, \omega) = \frac{\omega(p)}{p^s}. \]
Then
\[ a_p(s, \omega) - b_p(s, \omega) = \sum_{a=2}^{\infty} \frac{\omega^a(p)}{pa^s} \ll \frac{1}{p^2c}, \quad \sigma > \sigma_0. \]
Hence, the series
\[ \sum_{p \in P} |a_p(s, \omega) - b_p(s, \omega)| \quad (11) \]
is convergent for all $\omega \in \Omega_P$ with every $\sigma = c_0, c_0 > \sigma_0$, thus, uniformly convergent on compact subsets of the half-plane $\sigma > \sigma_0$. To prove the convergence for the series
\[ \sum_{p \in P} b_p(s, \omega), \]
we apply the same arguments as in the proof of Lemma 1. For fixed $\sigma > \sigma_0$, we have
\[ \mathbb{E}|b_p(\sigma, \omega)|^2 = \frac{1}{p^{2c_0}} \]
and for $p, q \in P$, $p \neq q$,
\[ \mathbb{E}b_p(\sigma, \omega)b_q(\sigma, \omega) = \frac{1}{p^{2c_0}q^{2c_0}} \int_{\Omega_P} \omega(p)\omega(q) \, dm_P = 0. \]
Thus, the series
\[ \sum_{p \in P} \mathbb{E}|b_p(\sigma, \omega)|^2 \log^2 p \]
is convergent, and the Rademacher theorem implies that the series
\[ \sum_{p \in P} b_p(\sigma, \omega) \]
converges for almost all \( \omega \in \Omega_P \). Hence, this series, for almost all \( \omega \in \Omega_P \), converges uniformly on compact subsets of the half-plane \( \sigma > \sigma_0 \). This, together with a convergence property of the series (11), shows that the series
\[ \sum_{p \in P} a_p(s, \omega), \]
for almost all \( \omega \in \Omega_P \), converges uniformly on compact subsets of the half-plane \( \sigma > \sigma_0 \), and it remains to prove the same for the series
\[ \sum_{p \in P} |a_p(s, \omega)|^2. \] (12)

Clearly, for all \( \omega \in \Omega_P \),
\[ |a_p(s, \omega)|^2 \ll \frac{1}{p^{2\sigma}}, \quad \sigma > \sigma_0. \]

Hence, the series (12), for all \( \omega \in \Omega_P \), converges uniformly on compact subsets of the half-plane \( \sigma > \sigma_0 \).\qed

3. Ergodicity

For \( \tau \in \mathbb{R} \), let
\[ \kappa_\tau = \left( p^{-i\tau} : p \in P \right), \]
and
\[ g_\tau(\omega) = \kappa_\tau \omega, \quad \omega \in \Omega_P. \]

Since the Haar measure \( m_P \) is invariant with respect to shifts by elements of \( \Omega_P \), i.e., for all \( A \in B(\Omega_P) \) and \( \omega \in \Omega_P \),
\[ m_P(A) = m_P(\omega A) = m_P(A \omega), \]
\( g_\tau(m) \) is a measurable measure preserving transformation on \( \Omega_P \). Thus, we have the one-parameter group \( G_\tau = \{ g_\tau : \tau \in \mathbb{R} \} \) of transformations of \( \Omega_P \). A set \( A \in B(\Omega_P) \) is called invariant with respect to \( G_\tau \) if, for every \( \tau \in \mathbb{R} \), the sets \( A \) and \( A_\tau = g_\tau(A) \) differ one from another at most by a set of \( m_P \)-measure zero. It is well known that all invariant sets form a \( \sigma \)-field which is a subfield of \( B(\Omega_P) \). The group \( G_\tau \) is called ergodic if its \( \sigma \)-field of invariant sets consists only of sets \( m_P \)-measure 0 or 1.

**Lemma 3.** Under the hypotheses of Theorem 2, the group \( G_\tau \) is ergodic.

**Proof.** Let \( A \in B(\Omega_P) \) be a fixed invariant set of \( G_\tau \). Denote by \( I_A(\omega) \) the indicator function of the set \( A \). Then, for almost all \( \omega \in \Omega_P \),
\[ I_A(g_\tau(\omega)) = I_A(\omega). \] (13)
Characters $\chi$ of the group $\Omega_P$ are of the form
\[
\chi(\omega) = \prod_{p \in P} \omega^{k_p(p)},
\] (14)
where $*$ indicates that only a finite number of integers $k_p$ are distinct from zero. Suppose that $\chi$ is a nontrivial character, i.e., $\chi(\omega) \not\equiv 1$ for all $\omega \in \Omega_P$. Then, we have
\[
\chi(g_\tau) = \prod_{p \in P} p^{-ik_p\tau} = \exp\left\{-i\tau \sum_{p \in P} k_p \log p\right\}.
\]
Since the set $L(P)$ is linearly independent over $\mathbb{Q}$, and $\chi$ is a nontrivial character,
\[
\sum_{p \in P} k_p \log p \neq 0.
\]
Thus, there exists a real number $a \neq 0$ such that
\[
\chi(g_\tau) = e^{-i\tau a}.
\]
Hence, there is $\tau_0 \in \mathbb{R}$ satisfying $\chi(g_{\tau_0}) \neq 1$.

