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On Coarse Isometries and Linear Isometries between Banach Spaces

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Abstract: Let X, Y be two Banach spaces and $f : X \rightarrow Y$ be a standard coarse isometry. In this paper, we first show a sufficient and necessary condition for the coarse left-inverse operator of general Banach spaces to admit a linearly isometric right inverse. Furthermore, by using the well-known simultaneous extension operator, we obtain an asymptotical stability result when Y is a space of continuous functions. In addition, we also prove that every coarse left-inverse operator does admit a linear isometric right inverse without other assumptions when Y is a L_p ($1 < p < \infty$) space, or both X and Y are finite dimensional spaces of the same dimension. Making use of the results mentioned above, we generalize several results of isometric embeddings and give a stability result of coarse isometries between Banach spaces.

Keywords: coarse isometry; linear isometry; $C(K)$ space; L_p space; finite dimensional space; Banach space

MSC: 46B04; 46B20



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1. Introduction

In this paper, we study the relationship between the coarse isometry and linear isometry of Banach spaces by using the coarse left-inverse operators. Throughout this article, we consider X and Y as real Banach spaces. We first recall the definitions of coarse isometry and coarse left-inverse operator.

Definition 1. Let $f : X \rightarrow Y$ be a mapping. Put

$$\varepsilon_f(s) = \sup_{w_1, w_2 \in X, \|w_1 - w_2\| \leq s} \{ \|f(w_1) - f(w_2)\| - \|w_1 - w_2\| \} \quad \text{for } s \geq 0.$$

- (1) f is said to be a coarse isometry if $\lim_{s \rightarrow \infty} \frac{\varepsilon_f(s)}{s} = 0$;
- (2) We say that f is an ε -isometry if $\varepsilon \equiv \lim_{s \rightarrow \infty} \varepsilon_f(s) < \infty$;
- (3) f is called an isometry if $\varepsilon_f(s) = 0$ for all $s \geq 0$;
- (4) f is called standard if $f(0) = 0$.

Definition 2. Let $f : X \rightarrow Y$ be a standard mapping, $\alpha > 0$, and let $T : Y_f \equiv \overline{\text{span}} f(X) \rightarrow X$ be a bounded linear operator satisfying $\|T\| = 1$ and

$$\|Tf(v) - v\| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X. \quad (1)$$

- (1) T is called a coarse left-inverse operator of f if f is a coarse isometry;
- (2) T is said to be an ε -left-inverse operator of f if f is an ε -isometry;
- (3) We say that T is a Fiegel operator of f if f is an isometry.

Isometry, ε -Isometry and Linear Isometry In 1932, Mazur and Ulam [1] gave a remarkable result which says that every standard surjective isometry $f : X \rightarrow Y$ must be linear. Further, in 1968, Figiel [2] showed that every standard isometry $f : X \rightarrow Y$ can admit a Figiel operator.

In 2003, Godefroy and Kalton [3] obtained a result concerning isometry and linear isometry with the help of the Figiel operator. They proved that the Figiel operator T has a linearly isometric right inverse if X is separable. That is, there is a linear isometry $U : X \rightarrow Y$ satisfying $TUv = v$ for all $v \in X$. However, even if X is a non-separable Hilbert space, they can construct a Banach space Y and a nonlinear isometry $f : X \rightarrow Y$ such that no subspace of Y can be linearly isomorphic to X . This implies that there cannot exist an isometric right inverse for T that preserves linearity. In 2015, Zhou et al. [4] obtained a necessary and sufficient condition for the existence of linear isometric right inverses of the Figiel operator. For more details on isometry and linear isometry, see [5–8] and references therein.

In 1945, the concept of ε -isometry was first introduced by Hyers and Ulam [9]. They posed the following question: given two Banach spaces X, Y and a positive constant α , is it possible to find a surjective linear isometry $U : X \rightarrow Y$ that corresponds to every standard surjective ε -isometry $f : X \rightarrow Y$, such that $\|f(x) - Ux\| < \alpha\varepsilon$? After the fifty years' efforts of many mathematicians (see, for example, [10–12]), Omladić and Šemrl [13] finally answered this question in the affirmative with the sharp estimate $\alpha = 2$.

In 2013, Cheng, Dong and Zhang [14] derived a remarkable result called the weak stability formula, which is inspired by the Figiel theorem [2] and other important results on non-surjective ε -isometries (see [15,16]). It has attracted considerable attention from many researchers (see [5,14,17–21]); for instance, by means of the invariant mean techniques and the weak stability formula, Cheng and Zhou [5] showed that, if Y is reflexive, then there exists a linear isometry $U : X \rightarrow Y$ for every standard ε -isometry $f : X \rightarrow Y$. Further, by using the ε -left-inverse operators, Zhou et al. [22] investigated the connection between the ε -isometry and linear isometry of general Banach spaces.

Coarse Isometry and Linear Isometry The large perturbation function, first studied by Lindenstrauss and Szankowski [23] in 1985, is defined as

$$\psi_f(s) = \sup_{w_1, w_2 \in X, \|f(w_1) - f(w_2)\| \wedge \|w_1 - w_2\| \leq s} \{ \|f(w_1) - f(w_2)\| - \|w_1 - w_2\| \} \quad \text{for } s \geq 0,$$

and the following asymptotic stability result is obtained.

Theorem 1. (Lindenstrauss and Szankowski) Let $f : X \rightarrow Y$ be a surjective standard coarse isometry. If

$$\int_1^\infty \frac{\psi_f(s)}{s^2} ds < \infty, \quad (2)$$

then there exists a linear surjective isometry $U : X \rightarrow Y$ such that

$$\|f(v) - Uv\| = o(\|v\|) \quad \text{as } \|v\| \rightarrow \infty. \quad (3)$$

Meanwhile, they constructed two non-isometric uniformly convex spaces X, Y and a coarse isometry $f : X \rightarrow Y$ with $\int_1^\infty \frac{\psi_f(s)}{s^2} ds = \infty$. This reveals that a coarse isometric embedding does not imply an isometric embedding when (2) is invalid. In addition, Benyamini and Lindenstrauss showed that conclusion (3) may fail when $\int_1^\infty \frac{\psi_f(s)}{s^2} ds = \infty$ even for coarse isometries of the Euclidean plane $\mathbb{R}^2$ onto itself (see [24], on page 367). Further, Dolinar [25] proved that Theorem 1 also holds if the condition $\int_1^\infty \frac{\psi_f(s)}{s^2} ds < \infty$ is substituted by a weaker condition

$$\int_1^\infty \frac{\varepsilon_f(s)}{s^2} ds < \infty. \quad (4)$$

This implies that, in order to establish the connection between coarse isometry and linear isometry, the integral convergence condition (4) is essential.

In 2019, Cheng et al. [26] first gave a stability result of non-surjective coarse isometries between uniformly convex spaces. Recently, Sun and Zhang [27] studied the relationship between weak stability and the stability of coarse isometries of $L_p(1 < p < \infty)$ spaces. Further, they also proved that the surjective assumption in Theorem 1 can be removed when X and Y are both finite dimensional spaces with equal dimensions (see [28], on page 1496).

