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Article

Four Families of Summation Formulas for ${}_4F_3(1)$ with Application

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Abstract: A collection of functions organized according to their indexing based on non-negative integers is grouped by the common factor of fixed integer N. This grouping results in a summation of N series, each consisting of functions partitioned according to this modulo N rule. Notably, when N is equal to two, the functions in the series are divided into two subseries: one containing even-indexed functions and the other containing odd-indexed functions. This partitioning technique is widely utilized in the mathematical literature and finds applications in various contexts, such as in the theory of hypergeometric series. In this paper, we employ this partitioning technique to establish four distinct families of summation formulas for ${}_4F_3(1)$ hypergeometric series. Subsequently, we leverage these summation formulas to introduce eight categories of integral formulas. These integrals feature compositions of Beta function-type integrands and ${}_3F_2(x)$ hypergeometric functions. Additionally, we highlight that our primary summation formulas can be used to derive some well-known summation results.

Keywords: gamma function; beta function; generalized hypergeometric series; Gauss's summation formula ${}_2F_1(1)$; Kummmer's summation formula ${}_2F_1(-1)$; integral formulas

MSC: 35B40; 35L70; 33C20; 33C60; 33C70; 33C65

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1. Introduction and Preliminaries

The generalized hypergeometric function (or series) ${}_\mu F_\nu$ ($\mu,\, \nu \in \mathbb{N}_0$), extending the Gaussian hypergeometric series ${}_2F_1$, is formally defined as (refer to, for instance, [1–5])

$$\begin{aligned} {}_\mu F_\nu\left[\begin{matrix}\xi_1,\,\ldots,\,\xi_\mu\,;\\ \tau_1,\,\ldots,\,\tau_\nu\,;\end{matrix}\, z\right] &= \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{\mu}(\xi_j)_k}{\prod_{j=1}^{\nu}(\tau_j)_k}\frac{z^k}{k!} \\ &= {}_\mu F_\nu(\xi_1,\,\ldots,\,\xi_\mu;\,\tau_1,\,\ldots,\,\tau_\nu;\,z), \end{aligned} \tag{1}$$

where $(\xi)_\eta$ denotes the Pochhammer symbol for complex ξ and η, defined in terms of Gamma function Γ (refer to, for instance, [5] (p. 2 and p. 5)) by

$$\begin{aligned} (\xi)_\eta &= \frac{\Gamma(\xi+\eta)}{\Gamma(\xi)} \quad (\xi+\eta \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0},\, \eta\in\mathbb{C}\setminus\{0\}) \\ &= \begin{cases} 1 & (\eta = 0,\, \xi\in\mathbb{C}), \\ \xi(\xi+1)\cdots(\xi+k-1) & (\eta = k\in\mathbb{N},\, \xi\in\mathbb{C}), \end{cases} \end{aligned} \tag{2}$$

with the understanding that $(0)_0 = 1$. In this context, as well as in other situations, an empty product is interpreted as being equal to one. It is assumed that variable z, numerator parameters $\xi_1, \ldots, \xi_\mu$, and denominator parameters $\tau_1, \ldots, \tau_\nu$ adopt complex values with the condition that

$$(\tau_j \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0};\ j = 1, \ldots, \nu). \tag{3}$$

Furthermore, in this context and elsewhere, we let $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ represent the sets of complex numbers, integers, and positive integers, respectively. Additionally, we define $\mathbb{N}_0$ as the union of $\mathbb{N}$ and $\{0\}$, and denote $\mathbb{Z}_{\leqslant \eta}$ as the set of integers less than or equal to a specified integer $\eta \in \mathbb{Z}$.

If any numerator parameter is a non-positive integer, the ${}_\mu F_\nu$ series terminates. Assuming neither numerator nor denominator parameters are non-positive integers, the behavior of the ${}_\mu F_\nu$ series (1) is characterized as follows:

(i) It diverges for all $z \in \mathbb{C} \setminus \{0\}$, if $\mu > \nu + 1$.

(ii) It converges for all $z \in \mathbb{C}$, if $\mu \leqslant \nu$.

(iii) It converges for $|z| < 1$ and diverges for $|z| > 1$ if $\mu = \nu + 1$.

(iv) It converges absolutely for $|z| = 1$ if $\mu = \nu + 1$ and $\Re(\varpi) > 0$.

(v) It converges conditionally for $|z| = 1$ $(z \neq 1)$ if $\mu = \nu + 1$ and $-1 < \Re(\varpi) \leqslant 0$.

