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New Summation and Integral Representations for 2-Variable (p,q)-Hermite Polynomials

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Abstract: In this paper, we introduce and study new features for 2-variable (p,q)-Hermite polynomials, such as the (p,q)-diffusion equation, (p,q)-differential formula and integral representations. In addition, we establish some summation models and their (p,q)-derivatives. Certain parting remarks and nontrivial examples are also provided.

Keywords: (p,q)-calculus; 2-variable (p,q)-Hermite polynomials; shift operator for polynomials; (p,q)-differential equation summation formulas; integral representations

MSC: 11B83; 33C10; 33E20

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1. Introduction

Applications of the (p,q)-calculus can be found in other disciplines including quantum mechanics, statistical physics and number theory. In particular, it has been used to study the quantum group $SU_{p,q}(2)$, which is a deformation of the Lie group $SU(2)$ and has applications in quantum field theory. In fact, all the (p,q)-calculus results decrease into the corresponding results to the q-calculus when $p = 1$. Similar (p,q)-calculus results additionally decrease to the corresponding outcomes of ordinary calculus when $p = 1$ and $q \to 1$. Therefore the (p,q)-calculus is a real generalization of the q-calculus that allows for non-commutative calculus. It has been used to define (p,q)-analogues of various special functions and polynomials and has applications in many areas of mathematics and physics. Recently, (p,q)-calculus got attention from the researchers of various fields of mathematics and physics. Presented and thoroughly investigated are the (p,q)-analogues of several conventional special functions, including the Hermite polynomials, Bernoulli polynomials, Euler polynomials, Beta function, Gamma function, generalized bivariate (p,q)-Bernoulli–Fibonacci polynomials and generalized bivariate (p,q)-Bernoulli–Lucas polynomials, family of (p,q)-hybrid polynomials and (p,q)-sine and (p,q)-cosine Fubini polynomials; for more details, we refer the readers to [1–9] and references therein.

Hermite polynomials are certain of the greatest significant and historical orthogonal special functions from classical times, and they have been used extensively. It is the collection of differential equation solutions with an oscillator of harmonics that match the Schrödinger equation in quantum mechanics. These polynomials are very important for considering classical boundary-value problems in parabolic regions with parabolic coordinates. Hermite polynomials are also found in the field of signal processing as Hermitian wavelets in the wavelet transform analysis probability, combinatorics as a manifestation of an Appell series observing the umbral calculus, and numerical computation. The range of applications whose mathematical description is based on polynomials is very wide in approximation theory. For further information, the interested reader may consult the research papers [10–17].

Hermite polynomials are the series solutions of Hermite differential equations. Recall that the classical Hermite polynomials $H_n(x)$ (see, e.g., [18]) are defined by means of the following generating function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

and the series definition was stated as follows:

$$H_n(x) = n!\sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!}(2x)^{n-2k}.$$

Many authors have worked on its generalization in various directions of the famous Hermitian polynomials. The 2-variable Hermite polynomials $H_n(x,y)$ (see [19]) are defined by means of the following generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}$$

and the series definition was stated as follows:

$$H_n(x,y) = n!\sum_{k=0}^{[n/2]} \frac{1}{k!(n-2k)!}x^{n-2k}y^k \qquad \text{for } y \neq 0,$$

and

$$H_n(x,0) = x^n.$$

It is known that these polynomials satisfy the following parabolic equation (Heat equation) (see [10,20–22]):

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y},$$

which is called the diffusion equation of the 2-variable Hermite polynomials $H_n(x,y)$ (see [22]). In 2021, Raza, Fadel, Nisar and Zakarya [5] introduced and studied the 2-variable (p,q)-Hermite polynomials $H_{n,p,q}(x,y)$. They used the following generating function to generate 2-variable (p,q)-Hermite polynomials $H_{n,p,q}(x,y)$:

$$e_{p,q}(xt)\ e_{p,q}(yt^2) = \sum_{n=0}^{\infty} H_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!} \tag{1}$$

and series definition was stated as follows:

$$H_{n,p,q}(x,y) = [n]_{p,q}!\sum_{k=0}^{[n/2]} \frac{(x)_{p,q}^{n-2k}(y)_{p,q}^k}{[n-2k]_{p,q}![k]_{p,q}!}. \tag{2}$$

Some important properties and relations for the 2-variable (p,q)-Hermite polynomials $H_{n,p,q}(x,y)$ were established; for more details, we refer the readers to [5].

In this work, we devote our attention to investigate new various features of 2-variable (p,q)-Hermite polynomials and introduce some new ones. The features of 2-variable (p,q)-Hermite polynomials, such as differential equations, integral and summation representations are studied and established in Sections 3 and 4. Finally, in Section 5, we will conclude this paper by highlighting some significant research directions that could be studied in the future.