In this article, we mainly study the relationship between the coarse isometry and linear isometry of varieties of Banach spaces by using the coarse left-inverse operators. This paper has the following structure.

In Section 2, inspired by Zhou et al. [22], we prove that the existence of linearly isometric right inverses for coarse left-inverse operators is closely related to a continuous linear projection with respect to weak star topology under the integral convergence condition (4). Using this result, we generalize an isometry result which was established by Zhou et al. [4] and obtain a stability result of coarse isometries.

For a standard coarse isometry $f : X \rightarrow C(K)$, where K is a compact Hausdorff space, in Section 3, we show that, with condition (4), if $T^*(B_{X^*}) \cap (\text{ext}(B_{C(K)^*})|_{Y_f}) \neq \emptyset$, then there exist a non-empty closed subset $\tilde{K}$ contained in K and a linear isometry $g : X \rightarrow C(\tilde{K})$ satisfying

$$\|f(v)|_{\tilde{K}} - g(v)\| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

In addition, if K is metrizable, we obtain that $E \circ g : X \rightarrow C(K)$ is a linear isometry and

$$\|E(f(v)|_{\tilde{K}}) - E(g(v))\| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X,$$

where $E : C(\tilde{K}) \rightarrow C(K)$ is a simultaneous extension operator. These results generalize several known results in [29,30].

Let (Ω, Σ, μ) be a measure space. In Section 4, by means of the properties of a free ultrafilter on $\mathbb{N}$, we prove that for a standard coarse isometry $f : X \rightarrow Y$ where Y is the $L_p(\Omega, \Sigma, \mu)(1 < p < \infty)$ space or $\dim X = \dim Y < \infty$, every coarse left-inverse operator of f corresponds to a linearly isometric right inverse. The conditions about continuous linear projection with respect to weak star topology and (4) can be removed.

In this paper, all symbols are standard. Y_f denotes $\overline{\text{span}} f(X)$ and X^* is the dual space of X ; the letters S_X and B_X denote the unit sphere and the closed unit ball of X , respectively. The symbol T^* represents the conjugate operator of a bounded linear operator T and $\mathcal{R}(T^*)$ denotes the range of T^* . In addition, we say that $R : X^* \rightarrow Y^*$ is w^* -to- w^* continuous if R is continuous with respect to the weak star topology on X^* and Y^* .

2. Right Inverse of General Banach Spaces

In the following, we give two lemmas.

Lemma 1. *Let $f : X \rightarrow Y$ be a standard coarse isometry and let T be a coarse left-inverse operator of f . Then, $T^* : X^* \rightarrow Y_f^*$ is a w^* -to- w^* continuous linear isometry.*

Proof. Given $z^* \in S_{X^*}$, we have $\|T^*z^*\| \leq 1$ because of $\|T\| = \|T^*\| = 1$. On the other hand, to each $\delta > 0$ corresponds a point x_0 in S_X so that $\langle z^*, x_0 \rangle > 1 - \delta$. Due to (1), we obtain

$$\|T \frac{f(nx_0)}{n} - x_0\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then,

$$|\langle T^*z^*, \frac{f(nx_0)}{n} \rangle - \langle z^*, x_0 \rangle| = |\langle z^*, T \frac{f(nx_0)}{n} - x_0 \rangle| \leq \|T \frac{f(nx_0)}{n} - x_0\| \rightarrow 0.$$

So, there exists $M \in \mathbb{N}$ so that $|\langle T^*z^*, \frac{f(nx_0)}{n} \rangle| > 1 - \delta$ for all $n \geq M$. This implies $\|T^*z^*\| \geq 1 - \delta$. In fact, if $\|T^*z^*\| < 1 - \delta$, then $|\langle T^*z^*, v \rangle| < 1 - \delta$ for each $v \in B_{Y_f}$. It follows that there exists $\alpha > 0$ such that $|\langle T^*z^*, (1 + \alpha)v \rangle| < 1 - \delta$ for each $v \in B_{Y_f}$. Since f is a standard coarse isometry, we have

$$\left\| \frac{f(nx_0)}{n} - \|x_0\| \right\| \leq \frac{\varepsilon_f(n\|x_0\|)}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (5)$$

This implies that $\frac{f(n_0x_0)}{n_0} \in B_{Y_f}(0, 1 + \alpha)$ for some $n_0 > M$. Then, $|\langle T^*z^*, \frac{f(n_0x_0)}{n_0} \rangle| < 1 - \delta$; this is a contradiction. Since δ is arbitrary, we have $\|T^*z^*\| \geq 1$. Consequently, $\|T^*z^*\| = 1 = \|z^*\|$. This means that $T^* : X^* \rightarrow Y_f^*$ is a w^* -to- w^* continuous linear isometry. $\square$

The following lemma presents a stability result for functional equations, known as the Hyers–Ulam–Rassias stability, which was established by Găvruta [31]. For more information on this topic, see [32,33] and related references.

Lemma 2. (Găvruta) Let G be an Abelian group, Y be a Banach space and let $\varphi : G \times G \rightarrow [0, \infty)$ satisfying

$$\Phi(w_1, w_2) = \sum_{k=0}^{\infty} \frac{\varphi(2^k w_1, 2^k w_2)}{2^{k+1}} < \infty \quad \text{for all } w_1, w_2 \in G. \quad (6)$$

If a function $h : G \rightarrow Y$ satisfies

$$\|h(w_1 + w_2) - (h(w_1) + h(w_2))\| \leq \varphi(w_1, w_2) \quad \text{for all } w_1, w_2 \in G,$$

then there is a unique additive function $A : G \rightarrow Y$ so that

$$\|h(u) - Au\| \leq \Phi(u, u) \quad \text{for all } u \in G,$$

where $Au = \lim_{n \rightarrow \infty} \frac{h(2^n u)}{2^n}$.

The following results of coarse isometries are inspired by Zhou et al. [22] (Theorem 2.1).

Proposition 1. Let $f : X \rightarrow Y$ be a standard coarse isometry and let T be a coarse left-inverse operator of f . If there is a linear isometry $U : X \rightarrow Y_f$ such that $TU = id_X$, then $T^*U^* : Y_f^* \rightarrow \mathcal{R}(T^*) \subseteq Y_f^*$ is a w^* -to- w^* continuous linear projection and $\|T^*U^*\| = 1$.