(vi) It diverges for $|z| = 1$ if $\mu = \nu + 1$ and $\Re(\varpi) \leqslant -1$,

where

$$\varpi := \sum_{j=1}^{\nu} \beta_j - \sum_{j=1}^{\mu} \alpha_j. \tag{4}$$

The renowned Gauss summation formula [6], expressed as

$$\begin{gathered} {}_2F_1(\kappa, \lambda\,;\, \mu\,;\, 1) = \frac{\Gamma(\mu)\,\Gamma(\mu - \kappa - \lambda)}{\Gamma(\mu - \kappa)\,\Gamma(\mu - \lambda)} \\ (\Re(\mu - \kappa - \lambda) > 0,\ \mu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}) \end{gathered} \tag{5}$$

has played a pivotal and pioneering role, notably in the realms of hypergeometric and generalized hypergeometric functions, as well as in related special functions. Since the emergence of (5), an extensive array of summation formulas for ${}_\mu F_\nu(z)$ with specific arguments z have been introduced through diverse and inventive techniques. For recent contributions including those by the third author, please refer to [7,8].

Among various methodologies, the approach of partitioning the set of nonnegative integers according to modulo N is commonly applied to series featuring functions Ψ_k (where $k \in \mathbb{N}_0$), as depicted by

$$\sum_{k=0}^{\infty} \Psi_k = \sum_{r=0}^{N-1} \sum_{k=0}^{\infty} \Psi_{kN+r}. \tag{6}$$

This technique finds widespread usage (see, for instance, [9,10]), particularly dividing the series into even and odd terms yields:

$$\sum_{k=0}^{\infty} \Psi_k = \sum_{k=0}^{\infty} \Psi_{2k} + \sum_{k=0}^{\infty} \Psi_{2k+1}. \tag{7}$$

Equations (6) and (7) are employed to derive certain identities involving generalized hypergeometric series and their extensions (refer to, for instance, [11–20]). Exton [13] considered the following two combinations:

$${}_{\nu+1}F_\nu \left[\begin{matrix} \xi_1, \xi_2, \ldots, \xi_{\nu+1}; \\ \tau_1, \tau_2, \ldots, \tau_\nu; \end{matrix} \, 1 \right] \pm {}_{\nu+1}F_\nu \left[\begin{matrix} \xi_1, \xi_2, \ldots, \xi_{\nu+1}; \\ \tau_1, \tau_2, \ldots, \tau_\nu; \end{matrix} \, -1 \right], \tag{8}$$

whose respective resulting series are

$$2\sum_{n=0}^{\infty}\frac{(\xi_1)_{2n}\cdots(\xi_{\nu+1})_{2n}}{(1)_{2n}(\tau_1)_{2n}\cdots(\tau_\nu)_{2n}} \tag{9}$$

and

$$2\,\frac{\prod\limits_{j=1}^{\nu+1}\xi_j}{\prod\limits_{j=1}^{\nu}\tau_j}\sum_{n=0}^{\infty}\frac{(\xi_1+1)_{2n}\cdots(\xi_{\nu+1}+1)_{2n}}{(2)_{2n}(\tau_1+1)_{2n}\cdots(\tau_\nu+1)_{2n}}. \tag{10}$$

Using the readily derived duplication formula,

$$(\eta)_{2k}=2^{2k}\left(\frac{\eta}{2}\right)_k\left(\frac{\eta+1}{2}\right)_k \quad (\eta\in\mathbb{C},\, k\in\mathbb{N}_0), \tag{11}$$

which is deduced by regrouping alternate factors in the expression $(\eta)_{2k}=\eta(\eta+1)\cdots(\eta+2k-1)$ and extracting two from each factor on the right-hand side, as in (9) and (10), respectively, the following can be obtained:

$$\begin{aligned}&{}_{\nu+1}F_\nu\left[\begin{matrix}\xi_1,\xi_2,\ldots,\xi_{\nu+1};\\ \tau_1,\tau_2,\ldots,\tau_\nu;\end{matrix}\;1\right]+{}_{\nu+1}F_\nu\left[\begin{matrix}\xi_1,\xi_2,\ldots,\xi_{\nu+1};\\ \tau_1,\tau_2,\ldots,\tau_\nu;\end{matrix}\;-1\right]\\ &=2\,{}_{2\nu+2}F_{2\nu+1}\left[\begin{matrix}\frac{\xi_1}{2},\frac{\xi_1+1}{2},\ldots,\frac{\xi_{\nu+1}}{2},\frac{\xi_{\nu+1}+1}{2};\\ \frac{1}{2},\frac{\tau_1}{2},\frac{\tau_1+1}{2},\ldots,\frac{\tau_\nu}{2},\frac{\tau_\nu+1}{2};\end{matrix}\;1\right]\end{aligned} \tag{12}$$