2. Preliminaries

In this section, we recall some basic definitions, notations and known results, which will be used and discussed further in this paper. The (p,q)-number $[\alpha]_{p,q}$ [23] is specified as

$$[\alpha]_{p,q} = \frac{p^\alpha - q^\alpha}{p-q} \qquad \text{for } 0 < q < p \leq 1 \text{ and } \alpha \in \mathbb{N}.$$

Presented is the value of the (p,q)-factorial [23]:

$$[m]_{p,q}! = \begin{cases} \prod_{r=1}^{m} [r]_{p,q}, & m \in \mathbb{N}, \\ 1, & m = 0. \end{cases}$$

The (p,q)-binomial coefficients [23] is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!} \qquad \text{for } k = 0,1,\ldots,n,$$

which for $p = 1$ and $q \to 1$, gives the following usual binomial coefficients [18]

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!} \qquad \text{for } k = 0,1,\ldots,n.$$

The raising (p,q)-power [1] is defined by

$$(x \oplus a)^n_{p,q} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{p,q} p^{\binom{m}{2}} q^{\binom{n-m}{2}} x^m a^{n-m}. \tag{3}$$

In particular, when $a = 0$, Equation (3) yields

$$(x)^n_{p,q} = p^{\binom{n}{2}} x^n. \tag{4}$$

From the above equation, it is straightforward to prove that

$$(bx)^n_{p,q} = b^n (x)^n_{p,q}.$$

In q-calculus, there are two (p,q)-exponential functions, designated by $e_{p,q}(x)$ and $E_{p,q}(x)$, that have been described as [1]:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!} \tag{5}$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{x^n}{[n]_{p,q}!}, \tag{6}$$

alternatively. The following relationship between $e_{p,q}(x)$ and $E_{p,q}(x)$ is known (see [1]):

$$e_{p,q}(x) E_{p,q}(-x) = 1. \tag{7}$$

In [24], a function f has a specific formula to describe its (p,q)-derivative with respect to x is given by

$$D_{p,q,x} f(x) = \begin{cases} \frac{f(px) - f(qx)}{(p-q)x}, & x \neq 0, \\ f'(0), & x = 0. \end{cases}$$

More specifically, we have

$$D_{p,q,x}(f(x)g(x)) = f(px) D_{p,q,x} g(x) + g(qx) D_{p,q,x}(f(x)),$$

$$D_{p,q,x}x^n = [n]_{p,q}x^{n-1},$$

$$D_{p,q,x}e_{p,q}(\alpha x) = \alpha e_{p,q}(\alpha p x)$$

and

$$D_{p,q,x}E_{p,q}(\alpha x) = \alpha E_{p,q}(\alpha q x).$$

The following are the derivatives of the (p,q)-exponential functions that correspond to the m^{th} order (see [24]):

$$D^m_{p,q,x}e_{p,q}(\alpha x) = \alpha^n p^{\binom{m}{2}} e_{p,q}(\alpha p^m x) \qquad \text{for } m \geq 1 \tag{8}$$

and

$$D^m_{p,q,x}E_{p,q}(\alpha x) = \alpha^m q^{\binom{m}{2}} e_{p,q}(\alpha q^m x) \qquad \text{for } m \geq 1.$$

where symbol $D^m_{p,q,x}$ indicated the m^{th} (p,q)-derivative with regard to x.

Moreover, an expression (p,q) integral for f is provided via [24]:

$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{n=0}^{\infty}\frac{p^n}{q^{n+1}}f\left(\frac{p^n}{q^{n+1}}a\right) \qquad \text{for } \mid \frac{p}{q} \mid < 1. \tag{9}$$

From Equation (9), it is clear that

$$\int_a^b \Big(f(x)d_{p,q}x + g(x)d_{p,q}x\Big) = \int_a^b f(x)d_{p,q}x + \int_a^b g(x)d_{p,q}x. \tag{10}$$

Given is the (p,q)-definite integral of the (p,q)-derivative of a function f (see [24]):

$$\int_a^b D_{p,q}f(x)d_{p,q}x = f(b) - f(a). \tag{11}$$

In [5], the (p,q)-partial derivative of formula in terms of x and y is given by

$$D_{p,q,x}H_{n,p,q}(x,y) = [n]_{p,q}H_{n-1,p,q}(px,y), \qquad n \geq 1 \tag{12}$$

and

$$D_{p,q,y}\Big(H_{n,p,q}(x,y)\Big) = [n]_{p,q}[n-1]_{p,q}H_{n-2,p,q}(x,py), \qquad n \geq 2, \tag{13}$$

alternatively. According to [5], the pure recurrence relation for $H_{n,p,q}(x,y)$ is as described below:

$$\begin{aligned} &H_{n+1,p,q}(x,y) \\ &= xH_{n,p,q}(px,p^2y) + [n]_{p,q}y\Big(pH_{n-1,p,q}(qx,p^2y) + qH_{n-1,p,q}(qx,pqy)\Big) = 0 \quad \text{for } n \geq 1. \end{aligned} \tag{14}$$

