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Cross Curvature Solitons of Lorentzian Three-Dimensional Lie Groups

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Abstract: In this paper, we study left-invariant cross curvature solitons on Lorentzian three-dimensional Lie groups and classify these solitons.

Keywords: cross curvature soliton; lie groups; homogeneous spaces

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1. Introduction

The study of various geometric flows used to improve a given metric for a geometric object has been undertaken by many mathematicians and physicists. Important geometric flows are the Ricci flow, Yamabe flow, mean curvature flow, Ricci-harmonic flow, and cross-curvature flow. These flows are impressive subjects in mathematical physics and geometry. The special solutions for geometric flow are solitons. In fact, solitons are the self-similar solution to flow. R. Hamilton [1] presented the Ricci soliton as $\frac{1}{2}\mathcal{L}_X g + Ric = \lambda g$ for the first time, which is a natural extension of Einstein metrics. After that, many authors generalized this soliton and introduced other solitons corresponding to other geometric flows.

The goal of this study is to discuss three-dimensional homogeneous Lorentzian cross curvature solitons. Three-dimensional locally homogeneous Lorentzian manifolds can fall into one of two categories: they are either locally isometric to a three-dimensional Lie group with a Lorentzian left-invariant metric or locally symmetric.

Suppose that (M, g) is a three-dimensional manifold. We consider the tensor

$$P_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}, \tag{1}$$

where R is the scalar curvature and R_{ij} is the Ricci tensor of M. Set

$$P^{ij} = g^{ik}g^{jl}R_{kl} - \frac{1}{2}Rg^{ij}. \tag{2}$$

The cross curvature tensor is defined as follows, where V_{ij} is the inverse of P^{ij}:

$$h_{ij} = \frac{\det P^{kl}}{\det g^{kl}} V_{ij}. \tag{3}$$

In the pseudo-Riemannian case, if a vector field X on M and a constant λ exist such that

$$\mathcal{L}_X g + \lambda g = 2h, \tag{4}$$

then (M, g) is a cross curvature soliton. We mention that $\mathcal{L}_X g$ indicates the Lie derivative of g with regard to X, and h is the cross curvature tensor of g. A cross curvature soliton is an interesting type of solution to the cross curvature flow. It is actually the self-similar solution of the cross curvature flow [2,3]. A cross curvature soliton is stated as being either expanding, steady, or shrinking if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively.

It is worth noting that when $\mathcal{L}_X g$ is equal to zero, a cross curvature soliton is considered trivial. The cross curvature flow, which was introduced by Chow and Hamilton, represented a significant advancement in this field [4]. Buckland's groundbreaking work on the short-term existence of this flow should not be underestimated [5]. Additionally, Cao et al. conducted a comprehensive study on the non-negative cross curvature flow on locally homogeneous Riemannian three-dimensional manifolds, providing valuable insights into the long-term behavior of this flow [6,7]. For further information, please consult [8–10].

Also, other geometric solitons have been studied on locally homogeneous manifolds. For instance, it has been proven that Lie groups with a left-invariant Riemannian metric of dimension of four at most lack non-trivial homogeneous invariant Ricci solitons (see [11–14]), but there are three-dimensional Riemannian homogeneous Ricci solitons [15,16]. Lauret's work established that every algebraic Ricci soliton on a Lie group with left-invariant Riemannian metric is a homogeneous Ricci soliton [17], and Onda later extended this finding to the case of Lie groups with pseudo-Riemannian left-invariant metric [18]. Additionally, Calvaruso and Fino discovered the Ricci solitons on four-dimensional non-reductive homogeneous spaces [19]. Also, for some consequences of Ricci solitons on homogeneous manifolds, refer to [20,21].

The paper is arranged as follows. Section 2 will delve into essential concepts on three-dimensional Lie groups, which will be integral to the paper. Section 3 will present the key findings and their corresponding proofs.

2. Lorentzian Lie Groups in Dimension 3

The Bianchi classification provides a list of all real three-dimensional Lie algebras. This classification contains 11 classes, two of which contain a continuum-sized family of Lie algebras and nine of which contain a single Lie algebra. In the following, we offer a succinct introduction to unimodular and non-unimodular Lie groups in three dimensions. It is important to note that fully connected and simply structured three-dimensional Lorentzian homogeneous manifolds can exhibit either symmetry or a left-invariant Lorentzian metric as a Lie group [22].

2.1. Unimodular Lie Groups

Suppose that $\{e_1, e_2, e_3\}$ is an orthonormal basis of signature $(+ + -)$. We represent the Lorentzian vector product on $\mathbb{R}^3_1$, which is generated by the cross product $\times$, i.e.,

$$e_3 \times e_1 = e_2, \quad e_2 \times e_3 = e_1, \quad e_1 \times e_2 = -e_3.$$

The Lie algebra $\mathfrak{g}$ is defined by the Lie bracket $[\,,\,]$. It is important to note that the algebra is only unimodular if the endomorphism L, which is defined as $[Z, Y] = L(Z \times Y)$, is self-adjoint. Additionally, L is non-unimodular when it is not self-adjoint. By analyzing the various types of L, we can identify four distinct classes of unimodular three-dimensional Lie algebras [23].

Type Ia.

The Lie algebra corresponding to a diagonalizable endomorphism L with three real eigenvalues $\{\alpha, \beta, \gamma\}$ regarding an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+ + -)$ is presented by

$$(\mathfrak{g}_{Ia}): \qquad [e_2, e_3] = \alpha e_1, \quad [e_1, e_3] = -\beta e_2, \quad [e_1, e_2] = -\gamma e_3.$$

In this case, Lie groups G admitting a Lie algebra $\mathfrak{g}_{Ia}$ are listed in Table 1.

Table 1. Type Ia. Lie groups G admitting a Lie algebra $\mathfrak{g}_{Ia}$.

G	α	β	γ
$O(1,2)$ or $SL(2,\mathbb{R})$	+	+	+
$O(1,2)$ or $SL(2,\mathbb{R})$	+	−	−
$SO(3)$ or $SU(2)$	+	+	−
$E(2)$	+	+	0
$E(2)$	+	0	−
$E(1,1)$	+	−	0
$E(1,1)$	+	0	+
H_3	+	0	0
H_3	0	0	−
$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$	0	0	0

In this case, the Levi-Civita connection is specified by

$$(\nabla_{e_i}e_j) = \begin{pmatrix} 0 & -\frac{1}{2}(\gamma+\beta-\alpha)e_3 & -\frac{1}{2}(\gamma+\beta-\alpha)e_2 \\ \frac{1}{2}(\alpha-\beta+\gamma)e_3 & 0 & \frac{1}{2}(\gamma-\beta+\alpha)e_1 \\ \frac{1}{2}(\alpha+\beta-\gamma)e_2 & \frac{1}{2}(\gamma-\beta-\alpha)e_1 & 0 \end{pmatrix}.$$

Let ∇ be the Levi-Civita connection; by using the formula $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$, the only non-vanishing terms of the curvature tensor are presented by

$$\begin{aligned} R_{2332} &= \tfrac{1}{4}\Big(-\gamma^2-\beta^2+3\alpha^2+2\beta\gamma-2\alpha\gamma-2\alpha\beta\Big), \\ R_{1313} &= \tfrac{1}{4}\Big(\gamma^2-3\beta^2+\alpha^2+2\beta\gamma-2\alpha\gamma+2\alpha\beta\Big), \\ R_{1221} &= \tfrac{1}{4}\Big(-3\gamma^2+\beta^2+\alpha^2+2\beta\gamma+2\alpha\gamma-2\alpha\beta\Big), \end{aligned}$$

its Ricci tensor is expressed by

$$\begin{aligned} R_{11} &= -\tfrac{1}{2}\Big(\alpha^2-(\gamma-\beta)^2\Big), \\ R_{22} &= -\tfrac{1}{2}\Big(\beta^2-(\gamma-\alpha)^2\Big), \\ R_{33} &= \tfrac{1}{2}\Big(\gamma^2-(\beta-\alpha)^2\Big), \end{aligned}$$

and other components are 0. The Lie derivative of the metric, $\mathcal{L}_X g$, for an optional left-invariant vector field $X = \sum_{i=1}^{3} x_i e_i$ is given by

$$(\mathcal{L}_X g) = \begin{pmatrix} 0 & (\alpha-\beta)x_3 & (\gamma-\alpha)x_2 \\ (\alpha-\gamma)x_3 & 0 & (\beta-\gamma)x_1 \\ (\gamma-\alpha)x_2 & (\beta-\gamma)x_1 & 0 \end{pmatrix}.$$