and

$$\begin{aligned}&{}_{\nu+1}F_\nu\left[\begin{matrix}\xi_1,\xi_2,\ldots,\xi_{\nu+1};\\ \tau_1,\tau_2,\ldots,\tau_\nu;\end{matrix}\;1\right]-{}_{\nu+1}F_\nu\left[\begin{matrix}\xi_1,\xi_2,\ldots,\xi_{\nu+1};\\ \tau_1,\tau_2,\ldots,\tau_\nu;\end{matrix}\;-1\right]\\ &=2\,\frac{\prod\limits_{j=1}^{\nu+1}\xi_j}{\prod\limits_{j=1}^{\nu}\tau_j}\,{}_{2\nu+2}F_{2\nu+1}\left[\begin{matrix}\frac{\xi_1+1}{2},\frac{\xi_1}{2}+1,\ldots,\frac{\xi_{\nu+1}+1}{2},\frac{\xi_{\nu+1}}{2}+1;\\ \frac{3}{2},\frac{\tau_1+1}{2},\frac{\tau_1}{2}+1,\ldots,\frac{\tau_\nu+1}{2},\frac{\tau_\nu}{2}+1;\end{matrix}\;1\right].\end{aligned} \tag{13}$$

If one is familiar with the summation formulas for ${}_{\nu+1}F_\nu(1)$ and ${}_{\nu+1}F_\nu(-1)$ as presented in Equation (8), it becomes possible to derive summation formulas for ${}_{2\nu+2}F_{2\nu+1}(1)$ in Equations (12) and (13). Conversely, the reverse process is also viable. In this context, we bring attention to two specific summation formulas, as referenced in [13] (Equations (2.2) and (2.3))):

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}\frac{\xi}{2},\frac{\xi}{2}+\frac{1}{2},\frac{\tau}{2},\frac{\tau}{2}+\frac{1}{2};\\ \frac{1}{2},\frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2},1+\frac{\xi}{2}-\frac{\tau}{2};\end{matrix}\;1\right]\\ &=\frac{1}{2}\left(\frac{\Gamma(1+\xi-\tau)\Gamma(1-2\tau)}{\Gamma(1-\tau)\Gamma(1+\xi-2\tau)}+\frac{\Gamma(1+\xi-\tau)\Gamma(1+\frac{\xi}{2})}{\Gamma(1+\xi)\Gamma(1+\frac{\xi}{2}-\tau)}\right)\end{aligned} \tag{14}$$

and

$$\begin{aligned}&{}_4F_3\left[\begin{matrix}\frac{\xi}{2}+\frac{1}{2},\frac{\xi}{2}+1,\frac{\tau}{2}+\frac{1}{2},\frac{\tau}{2}+1;\\ \frac{3}{2},1+\frac{\xi}{2}-\frac{\tau}{2},\frac{3}{2}+\frac{\xi}{2}-\frac{\tau}{2};\end{matrix}\;1\right]\\ &=\frac{1+\xi-\tau}{2\xi\tau}\left(\frac{\Gamma(1+\xi-\tau)\Gamma(1-2\tau)}{\Gamma(1-\tau)\Gamma(1+\xi-2\tau)}-\frac{\Gamma(1+\xi-\tau)\Gamma(1+\frac{\xi}{2})}{\Gamma(1+\xi)\Gamma(1+\frac{\xi}{2}-\tau)}\right),\end{aligned} \tag{15}$$

provided that, for each series, the denominator parameters are in the set of complex numbers excluding non-positive integers, denoted as $\mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, and the real part of τ is less than $\frac{1}{2}$.

Choi and Rathie [12] initially derived twenty-two summation formulas for ${}_4F_3(1)$, with two directly corresponding to Equations (14) and (15), while the remaining twenty (ten each) closely linked to Equations (14) and (15). This paper endeavors to extend the scope of the summation formulas presented in Equations (14) and (15). Additionally, we aim to leverage these extended results to establish a series of integral formulas. These formulas involve integrands structured as products of functions falling into two distinct forms:

$$u^{\rho}(1-u)^{\sigma} \quad \text{and} \quad {}_3F_2(u). \tag{16}$$

For our needs, we also remember the subsequent families of summation formulas (refer to [21] (Theorems 3 and 4)):

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} \xi, \tau; \\ 1+\xi-\tau+j; \end{matrix} \; -1\right] \\ &= \frac{2^{-\xi}\Gamma(\frac{1}{2})\Gamma(1+\xi-\tau+j)\Gamma(\tau-j)}{\Gamma(\tau)\Gamma(\frac{\xi}{2}-\tau+\frac{j}{2}+\frac{1}{2})\Gamma(\frac{\xi}{2}-\tau+\frac{j}{2}+1)} \\ &\times \sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma(\frac{\xi}{2}-\tau+\frac{j}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{\xi}{2}-\frac{j}{2}+\frac{r}{2}+\frac{1}{2})} \end{aligned} \tag{17}$$

$$\left(j \in \mathbb{N}_0,\ \xi-\tau \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1-j},\ \frac{1+2}{2} \leqslant \Re(\tau) < 1+\frac{j}{2}\right)$$

