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On the Conjecture over Dimensions of Associated Lie Algebra to the Isolated Singularities

Naveed Hussain [1,*,†], Ahmad N. Al-Kenani [2,†] and Muhammad Asif [3,†]

1 Department of Mathematics and Statistics, University of Agriculture, Faisalabad 38000, Pakistan
2 Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia; analkenani@kau.edu.sa
3 Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54000, Pakistan; muhammadasif.jrw@gmail.com
* Correspondence: dr.nhussain@uaf.edu.pk; Tel.: +92-316-169-8297
† These authors contributed equally to this work.

Abstract: Lie algebra plays an important role in the study of singularity theory and other fields of sciences. Finding numerous invariants linked with isolated singularities has always been a primary interest in the field of classification theory of isolated singularities. Any Lie algebra that characterizes simple singularity produces a natural question. The study of properties such as to find the dimensions of newly defined algebra is a remarkable work. Hussain, Yau and Zuo have found a new class of Lie algebra $\mathcal{L}_k(V)$, $k \geq 1$, i.e., Der $(M_k(V), M_k(V))$ and proposed a conjecture over its dimension $\delta_k(V)$ for $k \geq 0$. Later, they proved it true for k up to $k = 1, 2, 3, 4, 5$. In this work, the main concern is whether it is true for a higher value of k. According to this, we first calculate the dimension of Lie algebra $\mathcal{L}_k(V)$ for $k = 6$ and then compute the upper estimate conjecture of fewnomial isolated singularities. Additionally, we also justify the inequality conjecture $\delta_{k+1}(V) < \delta_k(V)$ for $k = 6$. Our calculated results are innovative and serve as a new addition to the study of singularity theory.

Keywords: simple singularity; hypersurface singularity; derivation Lie algebra; inequality conjecture

MSC: 14B05; 32S05

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1. Introduction

A mathematical structure known as a Lie algebra is made up of vector spaces and a particular binary operation known as the Lie bracket, which is used to measure the "twist" between two infinitesimal transformations geometrically. Lie algebras are essential to many branches of mathematics and science. In quantum mechanics and particle physics, they are widely used, and in physical phenomena, they arise as symmetry groups. In this regard, Brieskorn established the relationship between Lie algebras and simple singularities. Recent studies have investigated relations between finite-dimensional solvable Lie algebras and isolated hypersurface singularities in complex analysis. In different areas of science and mathematics, singularities have arisen naturally. That is why singularity theory has become the interlinking path between the various applications of mathematics with its conclusive parts. As an example, it coordinates regular polyhedra theory and simple Lie algebra with the optimal caustics investigations, and relates knot theory to wave fronts of hyperbolic PDE, while it also connects commutative algebra to the theory of solid shapes. In most problems of singularity theory, the core aim is to determine the dependence of various physical phenomena while dealing with the geometric objects.

Singularity theory has become a remarkably new subject in mathematics. Thus, as an example, singularities can be considered as arising from a 3-space generic surface in orthogonal projections, which is considered an important classical interest. In 1979, their classification was completed. Singularity theory is viewed as a recent equivalence with differential calculus and it also describes its wide applicability and central position. It began with the basic discoveries of Mather, 1970 [1]; Milnor, 1970 [2]; Brieskorn, 1971 [3]; and Yau, 1982 [4] . Currently, after the development of various new results, more and more applications of this subject have been achieved. For example, solvable Lie algebras have not been fully understood as compared to simple Lie algebras. Therefore, there is a pressing need to build a connection between solvable Lie algebras and singularities. Yau and his teammates naturally established finite-dimensional solvable Lie algebras [5,6], which have led these findings to become fruitful in understanding solvable Lie algebras through a geometric point of view.

The germs of holomorphic functions are widely accepted to be at the origin of $\mathbb{C}^n$ and $\mathcal{O}_n$. Naturally, the $\mathcal{O}_n$ can be used to identify the algebra of n indeterminate power series. The well-known Mather–Yau theorem [7] is as follows: Let V_1 and V_2 be two isolated hypersurface singularities and $A(V_1)$ and $A(V_2)$ be the moduli algebras; then, $(V_1, 0) \cong (V_2, 0) \iff A(V_1) \cong A(V_2)$. Yau takes into account the Lie algebras derived from the moduli algebra $A(V) := \mathcal{O}_n/(g, \frac{\partial g}{\partial y_1}, \cdots, \frac{\partial g}{\partial y_n})$, where $L(V)$ is defined as $\mathrm{Der}(A(V), A(V))$ and V denotes the isolated hypersurface singularity. Lie algebra $L(V)$ is a famous solvable finite-dimensional algebra [5–10]. Yau's algebra of V is used in singularity theory to distinguish $L(V)$ from the other types of Lie algebra [11,12]. The complex analytical set of isolated hypersurface singularities [13] and the finite set of solvable dimensional Lie algebras (nilpotent) have many new natural connections that Hussain, Yau, Zuo, and their research associates have discovered in recent years [14]. They have presented three distinct methods for connecting isolated hypersurface singularities to Lie algebra. We defined this new k-th Yau algebra in [15].

These associations are helpful in understanding the solvable Lie algebra (nilpotent) from a geometric point of view. A substantial amount of work has been performed by Yau and his research collaborates since the 1980s [16–22]. Let a holomorphic function $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be associated to the isolated hypersurface singularity $(V, 0)$, where its multiplicity is denoted as $mult(g)$. The $\mathcal{L}_k(V)$ [23] is defined below.

Let ideal $\mathbb{J}_k(g)$ be generated by $< \frac{\partial^k g}{\partial y_{i_1} \cdots \partial y_{i_k}} \mid 1 \le i_1, \cdots, i_k \le n >$ and $mult(g) = m$, $1 \le k \le m$. Then, $M_k(V) := \mathcal{O}_n/(g + \mathbb{J}_1(g) + \cdots + \mathbb{J}_k(g))$ is defined as the k-th local algebra; $\mathcal{L}_k(V)$ is denoted as the derivations of Lie algebras and its dimension is denoted as $\delta_k(V)$. It is further noted that the $\mathcal{L}_k(V)$ is the generalization of the Yau algebra. Further details can be found in [23,24].

In [23], the sharp upper estimate and inequality conjectures are introduced in the following pattern:

Conjecture 1 ([23])**.** *Let $\delta_k(\{y_1^{\alpha_1} + \cdots + y_n^{\alpha_n} = 0\}) = h_k(\alpha_1, \cdots, \alpha_n)$, $0 \le k \le n$ and $(V, 0) = \{(y_1, y_2, \cdots, y_n) \in \mathbb{C}^n : g(y_1, y_2, \cdots, y_n) = 0\}, (n \ge 2)$ be an isolated singularity with weight type $(w_1, w_2, \cdots, w_n; 1)$. Then, $\delta_k(V) \le h_k(1/w_1, \cdots, 1/w_n)$.*

Conjecture 2 ([23])**.** *Using the above notations, suppose that $(V, 0)$ is defined by $g \in \mathcal{O}_n$, $n \ge 2$. Then*

$$\delta_{(k+1)}(V) < \delta_k(V), k \ge 1.$$

Conjecture 1 has been proven for binomial and trinomial singularities when $k = 1, 2, 3, 4$ [20,23,25] and Conjecture 2 has been proven for $k = 1, 2, 3$ [23,25].