Proof. According to Lemma 1, $T^* : X^* \rightarrow Y_f^*$ is a w^* -to- w^* continuous linear isometry. It follows that $T^*U^* : Y_f^* \rightarrow \mathcal{R}(T^*) \subseteq Y_f^*$ is w^* -to- w^* continuous with $\|T^*U^*\| = 1$. Moreover,

$$(T^*U^*)(T^*U^*) = T^*(TU)^*U^* = T^*(id_X)^*U^* = T^*U^*.$$

Thus, T^*U^* is a linear projection. The proof is completed. $\square$

Theorem 2. Let $f : X \rightarrow Y$ be a standard coarse isometry and let T be a coarse left-inverse operator of f . If there is a w^* -to- w^* continuous linear projection $P : Y_f^* \rightarrow \mathcal{R}(T^*)$ with $\|P\| = 1$ and

$$\int_1^\infty \frac{\varepsilon_f(s)}{s^2} ds < \infty,$$

then there is a linear isometry $U : X \rightarrow Y$ such that $TU = id_X$. Furthermore, if f is almost surjective, i.e.,

$$\liminf_{t \rightarrow \infty} d(ty, f(X))/|t| = 0 \quad \text{for all } y \in S_Y, \quad (7)$$

then the above linear isometric mapping U is unique and $P = T^*U^*$. In addition, if $P_1, P_2 : Y_f^* \rightarrow \mathcal{R}(T^*)$ are two w^* -to- w^* continuous linear projections with $\|P_1\| = \|P_2\| = 1$ and $P_1 \neq P_2$, then $U_{P_1} \neq U_{P_2}$.

Proof. Step I. First, we define the mapping $J : X \rightarrow Y_f$ with

$$\langle y^*, Jv \rangle = \langle Py^*, f(v) \rangle \quad \text{for all } v \in X, y^* \in Y_f^*. \quad (8)$$

Indeed, since $P : Y_f^* \rightarrow \mathcal{R}(T^*)$ is a w^* -to- w^* continuous linear projection, $\langle Py^*, f(v) \rangle$ is w^* -continuous for each $v \in X$. This entails that $Jv : Y_f^* \rightarrow \mathbb{R}$ is w^* -continuous. Then, $Jv \in Y_f$.

On the one hand, by the definition of J , we have

$$\|Jv\| = \sup_{y^* \in B_{Y_f^*}} |\langle y^*, Jv \rangle| = \sup_{y^* \in B_{Y_f^*}} |\langle Py^*, f(v) \rangle| \leq \|f(v)\| \|P\| \leq \|v\| + \varepsilon_f(\|v\|). \quad (9)$$

On the other hand, T^* is a linear isometry by Lemma 1. Then, for each $z^* \in B_{X^*}$,

$$\|Jv\| = \sup_{y^* \in B_{Y_f^*}} |\langle y^*, Jv \rangle| \geq |\langle T^*z^*, Jv \rangle| = |\langle P(T^*z^*), f(v) \rangle| = |\langle T^*z^*, f(v) \rangle| = |\langle z^*, Tf(v) \rangle|.$$

It follows from $|\langle z^*, Tf(v) - v \rangle| \leq \|Tf(v) - v\| \leq \alpha \varepsilon_f(\|v\|)$ and the inequality above that

$$\|Jv\| \geq \sup_{z^* \in B_{X^*}} |\langle z^*, Tf(v) \rangle| \geq \sup_{z^* \in B_{X^*}} |\langle z^*, v \rangle| - \alpha \varepsilon_f(\|v\|) = \|v\| - \alpha \varepsilon_f(\|v\|). \quad (10)$$

By comparing (9) with (10), we obtain

$$\|v\| - \alpha \varepsilon_f(\|v\|) \leq \|Jv\| \leq \|v\| + \varepsilon_f(\|v\|) \quad \text{for all } v \in X. \quad (11)$$

Next, we prove that $J : X \rightarrow Y_f$ satisfies the following functional equation

$$\|J(w_1 + w_2) - (Jw_1 + Jw_2)\| \leq 2\alpha(\varepsilon_f(2\|w_1\|) + \varepsilon_f(2\|w_2\|)) \quad \text{for all } w_1, w_2 \in X. \quad (12)$$

Given each $w_1, w_2 \in X$, we can choose $\phi \in S_{Y_f^*}^*$ such that

$$\langle \phi, J(w_1 + w_2) - (Jw_1 + Jw_2) \rangle = \|J(w_1 + w_2) - (Jw_1 + Jw_2)\|.$$

Since $T^* : X^* \rightarrow \mathcal{R}(T^*) = P(Y_f^*)$ is a surjective linear isometry, there exists $\psi \in X^*$ such that $T^*\psi = P\phi$ and $\|\psi\| = \|P\phi\| \leq \|P\| \|\phi\| = 1$. Because of (8), $\|Tf(u) - u\| \leq \alpha \varepsilon_f(\|u\|)$ and the increasing nature of $\varepsilon_f(s)$, we have

$$\begin{aligned} \langle \phi, J(w_1 + w_2) - (Jw_1 + Jw_2) \rangle &= \langle P\phi, f(w_1 + w_2) - (f(w_1) + f(w_2)) \rangle \\ &= \langle T^*\psi, f(w_1 + w_2) - (f(w_1) + f(w_2)) \rangle \\ &= \langle \psi, Tf(w_1 + w_2) - (Tf(w_1) + Tf(w_2)) \rangle \\ &= \langle \psi, Tf(w_1 + w_2) - (w_1 + w_2) \rangle - \\ &\quad \langle \psi, (Tf(w_1) - w_1) - (Tf(w_2) - w_2) \rangle \\ &\leq \|Tf(w_1 + w_2) - (w_1 + w_2)\| + \|Tf(w_1) - w_1\| \\ &\quad + \|Tf(w_2) - w_2\| \\ &\leq \alpha \varepsilon_f(\|w_1 + w_2\|) + \alpha \varepsilon_f(\|w_1\|) + \alpha \varepsilon_f(\|w_2\|) \\ &\leq \alpha \max\{\varepsilon_f(2\|w_1\|), \varepsilon_f(2\|w_2\|)\} + \alpha \varepsilon_f(\|w_1\|) + \alpha \varepsilon_f(\|w_2\|) \\ &\leq 2\alpha(\varepsilon_f(2\|w_1\|) + \varepsilon_f(2\|w_2\|)). \end{aligned}$$

Thus, (12) holds.

Step II. We show that the limit

$$Uu = \lim_{n \rightarrow \infty} \frac{J(2^n u)}{2^n}$$

exists for each $u \in X$ and that $U : X \rightarrow Y_f$ is a linear isometry with $TU = id_X$.