and

$$\begin{aligned} &{}_2F_1\left[\begin{matrix} \xi, \tau; \\ 1+\xi-\tau-j; \end{matrix} \; -1\right] \\ &= \frac{2^{-\xi}\Gamma(\frac{1}{2})\Gamma(1+\xi-\tau-j)}{\Gamma(\frac{\xi}{2}-\tau-\frac{j}{2}+\frac{1}{2})\Gamma(\frac{\xi}{2}-\tau-\frac{j}{2}+1)} \\ &\times \sum_{r=0}^{j}\binom{j}{r}\frac{\Gamma(\frac{\xi}{2}-\tau-\frac{j}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{\xi}{2}-\frac{j}{2}+\frac{r}{2}+\frac{1}{2})} \end{aligned} \tag{18}$$

$$\left(j \in \mathbb{N}_0,\ \xi-\tau \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1+j},\ \frac{1-j}{2} \leqslant \Re(\tau) < 1-\frac{j}{2}\right).$$

Note that when $j = 0$, both Equations (17) and (18) simplify to Kummer's summation theorem (refer to, for instance, [1] (p. 387, Entry 15.4.26), [2] (p. 68, Equation (1)), [3] (p. 9, Section 2.3, Equation (1)), [4] (p. 243), [5] (p. 351, Equation (3))).

$${}_2F_1\left[\begin{matrix} \xi, \tau; \\ 1+\xi-\tau; \end{matrix} \; -1\right] = \frac{\Gamma(1+\frac{\xi}{2})\Gamma(1+\xi-\tau)}{\Gamma(1+\xi)\Gamma(1+\frac{\xi}{2}-\tau)} \tag{19}$$

$$\left(\xi-\tau \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1},\ \frac{1}{2} \leqslant \Re(\tau) < 1\right).$$

We also remember the classical Beta function, as referenced in, for instance, [5] (p. 8)

$$B(\rho, \sigma) = \begin{cases} \displaystyle\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}\,\mathrm{d}u & (\Re(\rho) > 0,\ \Re(\sigma) > 0) \\ \dfrac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)} & (\rho, \sigma \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}). \end{cases} \tag{20}$$

2. Main Summation Formulae

This section establishes four classes of summation formulas for ${}_4F_3(1)$, as stated in Theorems 1 and 2.

Theorem 1. *Let $j \in \mathbb{N}_0$ and $\Re(\tau) < \frac{1}{2}(1+j)$. Also, assume that none of the denominator parameters are negative integers or zero. Then,*

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} \frac{\xi}{2}, \frac{\xi}{2}+\frac{1}{2}, \frac{\tau}{2}, \frac{\tau}{2}+\frac{1}{2}; \\ \frac{1}{2}, \frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}, 1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}; \end{matrix} 1\right] \\
&= \frac{\Gamma(1+\xi-\tau+j)\Gamma(1-2\tau+j)}{2\Gamma(1-\tau+j)\Gamma(1+\xi-2\tau+j)} \\
&\quad + \frac{2^{-\xi-1}\Gamma(\frac{1}{2})\Gamma(1+\xi-\tau+j)\Gamma(\tau-j)}{\Gamma(\tau)\Gamma(\frac{\xi}{2}-\tau+\frac{1}{2}j+\frac{1}{2})\Gamma(\frac{\xi}{2}-\tau+\frac{j}{2}+1)} \\
&\quad \times \sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma(\frac{\xi}{2}-\tau+\frac{j}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{\xi}{2}-\frac{j}{2}+\frac{r}{2}+\frac{1}{2})} =: \Delta_1
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} \frac{\xi}{2}+\frac{1}{2}, \frac{\xi}{2}+1, \frac{\tau}{2}+\frac{1}{2}, \frac{\tau}{2}+1; \\ \frac{3}{2}, 1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}, \frac{3}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}; \end{matrix} 1\right] \\
&= \frac{1+\xi-\tau+j}{2\xi\tau}\left\{\frac{\Gamma(1+\xi-\tau+j)\Gamma(1-2\tau+j)}{\Gamma(1-\tau+j)\Gamma(1+\xi-2\tau+j)}\right. \\
&\quad - \frac{2^{-\xi}\Gamma(\frac{1}{2})\Gamma(1+\xi-\tau+j)\Gamma(\tau-j)}{\Gamma(\tau)\Gamma(\frac{\xi}{2}-\tau+\frac{j}{2}+\frac{1}{2})\Gamma(\frac{\xi}{2}-\tau+\frac{j}{2}+1)} \\
&\quad \left.\times \sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma(\frac{\xi}{2}-\tau+\frac{j}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{\xi}{2}-\frac{j}{2}+\frac{r}{2}+\frac{1}{2})}\right\} =: \Delta_2.
\end{aligned} \tag{22}$$