Then,

$$R = \frac{1}{2}\Big(\gamma^2+\alpha^2+\beta^2-2\alpha\beta-2\alpha\gamma-2\beta\gamma\Big)$$

and

$$P^{11} = \frac{1}{4}\Big(-3\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma - 2\beta\gamma \Big),$$
$$P^{22} = \frac{1}{4}\Big(\alpha^2 - 3\beta^2 + \gamma^2 + 2\alpha\beta - 2\alpha\gamma + 2\beta\gamma \Big),$$
$$P^{33} = \frac{1}{4}\Big(-\alpha^2 - \beta^2 + 3\gamma^2 + 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma \Big),$$

and other components of P^{ij} are 0. Throughout the paper, we assume that (P^{ij}) is invertible. Therefore, the only non-vanishing terms of the cross curvature tensor are obtained as follows:

$$h_{11} = -\frac{1}{16}(\gamma^2 - 3\beta^2 + \alpha^2 + 2\beta\gamma - 2\alpha\gamma + 2\alpha\beta)(3\gamma^2 - \beta^2 - \alpha^2 - 2\beta\gamma - 2\alpha\gamma + 2\alpha\beta),$$
$$h_{22} = -\frac{1}{16}(\gamma^2 + \beta^2 - 3\alpha^2 - 2\beta\gamma + 2\alpha\gamma + 2\alpha\beta)(3\gamma^2 - \beta^2 - \alpha^2 - 2\beta\gamma - 2\alpha\gamma + 2\alpha\beta),$$
$$h_{33} = -\frac{1}{16}(\gamma^2 + \beta^2 - 3\alpha^2 - 2\beta\gamma + 2\alpha\gamma + 2\alpha\beta)(\gamma^2 - 3\beta^2 + \alpha^2 + 2\beta\gamma - 2\alpha\gamma + 2\alpha\beta).$$

Type Ib.

Suppose that L has complex eigenvalues $\gamma \pm i\beta$ and one real eigenvalue α. Then, by considering an orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+\,+\,-)$, we have

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \gamma & -\beta \\ 0 & \beta & \gamma \end{pmatrix}, \qquad \beta \neq 0,$$

then the related Lie algebra is provided by

$$(\mathfrak{g}_{Ib}): \qquad [e_2, e_3] = \alpha e_1, \ \ [e_1, e_3] = -\gamma e_2 - \beta e_3, \ \ [e_1, e_2] = \beta e_2 - \gamma e_3.$$

In this case, $G = O(1,2)$ or $G = SL(2,\mathbb{R})$ if $\alpha \neq 0$, while $G = E(1,1)$ if $\alpha = 0$. The Levi-Civita connection is specified by

$$(\nabla_{e_i} e_j) = \begin{pmatrix} 0 & \frac{1}{2}(\alpha - 2\gamma)e_3 & \frac{1}{2}(\alpha - 2\gamma)e_2 \\ \frac{\alpha}{2}e_3 - \beta e_2 & \beta e_1 & \frac{\alpha}{2}e_1 \\ \frac{\alpha}{2}e_2 + \beta e_3 & -\frac{\alpha}{2}e_1 & \beta e_1 \end{pmatrix}.$$

With respect to the basis $\{e_1, e_2, e_3\}$ the only non-vanishing terms of the curvature tensor are described by

$$R_{1231} = -(2\gamma - \alpha)\beta, \quad R_{2332} = \frac{3}{4}\alpha^2 - \alpha\gamma + \beta^2, \quad R_{1313} = R_{1221} = \frac{1}{4}(4\beta^2 + \alpha^2),$$

and its Ricci tensor is expressed by

$$Ric = \begin{pmatrix} -\frac{1}{2}(\alpha^2 + 4\beta^2) & 0 & 0 \\ 0 & \frac{1}{2}\alpha(\alpha - 2\gamma) & -\beta(\alpha - 2\gamma) \\ 0 & \beta(2\gamma - \alpha) & \frac{1}{2}\alpha(2\gamma - \alpha) \end{pmatrix}.$$

For an optional left-invariant vector field $X = \sum_{i=1}^{3} x_i e_i$, we obtain

$$(\mathcal{L}_X g) = \begin{pmatrix} 0 & \beta x_2 + (\alpha - \gamma)x_3 & \beta x_3 + (\gamma - \alpha)x_2 \\ \beta x_2 + (\alpha - \gamma)x_3 & -2\beta x_1 & 0 \\ \beta x_3 + (\gamma - \alpha)x_2 & 0 & -2\beta x_1 \end{pmatrix}.$$

Hence,

$$R = \frac{1}{2}\Big(\alpha^2 - 4\beta^2 - 4\alpha\gamma\Big)$$

and

$$(P^{ij}) = \begin{pmatrix} \frac{1}{4}(-3\alpha^2 - 4\beta^2 + 4\alpha\gamma) & 0 & 0 \\ 0 & \frac{1}{4}(\alpha^2 + 4\beta^2) & \beta(\alpha - 2\gamma) \\ 0 & \beta(\alpha - 2\gamma) & -\frac{1}{4}(\alpha^2 + 4\beta^2) \end{pmatrix}.$$

Let

$$A_1 = -\Big(\frac{1}{16}(\alpha^2 + 4\beta^2)^2 + \beta^2(\alpha - 2\gamma)^2\Big)$$

then,

$$(V_{ij}) = \frac{1}{A_1}\begin{pmatrix} -4\frac{\frac{1}{16}(\alpha^2+4\beta^2)^2+\beta^2(\alpha-2\gamma)^2}{-3\alpha^2-4\beta^2+4\alpha\gamma} & 0 & 0 \\ 0 & -\frac{1}{4}(\alpha^2 + 4\beta^2) & -\beta(\alpha - 2\gamma) \\ 0 & \beta(2\gamma - \alpha) & \frac{1}{4}(\alpha^2 + 4\beta^2) \end{pmatrix}.$$

Therefore, the cross curvature tensor is described by

$$(h_{ij}) = -\frac{1}{4}(-3\alpha^2 - 4\beta^2 + 4\alpha\gamma)\begin{pmatrix} -4\frac{\frac{1}{16}(\alpha^2+4\beta^2)^2+\beta^2(\alpha-2\gamma)^2}{-3\alpha^2-4\beta^2+4\alpha\gamma} & 0 & 0 \\ 0 & -\frac{1}{4}(\alpha^2 + 4\beta^2) & \beta(2\gamma - \alpha) \\ 0 & \beta(2\gamma - \alpha) & \frac{1}{4}(\alpha^2 + 4\beta^2) \end{pmatrix}.$$

Type II.

Suppose that the minimal polynomial of L has two roots, α and β, such that $(L - \alpha I)(L - \beta I)^2 = 0$ holds. So, regarding the orthonormal basis $\{e_1, e_2, e_3\}$ of signature $(+\,+\,-)$ we have

$$L = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta + \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \beta - \frac{1}{2} \end{pmatrix},$$

then the related Lie algebra is provided by

$$(\mathfrak{g}_{II}): \qquad [e_2, e_3] = \alpha e_1, \quad [e_1, e_3] = -\frac{1}{2}\beta e_3 - \Big(\frac{1}{2} + \beta\Big)e_2, \quad [e_1, e_2] = \Big(\frac{1}{2} - \beta\Big)e_3 + \frac{1}{2}e_2.$$

In this case, Lie groups admitting a Lie algebra $\mathfrak{g}_{II}$ are listed in Table 2.

Table 2. Lie groups admitting a Lie algebra $\mathfrak{g}_{II}$.