The main results of this paper is to prove Conjectures 1 and 2 for binomial and trinomial singularities for a particular value of k. At the end of Section 1, we list the key findings of this work. In Section 2, related definitions and proven results are given. The main results of this work are provided in Section 3. Moreover, in Section 3, we shall first prove some propositions, and then, using these findings, we shall prove the main theorems. The key findings of this paper is as follows:

Theorem 1. *For* $(V,0) = \{(y_1, y_2, \cdots, y_n) \in \mathbb{C}^n : y_1^{\alpha_1} + \cdots + y_n^{\alpha_n} = 0\}, (n \geq 2;\ \alpha_l \geq 7,\ 1 \leq j \leq n)$

$$\delta_6(V) = h_6(\alpha_1, \cdots, \alpha_n) = \sum_{j=1}^{n} \frac{\alpha_j - 7}{\alpha_j - 6} \prod_{l=1}^{n} (\alpha_l - 6).$$

Theorem 2. *For binomial singularity* $(V,0)$ *defined by* $g(y_1, y_2)$*, a weighted homogeneous polynomial with* $mult(g) \geq 8$*, and weight type* $(w_1, w_2; 1)$*,*

$$\delta_6(V) \leq h_6(\frac{1}{w_1}, \frac{1}{w_2}) = \sum_{l=1}^{2} \frac{\frac{1}{w_l} - 7}{\frac{1}{w_l} - 6} \prod_{j=1}^{2} (\frac{1}{w_j} - 6).$$

Theorem 3. *For binomial singularity* $(V,0)$ *defined by* $g(y_1, y_2)$*, a weighted homogeneous polynomial with* $mult(g) \geq 8$*, and weight type* $(w_1, w_2; 1)$*,*

$$\delta_6(V) < \delta_5(V).$$

Theorem 4. *For feunomial singularity* $(V,0)$ *defined by* $g(y_1, y_2, y_3)$*, a weighted homogeneous polynomial with* $mult(g) \geq 8$*, and weight type* $(w_1, w_2, w_3; 1)$*, then*

$$\delta_6(V) \leq h_6(\frac{1}{w_1}, \frac{1}{w_2}, \frac{1}{w_3}) = \sum_{l=1}^{3} \frac{\frac{1}{w_l} - 7}{\frac{1}{w_l} - 6} \prod_{j=1}^{3} (\frac{1}{w_j} - 6).$$

Theorem 5. *For trinomial singularity* $(V,0)$ *defined by* $g(y_1, y_2, y_3)$*, a weighted homogeneous polynomial with* $mult(g) \geq 8$*, and weight type* $(w_1, w_2, w_3; 1)$*, then*

$$\delta_6(V) < \delta_5(V).$$

2. Preliminaries

In this section, we shall give definitions and important results that are helpful in solving the problem.

Definition 1. *Let* $f(x_1, \cdots, x_n)$ *be a complex polynomial and* $V = \{f = 0\}$ *be a germ of an isolated hypersurface singularity at the origin in* $\mathbb{C}^n$*. Let* $A^k(V) = \mathcal{O}_n/(f, m^k J(f)), 1 \leq k \leq n$ *be a moduli algebra. Then,* $Der(A^k(V), A^k(V))$ *is defined as the derivation Lie algebra* $L^k(V)$*. The* $\lambda^k(V)$ *is the dimension of derivation Lie algebra* $L^k(V)$*.*

It is noted that when $k = 0$, the derivation Lie algebra is called Yau algebra.

Proposition 1. *Analytically, the series of Type A, Type B, and Type C is the homogeneous fewnomial isolated singularity g having* $mult(g) \geq 3$*, as follows:*

Type A: $y_1^{\alpha_1} + y_2^{\alpha_2} + \cdots + y_{n-1}^{\alpha_{n-1}} + y_n^{\alpha_n}$, $n \geq 1$,
Type B: $y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_3 + \cdots + y_{n-1}^{\alpha_{n-1}} y_n + y_n^{\alpha_n}$, $n \geq 2$,
Type C: $y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_3 + \cdots + y_{n-1}^{\alpha_{n-1}} y_n + y_n^{\alpha_n} y_1$, $n \geq 2$.

Corollary 1. *Analytically, the series A, B, and C is the homogeneous fewnomial isolated binomial singularity g having* $mult(g) \geq 3$*, as follows:*

A: $y_1^{\alpha_1} + y_2^{\alpha_2}$,
B: $y_1^{\alpha_1} y_2 + y_2^{\alpha_2}$,
C: $y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_1$.

Proposition 2 ([26])**.** *Analytically,* $g(y_1, y_2, y_3)$ *,* $mult(g) \geq 3$ *is*

Type 1: $y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3}$,
Type 2: $y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_3 + y_3^{\alpha_3}$,

Type 3: $y_1^{\alpha_1}y_2 + y_2^{\alpha_2}y_3 + y_3^{\alpha_3}y_1$,
Type 4: $y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3}y_1$,
Type 5: $y_1^{\alpha_1}y_2 + y_2^{\alpha_2}y_1 + y_3^{\alpha_3}$.

3. Proof of Theorems

Before proceeding to the proof of main theorems, first, we will prove some propositions.

Proposition 3. *For $(V,0)$ defined by $g = y_1^{\alpha_1} + y_2^{\alpha_2} + \cdots + y_n^{\alpha_n}$ ($\alpha_l \geq 8, l = 1,2,\cdots,n$) with weight type $(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \cdots, \frac{1}{\alpha_n}; 1)$,*

$$\delta_6(V) = \sum_{j=1}^{n} \frac{\alpha_j - 7}{\alpha_j - 6} \prod_{l=1}^{n} (\alpha_l - 6).$$

Proof. Let $\prod_{l=1}^{n}(\alpha_l - 6)$ be the dimension of moduli algebra $M_6(V)$ and monomial basis to be in the form

$$\{y_1^{l_1}y_2^{l_2}\cdots y_n^{l_n}, 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, \cdots, 0 \leq l_n \leq \alpha_n - 7\},$$

with the following relations:

$$y_1^{\alpha_1-6} = 0, y_2^{\alpha_2-6} = 0, y_3^{\alpha_3-6} = 0, \cdots, y_n^{\alpha_n-6} = 0. \tag{1}$$

This implies

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7}\sum_{l_2=0}^{\alpha_2-7}\cdots\sum_{l_n=0}^{\alpha_n-7} \alpha^i_{l_1,l_2,\cdots,l_n} y_1^{l_1}y_2^{l_2}\cdots y_n^{l_n},\ j = 1,2,\cdots,n.$$