Now, we deal with Fourier analysis on $\Omega_P$. Denote by $\hat{g}$ the Fourier transform of a function $g$, i.e.,
\[
\hat{g}(\chi) = \int_{\Omega_P} g(\omega) \chi(\omega) \, dm_P.
\]
In virtue of (13), we find
\[
\hat{I}_A(\chi) = \int_{\Omega_P} I_A(\omega) \chi(\omega) \, dm_P = \chi(g_{\tau_0}) \int_{\Omega_P} \chi(\omega) I_A(\omega) \, dm_P = \chi(g_{\tau_0}) \hat{I}_A(\chi).
\]
Hence, in view of inequality $\chi(g_{\tau_0}) \neq 1$, we obtain
\[
\hat{I}_A(\chi) = 0.
\] (15)

Consider the case of the trivial character $\chi_0$ of the group $\Omega_P$. We set $\hat{I}_A(\chi_0) = c$. Then, the orthogonality of characters implies that
\[
\tilde{c}(\chi) = \int_{\Omega_P} c(\chi) \chi(\omega) \, dm_P = c \int_{\Omega_P} \chi(\omega) \, dm_P = \begin{cases} c & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases}
\]
Therefore, using (15) yields the equality
\[
\hat{I}_A(\chi) = \tilde{c}(\chi).
\] (16)

It is well known that a function is completely determined by its Fourier transform. Thus, by (16), we have that for almost all $\omega \in \Omega_P$, $I_A(\omega) = c$. However, as $I_A(\omega)$ is the indicator function, it follows that $c = 0$ or $1$. In other words, for almost all $\omega \in \Omega_P$, $I_A(\omega) = 0$ or $I_A(\omega) = 1$. Thus, $m_P(A) = 0$ or $m_P(A) = 1$. The lemma is proved. \qed

We apply Lemma 3 for the estimation of the mean square for $\xi_P(s, \omega)$.

**Lemma 4.** Under hypotheses of Theorem 2, for fixed $\sigma < \sigma' < 1$ and almost all $\omega \in \Omega_P$,
\[
\int_{-T}^{T} \left| \xi_P(\sigma + it, \omega) \right|^2 \, dt \ll_P \sigma', \quad T \to \infty.
\]
Therefore, we can apply the classical Birkhoff–Khintchine ergodic theorem, see [27]. This gives, by (17),

\[
|\xi P(\sigma, \omega)|^2 = |\xi P(\sigma + it, \omega)|^2 = |\xi P(\sigma + it, \omega)|^2.
\]

We recall that a strongly stationary random process \(X(t, \omega), t \in T\), on \((\Omega, A, P)\) is called ergodic if its \(\sigma\)-field of invariant sets consists of sets of \(P\)-measure 0 or 1. Since the group \(G_t\) is ergodic, the stationary process \(|\xi P(\sigma + it, \omega)|^2\) is ergodic, for details, see [22]. Therefore, we can apply the classical Birkhoff–Khintchine ergodic theorem, see [27]. This gives, by (17),

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\xi P(\sigma + it, \omega)|^2 dt = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \xi P(\sigma, g_t(\omega)) dt = \mathbb{E}|\xi P(\sigma, \omega)|^2 < \infty.
\]

\(\square\)

4. Approximation in the Mean

In this section, we approximate the functions \(|\xi P(s)|\) and \(|\xi P(s, \omega)|\) by absolutely convergent Dirichlet series. Let \(\eta > 1 - \tilde{\sigma}\) be a fixed number, and, for \(m \in \mathbb{N}_p\) and \(n \in \mathbb{N}\),

\[
a_n(m) = \exp\left\{ -\left( \frac{m}{n} \right)^{\eta} \right\}.
\]

Then the series

\[
\xi P_n(s) = \sum_{m \in \mathbb{N}_p} \frac{a_n(m)}{m^s} \quad \text{and} \quad \xi P_n(s, \omega) = \sum_{m \in \mathbb{N}_p} \frac{a_n(m) \omega(m)}{m^s}, \quad \omega \in \Omega_p,
\]

are absolutely convergent for \(\sigma > \tilde{\sigma}\) and for every fixed \(n \in \mathbb{N}\). We will approximate \(|\xi P(s)|\) and \(|\xi P(s, \omega)|\) by \(|\xi P_n(s)|\) and \(|\xi P_n(s, \omega)|\), respectively, in the mean. Recall a metric in the space \(H(D)\) inducing its topology. Let \(\{K_l : l \in \mathbb{N}\} \subset D\) be a sequence of embedded compact sets such that

\[
D = \bigcup_{l=1}^{\infty} K_l,
\]

and every compact set \(K \subset D\) lies in some \(K_l\). Then

\[
\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \sup_{s \in K_l} |g_1(s) - g_2(s)| / \left( 1 + \sup_{s \in K_l} |g_1(s) - g_2(s)| \right), \quad g_1, g_2 \in H(D),
\]

is the desired metric in \(H(D)\).

In [24], the following statement has been obtained.
Lemma 5. Suppose that (1) is valid. Then
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho (\zeta_p (s + i \tau), \zeta_{p,n} (s + i \tau)) \, d\tau = 0.
\]

Denote by \( \Omega_{p,1} \) a subset of \( \Omega_p \) such that a product
\[
\prod_{p \in P} \left( 1 - \frac{\omega(p)}{p^s} \right)^{-1}
\]
converges uniformly on compact subsets of \( D \) for \( \omega \in \Omega_{p,1} \), and by \( \Omega_{p,2} \) a subset of \( \Omega_p \) such that, for \( \omega \in \Omega_{p,2} \), the estimate
\[
\int_{-T}^T |\zeta_p (\sigma + it, \omega)|^2 \, dt \ll \sigma T
\]
holds for \( \sigma > \sigma \). Then, by Lemmas 3 and 4, \( m_p (\Omega_p, j) = 1, j = 1, 2 \). Let
\[
\Omega_p = \Omega_{p,1} \cap \Omega_{p,2}.
\]

Then again \( m_p (\Omega_p) = 1 \).

Lemma 6. Under the hypotheses of Theorem 2, for \( \omega \in \Omega_p \) the equality
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \rho (\zeta_p (s + i \tau, \omega), \zeta_{p,n} (s + i \tau, \omega)) \, d\tau = 0
\]
holds.