$\boldsymbol{G}$	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$
$O(1,2)$ or $SL(2,\mathbb{R})$	$\neq 0$	$\neq 0$
$E(1,1)$	0	$\neq 0$
$E(1,1)$	<0	0
$E(2)$	>0	0
H_3	0	0

The Levi-Civita connection in this case is expressed by

$$(\nabla_{e_i}e_j) = \begin{pmatrix} 0 & -\frac{1}{2}(2\beta-\alpha)e_3 & \frac{1}{2}(\alpha-2\beta)e_2 \\ \frac{1}{2}(-1+\alpha)e_3 - \frac{1}{2}e_2 & \frac{1}{2}e_1 & \frac{1}{2}(-1+\alpha)e_1 \\ \frac{1}{2}e_3 + \frac{1}{2}(1+\alpha)e_2 & -\frac{1}{2}(1+\alpha)e_1 & \frac{1}{2}e_1 \end{pmatrix}.$$

With respect to the basis $\{e_1, e_2, e_3\}$, the only non-zero terms of the curvature tensor are described by

$$R_{1231} = \frac{1}{2}\alpha - \beta, \quad R_{2332} = \frac{1}{4}(-4\alpha\beta + 3\alpha^2),$$
$$R_{1313} = \frac{1}{4}(-4\beta + 2\alpha + \alpha^2), \quad R_{1221} = \frac{1}{4}(4\beta - 2\alpha + \alpha^2),$$

and its Ricci tensor is given by

$$Ric = \begin{pmatrix} -\frac{1}{2}\alpha^2 & 0 & 0 \\ 0 & \frac{1}{2}(-2\beta+\alpha)(1+\alpha) & \beta - \frac{1}{2}\alpha \\ 0 & -\frac{1}{2}\alpha + \beta & -\frac{1}{2}(2\beta-\alpha)(1-\alpha) \end{pmatrix}.$$

For $X = \sum_{i=1}^{3} x_i e_i$ as an optional left-invariant vector field, we obtain

$$(\mathcal{L}_X g) = \begin{pmatrix} 0 & \frac{1}{2}\big(x_2 + (2\alpha - 2\beta - 1)x_3\big) & \frac{1}{2}\big(x_3 + (2\beta - 2\alpha - 1)x_2\big) \\ \frac{1}{2}\big(x_2 + (2\alpha - 2\beta - 1)x_3\big) & -x_1 & x_1 \\ \frac{1}{2}\big(x_3 + (2\beta - 2\alpha - 1)x_2\big) & x_1 & -x_1 \end{pmatrix}.$$

Thus,

$$R = \frac{1}{2}\alpha^2 - 2\alpha\beta$$

and

$$(P^{ij}) = \begin{pmatrix} -\frac{3}{4}\alpha^2 + \alpha\beta & 0 & 0 \\ 0 & \frac{1}{2}\left(\frac{1}{2}\alpha^2 + \alpha - 2\beta\right) & \frac{1}{2}\alpha - \beta \\ 0 & \frac{1}{2}\alpha - \beta & -\frac{1}{2}\left(\frac{1}{2}\alpha^2 - \alpha + 2\beta\right) \end{pmatrix}.$$

Then,

$$(V_{ij}) = \frac{-16}{\alpha^4}\begin{pmatrix} -\frac{\frac{1}{16}\alpha^4}{-\frac{3}{4}\alpha^2+\alpha\beta} & 0 & 0 \\ 0 & -\frac{1}{2}\left(\frac{1}{2}\alpha^2-\alpha+2\beta\right) & -\frac{1}{2}\alpha+\beta \\ 0 & -\frac{1}{2}\alpha+\beta & \frac{1}{2}\left(\frac{1}{2}\alpha^2+\alpha-2\beta\right) \end{pmatrix}.$$

Therefore, the cross curvature tensor is described by

$$(h_{ij}) = -(-\frac{3}{4}\alpha^2+\alpha\beta)\begin{pmatrix} -\frac{\frac{1}{16}\alpha^4}{-\frac{3}{4}\alpha^2+\alpha\beta} & 0 & 0 \\ 0 & -\frac{1}{2}\left(\frac{1}{2}\alpha^2-\alpha+2\beta\right) & \beta-\frac{1}{2}\alpha \\ 0 & -\frac{1}{2}\alpha+\beta & \frac{1}{2}\left(\frac{1}{2}\alpha^2+\alpha-2\beta\right) \end{pmatrix}.$$

Type III.

Suppose that the minimal polynomial of L has one real root α such that $(L-\alpha I)^3=0$ holds. So, regarding the orthonormal basis $\{e_1,e_2,e_3\}$ of signature $(++-)$ we have

$$L = \begin{pmatrix} \alpha & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \alpha & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \alpha \end{pmatrix},$$

then the related Lie algebra is provided by

$$(\mathfrak{g}_{III}) : \begin{cases} [e_1,e_2] = -\frac{1}{\sqrt{2}}e_1-\alpha e_3, \quad [e_1,e_3] = -\frac{1}{\sqrt{2}}e_1-\alpha e_2, \\ [e_2,e_3] = \alpha e_1+\frac{1}{\sqrt{2}}e_2-\frac{1}{\sqrt{2}}e_3. \end{cases}$$

In this case, $G=O(1,2)$ or $G=SL(2,\mathbb{R})$ if $\alpha\neq 0$ and $G=E(1,1)$ if $\alpha=0$. The Levi-Civita connection in this case is expressed by

$$(\nabla_{e_i}e_j) = \begin{pmatrix} \frac{1}{\sqrt{2}}e_2-\frac{1}{\sqrt{2}}e_3 & -\frac{\alpha}{2}e_3-\frac{1}{\sqrt{2}}e_1 & -\frac{\alpha}{2}e_2-\frac{1}{\sqrt{2}}e_1 \\ \frac{\alpha}{2}e_3 & \frac{1}{\sqrt{2}}e_3 & \frac{1}{\sqrt{2}}e_2+\frac{\alpha}{2}e_1 \\ \frac{\alpha}{2}e_2 & \frac{1}{\sqrt{2}}e_3-\frac{\alpha}{2}e_1 & \frac{1}{\sqrt{2}}e_2 \end{pmatrix}.$$

The only non-zero terms of the curvature tensor are described by

$$R_{1223} = \frac{1}{\sqrt{2}}\alpha, \quad R_{1313} = 1-\frac{1}{4}\alpha^2, \quad R_{1231} = 1,$$
$$R_{2323} = \frac{1}{4}\alpha^2, \quad R_{1221} = \frac{1}{4}(4+\alpha^2).$$

and its Ricci tensor is expressed by

$$Ric = \begin{pmatrix} -\frac{1}{2}\alpha^2 & -\frac{1}{\sqrt{2}}\alpha & -\frac{1}{\sqrt{2}}\alpha \\ -\frac{1}{\sqrt{2}}\alpha & -\frac{1}{2}\alpha^2 - 1 & -1 \\ -\frac{1}{\sqrt{2}}\alpha & -1 & -1 + \frac{1}{2}\alpha^2 \end{pmatrix}.$$

For $X = \sum_{i=1}^{3} x_i e_i$ as an optional left-invariant vector field, the Lie derivative of the metric is presented by the following relation.

$$(\mathcal{L}_X g) = \frac{1}{\sqrt{2}} \begin{pmatrix} -2(x_3 + x_2) & x_1 & x_1 \\ x_1 & 2x_3 & x_3 - x_2 \\ x_1 & -x_2 + x_3 & -2x_2 \end{pmatrix}.$$

Hence,

$$R = -\frac{3}{2}\alpha^2$$

and

$$(P^{ij}) = \begin{pmatrix} \frac{1}{4}\alpha^2 & -\frac{1}{\sqrt{2}}\alpha & \frac{1}{\sqrt{2}}\alpha \\ -\frac{1}{\sqrt{2}}\alpha & \frac{1}{4}\alpha^2 - 1 & 1 \\ \frac{1}{\sqrt{2}}\alpha & 1 & -1 - \frac{1}{4}\alpha^2 \end{pmatrix}.$$

Thus,

$$(V_{ij}) = \frac{-16}{\alpha^4} \begin{pmatrix} -\frac{1}{4}\alpha^2 & \frac{-1}{\sqrt{2}}\alpha & \frac{-1}{\sqrt{2}}\alpha \\ \frac{-1}{\sqrt{2}}\alpha & -3 - \frac{1}{4}\alpha^2 & -3 \\ \frac{-1}{\sqrt{2}}\alpha & -3 & -3 + \frac{1}{4}\alpha^2 \end{pmatrix}.$$