Using (1), we may determine a derivation of $M_6(V)$ as

$$\begin{aligned}
\alpha^1_{0,l_2,l_3,,\cdots,l_n} &= 0; 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \cdots, 0 \leq l_n \leq \alpha_n - 7;\\
\alpha^2_{l_1,0,l_3,,\cdots,l_n} &= 0; 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \cdots, 0 \leq l_n \leq \alpha_n - 7;\\
\alpha^3_{l_1,l_2,0,\cdots,l_n} &= 0; 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, \cdots, 0 \leq l_n \leq \alpha_n - 7;\\
&\vdots\\
\alpha^n_{l_1,l_2,l_3,\cdots,l_{n-1},0} &= 0; 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, \cdots, 0 \leq l_{n-1} \leq \alpha_{n-1} - 7.
\end{aligned}$$

The derivation bases are of the form

$$\begin{aligned}
&y_1^{l_1}y_2^{l_2}\cdots y_n^{l_n}\partial_1,\ 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \cdots, 0 \leq l_n \leq \alpha_n - 7;\\
&y_1^{l_1}y_2^{l_2}\cdots y_n^{l_n}\partial_2,\ 0 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \cdots, 0 \leq l_n \leq \alpha_n - 7;\\
&y_1^{l_1}y_2^{l_2}\cdots y_n^{l_n}\partial_3,\ 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 7, 0 \leq l_4 \leq \alpha_4 - 7,\\
&\qquad 0 \leq l_5 \leq \alpha_5 - 7, 0 \leq l_6 \leq \alpha_6 - 7, \cdots, 0 \leq l_n \leq \alpha_n - 7;\\
&\vdots\\
&y_1^{l_1}y_2^{l_2}\cdots y_n^{l_n}\partial_n,\ 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7, \cdots, 1 \leq l_n \leq \alpha_n - 7.
\end{aligned}$$

This implies

$$\delta_6(V) = \sum_{j=1}^{n} \frac{\alpha_j - 7}{\alpha_j - 6} \prod_{l=1}^{n} (\alpha_l - 6).$$

□

Remark 1. *For weighted isolated homogeneous fewnomial singularity $(V,0)$ of Type A, defined by $g = y_1^{\alpha_1} + y_2^{\alpha_2}$ $(\alpha_1 \geq 8,\ \alpha_2 \geq 8)$ with weight type $(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}; 1)$, Proposition 3 implies that*

$$\delta_6(V) = 2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 84.$$

Proposition 4. *For isolated binomial singularity $(V,0)$ of Type B, defined by $g = y_1^{\alpha_1} y_2 + y_2^{\alpha_2}$ $(\alpha_1 \geq 7,\ \alpha_2 \geq 8)$ with weight type $(\frac{\alpha_2-1}{\alpha_1\alpha_2}, \frac{1}{\alpha_2}; 1)$,*

$$\delta_6(V) = 2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 87.$$

For $mult(g) \geq 8$,

$$2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 87 \leq \frac{2\alpha_1\alpha_2^2}{\alpha_2 - 1} - 13(\frac{\alpha_1\alpha_2}{\alpha_2 - 1} + \alpha_2) + 84.$$

Proof. Let $\alpha_1\alpha_2 - 6(\alpha_1 + \alpha_2) + 37$ be the dimension of

$$M_6(V) = \mathbb{C}\{y_1, y_2\}/(g_{y_1y_1y_1y_1y_1}, g_{y_2y_2y_2y_2y_2}, g_{y_1y_2y_2y_2y_2}, g_{y_1y_1y_2y_2y_2}, g_{y_1y_1y_1y_2y_2}, g_{y_1y_1y_1y_1y_2})$$

and monomial basis to be in the form

$$\{y_1^{l_1} y_2^{l_2}, 0 \leq l_1 \leq \alpha_1 - 7; 0 \leq l_2 \leq \alpha_2 - 7; y_1^{\alpha_1 - 6}\}. \tag{2}$$

This implies that

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7} \sum_{l_2=0}^{\alpha_2-7} \alpha^i_{l_1,l_2} y_1^{l_1} y_2^{l_2} + \alpha^i_{\alpha_1-6,0} y_1^{\alpha_1-6},\ i = 1, 2.$$

The basis of $\mathcal{L}_6(V)$ is

$$y_1^{l_1} y_2^{l_2} \partial_1, 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7; y_1^{l_1} y_2^{l_2} \partial_2, 0 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7;$$

$$y_2^{\alpha_2-7}\partial_1; y_1^{\alpha_1-6}\partial_1; y_1^{\alpha_1-6}\partial_2.$$

We thus obtain the following formula

$$\delta_6(V) = 2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 87.$$

Finally we need to show that

$$2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 87 \leq \frac{2\alpha_1\alpha_2^2}{\alpha_2 - 1} - 13(\frac{\alpha_1\alpha_2}{\alpha_2 - 1} + \alpha_2) + 84. \tag{3}$$

After solving (3), we have $\alpha_1(\alpha_2 - 10) + \alpha_2(\alpha_1 - 6) + 6 \geq 0$. □

Proposition 5. *For isolated binomial singularity $(V,0)$ of Type C , defined by $g = y_1^{\alpha_1} y_2 + y_2^{\alpha_2} y_1$ $(\alpha_1 \geq 7, \alpha_2 \geq 7)$ with weight type $(\frac{\alpha_2-1}{\alpha_1\alpha_2-1}, \frac{\alpha_1-1}{\alpha_1\alpha_2-1}; 1)$,*

$$\delta_6(V) = \begin{cases} 2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 90; & \alpha_1 \geq 8, \alpha_2 \geq 8 \\ \alpha_2 - 3; & \alpha_1 = 7, \alpha_2 \geq 7. \end{cases}$$

For $mult(g) \geq 8$,

$$2\alpha_1\alpha_2 - 13(\alpha_1 + \alpha_2) + 90 \leq \frac{2(\alpha_1\alpha_2 - 1)^2}{(\alpha_1 - 1)(\alpha_2 - 1)} - 13(\alpha_1\alpha_2 - 1)(\frac{\alpha_1 + \alpha_2 - 2}{(\alpha_1 - 1)(\alpha_2 - 1)}) + 84.$$

Proof. The moduli algebra has the dimension $\alpha_1\alpha_2 - 6(\alpha_1+\alpha_1)+38$. The monomial basis of

$$M_5(V) = \mathbb{C}\{y_1, y_2\}/(g_{y_1y_1y_1y_1y_1}, g_{y_2y_2y_2y_2y_2}, g_{y_1y_2y_2y_2y_2}, g_{y_1y_1y_2y_2y_2}, g_{y_1y_1y_1y_2y_2}, g_{y_1y_1y_1y_1y_2})$$

is in the form

$$\{y_1^{l_1}y_2^{l_2}, 0 \le l_1 \le \alpha_1 - 7; 0 \le l_2 \le \alpha_2 - 7; y_1^{\alpha_1-6}; y_2^{\alpha_2-6}\}. \tag{4}$$

This implies that

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7}\sum_{l_2=0}^{\alpha_2-7} \alpha^i_{l_1,l_2} y_1^{l_1}y_2^{l_2} + \alpha^i_{\alpha_1-6,0}y_1^{\alpha_1-6} + \alpha^i_{0,\alpha_2-6}y_2^{\alpha_2-6},\ i = 1,2.$$