Proof. Denote
\[
l_n (s) = \eta^{-1} \Gamma (\eta^{-1} s) n^s, \quad n \in \mathbb{N},
\]
where \( \Gamma (s) \) is the Euler gamma function. Then the classical Mellin formula
\[
\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma (z) b^{-z} \, dz = e^{-b}, \quad a, b > 0,
\]
implies, for \( m \in \mathcal{N}_p \),
\[
\frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} m^{-z} l_n (z) \, dz = \frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} \Gamma (z) \left( \frac{m}{n} \right)^{-z} \, dz = a_n (m).
\]

Therefore, for \( \sigma > \sigma \) and \( \omega \in \Omega_p \),
\[
\zeta_{p,n} (s, \omega) = \sum_{m \in \mathcal{N}_p} \frac{a_n (m) \omega (m)}{m^s} = \frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} \left( \sum_{m \in \mathcal{N}_p} \frac{\omega (m)}{m^{s+z}} \right) l_n (z) \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} \zeta_p (s + z, \omega) l_n (z) \, dz.
\]
The definition of the metric \( \rho \) implies that it is sufficient to show that, for every compact set \( K \subset D \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \sup_{s \in K} |\zeta \rho (s + i\tau, \omega) - \zeta \rho, n (s + i\tau, \omega)| \, d\tau = 0. \tag{19}
\]

Thus, let \( K \subset D \) be a compact set. Then there exists \( \epsilon > 0 \) satisfying for \( \sigma + it \in K \) the inequalities \( \sigma + \epsilon \leq \sigma \leq 1 - \epsilon/2 \). Take \( \eta = 1 \) and \( \eta_1 = \sigma - \epsilon/2 - \sigma \) with the above \( \sigma \). Then \( \eta_1 < 0 \) and \( \eta_1 > \sigma + \epsilon/2 - 1 + \epsilon/2 = \sigma - 1 + \epsilon > -1 \). Consequently, the integrand in (18) has only a simple pole \( z = 0 \) in the strip \( \eta_1 < \Re z < \eta \). Hence, the residue theorem and (18) show that, for \( s \in K \),

\[
\zeta \rho, n (s, \omega) - \zeta \rho (s, \omega) = \frac{1}{2\pi i} \int_{\eta_1 + i\infty}^{\eta_1 - i\infty} \zeta \rho (s + z, \omega) l_n (z) \, dz.
\]

Thus, for \( s \in K \),

\[
\zeta \rho, n (s + i\tau, \omega) - \zeta \rho (s + i\tau, \omega)
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta \rho \left( \sigma + \frac{\epsilon}{2} + i\tau + it + iu, \omega \right) l_n \left( \sigma + \frac{\epsilon}{2} - \sigma + iu \right) \, du
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta \rho \left( \sigma + \frac{\epsilon}{2} + i\tau + iu, \omega \right) l_n \left( \sigma + \frac{\epsilon}{2} - s + iu \right) \, du
\ll \int_{-\infty}^{\infty} \left| \zeta \rho \left( \sigma + \frac{\epsilon}{2} + i\tau + iu, \omega \right) \right| \sup_{s \in K} \left| l_n \left( \sigma + \frac{\epsilon}{2} - s + iu \right) \right| \, du. \tag{20}
\]

It is well known that, for the gamma-function \( \Gamma (\sigma + it) \), the estimate

\[
\Gamma (\sigma + it) \ll \exp \{ -c|t| \}, \quad c > 0,
\]

is valid uniformly for \( \sigma \in [\sigma_1, \sigma_2] \) with every \( \sigma_1 < \sigma_2 \). Therefore, (20) implies

\[
\frac{1}{T} \int_{0}^{T} \sup_{s \in K} \left| \zeta \rho (s + i\tau, \omega) - \zeta \rho, n (s + i\tau, \omega) \right| \, d\tau
\ll \int_{-\infty}^{\infty} \left( \frac{1}{T} \int_{0}^{T} \left| \zeta \rho \left( \sigma + \frac{\epsilon}{2} + i\tau + iu, \omega \right) \right| \, d\tau \right) \sup_{s \in K} \left| l_n \left( \sigma + \frac{\epsilon}{2} - s + iu \right) \right| \, du \overset{\text{def}}{=} I. \tag{22}
\]

By Lemma 4, for \( \omega \in \Omega \rho \),

\[
\int_{-T}^{T} \left| \zeta \rho \left( \sigma + \frac{\epsilon}{2} + i\tau, \omega \right) \right|^2 \, d\tau \ll \epsilon T.
\]

Hence,
\[
\frac{1}{T} \int_{0}^{T} |\tilde{\zeta}_{p}(\tilde{\sigma} + \frac{\epsilon}{2} + i\tau + iu, \omega)| d\tau \leq \left( \frac{1}{T} \int_{0}^{T} |\tilde{\zeta}_{p}(\tilde{\sigma} + \frac{\epsilon}{2} + i\tau + iu, \omega)|^2 d\tau \right)^{1/2} \\
\leq \left( \frac{1}{T} \int_{-|u|}^{T+|u|} |\tilde{\zeta}_{p}(\tilde{\sigma} + \frac{\epsilon}{2} + i\tau, \omega)|^2 d\tau \right)^{1/2} \\
\ll \left( \frac{T + |u|}{T} \right)^{1/2} \ll_{\epsilon} (1 + |u|)^{1/2}. \tag{23}
\]

In view of (21), for \( s \in K \),
\[
l_{n}\left( \tilde{\sigma} + \frac{\epsilon}{2} - s + iu \right) \ll n^{\epsilon + \epsilon/2 - \theta} \exp\{-c|u - t|\} \ll_{K} n^{-\epsilon/2} \exp\{-c_{1}|u|\}, \quad c_{1} > 0.
\]

This and (23) give
\[
I \ll_{\epsilon, K} n^{-\epsilon/2} \int_{-\infty}^{\infty} (1 + |u|)^{1/2} \exp\{-c_{1}|u|\} du \ll_{\epsilon, K} n^{-\epsilon/2},
\]
and (19) is proved. \( \square \)

5. Limit Theorems

In previous sections, we gave preparatory results for the proof of a limit theorem for \( \zeta_{p}(s) \) in the space of analytic functions \( H(D) \). In this section, we consider the weak convergence for \( \tilde{P}_{T,p} \),

\[
\tilde{P}_{T,p}(A) = \frac{1}{T} \text{meas}\{ \tau \in [0, T] : \zeta_{p}(s + i\tau) \in A \}
\]
and
\[
\tilde{P}_{T,p}(A) = \frac{1}{T} \text{meas}\{ \tau \in [0, T] : \zeta_{p}(s + i\tau, \omega) \in A \}
\]
as \( T \to \infty \), where \( A \in \mathcal{B}(H(D)), \omega \in \tilde{\Omega}_{p} \).