Therefore, the cross curvature tensor is described by

$$(h_{ij}) = -\frac{\alpha^2}{4} \begin{pmatrix} -\frac{1}{4}\alpha^2 & \frac{-1}{\sqrt{2}}\alpha & \frac{-1}{\sqrt{2}}\alpha \\ \frac{-1}{\sqrt{2}}\alpha & -3 - \frac{1}{4}\alpha^2 & -3 \\ \frac{-1}{\sqrt{2}}\alpha & -3 & -3 + \frac{1}{4}\alpha^2 \end{pmatrix}.$$

2.2. Non-Unimodular Lie Groups

Moving on, we will address the non-unimodular case. We will use the class $\mathfrak{G}$ to represent a set of solvable Lie algebras $\mathfrak{g}$ where, for any $x, y \in \mathfrak{g}$, $[x, y]$ is a linear combination of x and y. According to [24], the Lorentzian non-unimodular Lie algebras with non-constant sectional curvature that do not fall under class $\mathfrak{G}$ can be represented using the following relation in a suitable basis $\mathcal{E} = \{e_1, e_2, e_3\}$,

$$(\mathfrak{g}_{IV}) : [e_2, e_3] = \delta e_2 + \gamma e_1, \quad [e_1, e_3] = \beta e_2 + \alpha e_1, \quad [e_1, e_2] = 0,$$

we have $\delta + \alpha \neq 0$ and one of the next modes is established:

IV.1 $\mathcal{E}$ is orthonormal and $\langle e_3, e_3 \rangle = \langle e_2, e_2 \rangle = -\langle e_1, e_1 \rangle = 1$; also, the constants of structure satisfy $\beta\delta = \alpha\gamma$.

IV.2 $\mathcal{E}$ is orthonormal and $-\langle e_3, e_3 \rangle = \langle e_2, e_2 \rangle = \langle e_1, e_1 \rangle = 1$; also, the constants of structure satisfy $\beta\delta = -\alpha\gamma$.

IV.3 $\mathcal{E}$ is a pseudo-orthonormal basis and

$$\langle \cdot, \cdot \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

also, the constants of structure satisfy $\alpha\gamma = 0$.

Type IV.1.

In this case, the Levi-Civita connection is given by

$$(\nabla_{e_i} e_j) = \begin{pmatrix} \alpha e_3 & -\frac{\beta-\gamma}{2} e_3 & \alpha e_1 + \frac{\beta-\gamma}{2} e_2 \\ -\frac{\beta-\gamma}{2} e_3 & -\delta e_3 & -\frac{\beta-\gamma}{2} e_1 + \delta e_2 \\ -\frac{\beta+\gamma}{2} e_2 & -\frac{\beta+\gamma}{2} e_{31} & 0 \end{pmatrix}.$$

With respect to the basis $\mathcal{E}$, the only non-vanishing terms of the curvature tensor are described by

$$R_{2332} = \frac{1}{4}(-3\gamma^2 + \beta^2 + 2\beta\gamma + 4\delta^2),$$
$$R_{1313} = \frac{1}{4}(-3\beta^2 + 4\alpha^2 + 2\beta\gamma + \gamma^2),$$
$$R_{1212} = \frac{1}{4}(\gamma^2 + \beta^2 - 2\beta\gamma + 4\alpha\delta),$$

its Ricci tensor is expressed by

$$R_{11} = -\frac{1}{2}\big(-\gamma^2 + \beta^2 - 2(\delta\alpha + \alpha^2)\big),$$
$$R_{22} = \frac{1}{2}\big(-\beta^2 + \gamma^2 - 2(\delta^2 + \delta\alpha)\big),$$
$$R_{33} = \frac{1}{2}\big(-2(\delta^2 + \alpha^2) + (\gamma - \beta)^2\big),$$

and other components are 0. For $X = \sum_{i=1}^{3} x_i e_i$ as an optional left-invariant vector field, the Lie derivative of the metric $\mathcal{L}_X g$ is equal to

$$(\mathcal{L}_X g) = \begin{pmatrix} -2\alpha x_3 & (\beta-\gamma)x_3 & \alpha x_1 + \gamma x_2 \\ (\beta-\gamma)x_3 & 2\delta x_3 & -\beta x_1 - \delta x_2 \\ \alpha x_1 + \gamma x_2 & -\beta x_1 - \delta x_2 & 0 \end{pmatrix}.$$

Then,

$$R = \frac{1}{2}\big(\beta^2 - \gamma^2 - 4\alpha^2 - 4\delta^2 - 2\beta\gamma - 4\alpha\delta\big)$$

and

$$P^{11} = \frac{1}{4}\big(- \beta^2 + \gamma^2 - 4\delta^2 - 2\beta\gamma\big),$$

$$P^{22} = \frac{1}{4}\big(4\alpha^2 - 3\beta^2 + 3\gamma^2 + 2\beta\gamma\big),$$

$$P^{33} = \frac{1}{4}\big(3\gamma^2 + \beta^2 + 4\alpha\delta - 2\beta\gamma\big),$$

and other components are 0. Therefore, the only non-vanishing terms of the cross curvature tensor are described by

$$\begin{aligned} h_{11} &= -\frac{1}{16}\big(4\alpha^2 - 3\beta^2 + 3\gamma^2 + 2\beta\gamma\big)\big(\beta^2 + 3\gamma^2 - 2\beta\gamma + 4\alpha\delta\big), \\ h_{22} &= -\frac{1}{16}\big(- \beta^2 + \gamma^2 - 4\delta^2 - 2\beta\gamma\big)\big(\beta^2 + 3\gamma^2 - 2\beta\gamma + 4\alpha\delta\big), \\ h_{33} &= -\frac{1}{16}\big(- \beta^2 + \gamma^2 - 4\delta^2 - 2\beta\gamma\big)\big(4\alpha^2 - 3\beta^2 + 3\gamma^2 + 2\beta\gamma\big). \end{aligned}$$

Type IV.2.
The Levi-Civita connection of Type IV.2 concerning $\mathcal{E}$ is determined by

$$(\nabla_{e_i}e_j) = \begin{pmatrix} \alpha e_3 & \frac{\beta+\gamma}{2}e_3 & \alpha e_1 + \frac{\beta+\gamma}{2}e_2 \\ \frac{\gamma+\beta}{2}e_3 & \delta e_3 & \delta e_2 + \frac{\gamma+\beta}{2}e_1 \\ -\frac{-\gamma+\beta}{2}e_2 & \frac{-\gamma+\beta}{2}e_1 & 0 \end{pmatrix}.$$

With respect to the basis $\mathcal{E}$, the only non-vanishing terms of the curvature tensor are described by

$$\begin{aligned} R_{2323} &= \frac{1}{4}(-3\gamma^2 + \beta^2 - 2\beta\gamma - 4\delta^2), \\ R_{1331} &= \frac{1}{4}(3\beta^2 + 4\alpha^2 + 2\beta\gamma - \gamma^2), \\ R_{1212} &= -\frac{1}{4}(\gamma + \beta)^2 + \alpha\delta, \end{aligned}$$

and its Ricci tensor is expressed by

$$\begin{aligned} R_{11} &= \frac{1}{2}(-\gamma^2 + \beta^2 + 2(\delta\alpha + \alpha^2)) \\ R_{22} &= \frac{1}{2}(-\beta^2 + \gamma^2 + 2(\delta^2 + \delta\alpha)), \\ R_{33} &= -\frac{1}{2}(2(\delta^2 + \alpha^2) + (\gamma + \beta)^2), \end{aligned}$$

where other components of Ricci tensor are 0. For $X = \sum_{i=1}^{3} x_i e_i$ as an optional left-invariant vector field, we obtain

$$(\mathcal{L}_X g) = \begin{pmatrix} 2\alpha x_3 & (\beta + \gamma)x_3 & -\alpha x_1 - \gamma x_2 \\ (\beta + \gamma)x_3 & 2\delta x_3 & -\beta x_1 - \delta x_2 \\ -\alpha x_1 - \gamma x_2 & -\beta x_1 - \delta x_2 & 0 \end{pmatrix}.$$