The shift operators $L_{a,x}$ and $L_{a,y}$ for any (p,q)-function of two variables $f(x,y)$ are defined as [5]:

$$L_{a,x}f(x,y) = f(ax,y) \tag{15}$$

and

$$L_{a,y}f(x,y) = f(x,ay),$$

where a is a constant. From Equation (15), the shift operator $L_{a,x}$ satisfy the following properties:

$$L_{a,x}L_{b,x}f(x,y) = f(abx,y) = L_{ab,x}f(x,y). \tag{16}$$

In particular, for $a = b$, we have

$$L_{a^2,x}f(x,y) = f(a^2x,y) = L_{a,x}L_{a,x}f(x,y) = L^2_{a,x}f(x,y). \tag{17}$$

Since $L_{a,x}^{-1}$ is the reverse of the operator $L_{a,x}$, that is $L_{a,x}^{-1}\, L_{a,x} = I$, which I is an identity operator that produces $If(x,y) = f(x,y)$. Then, from Equation (15), we have

$$L_{a,x}^{-1} f(ax,y) = f(x,y).$$

Replacing ax by x in the above equation, we acquire

$$L_{a,x}^{-1}\, f(x,y) = f\left(\frac{1}{a}x, y\right) = f(a^{-1}x, y),$$

which on using Equation (15), gives

$$L_{a,x}^{-1}\, f(x,y) = L_{a^{-1},x}\, f(x,y).$$

Using induction method, Equations (16) and (17), gives

$$L_{a^r,x} f(x,y) = L_{a,x}^r f(x,y) \quad \text{for } r \in \mathbb{Z}.$$

In a similar way, it has been demonstrated that $L_{a,y}$ has the characteristics that follow:

$$L_{ab,y} f(x,y) = L_{a,y} L_{b,y} f(x,y)$$

and

$$L_{a^r,y} f(x,y) = L_{a,y}^r f(x,y) \quad \text{for } r \in \mathbb{Z}.$$

3. New Properties of 2-Variable (p,q)-Hermite Polynomials

During this part, we establish some features about 2-variable (p,q)-Hermite polynomials such as differential equations and integral representations. Further, from Equation (2), it can be easily verified that

$$H_{n,p,q}(ax, a^2y) = a^n H_{n,p,q}(x,y), \tag{18}$$

wherein a has a fixed value.

Applying the 2nd order (p,q)-partial derivative of both the sides of Equation (1) with respect to x, then using Equation (8) for $m = 2$, we get

$$pt^2 e_{p,q}(pxt) e_{p,q}(yt^2) = \sum_{n=0}^{\infty} D_{p,q,x}^2 H_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!}.$$

Again, applying the Equation (1) on the left part of previous equation, provides us

$$p \sum_{n=2}^{\infty} H_{n-2,p,q}(p^2x, y, z) \frac{t^n}{[n-2]_{p,q}!} = \sum_{n=0}^{\infty} D_{p,q,x}^2 H_{n,p,q}(x,y) \frac{t^n}{[n]_{p,q}!}.$$

Therefore, when the corresponding values of t from each aspect are compared, we obtain

$$D_{p,q,x}^2 H_{n,p,q}(x,y) = [n]_{p,q}[n-1]_{p,q} p\, H_{n-2,p,q}(p^2x, y) \qquad (n \geq 2). \tag{19}$$

In the context of Equations (13), (18) and (19) it is easy to verify that the 2V(p,q)HP $H_{n,p,q}(x,y)$ satisfy the following (p,q)-partial differential equation :

$$D_{p,q,x}^2 H_{n,p,q}(x, p^3y) = p\, D_{p,q,y} H_{n,p,q}(p^2x, p^2y), H_{n,p,q}(x, p^3y), \quad H_{n,p,q}(x,0) = (x)_{p,q}^n. \tag{20}$$

Equation (20), is called the (p,q)-analogue of diffusion equation, which for $p = 1$, gives the for the q-analogue of diffusion equation for 2-variable q-Hermite polynomials $H_{n,q}(x,y)$ [5]. Further, for $p = 1, q \to 1^-$, gives the classical diffusion equation for 2-variable Hermite polynomials $H_n(x,y)$ (see [10,22]).

Example 1. *The following correlations of (p,q)-diffusion equations for 2-variable (p,q)-Hermite polynomials can be obtained immediately by applying (20):*

$$D^2_{1,2/3,x}H_{2,1,2/3}(x,y) - D_{1,2/3,y}H_{2,1,2/3}(x,y) = 0,$$
$$D^2_{2/3,3/5,x}H_{2,2/3,3/5}(x,8/27y) - 2/3\, D_{2/3,3/5,y}H_{2,2/3,3/5}(4/9x,4/9y) = 0,$$
$$D^2_{3/4,4/5,x}H_{3,3/4,4/5}(x,16/64y) - 3/4\, D_{3/4,4/5,y}H_{3,3/4,4/5}(9/16x,9/16y) = 0.$$

Afterwards, we will demonstrate the next outcome:

Theorem 1. *The subsequent (p,q)-differential equation for (p,q)-Hermite polynomials of 2-variables holds true:*

$$yL^{-2}_{p,x}\, L_{q,x}\left(1+\left(\frac{q}{p}\right)^{\frac{n}{2}}L_{\sqrt{\frac{p}{q}},x}\right)H''_{n,p,q}(x,y) + pxH'_{n,p,q}(x,y) - [n]_{p,q}\, p^{2-n}L_{p,x}H_{n,p,q}(x,y) = 0, \tag{21}$$

in which $L_{\cdot,x}$ represents the shift operator described in (15).