The derivation bases are of the form

$$y_1^{l_1}y_2^{l_2}\partial_1, 1 \le l_1 \le \alpha_1 - 7, 0 \le l_2 \le \alpha_2 - 7; y_1^{l_1}y_2^{l_2}\partial_2, 0 \le l_1 \le \alpha_1 - 7, 1 \le l_2 \le \alpha_2 - 7;$$

$$y_2^{\alpha_2-7}\partial_1; y_2^{\alpha_2-6}\partial_1; y_1^{\alpha_1-6}\partial_1; y_2^{\alpha_2-6}\partial_2; y_1^{\alpha_1-7}\partial_2; y_1^{\alpha_1-6}\partial_2.$$

This implies that

$$\delta_6(V) = 2\alpha_1\alpha_2 - 13(\alpha_1+\alpha_2) + 90.$$

For $\alpha_1 = 7, \alpha_2 \ge 7$, we obtain $\mathcal{L}_6(V)$ as

$$y_2^{l_2}\partial_2, 1 \le l_2 \le \alpha_2 - 6; y_2^{\alpha_2-6}\partial_1; y_1\partial_1; y_1\partial_2.$$

Next, we will prove that

$$2\alpha_1\alpha_2 - 13(\alpha_1+\alpha_2) + 90 \le \frac{2(\alpha_1\alpha_2-1)^2}{(\alpha_1-1)(\alpha_2-1)} - 13(\alpha_1\alpha_2-1)\left(\frac{\alpha_1+\alpha_2-2}{(\alpha_1-1)(\alpha_2-1)}\right) + 84. \tag{5}$$

After solving (5), we have

$$\alpha_1\alpha_2^2[(\alpha_2-5)(\alpha_1-5) - \alpha_1(\alpha_2-8)] + \alpha_2^3 + 4\alpha_1^2\alpha_2 + 10\alpha_2^2(\alpha_1-6) + 6\alpha_1\alpha_2(\alpha_1-6)$$
$$+ 3\alpha_1^2(\alpha_2-6) + \alpha_1\alpha_2(\alpha_1-6) + 15\alpha_1 + 2(\alpha_2-6) \ge 0.$$

Similarly, Conjecture 1 holds true for $\alpha_1 = 7, \alpha_2 \ge 7$. □

Remark 2. *For fewnomial surface isolated singularity $(V,0)$ of type 1, defined by $g = y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3}$ $(\alpha_1 \ge 8, \alpha_2 \ge 8, \alpha_3 \ge 8)$ with weight type $(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}; 1)$, from Proposition 3, we obtain*

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 + 120(\alpha_1+\alpha_2+\alpha_3) - 19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) - 756.$$

Proposition 6. *For fewnomial surface isolated singularity $(V,0)$ of type 2, defined by $g = y_1^{\alpha_1}y_2 + y_2^{\alpha_2}y_3 + y_3^{\alpha_3}$ $(\alpha_1 \ge 7, \alpha_2 \ge 7, \alpha_3 \ge 8)$ with weight type $(\frac{1-\alpha_3+\alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3}, \frac{\alpha_3-1}{\alpha_2\alpha_3}, \frac{1}{\alpha_3}; 1)$,*

$$\delta_6(V) = \begin{cases} 3\alpha_1\alpha_2\alpha_3 - 19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) + 124(\alpha_1+\alpha_3) & \\ +120\alpha_2 - 807; & \alpha_1 \ge 7, \alpha_2 \ge 8, \alpha_3 \ge 8 \\ 2\alpha_1\alpha_3 - 9\alpha_1 - 11\alpha_3 + 47; & \alpha_1 \ge 7, \alpha_2 = 7, \alpha_3 \ge 8. \end{cases}$$

For $\alpha_1 \geq 7, \alpha_2 \geq 8, \alpha_3 \geq 8$, we need to prove the following inequality:

$$
\begin{aligned}
&3\alpha_1\alpha_2\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 124(\alpha_1 + \alpha_3) + 120\alpha_2 - 807 \leq \\
&\frac{3\alpha_1\alpha_2^2\alpha_3^3}{(1-\alpha_3+\alpha_2\alpha_3)(\alpha_3-1)} - 19(\frac{\alpha_1\alpha_2^2\alpha_3^2}{(1-\alpha_3+\alpha_2\alpha_3)(\alpha_3-1)} + \frac{\alpha_1\alpha_2\alpha_3^2}{1-\alpha_3+\alpha_2\alpha_3} + \frac{\alpha_2\alpha_3^2}{\alpha_3-1}) \\
&+ 120(\frac{\alpha_1\alpha_2\alpha_3}{1-\alpha_3+\alpha_2\alpha_3} + \frac{\alpha_2\alpha_3}{\alpha_3-1} + \alpha_3) - 756.
\end{aligned}
$$

Proof. The moduli algebra has the dimension $\alpha_1\alpha_2\alpha_3 - 6(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 37(\alpha_1 + \alpha_3) + 36\alpha_2 - 228$. The monomial basis of $M_6(V)$ is in the form

$$
\begin{aligned}
&\{y_1^{l_1}y_2^{l_2}y_3^{l_3}, 0 \leq l_1 \leq \alpha_1 - 7; 0 \leq l_2 \leq \alpha_2 - 7; 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-6}y_3^{l_3}, 0 \leq l_3 \leq \alpha_3 - 7; \\
&y_1^{l_1}y_3^{\alpha_3-6}, 0 \leq l_1 \leq \alpha_1 - 7\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
Dy_j = &\sum_{l_1=0}^{\alpha_1-7}\sum_{l_2=0}^{\alpha_2-7}\sum_{l_3=0}^{\alpha_3-7} \alpha^i_{l_1,l_2,l_3} y_1^{l_1}y_2^{l_2}y_3^{l_3} + \sum_{l_1=0}^{\alpha_1-7} \alpha^i_{l_1,0,\alpha_3-6} y_1^{l_1}y_3^{\alpha_3-6} + \sum_{l_3=0}^{\alpha_3-7} \alpha^i_{\alpha_1-6,0,l_3} y_3^{l_3}y_1^{\alpha_1-6}, \\
&i = 1, 2, 3.
\end{aligned}
$$

The derivation bases are of the form

$$
\begin{gathered}
y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_1,\ 1 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-6}y_3^{l_3}\partial_1,\ 0 \leq l_3 \leq \alpha_3 - 7, \\
y_2^{\alpha_2-7}y_3^{l_3}\partial_1,\ 1 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_2^{\alpha_2-6}\partial_1,\ 0 \leq l_1 \leq \alpha_1 - 7, \\
y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_2,\ 0 \leq l_1 \leq \alpha_1 - 7, 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{\alpha_1-6}y_3^{l_3}\partial_2,\ 0 \leq l_3 \leq \alpha_3 - 7, \\
y_1^{l_1}y_2^{\alpha_2-6}\partial_2,\ 0 \leq l_1 \leq \alpha_1 - 7; y_1^{l_1}y_3^{\alpha_3-7}\partial_2,\ 1 \leq l_1 \leq \alpha_1 - 7, \\
y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_3,\ 0 \leq l_1 \leq \alpha_1 - 7, 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 7, y_1^{l_1}y_2^{\alpha_2-6}\partial_3,\ 0 \leq l_1 \leq \alpha_1 - 7, \\
y_1^{\alpha_1-6}y_3^{l_3}\partial_3,\ 1 \leq l_3 \leq \alpha_3 - 7.
\end{gathered}
$$