Then,

$$R = (\alpha+\delta)^2 + \delta^2 + \alpha^2 + \frac{1}{2}(\gamma+\beta)^2$$

and

$$\begin{aligned} P^{11} &= \frac{1}{4}\big(-3\gamma^2 + \beta^2 - 4\delta^2 - 2\beta\gamma\big),\\ P^{22} &= \frac{1}{4}\big(-4\alpha^2 - 3\beta^2 + \gamma^2 - 2\beta\gamma\big),\\ P^{33} &= \frac{1}{4}\big(-\beta^2 - 2\beta\gamma - \gamma^2 + \alpha\gamma\big), \end{aligned}$$

and other components are 0. Therefore, the only non-vanishing terms of the cross curvature tensor are described by

$$\begin{aligned} h_{11} &= -\frac{1}{16}\big(-4\alpha^2 - 3\beta^2 + \gamma^2 - 2\beta\gamma\big)\big(-\beta^2 - \gamma^2 - 2\beta\gamma + \alpha\gamma\big),\\ h_{22} &= -\frac{1}{16}\big(\beta^2 - 3\gamma^2 - 4\delta^2 - 2\beta\gamma\big)\big(-\beta^2 - \gamma^2 - 2\beta\gamma + \alpha\gamma\big),\\ h_{33} &= -\frac{1}{16}\big(\beta^2 - 3\gamma^2 - 4\delta^2 - 2\beta\gamma\big)\big(-4\alpha^2 - 3\beta^2 + \gamma^2 - 2\beta\gamma\big). \end{aligned}$$

Type IV.3.
The Levi-Civita connection in this type is specified by

$$(\nabla_{e_i} e_j) = \begin{pmatrix} \alpha e_2 & \frac{\gamma}{2} e_2 & \alpha e_1 - \frac{\gamma}{2} e_3 \\ \frac{\gamma}{2} e_2 & 0 & \frac{\gamma}{2} e_1 \\ -\frac{\gamma}{2} e_3 - \beta e_2 & -\delta e_2 - \frac{\gamma}{2} e_1 & \delta e_3 - \beta e_1 \end{pmatrix}.$$

The only non-zero components of the curvature tensor are given by

$$R_{1213} = \tfrac{1}{4}\gamma^2,\ \ R_{1331} = \alpha^2 - \alpha\delta + \beta\gamma,\ \ R_{2332} = \tfrac{3}{4}\gamma^2,$$

and its Ricci tensor is expressed by

$$Ric = \begin{pmatrix} -\frac{1}{2}\gamma^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\gamma^2 \\ 0 & -\frac{1}{2}\gamma^2 & -(\beta + \alpha^2 - \alpha\delta) \end{pmatrix}.$$

For $X = \sum_{i=1}^{3} x_i e_i$ as an optional left-invariant vector field, we obtain

$$(\mathcal{L}_X g) = \begin{pmatrix} 2\alpha x_3 & \gamma x_3 & -\alpha x_1 - \gamma x_2 - \beta x_3 \\ \gamma x_3 & 0 & -\delta x_3 \\ -\alpha x_1 - \gamma x_2 - \beta x_3 & -\delta x_3 & 2(\beta x_1 + \delta x_2) \end{pmatrix}.$$

Thus, $R = \frac{1}{2}\gamma^2$.

$$(P^{ij}) = \begin{pmatrix} -\frac{3}{4}\gamma^2 & 0 & 0 \\ 0 & -(\alpha^2 - \alpha\delta + \beta) & -\frac{1}{4}\gamma^2 \\ 0 & -\frac{3}{4}\gamma^2 & 0 \end{pmatrix},$$

and

$$(V_{ij}) = \begin{pmatrix} -\frac{4}{3\gamma^2} & 0 & 0 \\ 0 & 0 & -\frac{4}{3\gamma^2} \\ 0 & -\frac{4}{\gamma^2} & \frac{16}{3\gamma^4}(\alpha^2 - \alpha\delta + \beta) \end{pmatrix}.$$

Therefore, the cross curvature tensor is given by

$$(h_{ij}) = \begin{pmatrix} \frac{3}{16}\gamma^4 & 0 & 0 \\ 0 & 0 & \frac{3}{16}\gamma^4 \\ 0 & \frac{9}{16}\gamma^4 & \frac{3}{4}(\alpha\delta - \beta - \alpha^2)\gamma^2 \end{pmatrix}.$$

According to the research conducted by Calvaruso in [25], there has been a significant study of three-dimensional Lorentzian locally conformally flat Lie groups. It has been proposed that these groups possess certain characteristics that are worth exploring further. From [25], we have the following proposition.

Proposition 1. *One of the defining characteristics of a Lorentzian three-dimensional Lie group (G, g) is that it is locally conformally flat if and only if one of the following conditions applies:*

(1) *(G, g) is locally symmetric and*
- *(1a) of Type Ia with $\gamma = \beta = \alpha$ or any cyclic permutation of $\beta = \alpha, \gamma = 0$*
- *(1b) of Type II with $\beta = \alpha = 0$*
- *(1c) of Type IV.1 with constant sectional curvature, or otherwise $\delta = \gamma = \beta = 0$ and $\alpha \neq 0$, or $\gamma = \alpha = \beta = 0$ and $\delta \neq 0$*
- *(1d) of Type IV.2 with constant sectional curvature, or otherwise $\delta = \gamma = \beta = 0$ and $\alpha \neq 0$, or $\gamma = \beta = \alpha = 0$ and $\delta \neq 0$*
- *(1e) of Type IV.3 and flat, or otherwise $\delta = \gamma = 0$ and $\alpha \neq 0$*
- *(1f) of Type $\mathfrak{G}$ and therefore of constant sectional curvature.*

(2) *(G, g) is not locally symmetric and*
- *(2a) of Type Ib with $\beta = \pm\sqrt{3}\gamma$ and $\alpha = -2\gamma$*
- *(2b) of Type III with $\alpha = 0$*
- *(2c) of Type IV.3 with $\alpha\delta(\alpha - \delta) \neq 0$ and $\gamma = 0$.*

3. Lorentzian Cross Curvature Solitons on Lorentzian 3-Dimensional Lie Groups

In this section, we will delve into the investigation of left-invariant solutions to (4) on the Lorentzian Lie groups that were examined in Section 2. Our aim is to solve the related equations completely and provide a comprehensive explanation of all left-invariant cross curvature solitons.

Theorem 1. *Suppose that $\mathfrak{g}$ indicate a Lorentzian unimodular three-dimensional Lie algebra of Type Ia. Then, the left-invariant cross curvature soliton on $\mathfrak{g}$ satisfies $\beta = \alpha = \gamma$, $\alpha \neq 0$, and $\lambda = \frac{1}{8}\alpha^2$, for all X. Also, as $\beta = \alpha = \gamma$, all vectors in $\mathfrak{g}$ are Killing.*

Proof. Considering (4), there is a cross curvature soliton of Type Ia if and only if the subsequent system of equations is satisfied:

$$\begin{aligned}
&(\beta-\alpha)x_3=(\alpha-\gamma)x_2=(\gamma-\beta)x_1=0,\\
&-\frac{1}{8}(\gamma^2-3\beta^2+\alpha^2+2\beta\gamma-2\alpha\gamma+2\alpha\beta)(3\gamma^2-\beta^2-\alpha^2-2\beta\gamma-2\alpha\gamma+2\alpha\beta)=\lambda,\\
&-\frac{1}{8}(\gamma^2+\beta^2-3\alpha^2-2\beta\gamma+2\alpha\gamma+2\alpha\beta)(-\alpha^2-\beta^2+3\gamma^2+2\alpha\beta-2\alpha\gamma-2\beta\gamma)=\lambda,\\
&-\frac{1}{8}(\gamma^2+\beta^2-3\alpha^2-2\beta\gamma+2\alpha\gamma+2\alpha\beta)(\gamma^2-3\beta^2+\alpha^2+2\beta\gamma-2\alpha\gamma+2\alpha\beta)=-\lambda.
\end{aligned} \tag{5}$$

The first equation in (5) indicates that $\alpha=\beta$ or $x_3=0$. We consider $\alpha=\beta$. Then, $(\gamma-\alpha)x_2=0$ yields $\gamma=\alpha$ or $x_2=0$. If $\gamma=\alpha$, then $\gamma=\beta$. Since the tensor (P^{ij}) is invertible, we conclude $\alpha\neq 0$. Thus, the last three equations in (5) reduce to $\lambda=\frac{1}{8}\alpha^2$. In this case, for any left-invariant vector field X, Equation (4) holds.