We start with a limit lemma on \( \Omega_{p} \). For \( A \in \mathcal{B}(\Omega_{p}) \), define
\[
p^{\Omega_{p}}_{T,p}(A) = \frac{1}{T} \text{meas}\{ \tau \in [0, T] : (p^{-\tau} : p \in \mathcal{P}) \in A \}.
\]

**Lemma 7.** Suppose that the set \( L(\mathcal{P}) \) is linearly independent over \( \mathbb{Q} \). Then \( p^{\Omega_{p}}_{T,p} \) converges weakly to the Haar measure \( m_{\mathcal{P}} \) as \( T \to \infty \).

**Proof.** In the proof of Lemma 3, we have seen that characters of the group \( \Omega_{p} \) are given by (14). Therefore, the Fourier transform \( F_{T,p}(k) \), \( k = (k_{p} : k_{p} \in \mathbb{Z}, p \in \mathcal{P}) \) of \( p^{\Omega_{p}}_{T,p} \) is defined by

\[
F_{T,p}(k) = \int_{\Omega_{p}} \prod_{p \in \mathcal{P}} \omega_{p}(p) d\tilde{P}_{T,p}^{\Omega_{p}} = \frac{1}{T} \int_{0}^{T} \left( \prod_{p \in \mathcal{P}} p^{-i\epsilon_{p}} \right) d\tau \\
= \frac{1}{T} \int_{0}^{T} \exp\left\{ -i\tau \sum_{p \in \mathcal{P}} k_{p} \log p \right\} d\tau. \tag{24}
\]
We have to show that
\[
\lim_{T \to \infty} F_{T,p}(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}
\] (25)

For this, we apply the linear independence of the set \(L(\mathcal{P})\). We have
\[
A_p(k) \overset{\text{def}}{=} \sum_{p \in \mathcal{P}} k_p \log p = 0
\]
if and only if \(k_p = 0\). Thus, (24),
\[
F_{T,p}(k) = \begin{cases} \frac{1}{T} \exp\{-iT A_p(k)\} & \text{if } k = 0, \\ \frac{1}{i T \exp\{-i T A_p(k)\}} & \text{otherwise}, \end{cases}
\]
and (25) take place. \(\square\)

The next lemma is devoted to the functions \(\zeta_{p,n}(s)\) and \(\zeta_{p,n}(s, \omega)\). For \(A \in \mathcal{B}(H(D))\), set
\[
P_{T,p,n}(A) = \frac{1}{T} \text{meas}\{\tau \in [0,T] : \zeta_{p,n}(s+i\tau) \in A\}
\]
and
\[
\hat{P}_{T,p,n}(A) = \frac{1}{T} \text{meas}\{\tau \in [0,T] : \zeta_{p,n}(s+i\tau, \omega) \in A\}.
\]

**Lemma 8.** Suppose that the set \(L(\mathcal{P})\) is linearly independent over \(\mathbb{Q}\). Then, on \((H(D), \mathcal{B}(H(D)))\) there exists a probability measure \(P_{p,n}\) such that both the measures \(P_{T,p,n}\) and \(\hat{P}_{T,p,n}\) converge weakly to \(P_{p,n}\) as \(T \to \infty\).

**Proof.** We use a property of the preservation of weak convergence under continuous mappings. Consider the mapping \(v_{p,n} : \Omega_p \to H(D)\) given by
\[
v_{p,n}(\omega) = \zeta_{p,n}(s, \omega).
\]

Since the series for \(\zeta_{p,n}(s, \omega)\) is absolutely convergent for \(s > \sigma\), the mapping \(v_{p,n}\) is continuous. Moreover, for \(A \in \mathcal{B}(H(D))\),
\[
P_{T,p,n}(A) = \frac{1}{T} \text{meas}\{\tau \in [0,T] : (p^{-it} : p \in \mathcal{P}) \in v_{p,n}^{-1} A\} = P_{T,p}^{-1}(v_{p,n}^{-1} A).
\]

Thus, denoting by \(P_{T,p,n}\) the measure given by the latter equality, we obtain that \(P_{T,p,n} = P_{T,p}^{-1} v_{p,n}^{-1}\). This equality continuity of \(v_{p,n}\), and the principle of preservation of weak convergence, see Theorem 5.1 of [28], show that \(P_{T,p,n}\) converges weakly to the measure \(Q_{p,n} \overset{\text{def}}{=} m_p v_{p,n}^{-1}\) as \(T \to \infty\).