We obtain

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 124(\alpha_1 + \alpha_3) + 120\alpha_2 - 807.$$

For $\alpha_1 \geq 7, \alpha_2 = 7, \alpha_3 \geq 8$, we obtain the following bases:

$$
\begin{gathered}
y_1^{l_1}y_3^{l_3}\partial_1,\ 1 \leq l_1 \leq \alpha_1 - 6, 0 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_2\partial_1,\ 0 \leq l_1 \leq \alpha_1 - 7, \\
y_1^{l_1}y_2\partial_2,\ 0 \leq l_1 \leq \alpha_1 - 7; y_1^{l_1}y_3^{\alpha_3-7}\partial_2,\ 1 \leq l_1 \leq \alpha_1 - 6, \\
y_1^{l_1}y_3^{l_3}\partial_3,\ 0 \leq l_1 \leq \alpha_1 - 6, 1 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_2\partial_3,\ 0 \leq l_1 \leq \alpha_1 - 7.
\end{gathered}
$$

We obtain

$$\delta_6(V) = 2\alpha_1\alpha_3 - 9\alpha_1 - 11\alpha_3 + 47.$$

For $\alpha_1 \geq 7, \alpha_2 \geq 8, \alpha_3 \geq 8$, we need to prove the following inequality:

$$
\begin{aligned}
&3\alpha_1\alpha_2\alpha_3 - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 124(\alpha_1 + \alpha_3) + 120\alpha_2 - 807 \leq \\
&\frac{3\alpha_1\alpha_2^2\alpha_3^3}{(1-\alpha_3+\alpha_2\alpha_3)(\alpha_3-1)} - 19(\frac{\alpha_1\alpha_2^2\alpha_3^2}{(1-\alpha_3+\alpha_2\alpha_3)(\alpha_3-1)} + \frac{\alpha_1\alpha_2\alpha_3^2}{1-\alpha_3+\alpha_2\alpha_3} + \frac{\alpha_2\alpha_3^2}{\alpha_3-1}) \\
&+ 120(\frac{\alpha_1\alpha_2\alpha_3}{1-\alpha_3+\alpha_2\alpha_3} + \frac{\alpha_2\alpha_3}{\alpha_3-1} + \alpha_3) - 756.
\end{aligned}
$$

After simplification, we obtain
$(\alpha_1-5)^3(\alpha_2-7)\alpha_3+(\alpha_2-6)\alpha_1\alpha_3((\alpha_3-5)(\alpha_1-7)+(\alpha_2-5)(\alpha_3-5))+\alpha_2(3\alpha_3-6)(\alpha_1-5)+\alpha_2(\alpha_1-4)+6\geq 0.$

Similarly, one can prove that for $\alpha_1\geq 7,\alpha_2=7,\alpha_3\geq 8$, Conjecture 1 holds true. □

Proposition 7. *For isolated fewnomial surface singularity $(V,0)$ of type 3 defined by $g=y_1^{\alpha_1}y_2+y_2^{\alpha_2}y_3+y_3^{\alpha_3}y_1$ ($\alpha_1\geq 7,\alpha_2\geq 7,\alpha_3\geq 7$) with weight type*

$$(\frac{1-\alpha_3+\alpha_2\alpha_3}{1+\alpha_1\alpha_2\alpha_3},\frac{1-\alpha_1+\alpha_1\alpha_3}{1+\alpha_1\alpha_2\alpha_3},\frac{1-\alpha_2+\alpha_1\alpha_2}{1+\alpha_1\alpha_2\alpha_3};1),$$

$$\delta_6(V)=\begin{cases}3\alpha_1\alpha_2\alpha_3+124(\alpha_1+\alpha_2+\alpha_3)-19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) & \\ -831; & \alpha_1\geq 8,\alpha_2\geq 8,\alpha_3\geq 8\\ 2\alpha_2\alpha_3-11\alpha_2-9\alpha_3+51; & \alpha_1=7,\alpha_2\geq 8,\alpha_3\geq 7\\ 2\alpha_1\alpha_3-9\alpha_1-11\alpha_3+51; & \alpha_1\geq 7,\alpha_2=7,\alpha_3\geq 7\\ 2\alpha_1\alpha_2-11\alpha_1-9\alpha_2+51; & \alpha_1\geq 8,\alpha_2\geq 8,\alpha_3=7\end{cases}$$

Assuming that $\alpha_1\geq 8,\alpha_2\geq 8,\alpha_3\geq 8$, then we need to prove following inequality:

$$\begin{aligned}&3\alpha_1\alpha_2\alpha_3+124(\alpha_1+\alpha_2+\alpha_3)-19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)-831\\ &\leq\frac{3(1+\alpha_1\alpha_2\alpha_3)^3}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_1+\alpha_1\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)}+120(\frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_3+\alpha_2\alpha_3}+\frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_1+\alpha_1\alpha_3}+\frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_2+\alpha_1\alpha_2})\\ &-19(\frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_1+\alpha_1\alpha_3)}+\frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_1+\alpha_1\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)}+\frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)})-756.\end{aligned}$$

Proof. Let $(\alpha_1\alpha_2\alpha_3-6(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)+37(\alpha_1+\alpha_2+\alpha_3)-234)$ be the dimension of $M_5(V)$ and the monomial basis to be in the form

$$\{y_1^{l_1}y_2^{l_2}y_3^{l_3},0\leq l_1\leq\alpha_1-7;0\leq l_2\leq\alpha_2-7;0\leq l_3\leq\alpha_3-7;y_1^{\alpha_1-6}y_3^{l_3},0\leq l_3\leq\alpha_3-7;\\ y_2^{l_2}y_3^{\alpha_3-6},0\leq l_2\leq\alpha_2-7;y_1^{l_1}y_2^{\alpha_2-6},0\leq l_1\leq\alpha_1-7\}.$$

This implies that

$$\begin{aligned}Dy_j=&\sum_{l_1=0}^{\alpha_1-7}\sum_{l_2=0}^{\alpha_2-7}\sum_{l_3=0}^{\alpha_3-7}\alpha^i_{l_1,l_2,l_3}y_1^{l_1}y_2^{l_2}y_3^{l_3}+\sum_{l_1=0}^{\alpha_1-7}\alpha^i_{l_1,\alpha_2-6,0}y_1^{l_1}y_2^{\alpha_2-6}+\sum_{l_3=0}^{\alpha_3-7}\alpha^i_{\alpha_1-6,0,l_3}y_1^{\alpha_1-6}y_3^{l_3}\\ &+\sum_{l_2=0}^{\alpha_2-7}\alpha^i_{0,l_2,\alpha_3-6}y_2^{l_2}y_3^{\alpha_3-6},\ i=1,2,3.\end{aligned}$$