Now, we consider $\alpha=\beta$ and $\gamma\neq\alpha$. Then, $\beta\neq\gamma$ and $x_2=0$. The equation $(\beta-\gamma)x_1=0$ yields $x_1=0$. In this case, the last three equations in (5) reduce to

$$\begin{cases}-\frac{1}{8}\gamma^2(3\gamma^2-4\alpha\gamma)=\lambda,\\[2ex] -\frac{1}{8}\gamma^4=-\lambda.\end{cases}$$

Since the tensor (P^{ij}) is invertible, we have $\gamma\neq 0$; this implies that $\gamma=\alpha$, which is a contradiction.

Now, assume that $\alpha\neq\beta$. Then, $x_3=0$. From equation $(\gamma-\alpha)x_2=0$, we infer $\gamma=\alpha$ or $x_2=0$. If $\gamma=\alpha$, then the last three equations in (5) reduce to

$$\begin{cases}-\frac{1}{8}\beta^2(3\beta^2+4\alpha\beta)=\lambda,\\[2ex] \frac{1}{8}\beta^4=\lambda.\end{cases}$$

This system implies that $\alpha=\beta$, which is a contradiction. Hence, this case cannot happen. We suppose that $\beta\neq\alpha$, $\alpha\neq\gamma$ and $x_2=0$. From $(\gamma-\beta)x_1=0$, we have $\beta=\gamma$ or $x_1=0$. Similarly, the case $\beta=\gamma$ cannot occur. Then, we have $\beta\neq\alpha$, $\alpha\neq\gamma$, and $\gamma\neq\beta$. Also, $x_1=x_2=x_3=0$. In this case, using the last three equations of (5), we obtain

$$\begin{cases}\alpha^2+\beta\gamma=\beta^2+\alpha\gamma,\\[2ex] \gamma^2+\alpha\beta=\beta^2+\alpha\gamma.\end{cases}$$

Since $\beta\neq\gamma$, $\alpha\neq\gamma$, and $\alpha\neq\beta$, this system has no solution. □

From Theorem 1 and Proposition 1, we conclude the next result.

Corollary 1. *If a Type Ia Lorentzian unimodular Lie group is a left-invariant cross curvature soliton, then it is locally conformally flat.*

Theorem 2. *A Type Ib Lorentzian unimodular Lie groups does not accept any left-invariant cross curvature soliton.*

Proof. Considering (4), there is a cross curvature soliton of Type Ib if and only if the subsequent system of equations is satisfied:

$$\begin{cases} \frac{1}{8}(\alpha^2+4\beta^2)^2+2\beta^2(\alpha-2\gamma)^2=\lambda, \\ \beta x_2+(\alpha-\gamma)x_3=0, \\ \beta x_3+(\gamma-\alpha)x_2=0, \\ \frac{1}{8}(-3\alpha^2-4\beta^2+4\alpha\gamma)(\alpha^2+4\beta^2)+2\beta x_1=\lambda, \\ \frac{1}{2}(-3\alpha^2-4\beta^2+4\alpha\gamma)\beta(\alpha-2\gamma)=0, \\ -\frac{1}{8}(-3\alpha^2-4\beta^2+4\alpha\gamma)(\alpha^2+4\beta^2)+2\beta x_1=-\lambda. \end{cases} \tag{6}$$

The fourth and the sixth equations of (6) give $4\beta x_1=0$. Since $\beta\neq 0$, we obtain $x_1=0$. The fifth equation of (6) gives $\alpha=2\gamma$ or $-3\alpha^2-4\beta^2+4\alpha\gamma=0$. If $-3\alpha^2-4\beta^2+4\alpha\gamma=0$, the fourth equation indicates that $\lambda=0$. Thus, the foremost equation gives $\beta=0$ and this is a contradiction. If $\alpha=2\gamma$, the first and the fourth equations yield $\frac{1}{8}(\alpha^2+4\beta^2)^2=\lambda$ and $-\frac{1}{8}(\alpha^2+4\beta^2)^2=\lambda$, respectively, which imply $\lambda=0$ and $\beta=0$, which is a contradiction. Hence, the system (6) has no solution. Therefore, no homogeneous cross curvature soliton of Type Ib exists. □

Theorem 3. *Consider the Lorentzian unimodular three-dimensional Lie algebra $\mathfrak{g}_{II}$ of Type II. Then, the left-invariant cross curvature soliton on $\mathfrak{g}_{II}$ satisfies*

$$\alpha=\beta\neq 0, \lambda=\frac{1}{8}\alpha^4, x_1=-\frac{1}{4}\alpha^3, x_2=x_3.$$

Proof. Considering (4), there is a cross curvature soliton of Type II if and only if the subsequent system of equations is satisfied:

$$\begin{cases} \frac{1}{8}\alpha^4=\lambda, \\ x_2+(2\alpha-2\beta-1)x_3=0, \\ x_3+(2\beta-2\alpha-1)x_2=0, \\ (\frac{1}{2}\alpha^2-\alpha+2\beta)(-\frac{3}{4}\alpha^2+\alpha\beta)+x_1=\lambda, \\ (\alpha-2\beta)(-\frac{3}{4}\alpha^2+\alpha\beta)-x_1=0, \\ -(\frac{1}{2}\alpha^2+\alpha-2\beta)(-\frac{3}{4}\alpha^2+\alpha\beta)+x_1=-\lambda. \end{cases} \tag{7}$$

The fourth and the sixth equations of (7) give $\lambda=\frac{1}{2}\alpha^2(-\frac{3}{4}\alpha^2+\alpha\beta)$. Substituting this into the first equation in (7), we obtain $\alpha=\beta$. Since (P^{ij}) is invertible, $\alpha\neq 0$. Then, Equation (7) implies that $x_2=x_3$ and $x_1=-\frac{1}{4}\alpha^3$. □

From Theorem 3 and Proposition 1, we conclude the next result.

Corollary 2. *If a Type II Lorentzian unimodular Lie group is locally conformally flat, it is not necessarily a left-invariant cross curvature soliton.*

Theorem 4. *Consider the Lorentzian unimodular three-dimensional Lie algebra $\mathfrak{g}_{III}$ of Type III. A left-invariant cross curvature soliton on $\mathfrak{g}_{III}$ satisfies*

$$x_1 = \frac{1}{2}\alpha^3, x_2 = -x_3 = -\frac{3\sqrt{2}}{4}\alpha^2, \lambda = \frac{1}{8}\alpha^4, \textit{and } \alpha \neq 0.$$

Proof. In case of Type III, Equation (4) becomes

$$\begin{cases} \frac{1}{8}\alpha^4 + \frac{2}{\sqrt{2}}(x_2 + x_3) = \lambda, \\ \frac{1}{2\sqrt{2}}\alpha^3 - \frac{1}{\sqrt{2}}x_1 = 0, \\ \frac{3}{2}\alpha^2 + \frac{1}{8}\alpha^4 - \frac{2}{\sqrt{2}}x_3 = \lambda, \\ \frac{3}{2}\alpha^2 - \frac{1}{\sqrt{2}}(x_3 - x_2) = 0, \\ \frac{3}{2}\alpha^2 - \frac{1}{8}\alpha^4 + \frac{2}{\sqrt{2}}x_2 = -\lambda. \end{cases} \tag{8}$$

The second equation of (8) implies that $x_1 = \frac{1}{2}\alpha^3$. The first and third equations of (8) give $\frac{3}{2}\alpha^2 - \frac{4}{\sqrt{2}}x_3 - \frac{2}{\sqrt{2}}x_2 = 0$. Also, the first and fifth equations of (8) give $\frac{3}{2}\alpha^2 + \frac{4}{\sqrt{2}}x_2 + \frac{2}{\sqrt{2}}x_3 = 0$. Thus, $x_3 = -x_2 = \frac{3\sqrt{2}}{4}\alpha^2$. □

Theorem 5. *Let $\mathfrak{g}$ indicate a Lorentzian non-unimodular three-dimensional Lie algebra of Type IV.1. Then, the left-invariant cross curvature solitons on $\mathfrak{g}$ satisfy one of the following facts:*

(1) $\beta = \alpha = 0$, $x_2 = x_3 = 0$, and $\lambda = \frac{9}{8}\gamma^4$ for all x_1, δ, and γ such that $\gamma \neq 0$ and $\delta^2 = \gamma^2$.