Define one more mapping \(\tilde{v}_{p,n} : \Omega_p \to H(D)\) by
\[
\tilde{v}_{p,n}(\tilde{\omega}) = \zeta_{p,n}(s, \omega \tilde{\omega}), \quad \tilde{\omega} \in \Omega_p.
\]

Then, repeating the above arguments, we find that \(\hat{P}_{T,p,n}\) converges weakly to \(Q_{p,n} \overset{\text{def}}{=} m_p v_{p,n}^{-1}\). Let \(v_{p}(\omega) = \omega \tilde{\omega}\). Then, by invariance of the measure \(m_p\), we have
\[
Q_{p,n} = m_p (v_{p,n} v_{p}^{-1})^{-1} = (m_p v_{p,n}^{-1}) v_{p,n}^{-1} = m_p v_{p,n}^{-1} = Q_{p,n}.
\]

Thus, \(P_{T,p,n}\) and \(\hat{P}_{T,p,n}\) converge weakly to the same measure \(Q_{p,n}\) as \(T \to \infty\). \(\square\)
Next, we study the family of probability measures \( \{ Q_{P,n} : n \in \mathbb{N} \} \). We recall some notions. A family of probability measures \( \{ P \} \) on \( (\Xi, B(\Xi)) \) is called tight if, for every \( \varepsilon > 0 \), there exists a compact set \( K \subset \Xi \) such that

\[ P(K) > 1 - \varepsilon \]

for all \( P \), and \( \{ P \} \) is relatively compact if every sequence \( \{ P_k \} \subset \{ P \} \) has a subsequence \( \{ P_{k_l} \} \) weakly convergent to a certain probability measure \( P \) on \( (\Xi, B(\Xi)) \) as \( k \to \infty \). By the classical Prokhorov theorem, see Theorem 6.1 of [28], every tight family of probability measures is relatively compact.

**Lemma 9.** Under the hypotheses of Theorem 2, the family \( \{ Q_{P,n} : n \in \mathbb{N} \} \) is relatively compact.

**Proof.** In view of the above remark, it suffices to prove the tightness of \( \{ Q_{P,n} \} \). Let \( K \subset D \) be a compact. Then, using the Cauchy integral formula and absolute convergence of the series for \( \zeta_{P,n}(s) \), we obtain \( \sigma_n > \tilde{\sigma} \)

\begin{equation}
\sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \sup_{s \in K} \frac{1}{T} \int_0^T \left| \sup_{m \in \mathbb{N}_P} \frac{a_n^2(m)}{m^{2/3}} \right|^2 d\tau \leq \sup_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}_P} \frac{1}{m^{2/3}} \sum_{k \in \mathbb{N}_P} V_k < \infty. \tag{26}
\end{equation}

Suppose that \( \zeta_T \) is a random variable on a certain probability space \( (\Xi, A, \mu) \) uniformly distributed in the interval \([0, T] \). Define the \( H(D) \)-valued random element

\[ Y_{T,P,n} = Y_{T,P,n}(s) = \zeta_{P,n}(s + i\zeta_T). \]

Then, denoting by \( \xrightarrow{D} \) the convergence in distribution by Lemma 8, we obtain

\[ Y_{T,P,n} \xrightarrow{D} Y_{P,n}. \tag{27} \]

where \( Y_{P,n}(s) \) is the \( H(D) \)-valued random element with the distribution \( Q_{P,n} \). Since the convergence in \( H(D) \) is uniform on compact sets, (27) implies

\[ \sup_{s \in K} |Y_{T,P,n}(s)| \xrightarrow{D} \sup_{s \in K} |Y_{P,n}(s)|. \tag{28} \]

Now, let \( K = K_l \), where \( \{ K_l \} \) is a sequence of compact sets of \( D \) from the definition of the metric \( \rho \). Fix \( \varepsilon > 0 \), and set \( R_l = 2^{l - 1} \sqrt{V_l} \) where \( V_l = V_{K_l} \). Therefore, relation (26), and the Chebyshev type inequality yield

\begin{align*}
\limsup_{T \to \infty} \mu \left\{ \sup_{s \in K_l} |Y_{T,P,n}(s)| > R_l \right\} & \leq \limsup_{n \to \infty} \frac{1}{T R_l} \int_0^T \sup_{s \in K_l} \left| \zeta_{P,n}(s + i\tau) \right|^2 d\tau \\
& \leq \limsup_{n \to \infty} \frac{1}{R_l} \left( \frac{1}{T} \int_0^T \sup_{s \in K_l} \left| \zeta_{P,n}(s + i\tau) \right|^2 d\tau \right)^{1/2} = \frac{\varepsilon}{2^l}. 
\end{align*}

Hence, in view of (28),

\[ \mu \left\{ \sup_{s \in K_l} |Y_{P,n}(s)| > R_l \right\} \leq \frac{\varepsilon}{2^l}. \tag{29} \]

Define the set

\[ H(\varepsilon) = \left\{ g \in H(D) : \sup_{s \in K_l} |g(s)| \leq R_l, l \in \mathbb{N} \right\}. \]
Then $H(\varepsilon)$ is a compact set in $H(D)$. Moreover, inequality (29) implies that
\[
\mu \{ Y_{P,n} \in H(\varepsilon) \} = 1 - \mu \{ Y_{P,n} \notin H(\varepsilon) \} \geq 1 - \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 - \varepsilon
\]
for all $n \in \mathbb{N}$. Since $Q_{P,n}$ is the distribution of $Y_{P,n}$, this shows that
\[
Q_{P,n}(H(\varepsilon)) \geq 1 - \varepsilon
\]
for all $n \in \mathbb{N}$. The lemma is proved. $\square$

Now, we are ready to consider the weak convergence for $P_{T,P}$ and $\tilde{P}_{T,P}$. For convenience, we recall one general statement.

**Proposition 1.** Suppose that a metric space $(\mathcal{X}, d)$ is separable, and the $\mathcal{X}$-valued random elements $x_{mn}$ and $y_n$, $m, n \in \mathbb{N}$ are defined on the same probability space $(\Xi, \mathcal{A}, \mu)$. Suppose that
\[
x_{mn} \xrightarrow{D} x_m, \quad x_m \xrightarrow{D} x,
\]
and, for every $\varepsilon > 0$,
\[
\lim_{m \to \infty} \limsup_{n \to \infty} \mu \{ d(x_{mn}, y_n) \geq \varepsilon \} = 0
\]
Then
\[
y_n \xrightarrow{D} x.
\]

**Proof.** The proposition is Theorem 4.2 of [28], where its proof is given. $\square$

**Lemma 10.** Under the hypotheses of Theorem 2, on $(H(D), \mathcal{B}(H(D)))$ there exists a probability measure $P_P$ such that both the measures $P_{T,P}$ and $\tilde{P}_{T,P}$ converge weakly to $P_P$ as $T \to \infty$.