Proof. Substituting n by $(n-1)$ into the expression (14) and after using Equation (18) in the resulting formula, we arrive at

$$\begin{aligned} &H_{n,p,q}(x,y) - xp^{n-1}H_{n-1,p,q}(x,y) \\ &\quad - [n-1]_{p,q}y\left(p^{n-1}H_{n-2,p,q}\left(\frac{q}{p}x,y\right) + q^{n-1}H_{n-2,p,q}\left(x,\frac{p}{q}y\right)\right) = 0, \quad n \geq 2. \end{aligned} \tag{22}$$

From Equation (15), we possess

$$H_{n-2,p,q}\left(\frac{q}{p}x,y\right) = L_{\frac{q}{p^3},x}H_{n-2,p,q}(p^2x,y). \tag{23}$$

Moreover, from Equations (15) and (18), we obtain

$$H_{n-2,p,q}\left(x,\frac{p}{q}y\right) = \left(\sqrt{\frac{p}{q}}\right)^{n-2}L_{\frac{\sqrt{q}}{p^2\sqrt{p}},x}H_{n-2,p,q}\left(p^2x,y\right). \tag{24}$$

Using Equations (15), (23) and (24) in Equation (22), we get

$$\begin{aligned} &H_{n,p,q}(x,y) - xp^{n-1}L_{p^{-1},x}H_{n-1,p,q}(px,y) - [n-1]_{p,q}y\Big(p^{n-1}L_{\frac{q}{p^3},x} \\ &\quad + q^{n-1}\left(\sqrt{\frac{p}{q}}\right)^{n-2}L_{\sqrt{\frac{q}{p}}p^{-2},x}\Big)H_{n-2,p,q}(p^2x,y) = 0, \quad n \geq 2. \end{aligned} \tag{25}$$

Now, using Equation (12) in Equation (25) and denoting $D_{p,q,x}H_{n,p,q}(x,y)$ by $H'_{n,p,q}(x,y)$, we get

$$\begin{aligned} &H_{n,p,q}(x,y) - \frac{xp^{n-1}}{[n]_{p,q}}\, L_{p^{-1},x}\, H'_{n,p,q}(x,y) - \frac{y}{p[n]_{p,q}}\Big(p^{n-1}L_{p^{-3}q,x} \\ &\quad + \frac{q^n}{p}\left(\sqrt{\frac{p}{q}}\right)^{n}L_{p^{\frac{-5}{2}}q^{\frac{-1}{2}},x}\Big)H''_{n,p,q}(x,y) = 0, \end{aligned}$$

or, equivalently,

$$yL_{p,x}^{-3}\left(p^n L_{q,x} + q^n \left(\sqrt{\frac{p}{q}}\right)^n L_{\sqrt{pq},x}\right) H''_{n,p,q}(x,y) \\ + xp^{n+1} L_{p,x}^{-1} H'_{n,p,q}(x,y) - p^2 [n]_{p,q} H_{n,p,q}(x,y) = 0.$$

It, with additional simplifying, leads to the claim (21). □

Example 2. *The following correlations of (p,q)-differential equations for 2-variable (p,q)-Hermite polynomials can be obtained immediately by applying (21):*

$$yL_{p,x}^{-2} L_{q,x}\left(1 + L_{\sqrt{\frac{p}{q}},x}\right) H''_{0,p,q}(x,y) = p^2 L_{p,x} H_{0,p,q}(x,y) - pxH'_{0,p,q}(x,y),$$

$$yL_{p,x}^{-2} L_{q,x}\left(1 + \left(\frac{q}{p}\right)^{\frac{1}{2}} L_{\sqrt{\frac{p}{q}},x}\right) H''_{1,p,q}(x,y) = [1]_{p,q}\, pL_{p,x} H_{1,p,q}(x,y) - pxH'_{1,p,q}(x,y),$$

$$yL_{p,x}^{-2} L_{q,x}\left(1 + \left(\frac{q}{p}\right) L_{\sqrt{\frac{p}{q}},x}\right) H''_{2,p,q}(x,y) = [2]_{p,q}\, L_{p,x} H_{2,p,q}(x,y) - pxH'_{2,p,q}(x,y),$$

$$yL_{p,x}^{-2} L_{q,x}\left(1 + \left(\frac{q}{p}\right)^{\frac{3}{2}} L_{\sqrt{\frac{p}{q}},x}\right) H''_{3,p,q}(x,y) = [3]_{p,q}\, p^{-1} L_{p,x} H_{3,p,q}(x,y) - pxH'_{3,p,q}(x,y),$$

$$yL_{p,x}^{-2} L_{q,x}\left(1 + \left(\frac{q}{p}\right)^{2} L_{\sqrt{\frac{p}{q}},x}\right) H''_{4,p,q}(x,y) = [4]_{p,q}\, p^{-2} L_{p,x} H_{4,p,q}(x,y) - pxH'_{4,p,q}(x,y),$$

$$yL_{p,x}^{-2} L_{q,x}\left(1 + \left(\frac{q}{p}\right)^{\frac{5}{2}} L_{\sqrt{\frac{p}{q}},x}\right) H''_{5,p,q}(x,y) = [5]_{p,q}\, p^{-3} L_{p,x} H_{5,p,q}(x,y) - pxH'_{5,p,q}(x,y).$$