(2) $\alpha \neq 0$, $\gamma = \frac{\beta\delta}{\alpha}$, $\lambda = \frac{1}{8}(\alpha^2 + 3\gamma^2 + 2\epsilon\alpha\gamma)$, $x_1 = -\frac{\gamma}{\alpha}x_2$, $\beta = \epsilon\alpha$, and $x_3 = 0$ for all δ, and x_2 such that and $\epsilon^2 = 1$.

(3) $\alpha \neq 0$, $\delta = \gamma = x_1 = x_3 = 0$, and $\lambda = \frac{1}{8}\beta^4$ such that $\alpha^2 = \beta^2$ for all x_2.

(4) $\alpha = \delta \neq 0$, $\beta = \gamma$, $\beta^2 \neq \alpha^2$, $x_1 = x_2 = x_3 = 0$, and $\lambda = \frac{1}{2}(2\alpha^2 + \beta^2)^2$.

Proof. Equation (4) yields

$$\begin{cases} (\beta - \gamma)x_3 = 0, \\ \alpha x_1 + \gamma x_2 = 0, \\ -\beta x_1 - \delta x_2 = 0, \\ -\frac{1}{8}\left(4\alpha^2 - 3\beta^2 + 3\gamma^2 + 2\beta\gamma\right)\left(\beta^2 + 3\gamma^2 - 2\beta\gamma + 4\alpha\delta\right) + 2\alpha x_3 = -\lambda, \\ -\frac{1}{8}\left(\gamma^2 - \beta^2 - 2\beta\gamma - 4\delta^2\right)\left(3\gamma^2 + \beta^2 + 4\alpha\delta - 2\beta\gamma\right) - 2\delta x_3 = \lambda, \\ \frac{1}{8}\left(\gamma^2 - \beta^2 - 2\beta\gamma - 4\delta^2\right)\left(- 3\beta^2 + 4\alpha^2 + 2\beta\gamma + 3\gamma^2\right) = -\lambda. \end{cases} \tag{9}$$

We first analyze the case $\alpha = 0$. In this case, $\delta \neq 0$ and $\beta = 0$. Since (P^{ij}) is invertible, we obtain $\gamma \neq 0$. The first equation of (9) indicates that $x_3 = 0$. By substituting $\alpha = \beta = x_3 = 0$ into the last three equations in (9), we obtain

$$\lambda = \frac{9}{8}\gamma^4, \ \ \lambda = -\frac{3}{8}\gamma^2(\gamma^2 - 4\delta^2),$$

then $\gamma^2 = \delta^2$. We obtain $x_2 = 0$ from the second equation of (9). Therefore, we have a left-invariant cross curvature soliton (1) in this case.

Now, let $\alpha \neq 0$; then, $\gamma = \frac{\beta\delta}{\alpha}$ and the second equation of (9) indicates $x_1 = -\frac{\gamma}{\alpha}x_2$, while its third equation reduces to $(\beta^2 - \alpha^2)\delta x_2 = 0$. If $\beta^2 = \alpha^2$, then $\beta = \epsilon\alpha$ and $\delta = \epsilon\gamma$, where $\epsilon^2 = 1$. The last three equations of (9) reduce to

$$\begin{cases} -\frac{1}{8}\left(\alpha^2 + 3\gamma^2 + 2\epsilon\alpha\gamma\right)^2 + 2\alpha x_3 = -\lambda, \\ -\frac{1}{8}\left(\alpha^2 + 3\gamma^2 + 2\epsilon\alpha\gamma\right)^2 - 2\delta x_3 = \lambda, \\ \frac{1}{8}\left(\alpha^2 + 3\gamma^2 + 2\epsilon\alpha\gamma\right)^2 = \lambda. \end{cases}$$

Therefore, $\alpha x_3 = 0$; since $\alpha \neq 0$, we obtain $x_3 = 0$ and, in this case, we have a left-invariant cross curvature soliton (2).

If $\delta = 0$, then $\beta\delta - \alpha\gamma = 0$ implies that $\gamma = 0$ and $x_1 = 0$. The last three equations of (9) reduce to

$$\begin{cases} -\frac{1}{8}\beta^2(4\alpha^2 - 3\beta^2) + 2\alpha x_3 = -\lambda, \\ \frac{1}{8}\beta^4 = \lambda, \\ \frac{1}{8}\beta^2(4\alpha^2 - 3\beta^2) = \lambda. \end{cases}$$

Thus, $\alpha x_3 = 0$ and since $\alpha \neq 0$, we obtain $x_3 = 0$. Also, $\alpha^2 = \beta^2$; in this case, we have a left-invariant cross curvature soliton (3).

If $\alpha \neq 0$, $\beta^2 \neq \alpha^2$, $\delta \neq 0$, and $x_2 = 0$, then $x_1 = 0$. The first equation now gives $\beta = \gamma$ or $x_3 = 0$. We assume that $x_3 = 0$, and by using the last three equations of (9), we have

$$\begin{cases} -\frac{1}{8}\left(4\alpha^2 - 3\beta^2 + 3\gamma^2 + 2\beta\gamma\right)\left(\beta^2 + 3\gamma^2 - 2\beta\gamma + 4\alpha\delta\right) = -\lambda, \\ -\frac{1}{8}\left(-\beta^2 + \gamma^2 - 4\delta^2 - 2\beta\gamma\right)\left(\beta^2 + 3\gamma^2 - 2\beta\gamma + 4\alpha\delta\right) = \lambda, \\ -\frac{1}{8}\left(-\beta^2 + \gamma^2 - 4\delta^2 - 2\beta\gamma\right)\left(4\alpha^2 - 3\beta^2 + 3\gamma^2 + 2\beta\gamma\right) = \lambda. \end{cases}$$

Since (P^{ij}) is invertible, we conclude

$$\begin{cases} \alpha^2 + \beta\gamma = \beta^2 + \alpha\delta, \\ \beta^2 + \delta^2 = \gamma^2 + \alpha^2. \end{cases} \tag{10}$$

Substituting $\gamma = \frac{\beta\delta}{\alpha}$ in (10) and using $\delta + \alpha \neq 0$, we obtain $\delta = \alpha$, $\beta = \gamma$, and $\lambda = \frac{1}{2}(2\alpha^2 + \beta^2)^2$. In this case, we have a left-invariant cross curvature soliton (4).

Now, we consider the case $\alpha \neq 0$, $\beta^2 \neq \alpha^2$, $\delta \neq 0$, $x_2 = 0$, $x_3 \neq 0$, and $\beta = \gamma$. The sixth equation of (9) implies that $\lambda = \frac{1}{2}(2\delta^2 + \beta^2)(2\alpha^2 + \beta^2)$, and by substituting it into the fourth and fifth equations of (9) we obtain $x_3 = \frac{1}{2\alpha}(2\alpha^2 + \beta^2)(\alpha\delta - \delta^2)$ and $x_3 = \frac{1}{2\delta}(2\delta^2 + \beta^2)(\alpha\delta - \alpha^2)$, respectively. Since $x_3 \neq 0$, we obtain $\alpha \neq \delta$; hence,

$$\frac{\delta}{\alpha}(2\alpha^2 + \beta^2) = -\frac{\alpha}{\delta}(2\delta^2 + \beta^2).$$

Thus, $\alpha = 0$, which is a contradiction. $\square$

From Theorem 5 and Proposition 1, we conclude the next result.

Corollary 3. *If a Type IV.1 Lorentzian non-unimodular Lie group is locally conformally flat, then it is not necessarily a left-invariant cross curvature soliton.*

Theorem 6. *Suppose that $\mathfrak{g}$ indicates a Lorentzian non-unimodular three-dimensional Lie algebra of Type IV.2. Then, the left-invariant cross curvature solitons on $\mathfrak{g}$ satisfy one of the following conditions:*

(1) $\beta = \alpha = 0$, $x_2 = x_3 = 0$, $\lambda = \frac{1}{8}\gamma^4$ *for all* x_1 *and* δ *such that* $\delta \neq 0$ *and* $\delta^2 = \gamma^2$.