**Proof.** Let $\xi_T$ be the same random variable as in the proof of Lemma 9. By Lemma 9, there exists a sequence $\{ Q_{P,nu} \} \subset \{ Q_{P,n} \}$ and the probability measure $Q_P$ on $(H(D), \mathcal{B}(H(D)))$ such that $Q_{P,nu}$ converges weakly to $Q_P$ as $m \to \infty$. In other words, in the notation of the proof of Lemma 9,
\[
Y_{P,nu} \xrightarrow{D} Q_P.
\]

On $(\Xi, \mathcal{A}, \mu)$, define one more $H(D)$-valued random element
\[
Y_{T,P} = Y_{T,P}(s) = \xi_P(s + i\xi_T).
\]
Then the application of Lemma 5 gives, for $\varepsilon > 0$,
\[
\lim_{m \to \infty} \limsup_{T \to \infty} \mu \{ \rho(Y_{T,P}, Y_{T,P,nu}) \geq \varepsilon \}
= \lim_{m \to \infty} \limsup_{T \to \infty} \frac{1}{T} \text{meas} \{ \tau \in [0, T] : \rho(\xi_P(s + i\tau), \xi_P(s + i\tau)) \geq \varepsilon \}
\leq \lim_{m \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} \rho(\xi_P(s + i\tau), \xi_P(s + i\tau)) \, d\tau = 0.
\]

This, and relations (27) and (30) show that all conditions of Proposition 1 are fulfilled. Thus, we have
\[
Y_{T,P} \xrightarrow{D} Q_P,
\]
(31)
\( P_{\tau,p} \) converges weakly to \( Q_p \) as \( T \to \infty \). Since the family \( \{ Q_{p,n} \} \) is relatively compact, relation (31), in addition, implies that

\[
Y_{p,n} \xrightarrow{D} Q_p, \quad n \to \infty.
\]  

(32)

It remains to prove weak convergence for \( \tilde{\hat{P}}_{T,p} \). On \((\Xi, \mathcal{A}, \mu)\), define the \( H(D) \)-valued random elements

\[
\tilde{\hat{Y}}_{T,p,n} = \tilde{\hat{Y}}_{T,p}(s) = \xi_p(s + iT, \omega)
\]

and

\[
\tilde{\hat{Y}}_{T,p} = \tilde{\hat{Y}}_{T,p}(s) = \zeta_p(s + iT, \omega).
\]

Lemma 8 implies the relation

\[
\tilde{\hat{Y}}_{T,p,n} \xrightarrow{D} Q_p, \quad T \to \infty,
\]

(33)

while, in view of Lemma 6, for \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \limsup_{T \to \infty} \mu \left\{ \rho \left( \tilde{\hat{Y}}_{T,p,n}, \tilde{\hat{Y}}_{T,p} \right) \geq \varepsilon \right\} 
\leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\varepsilon T} \int_0^T \rho(\zeta_p(s + i\tau, \omega), \zeta_{p,n}(s + i\tau, \omega)) \, d\tau = 0.
\]

This, (32), (33) and Lemma 10 yield the relation

\[
\tilde{\hat{Y}}_{T,p} \xrightarrow{D} Q_p,
\]

Thus, \( \tilde{\hat{P}}_{T,p} \), as \( T \to \infty \), also converges weakly to \( Q_p \). \( \square \)

It remains to identify the measure \( Q_p \). Denote by \( P_{\xi_p} \) the distribution of the random element \( \xi_p(s, \omega) \), i.e.,

\[
P_{\xi_p}(A) = m_p \{ \omega \in \Omega_p : \xi_p(s, \omega) \in A \}.
\]

**Theorem 3.** Under hypotheses of Theorem 2, \( P_{T,p} \) converges weakly to the measure \( P_{\xi_p} \) as \( T \to \infty \).

**Proof.** We will show that the limit measure \( Q_p \) in Lemma 10 coincides with \( P_{\xi_p} \).

We apply the equivalent of weak convergence of probability measures in terms of continuity sets, see Theorem 2.1 of [28]. Let \( A \) be a continuity set of the measure \( Q_p \), i.e., \( Q_p(\partial A) = 0 \), where \( \partial A \) denotes the boundary of \( A \). Then, Lemma 10 implies that

\[
\lim_{T \to \infty} \tilde{\hat{P}}_{T,p}(A) = Q_p(A).
\]  

(34)

On \((\Omega_p, \mathcal{B}(\Omega_p))\), define the random variable

\[
\xi_p(\omega) = \begin{cases} 0 & \text{if } \xi_p(s, \omega) \notin A, \\ 1 & \text{otherwise}. \end{cases}
\]

Return to the group \( G_T \) of Lemma 3. Since, by Lemma 3, the group \( G_T \) is ergodic, the process \( \xi(g_T(\omega)) \) is ergodic, and application of the Birkhoff–Khintchine theorem [27] gives

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \xi_p(g_T(\omega)) \, d\tau = \mathbb{E}_{\xi_p}(\omega)
\]

(35)
for almost all $\omega \in \Omega_P$. However, the definition of the random variable $\xi_T(\omega)$ implies that, for almost all $\omega \in \Omega_P$,

$$\frac{1}{T} \int_0^T \xi_P(g_{\tau}(\omega)) \, d\tau = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \xi_P(s, g_{\tau}(\omega)) \in A\} = \frac{1}{T} \text{meas}\{\tau \in [0, T] : \xi_P(s + i\tau, \omega) \in A\}.$$ 