Theorem 2. *Let $j \in \mathbb{N}_0$ and $\Re(\tau) < \frac{1}{2}(1-j)$. Also, suppose that none of the denominator parameters are negative integers or zero. Then,*

$$\begin{aligned}
&{}_4F_3\left[\begin{matrix} \frac{\xi}{2}, \frac{\xi}{2}+\frac{1}{2}, \frac{\tau}{2}, \frac{\tau}{2}+\frac{1}{2}; \\ \frac{1}{2}, \frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2}, 1+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2}; \end{matrix} 1\right] \\
&= \frac{\Gamma(1+\xi-\tau-j)\Gamma(1-2\tau-j)}{2\Gamma(1-\tau-j)\Gamma(1+\xi-2\tau-j)} \\
&\quad + \frac{2^{-\xi-1}\Gamma(\frac{1}{2})\Gamma(1+\xi-\tau-j)}{\Gamma(\frac{\xi}{2}-\tau-\frac{j}{2}+\frac{1}{2})\Gamma(\frac{\xi}{2}-\tau-\frac{j}{2}+1)} \\
&\quad \times \sum_{r=0}^{j}\binom{j}{r}\frac{\Gamma(\frac{\xi}{2}-\tau-\frac{j}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{\xi}{2}-\frac{j}{2}+\frac{r}{2}+\frac{1}{2})} =: \Delta_3
\end{aligned} \tag{23}$$

and

$$
{}_4F_3\left[\begin{matrix}\frac{\xi}{2}+\frac{1}{2},\ \frac{\xi}{2}+1,\ \frac{\tau}{2}+\frac{1}{2},\ \frac{\tau}{2}+1\,;\\ \frac{3}{2},\ 1+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2},\ \frac{3}{2}+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2}\,;\end{matrix}\ 1\right]
= \frac{1+\xi-\tau-j}{2\xi\tau}\left\{\frac{\Gamma(1+\xi-\tau-j)\Gamma(1-2\tau-j)}{\Gamma(1-\tau-j)\Gamma(1+\xi-2\tau-j)}\right.
- \frac{2^{-\xi}\Gamma(\frac{1}{2})\Gamma(1+\xi-\tau-j)}{\Gamma(\frac{\xi}{2}-\tau-\frac{j}{2}+\frac{1}{2})\Gamma(\frac{\xi}{2}-\tau-\frac{j}{2}+1)}
\times\left.\sum_{r=0}^{j}\binom{j}{r}\frac{\Gamma(\frac{\xi}{2}-\tau-\frac{j}{2}+\frac{r}{2}+\frac{1}{2})}{\Gamma(\frac{\xi}{2}-\frac{j}{2}+\frac{r}{2}+\frac{1}{2})}\right\} =: \Delta_4. \tag{24}
$$

Proof of Theorems 1 and 2. The derivation of results (21)–(24) is straightforward. Indeed, setting $\nu = 1$, $\xi_1 = \xi$, $\xi_2 = \tau$ and $\tau_1 = 1+\xi-\tau+j$ $(j \in \mathbb{N}_0)$ in Relations (12) and (13), we obtain

$$
{}_4F_3\left[\begin{matrix}\frac{\xi}{2},\ \frac{\xi}{2}+\frac{1}{2},\ \frac{\tau}{2},\ \frac{\tau}{2}+\frac{1}{2}\,;\\ \frac{1}{2},\ \frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2},\ 1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}\,;\end{matrix}\ 1\right]
= \frac{1}{2}\left\{{}_2F_1\left[\begin{matrix}\xi,\ \tau\,;\\ 1+\xi-\tau+j\,;\end{matrix}\ 1\right] + {}_2F_1\left[\begin{matrix}\xi,\ \tau\,;\\ 1+\xi-\tau+j\,;\end{matrix}\ -1\right]\right\} \tag{25}
$$

and

$$
{}_4F_3\left[\begin{matrix}\frac{\xi}{2}+\frac{1}{2},\ \frac{\xi}{2}+1,\ \frac{\tau}{2}+\frac{1}{2},\ \frac{\tau}{2}+1\,;\\ \frac{3}{2},\ 1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2},\ \frac{3}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}\,;\end{matrix}\ 1\right]
= \frac{1+\xi-\tau+j}{2\xi\tau}\left\{{}_2F_1\left[\begin{matrix}\xi,\ \tau\,;\\ 1+\xi-\tau+j\,;\end{matrix}\ 1\right] - {}_2F_1\left[\begin{matrix}\xi,\ \tau\,;\\ 1+\xi-\tau+j\,;\end{matrix}\ -1\right]\right\}. \tag{26}
$$

Now, using summation Theorems (5) and (17), respectively, in ${}_2F_1(1)$ and ${}_2F_1(-1)$ of the right members of (25) and (26), we obtain (21) and (22).