(2) $\delta = \alpha \neq 0$, $\beta = -4\alpha = -\gamma$, $\lambda = 2\alpha^4$, *and* $x_1 = x_2 = x_3 = 0$.

(3) $\delta = \alpha \neq 0$, $\beta = -\gamma$, $x_1 = x_2 = 0$, $\lambda = 2\beta^4$, *and* $x_3 = -\frac{1}{4}\alpha^2\beta - \alpha^3$.

(4) $\delta \neq 0$, $\alpha \neq 0$, $\gamma = -\beta = -4\delta$, $x_1 = x_2 = 0$, $\lambda = 2\beta^2\delta^2$, *and* $x_3 = -\alpha^2\delta - \alpha\delta^2$.

Proof. Equation (4) becomes

$$\begin{cases} (\beta+\gamma)x_3 = 0, \\ -\alpha x_1 - \gamma x_2 = 0, \\ -\beta x_1 - \delta x_2 = 0, \\ -\frac{1}{8}\big(-4\alpha^2 - 3\beta^2 + \gamma^2 - 2\beta\gamma\big)\big(-\beta^2 - \gamma^2 - 2\beta\gamma + \alpha\gamma\big) - 2\alpha x_3 = \lambda, \\ -\frac{1}{8}\big(\beta^2 - 3\gamma^2 - 4\delta^2 - 2\beta\gamma\big)\big(-\beta^2 - \gamma^2 - 2\beta\gamma + \alpha\gamma\big) - 2\delta x_3 = \lambda, \\ \frac{1}{8}\big(-3\gamma^2 + \beta^2 - 2\beta\gamma - 4\delta^2\big)\big(-3\beta^2 - 4\alpha^2 - 2\beta\gamma + \gamma^2\big) = \lambda. \end{cases} \tag{11}$$

First, we analyze the case $\alpha = 0$. Regarding this matter, $\delta \neq 0$ and $\beta = 0$. Also, we obtain $\gamma \neq 0$ since (P^{ij}) is invertible. The first equation of (11) implies that $x_3 = 0$. By substituting $\alpha = \beta = x_3 = 0$ into the last three equations of (11), we obtain

$$\lambda = \frac{1}{8}\gamma^4, \;\; \lambda = -\frac{1}{8}\gamma^2(3\gamma^2 + 4\delta^2),$$

then $\gamma^2 = \delta^2$. We obtain $x_2 = 0$ from the second equation of (11). Therefore, we have a left-invariant cross curvature soliton (1) in this case.

Now, let $\alpha \neq 0$. Then, $\gamma = -\frac{\beta\delta}{\alpha}$ and the second equation of (11) indicates that $x_1 = -\frac{\gamma}{\alpha}x_2$, while its third equation reduces to $\delta x_2 = 0$. If $\delta = 0$, then $\beta\delta + \alpha\gamma = 0$ implies that $\gamma = 0$ and $x_1 = 0$. The last three equations of (11) reduce to

$$\begin{cases} -\frac{1}{8}\beta^2(4\alpha^2 + 3\beta^2) + 2\alpha x_3 = -\lambda, \\ \frac{1}{8}\beta^4 = \lambda, \\ \frac{1}{8}\beta^2(4\alpha^2 + 3\beta^2) = \lambda. \end{cases}$$

Thus, $4\alpha^2 + 2\beta^2 = 0$ and $\alpha = 0$, which is a contradiction.

If $\alpha \neq 0$, $\delta \neq 0$, and $x_2 = 0$, then $x_1 = 0$. Now, the first equation gives $\beta = -\gamma$ or $x_3 = 0$. We assume that $x_3 = 0$, and by using the last three equations of (11), we have

$$\begin{cases} -\frac{1}{8}\big(-4\alpha^2-3\beta^2+\gamma^2-2\beta\gamma\big)\big(-\beta^2-\gamma^2-2\beta\gamma+\alpha\gamma\big)=\lambda, \\ -\frac{1}{8}\big(\beta^2-3\gamma^2-4\delta^2-2\beta\gamma\big)\big(-\beta^2-\gamma^2-2\beta\gamma+\alpha\gamma\big)=\lambda, \\ -\frac{1}{8}\big(\beta^2-3\gamma^2-4\delta^2-2\beta\gamma\big)\big(-4\alpha^2-3\beta^2+\gamma^2-2\beta\gamma\big)=-\lambda. \end{cases}$$

Since (P^{ij}) is invertible, we conclude

$$\begin{cases} \beta^2+\alpha^2=\gamma^2+\delta^2, \\ \beta^2+\alpha^2+\beta\gamma=\frac{1}{4}\alpha\gamma. \end{cases} \tag{12}$$

Substituting $\gamma=-\frac{\beta\delta}{\alpha}$ in (12) and using $\delta+\alpha\neq 0$, we obtain $\delta=\alpha$, $\gamma=-\beta$, $\gamma=4\alpha$, and $\lambda=2\alpha^4$. In this case, we have a left-invariant cross curvature soliton (2).

Now, we consider the case $\alpha\neq 0$, $\delta\neq 0$, $x_2=0$, $x_3\neq 0$, and $\beta=-\gamma$. Then, $x_1=0$ and the sixth equation of (11) implies that $\lambda=2\alpha^2\delta^2$; substituting it into the fourth and the fifth equations in (11), we obtain $x_3=-\frac{1}{4}\alpha^2\beta-\alpha\delta^2$ and $x_3=-\frac{1}{4}\alpha\beta\delta-\alpha^2\delta$, respectively. We obtain

$$\left(\frac{1}{4}\beta-\delta\right)(\alpha-\delta)=0.$$

If $\alpha=\delta$, then $x_3=-\frac{1}{4}\alpha^2\beta-\alpha^3$; in this case, we have a left-invariant cross curvature soliton satisfying (3).

If $\alpha\neq\delta$ and $\beta=4\delta$, then $x_3=-\alpha^2\delta-\alpha\delta^2$; in this case, we have a left-invariant cross curvature soliton satisfying (4). □

From Theorem 6 and Proposition 1, we conclude the next result.

Corollary 4. *If a Type IV.2 Lorentzian non-unimodular Lie group is locally conformally flat, then it is not necessarily a left-invariant cross curvature soliton.*

Theorem 7. *A Type IV.3 Lorentzian non-unimodular Lie group does not accept any left-invariant cross curvature soliton.*

Proof. Considering (4), there is a cross curvature soliton of Type IV.3 if and only if the subsequent system of equations is satisfied:

$$\begin{cases} \frac{3}{8}\gamma^4-2\alpha x_3=\lambda, \\ \gamma x_3=0, \\ -\alpha x_1-\gamma x_2-\beta x_3=0, \\ \lambda=0, \\ -\alpha x_1-\gamma x_2-\beta x_3+2\delta x_3=-\lambda, \\ -\frac{3}{2}\gamma^2(\alpha^2-\alpha\delta+\beta)-2(\beta x_1+\delta x_2)=0. \end{cases} \tag{13}$$

Since (P^{ij}) is invertible, $\gamma\neq 0$. The condition $\alpha\gamma=0$ yields $\alpha=0$. The first and the fourth equations of (13) imply that $\gamma=0$, which is a contradiction. Therefore, Lorentzian non-unimodular Lie groups do not accept any left-invariant cross curvature soliton. □

4. Conclusions

The main study of the paper is to classify left-invariant cross curvature solitons on Lorentzian three-dimensional Lie groups. Three-dimensional locally homogeneous Lorentzian manifolds are classified into seven classes. The first four classes—Type Ia, Type Ib, Type II, and Type III—are unimodular, and the last three classes—Type IV.1, Type IV.2, and Type IV.3—are non-unimodular. In any of such classes, we obtain the Levi-Civita connection, the Ricci tensor, the Lie derivation of the metric in the direction of the vector field X, and the cross curvature tensor. By solving the cross curvature soliton equation $\mathcal{L}_X g + \lambda g = 2h$, we show that Lorentzian unimodular Lie groups Types Ia, II, III and Lorentzian non-unimodular Lie groups of Types IV.1 and IV.2 admit a left-invariant cross curvature soliton, and Lorentzian unimodular Lie groups of type Ib and Lorentzian non-unimodular Lie groups of type IV.3 do not admit left-invariant cross curvature solitons.

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