We then derive the integral representations about 2-variable (p,q)-Hermite polynomials $H_{n,p,q}(x,y)$ in the following manner:

Theorem 2. *The definite (p,q)-integral of $H_{n,p,q}(x,y)$ with regard to x is as follows:*

$$\int_a^b H_{n,p,q}(x,y) d_{p,q}x = p\frac{H_{n+1,p,q}\left(\frac{b}{p},y\right) - H_{n+1,p,q}\left(\frac{a}{p},y\right)}{[n+1]_{p,q}}. \tag{26}$$

Proof. Using Equation (12), we get

$$\int_a^b H_{n,p,q}(x,y) d_{p,q}x = \frac{p}{[n+1]_{p,q}} \int_a^b D_{p,q,x} H_{n+1,p,q}\left(\frac{x}{p},y\right) d_{p,q}x,$$

which yields the assertion (26) when applying Equation (11) on the right hand side. □

Following that, we determine the subsequent outcome:

Theorem 3. *The definite (p,q)-integral of $H_{n,p,q}(x,y)$ with respect to y is as follows:*

$$\int_c^d H_{n,p,q}(x,y) d_{p,q}y = p\frac{H_{n+2,p,q}\left(x,\frac{d}{p}\right) - H_{n+2,p,q}\left(x,\frac{c}{p}\right)}{[n+1]_{p,q}[n+2]_{p,q}}. \tag{27}$$

Proof. Using Equation (13), we get

$$\int_a^b H_{n,p,q}(x,y)d_{p,q}y = \frac{p}{[n+1]_{p,q}[n+2]_{p,q}} \int_a^b D_{p,q,y}H_{n+2,p,q}\left(x, \frac{y}{p}\right)d_{p,q}y,$$

This, when applied to Equation (11) on the right side, yields assertion (27). □

We can obtain the following result in light of Theorems 2 and 3:

Corollary 1. *The formula that follows is the double (p,q)-integration of $H_{n,p,q}(x,y)$:*

$$\begin{aligned}&\int_c^d \int_a^b H_{n,p,q}(x,y)d_{p,q}x\, d_{p,q}y \\ &= \frac{[n]_{p,q}!p^2}{[n+3]_{p,q}!}\left(H_{n+3,p,q}\left(\frac{b}{p},\frac{d}{p}\right) + H_{n+3,p,q}\left(\frac{a}{p},\frac{c}{p}\right) - H_{n+3,p,q}\left(\frac{b}{p},\frac{c}{p}\right) - H_{n+3,p,q}\left(\frac{a}{p},\frac{d}{p}\right)\right).\end{aligned} \tag{28}$$

Proof. Integrating Equation (26) with regard to y by finding the limit through c through d and utilizing Equation (10), that we get

$$\begin{aligned}&\int_c^d \int_a^b H_{n,p,q}(x,y)d_{p,q}x\, d_{p,q}y \\ &= \frac{p}{[n+1]_{p,q}}\left(\int_c^d H_{n+1,p,q}\left(\frac{b}{p},y\right)d_{p,q}y - \int_c^d H_{n+1,p,q}\left(\frac{a}{p},y\right)d_{p,q}y\right),\end{aligned}$$

Hence, according to Equation (27), provides

$$\begin{aligned}&\int_c^d \int_a^b H_{n,p,q}(x,y)d_{p,q}x\, d_{p,q}y \\ &= \frac{p^2}{[n+1]_{p,q}[n+2]_{p,q}[n+3]_{p,q}}\left(H_{n+3,p,q}\left(\frac{b}{p},\frac{d}{p}\right) + H_{n+3,p,q}\left(\frac{a}{p},\frac{c}{p}\right) - H_{n+3,p,q}\left(\frac{b}{p},\frac{c}{p}\right) - H_{n+3,p,q}\left(\frac{a}{p}\right)\right).\end{aligned} \tag{29}$$

From Equation (29), we obtain the assertion (28). □

4. New Summation Models for $H_{n,p,q}(x,y)$ and (p,q)-Derivatives

In this section, we will create some summation models for the (p,q)-Hermite polynomials $H_{n,p,q}(x,y)$ and their (p,q)-derivatives. First, we get several summation models for 2V(p,q)HP by employing the generating function of $H_{n,p,q}(x,y)$ and the identities (5) and (7). The summing models for (p,q)-Hermite polynomials of 2-variables are the ones that follow:

Theorem 4. *The summation models for (p,q)-Hermite polynomials of 2-variables are given below:*

$$\sum_{r=0}^{[n/2]} \frac{q^{\binom{r}{2}}(-y)^r H_{n-2r,p,q}(x,y)}{[r]_{p,q}![n-2r]_{p,q}!} = \frac{p^{\binom{n}{2}}x^n}{[n]_{p,q}!}. \tag{30}$$

(1) If $n = 2m$ ($m \in \mathbb{N}$), then

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}(-x)^r H_{2m-r,p,q}(x,y)}{[r]_{p,q}![2m-r]_{p,q}!} = \frac{p^{\binom{m}{2}}y^m}{[m]_{p,q}!}. \tag{31}$$

(2) If $n = 2m+1$ ($m \in \mathbb{N}\cup\{0\}$), then

$$\sum_{r=0}^{2m+1} \frac{q^{\binom{r}{2}}(-x)^r H_{2m+1-r,p,q}(x,y)}{[r]_{p,q}![2m+1-r]_{p,q}!} = 0. \tag{32}$$

Proof. In the context of the Equation (7), it is clear

$$e_{p,q}(xt)\ e_{p,q}(yt^2)E_{p,q}(-yt^2) = e_{p,q}(xt),$$

which on using Equations (1), (5) and (6) gives

$$\sum_{n=0}^{\infty} H_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}\sum_{r=0}^{\infty}\frac{q^{\binom{r}{2}}(-y)^r t^{2r}}{[r]_{p,q}!} = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}x^n t^n}{[n]_{p,q}!},$$

or, equivalently,

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{H_{n,p,q}(x,y)q^{\binom{r}{2}}(-y)^r\ t^{n+2r}}{[n]_{p,q}![r]_{p,q}!} = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}x^n t^n}{[n]_{p,q}!},$$

which after utilizing the subsequent series arrangement method as in [18]:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{[n/2]} A(m,n-2m),$$

we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\frac{H_{n-2r,p,q}(x,y)q^{\binom{r}{2}}(-y)^r\ t^n}{[n-2r]_{p,q}![r]_{p,q}!} = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}x^n t^n}{[n]_{p,q}!}.$$

Therefore, when the corresponding values of t from each side are compared, yields the assertion (30).

Once more, utilizing Formula (7), the result is

$$E_{p,q}(-xt)\ e_{p,q}(xt)e_{p,q}(yt^2) = e_{p,q}(yt^2).$$

Utilizing Formulas (1), (5) and (6) in the above equation, we receive

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{q^{\binom{r}{2}}(-x)^r H_{n,p,q}(x,y)\ t^{n+r}}{[n]_{p,q}![r]_{p,q}!} = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}y^n t^{2n}}{[n]_{p,q}!},$$

which after utilizing the subsequent series arrangement method as in [18]:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k),$$

gives

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{q^{\binom{r}{2}}(-x)^r H_{n-r,p,q}(x,y)t^n}{[r]_{p,q}![n-r]_{p,q}!} = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}y^n t^{2n}}{[n]_{p,q}!}. \tag{33}$$

When each of the even as well as odd values of t from each side of Equation (33), are compared, we get the assertions (31) and (32). The proof is completed. □

Remark 1. *It is worth mentioning that the corresponding expression of the summation models, provided in Equation* (32)*, is as outlined below:*

$$\sum_{r=0}^{2m}\frac{q^{\binom{r}{2}}(-x)^r H_{2m+1-r,p,q}(x,y)}{[r]_{p,q}![2m+1-r]_{p,q}!} = \frac{q^{\binom{2m+1}{2}}x^{2m+1}}{[2m+1]_{p,q}!}. \tag{34}$$

We derive the subsequent summation models for the (p,q)-derivative of $H_{n,p,q}(x,y)$ from Theorem 1 and Remark 1:

Corollary 2. *The following summation models hold*

$$\sum_{r=0}^{[n/2]} \frac{q^{\binom{r}{2}}(-p^2y)^r D_{p,q,x} H_{n+1-2r,p,q}(x,p^2y)}{[r]_{p,q}![n+1-2r]_{p,q}!} = \frac{(px)^n_{p,q}}{[n]_{p,q}!}\ , \tag{35}$$

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}(-px)^r D_{p,q,x} H_{2m+1-r,p,q}(x,p^2y)}{[r]_{p,q}![2m+1-r]_{p,q}!} = \frac{(p^2y)^m_{p,q}}{[m]_{p,q}!} \tag{36}$$

and

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}(-px)^r D_{p,q,x} H_{2m+2-r,p,q}(x,p^2y)}{[r]_{p,q}![2m+2-r]_{p,q}!} = \frac{q^{\binom{2m+1}{2}}(px)^{2m+1}}{[2m+1]_{p,q}!}. \tag{37}$$

Proof. Using Equation (12), we get

$$D_{p,q,x} H_{n+1-2r,p,q}(x,p^2y) = [n+1-2r]_{p,q} H_{n-2r,p,q}(px,p^2y),$$

which on using Equation (18) and simplifying, gives

$$H_{n-2r,p,q}(x,y) = \frac{D_{p,q,x} H_{n+1-2r,p,q}(x,p^2y)}{p^{n-2r}[n+1-2r]_{p,q}}. \tag{38}$$