Using (5) and (18), the proof of (23) and (24) runs in parallel with that of (21) and (22). We omit the details. □

Remark 1. *The identities in Theorems 1 and 2 are pointed out to reduce to some known results as follows:*

(i) *The case* $j = 0$ *of Identities* (21) *or* (23) and (22) *or* (24) *yields, respectively, results* (14) *and* (15) *in [13].*

(ii) *Setting* $j = 0, 1, 2, 3, 4, 5$ *in Identities* (21) *and* (22)*, correspondence emerges to the results of [12] (Equations (2.1) and (2.2)) when* $j = 0, 1, 2, 3, 4, 5$.

(iii) *Setting* $j = 0, 1, 2, 3, 4, 5$ *in Identities* (23) *and* (24)*, correspondence emerges to the results of [12] (Equations (2.1) and (2.2)) when* $j = 0, -1, -2, -3, -4, -5$.

3. Application

In this section, leveraging the findings presented in Section 2, we assess eight classes of integrals. These integrals feature integrands linked to the structure of Beta function (20) and ${}_3F_2(u)$. Here, Δ_j $(j = 1, 2, 3, 4)$ mirrors the definitions outlined in Theorems 1 and 2.

Theorem 3. *Let $j \in \mathbb{N}_0$, $\Re(\xi) > 0$ and $\Re(\tau) < 1 + j$. Also, assume that none of the denominator parameters in ${}_3F_2(u)$ are negative integers or zero. Then,*

$$\int_0^1 u^{\frac{\xi}{2}-1}(1-u)^{\frac{j}{2}-\frac{\tau}{2}-\frac{1}{2}}\, {}_3F_2\left[\begin{matrix} \frac{\xi}{2}+\frac{1}{2}, \frac{\tau}{2}, \frac{\tau}{2}+\frac{1}{2}; \\ \frac{1}{2}, 1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}; \end{matrix} u\right] \mathrm{d}u = \frac{\Gamma(\frac{\xi}{2})\,\Gamma(\frac{1}{2}-\frac{\tau}{2}+\frac{j}{2})}{\Gamma(\frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2})}\,\Delta_1 \tag{27}$$

and

$$\int_0^1 u^{\frac{\xi}{2}-1}(1-u)^{\frac{j}{2}-\frac{\tau}{2}}\, {}_3F_2\left[\begin{matrix} \frac{\xi}{2}+\frac{1}{2}, \frac{\tau}{2}, \frac{\tau}{2}+\frac{1}{2}; \\ \frac{1}{2}, \frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}; \end{matrix} u\right] \mathrm{d}u = \frac{\Gamma(\frac{\xi}{2})\,\Gamma(1-\frac{\tau}{2}+\frac{j}{2})}{\Gamma(1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2})}\,\Delta_1. \tag{28}$$

Proof of (27). We consider the left member denoted by $\mathcal{L}$ in Equation (27). We proceed by expanding ${}_3F_2(u)$ in a series, as illustrated in Equation (1), and then interchange the order of integration and summation within the resulting expression of $\mathcal{L}$. To justify this process, we adopt the perspective of regarding the integral as a Lebesgue integral. Notably, the series involved converges absolutely on interval $[0, 1]$ under specified restrictions, thereby allowing each term in the series, including the other integrand function of the Beta function form, to be assumed nonnegative. Consequently, we apply Lévi's convergence theorem (refer to [22] (p. 268, Theorem 10.25)) to yield

$$\mathcal{L} = \sum_{n=0}^{\infty} \frac{(\frac{\xi}{2}+\frac{1}{2})_n(\frac{\tau}{2})_n(\frac{\tau}{2}+\frac{1}{2})_n}{(\frac{1}{2})_n(1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2})_n\, n!} \int_0^1 u^{\frac{\xi}{2}+n-1}(1-u)^{\frac{1}{2}+\frac{j}{2}-\frac{\tau}{2}-1}\,\mathrm{d}u. \tag{29}$$

Using (20) to evaluate the integral in (29) and employing (2) in the resulting expression, we obtain

$$\mathcal{L} = \frac{\Gamma(\frac{\xi}{2})\Gamma(\frac{1}{2}-\frac{\tau}{2}+\frac{j}{2})}{\Gamma(\frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2})} \sum_{n=0}^{\infty} \frac{(\frac{\xi}{2})_n(\frac{\xi}{2}+\frac{1}{2})_n(\frac{\tau}{2})_n(\frac{\tau}{2}+\frac{1}{2})_n}{(\frac{1}{2})_n(\frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2})_n(1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2})_n\, n!},$$

which, in terms of (1), produces

$$\mathcal{L} = \frac{\Gamma(\frac{\xi}{2})\Gamma(\frac{1}{2}-\frac{\tau}{2}+\frac{j}{2})}{\Gamma(\frac{1}{2}+\frac{a}{2}-\frac{\tau}{2}+\frac{j}{2})}\, {}_4F_3\left[\begin{matrix} \frac{\xi}{2}, \frac{\xi}{2}+\frac{1}{2}, \frac{\tau}{2}, \frac{\tau}{2}+\frac{1}{2}; \\ \frac{1}{2}, \frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}, 1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}; \end{matrix} 1\right]. \tag{30}$$