The derivation bases are of the form

$$\begin{gathered}y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_1,\ 1\leq l_1\leq\alpha_1-7,0\leq l_2\leq\alpha_2-7,0\leq l_3\leq\alpha_3-7;y_2^{l_2}y_3^{\alpha_3-6}\partial_1,\ 0\leq l_2\leq\alpha_2-7,\\ y_2^{\alpha_2-7}y_3^{l_3}\partial_1,\ 1\leq l_3\leq\alpha_3-7;y_1^{l_1}y_2^{\alpha_2-6}\partial_1,\ 0\leq l_1\leq\alpha_1-7;y_1^{\alpha_1-6}y_3^{l_3}\partial_1,\ 0\leq l_3\leq\alpha_3-7,\\ y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_2,\ 0\leq l_1\leq\alpha_1-7,1\leq l_2\leq\alpha_2-7,0\leq l_3\leq\alpha_3-7;y_1^{\alpha_1-6}y_3^{l_3}\partial_2,\ 0\leq l_3\leq\alpha_3-7,\\ y_1^{l_1}y_2^{\alpha_2-6}\partial_2,\ 0\leq l_1\leq\alpha_1-7;y_1^{l_1}y_3^{\alpha_3-7}\partial_2,\ 1\leq l_1\leq\alpha_1-7;y_2^{l_2}y_3^{\alpha_3-6}\partial_2,\ 0\leq l_2\leq\alpha_2-7,\\ y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_3,\ 0\leq l_1\leq\alpha_1-7,0\leq l_2\leq\alpha_2-7,1\leq l_3\leq\alpha_3-7;y_1^{l_1}y_2^{\alpha_2-6}\partial_3,\ 0\leq l_1\leq\alpha_1-7,\\ y_1^{\alpha_1-7}y_2^{l_2}\partial_3,\ 1\leq l_2\leq\alpha_2-7;y_2^{l_2}y_3^{\alpha_3-6}\partial_3,\ 0\leq l_2\leq\alpha_2-7;y_1^{\alpha_1-4}y_3^{l_3}\partial_3,\ 0\leq l_3\leq\alpha_3-7.\end{gathered}$$

Therefore, we have

$$\delta_6(V)=3\alpha_1\alpha_2\alpha_3+124(\alpha_1+\alpha_2+\alpha_3)-19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)-831.$$

In the case of $\alpha_1 = 7, \alpha_2 \geq 8, \alpha_3 \geq 7$, we obtain the bases as

$$y_2^{\alpha_2-7}y_3^{l_3}\partial_1,\ 1 \leq l_3 \leq \alpha_3 - 6; y_1y_3^{l_3}\partial_1,\ 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-6}\partial_1; y_3^{\alpha_3-6}\partial_2,$$
$$y_1y_3^{l_3}\partial_2,\ 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-6}\partial_2; y_2^{l_2}y_3^{l_3}\partial_2,\ 1 \leq l_2 \leq \alpha_2 - 7, 0 \leq l_3 \leq \alpha_3 - 6,$$
$$y_2^{l_2}y_3^{l_3}\partial_3,\ 0 \leq l_2 \leq \alpha_2 - 7, 1 \leq l_3 \leq \alpha_3 - 6, y_1y_3^{l_3}\partial_3,\ 0 \leq l_3 \leq \alpha_3 - 7; y_2^{\alpha_2-6}\partial_3.$$

Therefore, we have

$$\delta_6(V) = 2\alpha_2\alpha_3 - 11\alpha_2 - 9\alpha_3 + 51.$$

Similarly, we can obtain bases for $\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 = 7$ and $\alpha_1 \geq 7, \alpha_2 = 7, \alpha_3 \geq 7$.

For $\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 8$, we need to prove following inequality:

$$\begin{aligned}&3\alpha_1\alpha_2\alpha_3 + 124(\alpha_1 + \alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 851\\ &\leq \frac{3(1+\alpha_1\alpha_2\alpha_3)^3}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_1+\alpha_1\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)} + 120(\frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_3+\alpha_2\alpha_3} + \frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_1+\alpha_1\alpha_3} + \frac{1+\alpha_1\alpha_2\alpha_3}{1-\alpha_2+\alpha_1\alpha_2})\\ &- 19(\frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_1+\alpha_1\alpha_3)} + \frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_1+\alpha_1\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)} + \frac{(1+\alpha_1\alpha_2\alpha_3)^2}{(1-\alpha_3+\alpha_2\alpha_3)(1-\alpha_2+\alpha_1\alpha_2)}) - 756.\end{aligned}$$

After simplification, we obtain

$$\begin{aligned}&4(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3) + \alpha_1(\alpha_2 - 7) + \alpha_2(\alpha_3 - 7) + \alpha_3(\alpha_1 - 7) + 4\alpha_1^2[\alpha_2(\alpha_3 - 7) + \alpha_3(\alpha_2 - 7)]\\ &+ 3\alpha_2^2[\alpha_1(\alpha_3 - 6) + \alpha_3(\alpha_1 - 7)] + 5\alpha_3^2[\alpha_1(\alpha_2 - 7) + \alpha_2(\alpha_1 - 6)] + 2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + 3(\alpha_1^3\alpha_2 +\\ &\alpha_2^3\alpha_3 + \alpha_3^3\alpha_1) + 2\alpha_1^2\alpha_2^2\alpha_3^2 + 5(\alpha_1\alpha_2^2\alpha_3 + \alpha_1\alpha_2\alpha_3^2) + 2\alpha_1^2\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_3[2\alpha_1 - 11] + \alpha_1^3\alpha_2\alpha_3^2(\alpha_3 -\\ &7)(\alpha_2 - 7) + \alpha_1^2\alpha_3^2(\alpha_3 - 7)(\alpha_1\alpha_2 - 7) + \alpha_1^2\alpha_2\alpha_3^2(\alpha_3 + \alpha_2 - 8) + 3\alpha_1\alpha_2\alpha_3^3(\alpha_1 - 7) + \alpha_1^2\alpha_2^3\alpha_3(\alpha_3 -\\ &7)(\alpha_1 - 6) + \alpha_1^2\alpha_2^2(\alpha_1 - 7)(\alpha_2\alpha_3 - 6) + \alpha_1^3\alpha_2\alpha_3(\alpha_2 - 7) + \alpha_1^2\alpha_2^2\alpha_3(\alpha_1 - 6 + (\alpha_3 - 7)) + \alpha_1\alpha_2^2\alpha_3^3\\ &(\alpha_2 - 7)(\alpha_1 - 6) + \alpha_2^2\alpha_3^2(\alpha_2 - 7)(\alpha_1\alpha_3 - 7) + 12 \geq 0.\end{aligned}$$

Similarly, using Conjecture 1, we can prove for $\alpha_1, \alpha_3 \geq 7, \alpha_2 = 7; \alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 = 7$ and $\alpha_1 = 7, \alpha_2 \geq 8, \alpha_3 \geq 7$. $\square$

Proposition 8. *For isolated fewnomial surface singularity $(V, 0)$ of type 4, defined by $g = y_1^{\alpha_1} + y_2^{\alpha_2} + y_3^{\alpha_3}y_2$ ($\alpha_1 \geq 8, \alpha_2 \geq 8, \alpha_3 \geq 7$) with weight type $(\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{\alpha_2-1}{\alpha_2\alpha_3}; 1)$,*