Using Equation (38) in Equation (30), we get the assertion (35). Similarly, using Equations (12) and (18), we get

$$H_{2m-r,p,q}(x,y) = \frac{D_{p,q,x} H_{2m+1-r,p,q}(x,p^2y)}{p^{2m-r}[2m+1-r]_{p,q}}. \tag{39}$$

We acquire the claim (36) by combining Equations (4) and (39) in Equation (31). We can deduce from Equation (39) that

$$H_{2m+1-r,p,q}(x,y) = \frac{D_{p,q,x} H_{2m+2-r,p,q}(x,p^2y)}{p^{2m+1-r}[2m+2-r]_{p,q}}. \tag{40}$$

Using Equation (40) in Equation (34), we get the assertion (37). □

Remark 2. *The subsequent alternatives are obtained by applying Equation (18) to the right-side of Equations (36) and (37):*

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}(-x)^r D_{p,q,x} H_{2m+1-r,p,q}(\frac{x}{p},y)}{[r]_{p,q}![2m+1-r]_{p,q}!} = \frac{(y)^m_{p,q}}{p[m]_{p,q}!}$$

and

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}(-x)^r D_{p,q,x} H_{2m+2-r,p,q}(\frac{x}{p},y)}{[r]_{p,q}![2m+1-r]_{p,q}!} = \frac{q^{\binom{2m+1}{2}}(x)^{2m+1}}{[2m+1]_{p,q}!},$$

respectively.

Likewise, we obtain the subsequent summation expressions for (p,q)-derivative of 2V(p,q)HP $H_{n,p,q}(x,y)$ with regard to y utilising Equations (13) and (18) in Equations (30), (31) and (34):

Corollary 3. *The summation models for the (p,q)-derivative of $H_{n,p,q}(x,y)$ are listed below:*

$$\sum_{r=0}^{[n/2]} \frac{q^{\binom{r}{2}}(-p^2y)^r D_{p,q,y} H_{n-2r+2,p,q}(px,py)}{[r]_{p,q}![n-2r+2]_{p,q}!} = \frac{(px)^n_{p,q}}{[n]_{p,q}!},$$

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}\left(-\sqrt{p}x\right)^{r} D_{p,q,y} H_{2m+2-r,p,q}\left(\sqrt{p}x,y\right)}{[r]_{p,q}![2m+2-r]_{p,q}!} = \frac{(py)^{m}_{p,q}}{[m]_{p,q}!}$$

and

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}\left(-\sqrt{p}x\right)^{r} D_{p,q,y} H_{2m+3-r,p,q}\left(\sqrt{p}x,y\right)}{[r]_{p,q}![2m+3-r]_{p,q}!} = \frac{q^{\binom{2m+1}{2}}\left(\sqrt{p}x\right)^{2m+1}}{[2m+1]_{p,q}!}.$$

Example 3. *The following correlations of (p,q)-summation models for 2-variable (p,q)-Hermite polynomials can be obtained immediately by applying (30) and (35):*

$$\frac{H_{4,p,q}(x,y)}{[4]_{p,q}!} - \frac{yH_{2,p,q}(x,y)}{[2]_{p,q}!} + \frac{qy^2}{[2]_{p,q}!} = \frac{p^6x^4}{[4]_{p,q}!},$$

$$\frac{D_{p,q,x}H_{5,p,q}(x,p^2y)}{[5]_{p,q}!} - \frac{p^2yD_{p,q,x}H_{3,p,q}(x,p^2y)}{[3]_{p,q}!} + \frac{p^4y^2D_{p,q,x}H_{1,p,q}(x,p^2y)}{[2]_{p,q}!} = \frac{(px)^4_{p,q}}{[4]_{p,q}!},$$

5. Recommendations for Future Research

We conclude this article by highlighting some important research directions that could be considered in the future.

Since in (p,q)-calculus, we have two types of exponential functions, we exploit this opportunity to introduce some other kinds of 2-variable(p,q)-Hermite polynomials. In view of the following identity (see [25,26]):

$$e_{p,q}(a)\ E_{p,q}(b) = \sum_{n=0}^{\infty} \frac{(a \oplus b)^n_{p,q}}{[n]_{p,q}!}, \tag{41}$$

we have

$$e_{p,q}(xt)\ E_{p,q}(yt^2) = \sum_{n=0}^{\infty} \frac{(xt \oplus yt^2)^n_{p,q}}{[n]_{p,q}!}, \tag{42}$$

We now create another form of the two-variable (p,q)-Hermite polynomials $\mathcal{H}_{n,p,q}(x,y)$ in the following manner:

$$e_{p,q}(xt)\ E_{p,q}(yt^2) = \sum_{n=0}^{\infty} \mathcal{H}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}. \tag{43}$$