We observe that expression ${}_4F_3(1)$ in Equation (21) coincides with that in Equation (21) and can be represented using Gamma functions, akin to Δ_1. This establishes the validity of (27). □

The proofs of the remaining Identities (28), (31)–(36) follow a similar pattern to that of (27) and can be derived accordingly. However, for brevity, their proofs are omitted here.

Theorem 4. *Let $j \in \mathbb{N}_0$, $\Re(\xi) > -1$ and $\Re(\tau) < 1 + j$. Also, assume that none of the denominator parameters in ${}_3F_2(u)$ are negative integers or zero. Then,*

$$\begin{aligned}\int_0^1 u^{\frac{\xi}{2}-\frac{1}{2}}(1-u)^{\frac{j}{2}-\frac{\tau}{2}-\frac{1}{2}}\, {}_3F_2\left[\begin{matrix}\frac{\xi}{2}+1, \frac{\tau}{2}+\frac{1}{2}, \frac{\tau}{2}+1\,; \\ \frac{3}{2}, \frac{3}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}\,;\end{matrix}\ u\right] \mathrm{d}u \\ = \frac{\Gamma(\frac{\xi}{2}+\frac{1}{2})\Gamma(\frac{1}{2}-\frac{\tau}{2}+\frac{j}{2})}{\Gamma(1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2})}\,\Delta_2\end{aligned} \tag{31}$$

and

$$\begin{aligned}\int_0^1 u^{\frac{\xi}{2}-\frac{1}{2}}(1-u)^{\frac{j}{2}-\frac{\tau}{2}}\, {}_3F_2\left[\begin{matrix}\frac{\xi}{2}+1, \frac{\tau}{2}+\frac{1}{2}, \frac{\tau}{2}+1\,; \\ \frac{3}{2}, 1+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2}\,;\end{matrix}\ u\right] \mathrm{d}u \\ = \frac{\Gamma(\frac{\xi}{2}+\frac{1}{2})\Gamma(1-\frac{\tau}{2}+\frac{j}{2})}{\Gamma(\frac{3}{2}+\frac{\xi}{2}-\frac{\tau}{2}+\frac{j}{2})}\,\Delta_2.\end{aligned} \tag{32}$$

Theorem 5. *Let $j \in \mathbb{N}_0$, $\Re(\xi) > 0$ and $\Re(\tau) < 1 - j$. Also, assume that none of the denominator parameters in ${}_3F_2(u)$ are negative integers or zero. Then,*

$$\begin{aligned}\int_0^1 u^{\frac{\xi}{2}-1}(1-u)^{-\frac{1}{2}-\frac{j}{2}-\frac{\tau}{2}}\, {}_3F_2\left[\begin{matrix}\frac{\xi}{2}+\frac{1}{2}, \frac{\tau}{2}, \frac{\tau}{2}+\frac{1}{2}\,; \\ \frac{1}{2}, 1+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2}\,;\end{matrix}\ u\right] \mathrm{d}u \\ = \frac{\Gamma(\frac{\xi}{2})\Gamma(\frac{1}{2}-\frac{\tau}{2}-\frac{j}{2})}{\Gamma(\frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2})}\,\Delta_3\end{aligned} \tag{33}$$

and

$$\begin{aligned}\int_0^1 u^{\frac{\xi}{2}-1}(1-u)^{-\frac{j}{2}-\frac{\tau}{2}}\, {}_3F_2\left[\begin{matrix}\frac{\xi}{2}+\frac{1}{2}, \frac{\tau}{2}, \frac{\tau}{2}+\frac{1}{2}\,; \\ \frac{1}{2}, \frac{1}{2}+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2}\,;\end{matrix}\ u\right] \mathrm{d}u \\ = \frac{\Gamma(\frac{\xi}{2})\Gamma(1-\frac{\tau}{2}-\frac{j}{2})}{\Gamma(1+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2})}\,\Delta_3.\end{aligned} \tag{34}$$