Firstly, we prove that $\lim_{n \rightarrow \infty} \frac{J(2^n u)}{2^n}$ exists for all $u \in X$. Let

$$\varphi(w_1, w_2) = 2\alpha(\varepsilon_f(2\|w_1\|) + \varepsilon_f(2\|w_2\|)) \quad \text{and} \quad \Phi(w_1, w_2) = \sum_{k=0}^{\infty} \frac{\varphi(2^k w_1, 2^k w_2)}{2^{k+1}}.$$

Since $\int_1^{\infty} \frac{\varepsilon_f(s)}{s^2} ds < \infty$ and, for each nonzero element $u \in X$, there exists $n_0 \in \mathbb{N}$ such that

$$\int_1^{\infty} \frac{\varepsilon_f(s)}{s^2} ds \geq \sum_{k=n_0}^{\infty} \int_{2^k \|u\|}^{2^{k+1} \|u\|} \frac{\varepsilon_f(s)}{s^2} ds \geq \sum_{k=n_0}^{\infty} \frac{\varepsilon_f(2^k \|u\|)}{2^{k+1} \|u\|}.$$

This implies

$$\sum_{k=0}^{\infty} \frac{\varepsilon_f(2^k \|u\|)}{2^{k+1}} < \infty \quad \text{for all } u \in X.$$

Therefore, $\Phi(w_1, w_2) < \infty$ and $\Phi(u, u) = \sum_{k=0}^{\infty} \frac{\alpha \varepsilon_f(2^{k+1} \|u\|)}{2^{k+1}}$ for each $u, w_1, w_2 \in X$. Due to (12) and Lemma 2, there is an additive mapping $U : X \rightarrow Y_f$ satisfying

$$Uu = \lim_{n \rightarrow \infty} \frac{J(2^n u)}{2^n}$$

and

$$\|Ju - Uu\| \leq \Phi(u, u) = \sum_{k=0}^{\infty} \frac{\alpha \varepsilon_f(2^{k+1} \|u\|)}{2^{k+1}} \quad \text{for all } u \in X. \quad (13)$$

In what follows, we will prove that U is a linear isometry and $TU = id_X$. Note that, for each $w_1, w_2 \in X$,

$$\|Jw_1 - Jw_2\| = \sup_{y^* \in B_{Y_f^*}} |\langle y^*, Jw_1 - Jw_2 \rangle| = \sup_{y^* \in B_{Y_f^*}} |\langle Py^*, f(w_1) - f(w_2) \rangle| \leq \|f(w_1) - f(w_2)\|.$$

Then,

$$\begin{aligned} \|Uw_1 - Uw_2\| &= \lim_{n \rightarrow \infty} \left\| \frac{J(2^n w_1) - J(2^n w_2)}{2^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left\| \frac{f(2^n w_1) - f(2^n w_2)}{2^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \frac{\|2^n w_1 - 2^n w_2\| + \varepsilon_f(2^n \|w_1 - w_2\|)}{2^n} \\ &= \|w_1 - w_2\|. \end{aligned}$$

This entails that U is a 1-Lipschitz mapping and then U is a continuous linear operator. From (11),

$$\|Uu\| = \lim_{n \rightarrow \infty} \frac{\|J(2^n u)\|}{2^n} \geq \lim_{n \rightarrow \infty} \frac{\|2^n u\| - \alpha \varepsilon_f(2^n \|u\|)}{2^n} = \|u\| \quad \text{for each } u \in X.$$

Consequently, U is a linear isometry. Moreover, by means of the definition of T and J , we obtain that, for each $u \in X$,

$$\begin{aligned} \|TUu - u\| &= \lim_{n \rightarrow \infty} \|T \frac{J(2^n u)}{2^n} - u\| \\ &= \lim_{n \rightarrow \infty} \sup_{z^* \in B_{X^*}} |\langle z^*, T \frac{J(2^n u)}{2^n} - u \rangle| \\ &= \lim_{n \rightarrow \infty} \sup_{z^* \in B_{X^*}} |\langle T^* z^*, \frac{J(2^n u)}{2^n} \rangle - \langle z^*, u \rangle| \\ &= \lim_{n \rightarrow \infty} \sup_{z^* \in B_{X^*}} |\langle P(T^* z^*), \frac{f(2^n u)}{2^n} \rangle - \langle z^*, u \rangle| \\ &= \lim_{n \rightarrow \infty} \sup_{z^* \in B_{X^*}} |\langle T^* z^*, \frac{f(2^n u)}{2^n} \rangle - \langle z^*, u \rangle| \\ &= \lim_{n \rightarrow \infty} \sup_{z^* \in B_{X^*}} |\langle z^*, T \frac{f(2^n u)}{2^n} \rangle - \langle z^*, u \rangle| \\ &\leq \lim_{n \rightarrow \infty} \|T \frac{f(2^n u)}{2^n} - u\| \leq \lim_{n \rightarrow \infty} \frac{\alpha \varepsilon_f(2^n \|u\|)}{2^n} = 0. \end{aligned}$$

Then, $TU = id_X$.

Step III. We prove that $P = T^*U^*$ under the almost surjective assumption condition (7). Given $y \in S_{Y_f}$, by (7), there exist $\{t_n\} \subset \mathbb{R}$ with $|t_n| \nearrow \infty$ and $\{v_n\} \subset X$ satisfying

$$\lim_{n \rightarrow \infty} d(t_n y, f(v_n)) / |t_n| = 0.$$

It follows that $\lim_{n \rightarrow \infty} \frac{f(v_n)}{t_n} = y$ and then

$$Y_f \equiv \overline{\text{span}} f(X) = Y. \quad (14)$$

Since f is a coarse isometry and $\lim_{n \rightarrow \infty} \|f(v_n)\| = \infty$, we have $\lim_{n \rightarrow \infty} \|v_n\| = \infty$ and $\lim_{n \rightarrow \infty} \frac{\|f(v_n)\|}{\|v_n\|} = 1$. Note that, for each $y^* \in S_{Y_f^*}$ and $u \in X$,

$$\begin{aligned} |\langle (T^*U^* - P)y^*, f(u) \rangle| &= |\langle U^*y^*, Tf(u) \rangle - \langle Py^*, f(u) \rangle| \\ &\leq |\langle U^*y^*, Tf(u) - u \rangle| + |\langle U^*y^*, u \rangle - \langle Py^*, f(u) \rangle| \\ &\leq \|Tfu - u\| + |\langle y^*, Uu \rangle - \langle y^*, Ju \rangle| \\ &\leq \alpha \varepsilon_f(\|u\|) + \|Uu - Ju\| \\ &\leq \alpha \varepsilon_f(\|u\|) + \sum_{k=0}^{\infty} \frac{\alpha \varepsilon_f(2^{k+1}\|u\|)}{2^{k-1}}. \end{aligned}$$

Since $\int_1^{\infty} \frac{\varepsilon_f(s)}{s^2} ds < \infty$, the above inequality implies that

$$|\langle (T^*U^* - P)y^*, \frac{f(v_n)}{\|v_n\|} \rangle| \leq \frac{\alpha \varepsilon_f(\|v_n\|)}{\|v_n\|} + \sum_{k=0}^{\infty} \frac{\alpha \varepsilon_f(2^{k+1}\|v_n\|)}{2^{k-1}\|v_n\|} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Consequently,

$$\begin{aligned} |\langle (T^*U^* - P)y^*, y \rangle| &= \lim_{n \rightarrow \infty} |\langle (T^*U^* - P)y^*, \frac{f(v_n)}{t_n} \rangle| \\ &= \lim_{n \rightarrow \infty} \left(|\langle (T^*U^* - P)y^*, \frac{f(v_n)}{\|v_n\|} \rangle| \cdot \frac{\|v_n\|}{\|f(v_n)\|} \cdot \frac{\|f(v_n)\|}{|t_n|} \right) \\ &= 0. \end{aligned}$$

Then, $T^*U^*y^* = Py^*$ for each $y^* \in S_{Y_f^*}$. Thus, $P = T^*U^*$.