Using Equation (41) in Equation (43), we get

$$\sum_{n=0}^{\infty} \frac{(xt \oplus yt^2)^n_{p,q}}{[n]_{p,q}!} = \sum_{n=0}^{\infty} \mathcal{H}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!}. \tag{44}$$

Using Equation (3) and the subsequent series rearrangement technique (see [18]), the left side of Equation (44) is expanded as as follows:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(n-k,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(n-2k,k)$$

and then comparing the equal powers of t from the both sides of the resultant equation, gives the following series definition of 2V$(p,q)\mathcal{H}$P $\mathcal{H}_{n,p,q}(x,y)$:

$$\mathcal{H}_{n,p,q}(x,y) = [n]_{p,q}! \sum_{k=0}^{[n/2]} \frac{x^k (y)^{n-2k}_{p,q} q^{\binom{k}{2}}}{[n-2k]_{p,q}![k]_{p,q}!}.$$

Applying the (p,q)-partial derivative of each side of Equation (43) with regard to x, then once more using Equation (43), then comparing the equal values of t of each of the sides of the resulting equation, we receive

$$D_{p,q,x}\mathcal{H}_{n,p,q}(x,y) = [n]_{p,q}\mathcal{H}_{n-1,p,q}(px,y) \quad \text{for } n \geq 1.$$

Likewise, taking the (p,q)-partial derivative of each side of Equation (43) with respect to y, then done once again using Equation (43) and matching the identical values of t from the two sides of the resulting equation, we obtain

$$D_{p,q,y}\Big(\mathcal{H}_{n,p,q}(x,y)\Big) = [n]_{p,q}[n-1]_{p,q}\mathcal{H}_{n-2,p,q}(x,qy) \quad \text{for } n \geq 1.$$

We believe that the above newly discovered results will help us obtain new results such as recurrence relations, (p,q)-differential equation, summation formulas and integral representations related to other kinds of 2-variable (p,q)-Hermite polynomials in future studies.

6. Conclusions

In this paper, we establish new various features of 2-variable (p,q)-Hermite polynomials, such as diffusion equation, differential equations, integral and summation representations as follows:

- **(p,q)-Diffusion equation** (see Equation (20)):

$$D^2_{p,q,x}H_{n,p,q}(x,p^3y) = p\, D_{p,q,y}H_{n,p,q}(p^2x,p^2y).$$

- **(p,q)-Differential equation** (see Theorem 1):
The subsequent (p,q)-differential equation for (p,q)-Hermite polynomials of 2-variables holds true:

$$yL^{-2}_{p,x}\, L_{q,x}\left(1+\left(\frac{q}{p}\right)^{\frac{n}{2}} L_{\sqrt{\frac{p}{q}},x}\right)H''_{n,p,q}(x,y) + pxH'_{n,p,q}(x,y) - [n]_{p,q}\, p^{2-n}L_{p,x}H_{n,p,q}(x,y) = 0,$$

in which $L_{\cdot,x}$ represents the shift operator described in (15).

- **Integral representations** (see Theorems 2 and 3):
(i) The definite (p,q)-integral of $H_{n,p,q}(x,y)$ with regard to x is as follows:

$$\int_a^b H_{n,p,q}(x,y)d_{p,q}x = p\frac{H_{n+1,p,q}\left(\frac{b}{p},y\right) - H_{n+1,p,q}\left(\frac{a}{p},y\right)}{[n+1]_{p,q}}.$$

(ii) The definite (p,q)-integral of $H_{n,p,q}(x,y)$ with respect to y is as follows:

$$\int_c^d H_{n,p,q}(x,y)d_{p,q}y = p\frac{H_{n+2,p,q}\left(x,\frac{d}{p}\right) - H_{n+2,p,q}\left(x,\frac{c}{p}\right)}{[n+1]_{p,q}[n+2]_{p,q}}.$$

- **Summation representations** (see Theorem 4):
The summation models for (p,q)-Hermite polynomials of 2-variables are given below:

$$\sum_{r=0}^{[n/2]} \frac{q^{\binom{r}{2}}(-y)^r H_{n-2r,p,q}(x,y)}{[r]_{p,q}![n-2r]_{p,q}!} = \frac{p^{\binom{n}{2}}x^n}{[n]_{p,q}!}.$$

(1) If $n = 2m$ ($m \in \mathbb{N}$), then

$$\sum_{r=0}^{2m} \frac{q^{\binom{r}{2}}(-x)^r H_{2m-r,p,q}(x,y)}{[r]_{p,q}![2m-r]_{p,q}!} = \frac{p^{\binom{m}{2}} y^m}{[m]_{p,q}!}.$$

(2) If $n = 2m+1$ ($m \in \mathbb{N} \cup \{0\}$), then

$$\sum_{r=0}^{2m+1} \frac{q^{\binom{r}{2}}(-x)^r H_{2m+1-r,p,q}(x,y)}{[r]_{p,q}![2m+1-r]_{p,q}!} = 0.$$

As applications, some new features for 2-variable (p,q)-Hermite polynomials are presented in Sections 3 and 4.

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