Thus, by (34),

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \xi_P(g_{\tau}(\omega)) \, d\tau = Q_P(A). \quad (36)$$

Moreover,

$$\mathbb{E}_\xi(\omega) = \int_{\Omega_P} \xi_P(\omega) \, d\mu_P = P_{\xi_P}(A).$$

This, (35) and (36) prove that $Q_P(A) = P_{\xi_P}(A)$ for all continuity sets $A$ of the measure $Q_P$. It is well known that all continuity sets constitute a determining class. Hence, we have $Q_P = P_{\xi_P}$, and the theorem is proved. □

6. Support

For the proof of Theorem 2, the explicitly given support of the measure $P_{\xi_P}$ is needed. We recall that the support of $P_{\xi_P}$ is a minimal closed set $S_{\xi_P} \subset H(D)$ such that $P_{\xi_P}(S_{\xi_P}) = 1$. Every open neighbourhood of elements $S_{\xi_P}$ has a positive $P_{\xi_P}$-measure.

Define the set

$$S_{\xi_P} = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

**Proposition 2.** Under the hypotheses of Theorem 2, the support of the measure $P_{\xi_P}$ is the set $S_{\xi_P}$.

A proof of Proposition 2 is similar to that in the case of the Riemann zeta-function. Therefore, we will state without proof only the lemmas because their proofs word for word coincide with analogical assertions from [22].

We start with some estimations over generalized primes $p \in \mathcal{P}$.

**Lemma 11.** Suppose that the estimate (5) is valid. Then, for $x \to \infty$,

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + a + O\left(x^{\beta_2 - 1}\right),$$

where $a$ is a constant, and $0 \leq \beta_2 < 1$.

**Proof.** We have

$$\psi_1(x) \overset{\text{def}}{=} \sum_{p \leq x} \log p = \psi(x) - \sum_{p' \leq x} \sum_{2^{\alpha} \leq (\log x) / (\log 2)} \log p$$

$$= \psi(x) + O\left(\psi\left(x^{1/2}\right) \log x\right) = x + r(x),$$

where

$$r(x) = O\left(x^{\beta_2 \log x}\right)$$

with

$$\beta_2 = \max\left(\beta_1, \frac{1}{2}\right).$$
From this, by partial summation, we obtain

\[
\sum_{p \leq x} \frac{1}{p} = \frac{1}{x \log x} \sum_{p \leq x} \log p + \int_{p_1}^{x} \left( \frac{1}{u^2 \log u} + \frac{1}{u^2 \log^2 u} \right) \psi_1(u) \, du
\]

\[
= \frac{1}{\log x} + \log \log x - \frac{1}{\log x} + c_1 + \int_{p_1}^{x} \left( \frac{1}{u^2 \log u} + \frac{1}{u^2 \log^2 u} \right) r(u) \, du
\]

\[
= \log \log x + c_1 + \int_{x}^{\infty} \left( \frac{1}{u^2 \log u} + \frac{1}{u^2 \log^2 u} \right) r(u) \, du
\]

\[
- \int_{x}^{\infty} \left( \frac{1}{u^2 \log u} + \frac{1}{u^2 \log^2 u} \right) u r(u) \, du
\]

\[
= \log \log x + c_2 + O \left( \int_{x}^{\infty} u^{\beta_1 - 2} \, du \right) = \log \log x + c_2 + O \left( x^{\beta_2 - 1} \right).
\]

\(\square\)

In what follows, we will use some properties of functions of exponential type. We recall a function \(g(s)\) analytic in the region \(|\arg s| \leq \theta_0, 0 < \theta_0 \leq \pi\) is of exponential type if uniformly in \(\theta, \theta \leq \theta_0\),

\[
\limsup_{r \to \infty} \frac{\log |g(re^{i\theta})|}{r} < \infty.
\]

**Lemma 12.** Suppose that \(g(s)\) is an entire function of exponential type, (5) holds, and

\[
\limsup_{r \to \infty} \frac{\log |g(r)|}{r} > -1.
\]

Then

\[
\sum_{p \in P} |g(\log p)| = \infty.
\]

**Proof.** We use the formula of Lemma 11, and repeat word for word the proof of Theorem 6.4.14 of [22]. \(\square\)

Let \(s \in D\), and \(|a_p| = 1\). For brevity, we set

\[
g_P(s, a_p) = \log \left( 1 - \frac{a_p}{p^s} \right), \quad p \in P,
\]

where

\[
\log \left( 1 - \frac{a_p}{p^s} \right) = - \frac{a_p}{p^s} - \frac{a_p^2}{2p^{2s}} - \cdots.
\]

**Lemma 13.** Suppose that (5) holds. Then the set of all convergent series

\[
\sum_{p \in P} g_P(s, a_p)
\]

is dense in the space \(H(D)\).

**Proof.** The object connected to the system \(P\) is only Lemma 12. Other arguments of the proof are the same as those applied in the proof of Lemma 6.5.4 from [22]. \(\square\)
Recall that the support of the distribution of a random element $X$ is called a support of $X$, and is denoted by $S_X$.

For convenience, we state a lemma on the support of a series of random elements.