Step IV. We first prove that the linear isometric mapping $U : X \rightarrow Y_f$ is unique. If not, there are two linear isometries $U_1, U_2 : X \rightarrow Y_f$ with $U_1 \neq U_2$ such that $T^*U_1^* = P = T^*U_2^*$. We can find $x \in X$ with $U_1x \neq U_2x$. Then, there exists $y^* \in Y_f^*$ such that $\langle y^*, U_1x \rangle \neq \langle y^*, U_2x \rangle$. This implies $\langle U_1^*y^*, x \rangle \neq \langle U_2^*y^*, x \rangle$ and then $T^*U_1^*y^* \neq T^*U_2^*y^*$ since T^* is a linear isometry. This is contrary to $T^*U^* = P = T^*V^*$.

In addition, suppose that $P_1, P_2 : Y_f^* \rightarrow \mathcal{R}(T^*)$ are two w^* -to- w^* continuous linear projections with $\|P_1\| = \|P_2\| = 1$ and $P_1 \neq P_2$, then $T^*U_{P_1}^* = P_1 \neq P_2 = T^*U_{P_2}^*$. This entails that $U_{P_1}^* \neq U_{P_2}^*$ and then $U_{P_1} \neq U_{P_2}$. $\square$

In particular, if $f : X \rightarrow Y$ is an ε -isometry for some $\varepsilon \geq 0$, Theorem 2 also holds when we weaken the almost surjective condition (7) to $\overline{\text{co}}[f(X) \cup -f(X)] = Y$. For the proof one can refer to Zhou et al. (see [22], on page 756). Furthermore, if $\varepsilon_f(s) = 0$ for all $s \geq 0$, f is a standard isometric mapping. We obtain the following result without the surjectivity condition (7).

Corollary 1. [4] (Theorem 2.1) Suppose that $f : X \rightarrow Y$ is a standard isometry and that T is a Figiel operator of f .

(1) If $U : X \rightarrow Y_f$ is a linear isometry with $TU = \text{id}_X$, then $T^*U^* : Y_f^* \rightarrow \mathcal{R}(T^*)$ is a w^* -to- w^* continuous linear projection and $\|T^*U^*\| = 1$.

(2) If $P : Y_f^* \rightarrow \mathcal{R}(T^*)$ is a w^* -to- w^* continuous linear projection with $\|P\| = 1$, then there exists a unique linear isometry $U : X \rightarrow Y_f$ such that

$$TU = \text{id}_X \quad \text{and} \quad P = T^*U^*.$$

Furthermore, if $P_1, P_2 : Y_f^* \rightarrow \mathcal{R}(T^*)$ are two w^* -to- w^* continuous linear projections with $\|P_1\| = \|P_2\| = 1$ and $P_1 \neq P_2$, then $U_{P_1} \neq U_{P_2}$.

Proof. (1) It can be proved by Proposition 1.

(2) By the proof of Theorem 2, we just need to verify $P = T^*U^*$ without the almost surjective assumption condition (7). For each $y^* \in S_{Y_f^*}$ and $v \in X$,

$$\begin{aligned} |\langle (T^*U^* - P)y^*, f(v) \rangle| &= |\langle U^*y^*, Tf(v) \rangle - \langle Py^*, f(v) \rangle| \\ &\leq |\langle U^*y^*, Tf(v) - v \rangle| + |\langle U^*y^*, v \rangle - \langle Py^*, f(v) \rangle| \\ &\leq \|Tf(v) - v\| + |\langle y^*, Uv \rangle - \langle y^*, Jv \rangle| \\ &\leq \alpha \varepsilon_f(\|v\|) + \|Uv - Jv\| \\ &\leq \alpha \varepsilon_f(\|v\|) + \sum_{k=0}^{\infty} \frac{\alpha \varepsilon_f(2^{k+1}\|v\|)}{2^{k-1}} \\ &= 0. \end{aligned}$$

This implies $|\langle (T^*U^* - P)y^*, y \rangle| = 0$ for all $y \in Y_f$. Thus, $P = T^*U^*$. $\square$

This section concludes with a stability result for coarse isometries using Theorem 2.

Theorem 3. Let $f : X \rightarrow Y$ be a standard coarse isometry with (7) and let T be a coarse left-inverse operator of f . If

$$\int_1^\infty \frac{\varepsilon_f(s)}{s^2} ds < \infty,$$

then there is a surjective linear isometry $U : X \rightarrow Y$ such that

$$\|f(v) - Uv\| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

Proof. Firstly, we prove that T^* is surjective, i.e., $\mathcal{R}(T^*) = Y_f^*$. Otherwise, $\mathcal{R}(T^*)$ is a proper w^* -closed subspace of Y_f^* by Lemma 1. We choose $\psi \in S_{Y_f^*} \setminus \mathcal{R}(T^*)$ and $y \in S_{Y_f}$ so that

$$\langle \phi, y \rangle = 0 \quad \text{for all } \phi \in \mathcal{R}(T^*) \quad \text{and} \quad \langle \psi, y \rangle > 0.$$

Due to (7), there exist two sequences $\{v_n\} \subset X$ and $\{t_n\} \subset \mathbb{R}$ with $\lim_{n \rightarrow \infty} |t_n| = \infty$ so that

$$\lim_{n \rightarrow \infty} \frac{\|t_n y - f(v_n)\|}{|t_n|} = 0.$$

This entails that $\lim_{n \rightarrow \infty} \frac{\|f(v_n)\|}{|t_n|} = 1$. For the sequence $\{v_n\}$ above, let $z_n^* \in S_{X^*}$ with $\langle z_n^*, v_n \rangle = \|v_n\|$ and let $\phi_n = T^* z_n^*$. Then,

$$\begin{aligned} \alpha \varepsilon_f(\|v_n\|) &\geq \|Tf(v_n) - v_n\| \geq \langle z_n^*, v_n \rangle - \langle \phi_n, f(v_n) \rangle \\ &\geq \|v_n\| - \langle \phi_n, f(v_n) - t_n y \rangle - \langle \phi_n, t_n y \rangle \\ &\geq \|v_n\| - \|f(v_n) - t_n y\|. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{\|f(v_n)\|}{\|v_n\|} = 1$, we have $\lim_{n \rightarrow \infty} \frac{\|v_n\|}{|t_n|} = 1$. Thus,

$$\frac{\alpha \varepsilon_f(\|v_n\|)}{\|v_n\|} \geq 1 - \frac{\|f(v_n) - t_n y\|}{\|v_n\|}.$$

Putting $n \rightarrow \infty$ in the inequality above,

$$0 \geq 1 - \lim_{n \rightarrow \infty} \frac{\|f(v_n) - t_n y\|}{\|v_n\|} = 1 - \lim_{n \rightarrow \infty} \frac{\|f(v_n) - t_n y\|}{|t_n|} = 1.$$

This is a contradiction. Then, $Y_f^* = \mathcal{R}(T^*)$.