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 + 124\alpha_1 + 120(\alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 781.$$

Assuming that $mult(g) \geq 8$, we need to prove the following inequality:

$$\begin{aligned}&3\alpha_1\alpha_2\alpha_3 + 124\alpha_1 + 120(\alpha_2 + \alpha_3) - 19(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) - 781 \leq \frac{3\alpha_2^2\alpha_1\alpha_3}{\alpha_2 - 1} + 120(\alpha_1 + \alpha_2 +\\ &\frac{\alpha_2\alpha_3}{\alpha_2 - 1}) - 19(\alpha_1\alpha_2 + \frac{\alpha_1\alpha_2\alpha_3}{\alpha_2 - 1} + \frac{\alpha_2^2\alpha_3}{\alpha_2 - 1}) - 756.\end{aligned}$$

Proof. Let $(\alpha_1\alpha_2\alpha_3 - 6(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 36(\alpha_2 + \alpha_3) + 37\alpha_1 - 222)$ be the dimension of $M_6(V)$ and the monomial basis be in the form

$$\{y_1^{l_1}y_2^{l_2}y_3^{l_3}, 0 \leq l_1 \leq \alpha_1 - 7; 0 \leq l_2 \leq \alpha_2 - 7; 0 \leq l_3 \leq \alpha_3 - 7; y_1^{l_1}y_3^{\alpha_3-6}, 0 \leq l_2 \leq \alpha_1 - 7\}.$$

This implies that

$$Dy_j = \sum_{l_1=0}^{\alpha_1-7}\sum_{l_2=0}^{\alpha_2-7}\sum_{l_3=0}^{\alpha_3-7} \alpha_{l_1,l_2,l_3}^i y_1^{l_1}y_2^{l_2}y_3^{l_3} + \sum_{l_1=0}^{\alpha_1-7} \alpha_{l_1,0,\alpha_3-6}^i y_1^{l_1}y_3^{\alpha_3-6},\ i = 1, 2, 3.$$

The derivation bases are of the form

$$y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_1,\ 1\le l_1\le\alpha_1-7, 0\le l_2\le\alpha_2-7, 0\le l_3\le\alpha_3-7; y_1^{l_1}y_3^{\alpha_3-6}\partial_1,\ 1\le l_1\le\alpha_1-7,$$
$$y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_2,\ 1\le l_1\le\alpha_1-7, 1\le l_2\le\alpha_2-7, 0\le l_3\le\alpha_3-7; y_1^{l_1}y_3^{\alpha_3-6}\partial_2,\ 0\le l_1\le\alpha_1-7,$$
$$y_2^{l_2}y_3^{l_3}\partial_2,\ 1\le l_2\le\alpha_2-7, 0\le l_3\le\alpha_3-7, y_1^{l_1}y_2^{\alpha_2-7}\partial_3,\ 0\le l_1\le\alpha_1-7$$
$$y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_3,\ 0\le l_1\le\alpha_1-7, 0\le l_2\le\alpha_2-7, 1\le l_3\le\alpha_3-7, y_1^{l_1}y_3^{\alpha_3-6}\partial_3,\ 0\le l_1\le\alpha_1-7.$$

Therefore, we have

$$\delta_6(V)=3\alpha_1\alpha_2\alpha_3+124\alpha_1+120(\alpha_2+\alpha_3)-19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)-781.$$

Next, we also need to show that when $\alpha_1\ge 8,\alpha_2\ge 8,\alpha_3\ge 7$, then

$$3\alpha_1\alpha_2\alpha_3+124\alpha_1+120(\alpha_2+\alpha_3)-19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)-781\le\frac{3\alpha_2^2\alpha_1\alpha_3}{\alpha_2-1}+120(\alpha_1+\alpha_2+\frac{\alpha_2\alpha_3}{\alpha_2-1})-19(\alpha_1\alpha_2+\frac{\alpha_1\alpha_2\alpha_3}{\alpha_2-1}+\frac{\alpha_2^2\alpha_3}{\alpha_2-1})-756.$$

From the above inequality, we obtain

$$\frac{\alpha_1\alpha_3(2\alpha_2-12)}{\alpha_2-7}+\alpha_2\alpha_3+\alpha_3(\alpha_2-5)+\frac{6\alpha_3}{\alpha_2-6}+\frac{\alpha_1[\alpha_2(\alpha_3-6)+7]}{\alpha_2-6}\ge 0.$$

□

Proposition 9. *For isolated fewnomial surface singularity $(V,0)$ of type 5 defined by $g=y_1^{\alpha_1}y_2+y_2^{\alpha_2}y_1+y_3^{\alpha_3}$ $(\alpha_1\ge 7,\alpha_2\ge 7,\alpha_3\ge 8)$ with weight type $(\frac{\alpha_2-1}{\alpha_1\alpha_2-1},\frac{\alpha_1-1}{\alpha_1\alpha_2-1},\frac{1}{\alpha_3};1)$,*

$$\delta_6(V)=\begin{cases}3\alpha_1\alpha_2\alpha_3+120(\alpha_1+\alpha_2)+128\alpha_3-19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\\ -806; & \alpha_1\ge 8,\alpha_2\ge 8,\alpha_3\ge 8\\ 2\alpha_2\alpha_3-13\alpha_2-8\alpha_3+53; & \alpha_1=7,\alpha_2\ge 7,\alpha_3\ge 8\end{cases}$$

Assuming that $\alpha_1\ge 8,\alpha_2\ge 8,\alpha_3\ge 8$, then we need to prove the following inequality:

$$\begin{aligned}&3\alpha_1\alpha_2\alpha_3+120(\alpha_1+\alpha_2)+128\alpha_3-19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)-806\\ &\le\frac{3\alpha_3(\alpha_1\alpha_2-1)^2}{(\alpha_2-1)(\alpha_1-1)}+120(\frac{\alpha_1\alpha_2-1}{\alpha_2-1}+\frac{\alpha_1\alpha_2-1}{\alpha_1-1}+\alpha_3)-19(\frac{(\alpha_1\alpha_2-1)^2}{(\alpha_2-1)(\alpha_1-1)}+\frac{\alpha_3(\alpha_1\alpha_2-1)}{\alpha_1-1}+\frac{\alpha_3(\alpha_1\alpha_2-1)}{\alpha_2-1})\\ &-756.\end{aligned}$$

Proof. Let $\alpha_1\alpha_2\alpha_3-6(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)+36(\alpha_1+\alpha_2)+38\alpha_3-228$ be the number of dimension of $M_6(V)$ and the monomial basis be in the form

$$\{y_1^{l_1}y_2^{l_2}y_3^{l_3}, 0\le l_1\le\alpha_1-7; 0\le l_2\le\alpha_2-7; 0\le l_3\le\alpha_3-7; y_1^{\alpha_1-6}y_3^{l_3}, 0\le l_3\le\alpha_3-7;$$
$$y_2^{\alpha_2-6}y_3^{l_3}, 0\le l_3\le\alpha_3-7\}.$$