**Lemma 14.** Let $\{\xi_m\}$ be a sequence of independent $H(D)$-valued random elements on a certain probability space $(\Xi, A, \mu)$; the series
\[ \sum_{m=1}^{\infty} \xi_m \]
is convergent almost surely. Then, the support of the sum of this series is the closure of the set of all $g \in H(D)$ which may be written as a convergent series
\[ g = \sum_{m=1}^{\infty} g_m, \quad g_m \in S_{\xi_m}. \]

**Proof.** The lemma is Theorem 1.7.10 of [22], where its proof is given. □

**Proof of Proposition 2.** By the definition, $\{\omega(p) : p \in \mathcal{P}\}$ is a sequence of independent complex-valued random variables. Therefore, $\{g_P(s, \omega(p))\}$ is a sequence of independent $H(D)$-valued random elements. Since the support of each $\omega(p)$ is the unit circle, the support of $g_P(s, \omega(p))$ is the set
\[ \left\{ g \in H(D) : g(s) = -\log \left( 1 - \frac{a}{p^s} \right), |a| = 1 \right\}. \]

Therefore, in view of Lemma 14, the support of the $H(D)$-valued random element
\[ \log \xi_P(s, \omega) = -\sum_{p \in \mathcal{P}} \log \left( 1 - \frac{\omega(p)}{p^s} \right) \]
is the closure of the set of all convergent series
\[ \sum_{p \in \mathcal{P}} g_P(s, a_p) \]
with $|a_p| = 1$. By Lemma 13, the set of the latter series is dense in $H(D)$. Define $u : H(D) \to H(D)$ by $u(g) = e^g$, $g \in H(D)$. The mapping $u$ is continuous, $u(\log \xi_P(s, \omega)) = \xi_P(s, \omega)$ and $u(H(D)) = S_P \setminus \{0\}$. This shows that $S_P \setminus \{0\}$ lies in the support of $\xi_P(s, \omega)$. Since the support is a closed set, we obtain that the support of $\xi_P(s, \omega)$ contains the closure of $S_P \setminus \{0\}$, i.e.,
\[ S_{\xi_P} \supset S_P. \tag{37} \]

On the other hand, the random element $\xi_P(s, \omega)$ is convergent for almost all $\omega \in \Omega_P$, a product of non-zeros multipliers. Therefore, by the classical Hurwitz theorem, see [29],
\[ S_{\xi_P} \subset S_P. \]

This inclusion together with (37) proves the proposition. □

### 7. Proof of Universality

In this section, we prove Theorem 2. Its proof is based on Theorem 3, Proposition 2 and the Mergelyan theorem [30] on the approximation of analytic functions by polynomials on compact sets with connected complements.

**Proof of Theorem 2.** Let $p(s)$ be a polynomial, $K$ and $\epsilon$ defined in Theorem 2, and
\[ G_\epsilon = \left\{ g \in H(D) : \sup_{s \in K} \left| g(s) - e^{p(s)} \right| < \frac{\epsilon}{2} \right\}. \]
Then, the set $G_\varepsilon$ is an open neighborhood of an element $e^{\theta(s)} \in S_P$. Since, in view of Proposition 2, $S_P$ is the support of the measure $P_{s_P}$, by a property of supports, we have
\[ P_{s_P}(G_\varepsilon) > 0. \] 
(38)

Since $f(s) \in H_0(K)$, we may apply the mentioned Mergelyan theorem and choose the polynomial $p(s)$ satisfying
\[ \sup_{s \in K} |f(s) - e^{\theta(s)}| < \frac{\varepsilon}{2}. \]
This shows that the set $G_\varepsilon$ lies in
\[ \tilde{G}_\varepsilon = \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon \right\}. \]

Thus, by (38), we have
\[ P_{s_P}(\tilde{G}_\varepsilon) > 0. \] 
(39)

Theorem 3 and the equivalent of weak convergence in terms of open sets yield
\[ \liminf_{I \to \infty} P_{s_P}(\tilde{G}_\varepsilon) \geq P_{s_P}(\tilde{G}_\varepsilon). \]

This, (39), and the definitions of $P_{s_P}$ and $\tilde{G}_\varepsilon$ prove the first statement of the theorem.

To prove the second statement of the theorem, we observe that the boundary $\partial \tilde{G}_\varepsilon$ of the set $\tilde{G}_\varepsilon$ lies in the set
\[ \left\{ g \in H(D) : \sup_{s \in K} |g(s) - f(s)| = \varepsilon \right\}. \]

Hence, the boundaries $\partial \tilde{G}_{\varepsilon_1}$ and $\partial \tilde{G}_{\varepsilon_2}$ do not intersect for different positive $\varepsilon_1$ and $\varepsilon_2$. Therefore, $P_{s_P}(\partial \tilde{G}_\varepsilon) > 0$ for countably many $\varepsilon > 0$. In other words, the set $\tilde{G}_\varepsilon$ is a continuity set of the measure $P_{s_P}$ for all but at most countably many $\varepsilon > 0$. This, (39), Theorem 3 and the equivalent of weak convergence in terms of continuity sets prove the second statement of the theorem.

8. Conclusions

In the paper, we considered the set $P$ of generalized prime numbers satisfying
\[ \sum_{m \leq x \atop m \in \mathbb{N}_P} 1 = ax + O \left( x^{\beta} \right), \quad a > 0, \ 0 \leq \beta < 1, \]
and
\[ \sum_{m \leq x \atop m \in \mathbb{N}_P} \Lambda_P(m) = x + O \left( x^{\beta_1} \right), \quad 0 \leq \beta_1 < 1, \]
where $\mathbb{N}_P$ is the set of generalized integers and $\Lambda_P(m)$ is the generalized von Mangoldt function corresponding to the set $P$. Assuming that the set $\{ \log p : p \in P \}$ is linearly independent over $\mathbb{Q}$, and the Beurling zeta-function
\[ \zeta_P(s) = \sum_{m \in \mathbb{N}_P} \frac{1}{m^s}, \quad s = \sigma + it, \ \sigma > 1, \]
has the bounded mean square for $\sigma > \hat{\sigma}$ with some $\hat{\beta} < \hat{\sigma} < 1$, we obtained universality of $\zeta_P(s)$, i.e., that every non-vanishing analytic function can be approximated by shifts $\zeta_P(s + i\tau)$, $\tau \in \mathbb{R}$.
In the future, we are planning to obtain a more complicated discrete version of Theorem 2, i.e., to prove the approximation of analytic functions by discrete shifts \( \zeta_p(s + ikh) \), \( h > 0, k \in \mathbb{N}_0 \).

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