Let $P = id_{Y_f^*}$. Note that $P : Y_f^* \rightarrow Y_f^*$ is a w^* -to- w^* continuous linear projection with $\|P\| = 1$. By Theorem 2 and (14), there is a linear isometry $U : X \rightarrow Y_f = Y$ such that

$$TU = id_X \quad \text{and} \quad id_{Y^*} = P = T^* U^*.$$

This implies $U = T^{-1}$. Consequently, $U : X \rightarrow Y$ is surjective and

$$\|f(v) - Uv\| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

□

3. Right Inverse of Spaces of Continuous Functions

In this section, K is a compact Hausdorff space and $C(K)$ represents the set of all continuous functions on K with real values, equipped with the supremum norm. For each $s \in K$, let δ_s be the point mass at t ; then, $\langle \delta_s, \psi \rangle = \psi(s)$ for all $\psi \in C(K)$. We use $\text{ext}(B_X)$ to denote the sets of all extreme points of B_X . Note that $\text{ext}(B_{C(K)^*}) = \{\pm \delta_s : s \in K\}$. Given a topological subspace $\tilde{K} \subseteq K$ and $\psi \in C(K)$, the notion $\psi|_{\tilde{K}}$ denotes the restriction of ψ to $\tilde{K}$.

First, we give a weaker condition than the existence of a continuous linear projection with respect to the weak star topology.

Proposition 2. Suppose that $f : X \rightarrow C(K)$ is a standard coarse isometry and that T is a coarse left-inverse operator of f . If there is a w^* -to- w^* continuous linear projection $P : Y_f^* \rightarrow \mathcal{R}(T^*)$, then $T^*(B_{X^*}) \cap (\text{ext}(B_{C(K)^*})|_{Y_f}) \neq \emptyset$.

Proof. Since $P : Y_f^* \rightarrow \mathcal{R}(T^*)$ is a w^* -to- w^* continuous linear projection, there exists a w^* -closed subspace M of Y_f^* such that $T^*(X^*) \oplus M = Y_f^*$. If $T^*(B_{X^*}) \cap (\text{ext}(B_{C(K)^*})|_{Y_f}) = \emptyset$, it follows from Lemma 1 that $\{\pm \delta_s|_{Y_f}\} \subset M$. This entails that $M = Y_f^*$ by the Krein–Milman theorem. Then, $T^*(X^*) = \{0\}$; this is a contradiction. $\square$

Theorem 4. Let $f : X \rightarrow C(K)$ be a standard coarse isometry and let T be a coarse left-inverse operator of f . If $T^*(B_{X^*}) \cap (\text{ext}(B_{C(K)^*})|_{Y_f}) \neq \emptyset$ and

$$\int_1^\infty \frac{\varepsilon_f(s)}{s^2} ds < \infty,$$

then there is a non-empty closed subset $\tilde{K} \subset K$ and a linear isometry $g : X \rightarrow C(\tilde{K})$ so that

$$\|f(v)|_{\tilde{K}} - g(v)\| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

Proof. Since T is a coarse left-inverse operator,

$$\sup_{x^* \in B_{X^*}} |\langle x^*, v \rangle - \langle T^* x^*, f(v) \rangle| = \sup_{x^* \in B_{X^*}} |\langle x^*, Tf(v) - v \rangle| = \|Tf(v) - v\| \leq \alpha \varepsilon_f(\|v\|)$$

for all $v \in X$. Note that $T^*(B_{X^*}) \cap (\text{ext}(B_{C(K)^*})|_{Y_f}) \neq \emptyset$; there exist $z^* \in B_{X^*}$, $s \in K$ and $\lambda \in \{\pm 1\}$ such that $T^*(\lambda z^*) = \delta_s|_{Y_f}$. It follows that

$$|\langle \lambda z^*, v \rangle - \langle \delta_s, f(v) \rangle| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X. \quad (15)$$

Let

$$\tilde{K} = \{t \in K : \exists z^* \in B_{X^*} \text{ and } \lambda \in \{\pm 1\} \text{ so that } |\langle \lambda z^*, v \rangle - \langle \delta_t, f(v) \rangle| \leq \alpha \varepsilon_f(\|v\|) \text{ for } \forall v \in X\}. \quad (16)$$

From the discussion above, we have $\tilde{K} \neq \emptyset$. Next, we prove that $\tilde{K}$ is closed. Indeed, for each net $\{t_\alpha\}_{\alpha \in \Lambda} \subset \tilde{K}$ with $t_\alpha \rightarrow t_0 \in K$, there exist a net $\{z_\alpha^*\}_{\alpha \in \Lambda} \subset B_{X^*}$ and $\lambda_\alpha \in \{\pm 1\}$ such that

$$|\langle \lambda_\alpha z_\alpha^*, v \rangle - \langle \delta_{t_\alpha}, f(v) \rangle| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

Since B_{X^*} is w^* -compact, there exist a subnet $\{\lambda_\beta z_\beta^*\}_{\beta \in \Lambda'}$ of $\{\lambda_\alpha z_\alpha^*\}_{\alpha \in \Lambda}$ and $z_0^* \in B_{X^*}$ such that $\lambda_\beta z_\beta^* \xrightarrow{w^*} z_0^*$. Therefore, for each $v \in X$,

$$|\langle z_0^*, v \rangle - \langle \delta_{t_0}, f(v) \rangle| \leq \alpha \varepsilon_f(\|v\|).$$

Note that z_0^* and t_0 are independent of x . This implies $t_0 \in \tilde{K}$. Hence, $\tilde{K} \subset K$ is closed.

Given $s \in \tilde{K}$, there exist $z^* \in B_{X^*}$ and $\lambda \in \{\pm 1\}$ such that, for each $w_1, w_2 \in X$,

$$\begin{aligned} & |\langle \delta_s, f(w_1 + w_2) - (f(w_1) + f(w_2)) \rangle| \\ & \leq |\langle \delta_s, f(w_1 + w_2) \rangle - \langle \lambda z^*, w_1 + w_2 \rangle| + |\langle \delta_s, f(w_1) \rangle - \langle \lambda z^*, w_1 \rangle| + |\langle \delta_s, f(w_2) \rangle - \langle \lambda z^*, w_2 \rangle| \\ & \leq \alpha \varepsilon_f(\|w_1 + w_2\|) + \alpha \varepsilon_f(\|w_1\|) + \alpha \varepsilon_f(\|w_2\|) \\ & \leq \alpha \max\{\varepsilon_f(2\|w_1\|), \varepsilon_f(2\|w_2\|)\} + \alpha \varepsilon_f(\|w_1\|) + \alpha \varepsilon_f(\|w_2\|) \\ & \leq 2\alpha(\varepsilon_f(2\|w_1\|) + \varepsilon_f(2\|w_2\|)). \end{aligned}$$

Thus,

$$\begin{aligned} & \|f(w_1 + w_2)|_{\tilde{K}} - (f(w_1)|_{\tilde{K}} + f(w_2)|_{\tilde{K}})\| \\ &= \sup_{s \in \tilde{K}} |\langle \delta_s, f(w_1 + w_2) - (f(w_1) + f(w_2)) \rangle| \\ &\leq 2\alpha(\varepsilon_f(2\|w_1\|) + \varepsilon_f(2\|w_2\|)) \quad \text{for all } w_1, w_2 \in X. \end{aligned}$$

According to the proof of Step II in Theorem 2 and Lemma 2, we can conclude that the function $g : X \rightarrow C(\tilde{K})$, defined as

$$g(v) = \lim_{n \rightarrow \infty} \frac{f(2^n v)|_{\tilde{K}}}{2^n}, \quad (17)$$

is a continuous linear operator with $\|g\| \leq 1$.