This implies that

$$Dy_j=\sum_{l_1=0}^{\alpha_1-7}\sum_{l_2=0}^{\alpha_2-7}\sum_{l_3=0}^{\alpha_3-7}\alpha^i_{l_1,l_2,l_3}y_1^{l_1}y_2^{l_2}y_3^{l_3}+\sum_{l_3=0}^{\alpha_3-7}\alpha^i_{\alpha_1-6,0,l_3}y_1^{\alpha_1-6}y_3^{l_3}+\sum_{l_3=0}^{\alpha_3-7}\alpha^i_{0,\alpha_2-6,l_3}y_2^{\alpha_2-6}y_3^{l_3},$$
$$i=1,2,3.$$

The derivation bases are of the form

$$y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_1,\ 1\le l_1\le \alpha_1-7, 0\le l_2\le \alpha_2-7, 0\le l_3\le \alpha_3-7; y_1^{\alpha_1-6}y_3^{l_3}\partial_1,\ 0\le l_3\le \alpha_3-7,$$
$$y_2^{\alpha_2-6}y_3^{l_3}\partial_1,\ 0\le l_3\le \alpha_3-7; y_2^{\alpha_2-7}y_3^{l_3}\partial_1,\ 0\le l_3\le \alpha_3-7,$$
$$y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_2,\ 0\le l_1\le \alpha_1-7, 1\le l_2\le \alpha_2-7, 0\le l_3\le \alpha_3-7; y_1^{\alpha_1-6}y_3^{l_3}\partial_2,\ 0\le l_3\le \alpha_3-7,$$
$$y_2^{\alpha_2-6}y_3^{l_3}\partial_2,\ 0\le l_3\le \alpha_3-7; y_1^{\alpha_1-7}y_3^{l_3}\partial_2,\ 0\le l_3\le \alpha_3-7,$$
$$y_1^{l_1}y_2^{l_2}y_3^{l_3}\partial_3,\ 0\le l_1\le \alpha_1-7, 0\le l_2\le \alpha_2-7, 1\le l_3\le \alpha_3-7; y_1^{\alpha_1-6}y_3^{l_3}\partial_3,\ 1\le l_3\le \alpha_3-7,$$
$$y_2^{\alpha_2-6}y_3^{l_3}\partial_3,\ 1\le l_3\le \alpha_3-7.$$

Therefore, we have

$$\delta_6(V) = 3\alpha_1\alpha_2\alpha_3 + 120(\alpha_1+\alpha_2) + 128\alpha_3 - 19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) - 806.$$

For $\alpha_1 = 7, \alpha_2 \ge 7, \alpha_3 \ge 8$, we obtain the following bases:

$$y_2^{l_2}y_3^{l_3}\partial_2,\ 1\le l_2\le \alpha_2-7, 0\le l_3\le \alpha_3-7; y_2^{\alpha_2-6}y_3^{l_3}\partial_1,\ 0\le l_3\le \alpha_3-7,$$
$$y_1y_3^{l_3}\partial_1,\ 0\le l_3\le \alpha_3-7; y_2^{\alpha_2-6}y_3^{l_3}\partial_2,\ 0\le l_3\le \alpha_3-7,$$
$$y_2^{l_2}y_3^{l_3}\partial_3,\ 0\le l_2\le \alpha_2-7, 1\le l_3\le \alpha_3-7; y_1y_3^{l_3}\partial_2,\ 0\le l_3\le \alpha_3-7,$$
$$y_1y_3^{l_3}\partial_3,\ 1\le l_3\le \alpha_3-7.$$

We have

$$\delta_6(V) = 2\alpha_2\alpha_3 - 13\alpha_2 - 8\alpha_3 + 53.$$

Next, we need to show that when $\alpha_1 \ge 8, \alpha_2 \ge 8, \alpha_3 \ge 8$, then

$$3\alpha_1\alpha_2\alpha_3 + 120(\alpha_1+\alpha_2) + 128\alpha_3 - 19(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) - 806 \le \frac{3\alpha_3(\alpha_1\alpha_2-1)^2}{(\alpha_2-1)(\alpha_1-1)}$$
$$+120\left(\frac{\alpha_1\alpha_2-1}{\alpha_2-1}+\frac{\alpha_1\alpha_2-1}{\alpha_1-1}+\alpha_3\right) - 19\left(\frac{(\alpha_1\alpha_2-1)^2}{(\alpha_2-1)(\alpha_1-1)}+\frac{\alpha_3(\alpha_1\alpha_2-1)}{\alpha_1-1}+\frac{\alpha_3(\alpha_1\alpha_2-1)}{\alpha_2-1}\right) - 756.$$

After simplification, we obtain

$$\begin{aligned}&\alpha_1(\alpha_1-7)(\alpha_2-6)(\alpha_3+(\alpha_1-5)\alpha_2(\alpha_2-7)\alpha_3) + \alpha_1^2(\alpha_3-6)(\alpha_2-5) + \alpha_2^2\alpha_1 + 4\alpha_1(\alpha_2-6)\\ &+4\alpha_2(\alpha_1-6) + 4\alpha_3(\alpha_1-5) + 12\alpha_1\alpha_2 + 14\alpha_1\alpha_3 + 5\alpha_2\alpha_3 + 22\alpha_2 + \alpha_1\alpha_2(\alpha_1-6)\\ &+(\alpha_1-5)\alpha_2(\alpha_2-6)(\alpha_3-5) + (\alpha_1-6)(\alpha_3-7) + 22 \ge 0.\end{aligned}$$

Similarly, for $\alpha_1 = 7, \alpha_2 \ge 7, \alpha_3 \ge 8$, Conjecture 1 also holds true. □

Proof of Theorem 1. The proof of Theorem 1 is the consequence of Proposition 3. □

Proof of Theorem 2. Propositions 4 and 5 and Remark 1 imply Theorem 2 as a corollary. □

Proof of Theorem 3. Propositions 4–6 and Remark 3 of [23], along with Remark 1 and Propositions 4 and 5, imply that the inequality $\delta_6(V) < \delta_5(V)$ holds true for binomial singularities. □

Proof of Theorem 4. Theorem 4 is the consequence of Remark 2 and Propositions 6–9. □

Proof of Theorem 5. Propositions 6–9 and Remark 4 of [23], along with Remark 2 and Propositions 6–9, imply that the inequality $\delta_6(V) < \delta_5(V)$ holds true for trinomial singularities. □

4. Conclusions

Finding the dimension of an algebra is critical for studying its applications. This work is the partial proof of the conjecture over dimensions $\delta_k(V)$ of k-th Yau Algebra. First of all, we determine the general formula for the dimension of fewnomial isolated singularities for $\mathcal{L}_6(V)$ in Proposition 3. Then, this result determines the formulas for the weighted binomial isolated singularities of all the three types listed in Corollary 1 for Propositions 2, 4, and 5. Then, using the general formula for the dimension $\delta_6(V)$ of $\mathcal{L}_6(V)$, we determine the formulas for weighted binomial isolated singularities of Propositions 6–9 of all the five types listed in Proposition 2. Based on all these findings and the dimension of $\mathcal{L}_5(V)$ determined in our previous work [27]: we prove the inequality conjecture $\delta_6(V) < \delta_5(V)$ for binomial and trinomial isolated singularities in Theorems 3 and 5.

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