Theorem 6. *Let $j \in \mathbb{N}_0$, $\Re(\xi) > -1$ and $\Re(\tau) < 1 - j$. Also, assume that none of the denominator parameters in ${}_3F_2(u)$ are negative integers or zero. Then,*

$$\begin{aligned}\int_0^1 u^{\frac{\xi}{2}-\frac{1}{2}}(1-u)^{-\frac{1}{2}-\frac{j}{2}-\frac{\tau}{2}}\, {}_3F_2\left[\begin{matrix}\frac{\xi}{2}+1, \frac{\tau}{2}+\frac{1}{2}, \frac{\tau}{2}+1\,; \\ \frac{3}{2}, \frac{3}{2}+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2}\,;\end{matrix}\ u\right] \mathrm{d}u \\ = \frac{\Gamma(\frac{\xi}{2}+\frac{1}{2})\Gamma(\frac{1}{2}-\frac{\tau}{2}-\frac{j}{2})}{\Gamma(1+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2})}\,\Delta_4\end{aligned} \tag{35}$$

and

$$\int_0^1 u^{\frac{\xi}{2}-\frac{1}{2}}(1-u)^{-\frac{j}{2}-\frac{\tau}{2}}\, {}_3F_2\left[\begin{matrix}\frac{\xi}{2}+1,\ \frac{\tau}{2}+\frac{1}{2},\ \frac{\tau}{2}+1\,;\\ \frac{3}{2},\ 1+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2}\,;\end{matrix}\ u\right] \mathrm{d}u = \frac{\Gamma(\frac{\xi}{2}+\frac{1}{2})\Gamma(1-\frac{\tau}{2}-\frac{j}{2})}{\Gamma(\frac{3}{2}+\frac{\xi}{2}-\frac{\tau}{2}-\frac{j}{2})}\,\Delta_4. \tag{36}$$

Remark 2. *We recall the following identity, for instance, as described in [2] (Theorem 38, p. 104)): we suppose $t > 0$, $\min\{\Re(\rho), \Re(\sigma)\} > 0$, and $k \in \mathbb{N}$. Within the region where the series converges, we find*

$$\int_0^t u^{\rho-1}(t-u)^{\sigma-1}{}_pF_q\left[\begin{matrix}\xi_1, \ldots, \xi_p\,;\\ \tau_1, \ldots, \tau_q\,;\end{matrix}\ cu^k\right] du = B(\rho,\sigma)\, t^{\rho+\sigma-1}\ {}_{p+k}F_{q+k}\left[\begin{matrix}\xi_1, \ldots, \xi_p,\ \frac{\rho}{k},\ \frac{\rho+1}{k},\ \ldots,\ \frac{\rho+k-1}{k};\\ \tau_1, \ldots, \tau_q,\ \frac{\rho+\sigma}{k},\ \frac{\rho+\sigma+1}{k},\ \ldots,\ \frac{\rho+\sigma+k-1}{k};\end{matrix}\ ct^k\right]. \tag{37}$$

A reviewer has pointed out that the instance where $k = 1$ in Equation (37) is given in Entry 2.22.2.-1 of the second edition, released in 2003, of the Russian version found in reference [23]. By employing this particular instance, it is straightforward to confirm that the left-sided integrals in Equations (27), (28) and (31)–(36) result in the corresponding ${}_4F_3(1)$. The case of (37) when $k = 1$, $p = 3$, $q = 2$, $t = 1$, and $c = 1$, using (20), produces

$$\int_0^1 u^{\rho-1}(1-u)^{\sigma-1}{}_3F_2\left[\begin{matrix}\xi_1,\ \xi_2,\ \xi_3\,;\\ \tau_1,\ \tau_2\,;\end{matrix}\ u\right] du = \frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma(\rho+\sigma)}\ {}_4F_3\left[\begin{matrix}\xi_1,\ \xi_2,\ \xi_3,\ \rho;\\ \tau_1,\ \tau_2,\ \rho+\sigma;\end{matrix}\ 1\right]. \tag{38}$$

Formula (38) is then applied to evaluate the left-sided integrals in Equations (27), (28) and (31)–(36) in terms of Gamma functions.

4. Concluding Remarks and Observations

Applying the elementary yet useful Identities (6) and (7), which are partitions of the set of non-negative integers into modulo N and modulo 2 and appear ubiquitously in the mathematical literature, to the generalized hypergeometric series, we established four classes of summation formulae for series ${}_4F_3(1)$ with the help of the classical Gauss's summation theorem (5) and Generalizations (17) and (18) of the classical Kummer's summation theorem (19). Then, we applied those summation formulas to evaluate eight classes of integrals whose integrands are composed of the form of Beta function (20) and ${}_3F_2(u)$. Also, it was easy to find that the four formulas in Theorems 1 and 2 are not interchangeable by simply modifying the parameters, and neither are the eight identities in Theorems 3–6. That is, by means of modification of parameters, the four summation formulas in Theorems 1 and 2, and the eight integral formulas in Theorems 3–6 are mutually independent.

Moreover, the four families of summation formulas and the eight classes of integral formulas, elegantly expressed in terms of Gamma functions, represent novel contributions to the mathematical landscape (refer to, for instance, [24,25], ([26], Entry 7.512-12), ([23], Entry 2.22.2.-1).

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