Now, we shall prove that $g : X \rightarrow C(\tilde{K})$ is an isometric mapping. Let $v \in X$; there exists $z^* \in S_{X^*}$ such that $\langle z^*, v \rangle = \|v\|$. Due to (15), there exist $s \in \tilde{K}$ and $\lambda \in \{\pm 1\}$ such that

$$|\langle \lambda z^*, v \rangle - \langle \delta_s, f(v) \rangle| \leq \alpha \varepsilon_f(\|v\|).$$

This implies that

$$|\langle z^*, v \rangle - \langle \lambda \delta_s, \frac{f(2^n v)}{2^n} \rangle| \leq \frac{\alpha \varepsilon_f(2^n \|v\|)}{2^n}.$$

Then, $|\langle z^*, v \rangle - \langle \lambda \delta_s, g(v) \rangle| = 0$. This implies that $\|g(v)\| \geq |\langle \lambda \delta_s, g(v) \rangle| = |\langle z^*, v \rangle| = \|v\|$. Therefore, $g : X \rightarrow C(\tilde{K})$ is a linear isometry.

Finally, we prove that

$$\|f(v)|_{\tilde{K}} - g(v)\| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

According to (16), for each $s \in \tilde{K}$, there exist $z^* \in B_{X^*}$ and $\lambda \in \{\pm 1\}$ such that

$$|\langle \lambda z^*, v \rangle - \langle \delta_s, f(v) \rangle| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X. \quad (18)$$

Then,

$$\lim_{n \rightarrow \infty} |\langle \lambda z^*, v \rangle - \langle \delta_s, \frac{f(2^n v)}{2^n} \rangle| \leq \lim_{n \rightarrow \infty} \alpha \frac{\varepsilon_f(2^n \|v\|)}{2^n} = 0 \quad \text{for all } v \in X.$$

This entails that

$$\langle \lambda z^*, v \rangle = \lim_{n \rightarrow \infty} \langle \delta_s, \frac{f(2^n v)}{2^n} \rangle = \langle \delta_s, g(v) \rangle. \quad (19)$$

Combining (18) and (19), we have

$$\|f(v)|_{\tilde{K}} - g(v)\| = \sup_{s \in \tilde{K}} |\langle \delta_s, f(v) - g(v) \rangle| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

□

If $f : X \rightarrow C(K)$ is an isometry, T is a Figiel operator of f . By Theorem 2.4 in reference [21], for each $z^* \in B_{X^*}$, there exist $s \in K$ and $\lambda \in \{\pm 1\}$ satisfying $\langle \lambda z^*, v \rangle = \langle \delta_s, f(v) \rangle$ for all $v \in X$. This entails that all conditions of Theorem 4 are satisfied naturally when f is a standard isometry. At the same time, the linear isometry $g(v) = \lim_{n \rightarrow \infty} \frac{f(2^n v)|_{\tilde{K}}}{2^n}$ is actually the restriction of f to $\tilde{K}$, that is, $g(v) = f(v)|_{\tilde{K}}$. Then, we obtain the following result by Theorem 4.

Corollary 2. [29] (Theorem 1.1) Let $f : X \rightarrow C(K)$ be a standard isometry. Then, there is a non-empty closed subset $\tilde{K} \subseteq K$ such that $f|_{\tilde{K}} : X \rightarrow C(\tilde{K})$ is a linear isometry.

In the following, we recall the definition of simultaneous extension operators.

Definition 3. Let $E : C(\tilde{K}) \rightarrow C(K)$ be a continuous linear operator, where $\tilde{K} \subseteq K$ is a non-empty closed subset. We say that E is a simultaneous extension operator if, for every $g \in C(\tilde{K})$, $E(g)$ is an extension of g from $\tilde{K}$ to K .

Lemma 3. [24] (Theorem 1.21) Let $K, \tilde{K}$ be as above and assume that $\tilde{K}$ is metrizable. Then, there exists a simultaneous extension operator $E : C(\tilde{K}) \rightarrow C(K)$ with $\|E\| = 1$ and $E(1) = 1$.

Note that the operator E in Lemma 3 is a linear isometry if $\tilde{K}$ is metrizable. But Lemma 3 is not necessarily true when $\tilde{K}$ is not metrizable, see [34,35]. Then, the following result is presented.

Theorem 5. Let K be a compact metric space. Suppose that $X, f, T, \tilde{K}, g$ are as in Theorem 4, and that E corresponds to Lemma 3. Then, $E \circ g : X \rightarrow C(K)$ is a linear isometry and

$$\|E(f(v)|_{\tilde{K}}) - E(g(v))\| = \|f(v)|_{\tilde{K}} - g(v)\| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

From Theorem 5 and Corollary 2, the following result, established by Villa, can be derived.

Corollary 3. [30] (Theorem 2) Assume that K is a compact metric space and that $f : X \rightarrow C(K)$ is a standard isometry. Then, there is a non-empty closed subset $\tilde{K} \subseteq K$ such that $E \circ (f|_{\tilde{K}}) : X \rightarrow C(K)$ is a linear isometry.

4. Right Inverse of $L_p(\Omega, \Sigma, \mu)$ ($1 < p < \infty$) Spaces and Finite Dimensional Spaces

In this section, we will show that every coarse left-inverse operator has a linearly isometric right inverse without other additional conditions when target space Y is the $L_p(\Omega, \Sigma, \mu)$ ($1 < p < \infty$) space or $\dim X = \dim Y < \infty$, where (Ω, Σ, μ) is a measure space.

To begin with, we give the following lemma of $L_p(\Omega, \Sigma, \mu)$ spaces which was proved by Sun and Zhang (see [27], on pages 10–11).

Lemma 4. [27] (Theorem 2.12) Let (Ω, Σ, μ) be a measure space. Assume that $f : X \rightarrow L_p(\Omega, \Sigma, \mu)$ ($1 < p < \infty$) is a standard coarse isometry and that for each $z^* \in S_{X^*}$ there exists $\varphi \in S_{Y^*}$ satisfying

$$|\langle z^*, u \rangle - \langle \varphi, f(u) \rangle| = o(\|u\|) \quad \text{as } \|u\| \rightarrow \infty. \quad (20)$$

Then, there exists a linear isometry $U : X \rightarrow L_p(\Omega, \Sigma, \mu)$ such that

$$\|f(u) - Uu\| = o(\|u\|) \quad \text{as } \|u\| \rightarrow \infty.$$

By using Lemma 4 above, the following result can be shown.

Theorem 6. Let (Ω, Σ, μ) be a measure space. Assume that $f : X \rightarrow L_p(\Omega, \Sigma, \mu)$ ($1 < p < \infty$) is a standard coarse isometry and that T is a coarse left-inverse operator of f . Then, there exists a linear isometry $U : X \rightarrow L_p(\Omega, \Sigma, \mu)$ such that

$$TU = id_X.$$

Proof. Since T is a coarse left-inverse operator of f ,

$$\sup_{z^* \in S_{X^*}} |\langle z^*, Tf(v) - v \rangle| = \|Tf(v) - v\| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

Then, for each $z^* \in S_{X^*}$,

$$|\langle T^*z^*, f(v) \rangle - \langle z^*, v \rangle| = |\langle z^*, Tf(v) - v \rangle| \leq \alpha \varepsilon_f(\|v\|) \quad \text{for all } v \in X.$$

Put $\varphi = T^*z^*$. Then, $\|\varphi\| = \|z^*\| = 1$ by Lemma 1 and then (20) holds. By Lemma 4, we obtain a linear isometry $U : X \rightarrow L_p(\Omega, \Sigma, \mu)$ with

$$Uv = \lim_{r \rightarrow \infty} \frac{f(rv)}{r} \quad \text{for all } v \in X.$$

Then,

$$\begin{aligned} \|TUv - v\| &= \lim_{r \rightarrow \infty} \left\| T \frac{f(rv)}{r} - v \right\| \\ &= \lim_{r \rightarrow \infty} \frac{\|Tf(rv) - rv\|}{|r|} \\ &\leq \lim_{r \rightarrow \infty} \frac{\alpha \varepsilon_f(\|rv\|)}{|r|} = 0 \quad \text{for all } v \in X. \end{aligned}$$

□

For the case of finite dimensional spaces, we first recall some definitions of free ultrafilters.

Definition 4. Let Γ be a non-empty set and let $\mathcal{U}$ be a family of subsets of Γ .

(i) We say that $\mathcal{U}$ is a free ultrafilter on Γ if the following conditions are satisfied. (1) $\emptyset \notin \mathcal{U}$; (2) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$; (3) if $B \subset A \subset \Gamma$ and $B \in \mathcal{U}$, then $A \in \mathcal{U}$; (4) $\bigcap_{A \in \mathcal{U}} A = \emptyset$; (5) if $A \subset \Gamma$, then either $A \in \mathcal{U}$ or $\Gamma \setminus A \in \mathcal{U}$.

(ii) Given a topological space $\mathcal{D}$, we say a mapping $h : \Gamma \rightarrow \mathcal{D}$ is $\mathcal{U}$ -convergent to some $w \in \mathcal{D}$ and denote

$$\lim_{\mathcal{U}} h = w,$$

if $h^{-1}(W) \in \mathcal{U}$ for every neighborhood W of w .

The following property is fundamental for free ultrafilters.

Proposition 3. Let $\Gamma, \mathcal{U}$ and $\mathcal{D}$ be as in Definition 4. If the topological space $\mathcal{D}$ is compact, then every mapping $h : \Gamma \rightarrow \mathcal{D}$ is $\mathcal{U}$ -convergent.

For more details of free ultrafilters we refer to [36]. In what follows, we will be interested in $\Gamma = \mathbb{N}$.

Theorem 7. Let $f : X \rightarrow Y$ be a standard coarse isometry with $\dim X = \dim Y = m < \infty$ and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Suppose that $T : Y_f \equiv \overline{\text{span}} f(X) \rightarrow X$ is a linear operator satisfying $\|T\| = 1$ and

$$\lim_{\mathcal{U}} \left\| T \frac{f(ne_k)}{n} - e_k \right\| = 0 \quad \text{for all } k \in \{1, 2, \dots, m\}, \quad (21)$$

where $\{e_k\}_{k=1}^m$ is a basis. Then, there is a surjective linear isometry $U : X \rightarrow Y$ such that

$$TU = id_X.$$

Proof. Given $v \in X$, since f is a standard coarse isometry, we obtain

$$\left\| \frac{f(nv)}{n} \right\| \leq \frac{\varepsilon_f(|n| \|v\|)}{|n|} + \|v\| \rightarrow \|v\| \quad \text{as } n \rightarrow \infty.$$

This implies $\{\frac{f(nv)}{n}\}$ is bounded and then $\{\frac{f(nv)}{n}\}$ is relatively compact. Thus, the mapping $U : X \rightarrow Y$ defined by

$$Uv = \lim_{\mathcal{U}} \frac{f(nv)}{n} \quad (22)$$

is an isometry. Indeed, for each $v, w \in X$,

$$\|Uv - Uw\| - \|v - w\| = \lim_{\mathcal{U}} \frac{\|f(nv) - f(nw)\| - \|nv - nw\|}{n} \leq \lim_{\mathcal{U}} \frac{\varepsilon_f(n\|v - w\|)}{n} = 0.$$

Since X is complete, $U(X)$ is a closed subspace of Y . According to the domain invariance theorem, $U(X)$ is also an open subspace of Y and then $U(X) = Y$. This means that U is a surjective standard isometry and then U is linear. Combining (21) and (22), we have

$$T U e_k = e_k \quad \text{for all } k \in \{1, 2, \dots, m\}.$$

Then, $TU = id_X$. The proof is completed. $\square$

Remark 1. Note that (21) is weaker than the condition that T is a coarse left-inverse operator of f .

Corollary 4. Let $f : X \rightarrow Y$ be a standard coarse isometry with $\dim X = \dim Y = m < \infty$ and let T be a coarse left-inverse operator of f . Then, there is a surjective linear isometry $U : X \rightarrow Y$ such that $TU = id_X$.

5. Conclusions

Lindenstrauss and Szankowski [23] noted that a coarse isometry between infinite-dimensional uniformly convex spaces cannot result in a linear isometry without additional conditions. Therefore, when studying coarse isometric embeddings and linear isometric embeddings, an additional assumption is usually made. This paper explores the relationship between coarse isometries and linear isometries of various Banach spaces under the assumption of coarse left-inverse operators. Our conclusions generalize some well-known isometric results, for example, see Corollary 1 ([4] Theorem 2.1), Corollary 2 ([29] Theorem 1.1) and Corollary 3 ([30] Theorem 2).

As a future research direction, we suggest the following:

1. What is the class of coarse isometries that has a coarse left-inverse operator?
2. If $f : X \rightarrow Y$ is a coarse isometry, is there another condition that guarantees the existence of a linear isometry from X to Y ?
3. If $f : X \rightarrow Y$ is an ε -isometry, does there exist an isometry from X to Y ?
4. We will extend our results to the case where Y is a Bochner space.

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