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Efficient Numerical Solutions for Fuzzy Time Fractional Convection Diffusion Equations Using Two Explicit Finite Difference Methods

Areen Al-Khateeb

Department of Mathematics, Faculty of Science and Technology, Jadara University, Irbid 21110, Jordan; areen.k@jadara.edu.jo

Abstract: In this study, we explore fractional partial differential equations as a more generalized version of classical partial differential equations. These fractional equations have shown promise in providing improved descriptions of certain phenomena under specific circumstances. The main focus of this paper comprises the development, analysis, and application of two explicit finite difference schemes to solve an initial boundary value problem involving a fuzzy time fractional convection–diffusion equation with a fractional order in the range of $0 \leq \xi \leq 1$. The uniqueness of this problem lies in its consideration of fuzziness within both the initial and boundary conditions. To handle the uncertainty, we propose a computational mechanism based on the double parametric form of fuzzy numbers, effectively converting the problem from an uncertain format to a crisp one. To assess the stability of our proposed schemes, we employ the von Neumann method and find that they demonstrate unconditional stability. To illustrate the feasibility and practicality of our approach, we apply the developed scheme to a specific example.

Keywords: Caputo formula; explicit schemes; double parametric form; fuzzy time fractional convection–diffusion equation

MSC: 35R13; 35R11

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1. Introduction

Fractional differential equations (FDEs) have garnered significant interest due to their wide range of real-life applications in chemistry, physics, biology, engineering, and other fields [1–13]. In conventional studies on natural processes modeled by FDEs, the vague parameters are considered to be precise and well defined. However, in reality, these components and parameters may be subject to vagueness and uncertainty arising from measurement errors in the experimental part, resulting in what are known as fuzzy fractional differential equations (FFDEs). As of late, considerable attention has been devoted to exploring and understanding fuzzy FFDEs, as evidenced by various noteworthy contributions found in earlier research.

Previous studies [14–18] have addressed various theoretical aspects of FFDEs, such as stability, existence, and uniqueness. Agarwal et al. [14] investigated and discussed the solution concept of FFDEs, while Lupulescu and Arshad [15] utilized the main concept that was developed by Agarwal et al. [14] to investigate the existence and uniqueness of solutions for initial value problems involving FFDEs. Salahshour et al. [16] considered FFDEs and demonstrated the existence and uniqueness of solutions using the contraction principle. Furthermore, Long et al. [17] established theoretical aspects, such as the uniqueness and existence of the lower and upper fuzzy solutions of FFDEs. Souahi et al. [18] confirmed the theoretical aspects of high-order FFDEs. In addition, Long et al. [17] proved the theoretical aspects of fuzzy fractional partial differential equations (FFPDEs). Various solution techniques for FFPDEs have been explored in recent research [19–26]. Salahshour et al. [19]

employed a Laplace transform approach to solve FFPDEs, while Allahviranloo et al. [20] discussed and implemented an explicit scheme to obtain numerical solutions for the same equations. Mazandarani and Kamyad [21] utilized the modified fractional Euler method to solve the fuzzy fractional initial value problem (FFIVP) with Caputo fractional derivatives. Ahmadian et al. [22] employed the Tau method, and Jafarian et al. [23] used a Laplace transformation method to solve FFPDEs with Caputo fractional derivatives. Furthermore, Raj et al. [24] applied the numerical method known as the Adams–Bashforth multistep method to derive a solution for FFPDEs. Ahmadian et al. [25] proposed a solution for FFPDEs using the shifted Chebyshev matrix. Also, Salah et al. [26] successfully employed the homotopy analysis transform technique for deriving analytical solutions applicable to fuzzy fractional equations related to heat or diffusion phenomena. Shah et al. [27] presented a new fuzzy transform iterative method to FTFCDE using generalized Hukuhara differentiability to handle time fractional derivatives. Multiple numerical examples have been examined to demonstrate the effectiveness and significance of the suggested approach in determining whether it offers a viable solution to the given problem. The simulation outcomes indicate that the fuzzy new iterative transform method is a highly effective means of meticulously and precisely investigating the behavior of a proposed approach. Saadeh [28] facilitated the application of finite-difference techniques for solving an FTFCDE of order β within the range [0, 1] under the initial conditions considered suitable. The fractional derivative is defined in the Riemann–Liouville sense. The key innovation in this approach lies in introducing an alternative spatial discretization method for the fractional derivative through the use of the fractional Grünwald formula. The numerical results have been analyzed to assess the precision of the proposed method and to compare it under various conditions. It has been found that the proposed methodologies are straightforward, precise, and applicable to a wide spectrum of spatial fractional models encountered in natural sciences.

The authors employed a single parametric form to devise a finite difference scheme in order to solve the fuzzy time in investigating the solution to the FTFCDE using the double parametric form. The paper will present two different explicit schemes for solving this equation and will incorporate the approach introduced by Chakraverty and Tapaswini in handling uncertainties, as mentioned earlier.

2. Fuzzy Environmental Considerations in the Context of the TFCDE

The one-dimensional FTFCDE presented in [29], should be taken into account, as follows:

$$\begin{gathered}\frac{\partial^{\xi}\widetilde{w}(x,\tau,\xi)}{\partial^{\xi}\tau} = -\widetilde{v}(x)\frac{\partial\widetilde{w}(x,\tau)}{\partial x} - \widetilde{d}(x)\frac{\partial^2\widetilde{w}(x,\tau)}{\partial x^2} + \widetilde{u}(x)\,, \quad 0 \le \xi \le 1\,,\, 0 < x < l,\, \tau > 0 \\ \widetilde{w}(x,0) = \widetilde{g}(x)\,,\, \widetilde{w}(0,\tau) = \widetilde{f},\, \widetilde{w}(l,\tau) = \widetilde{z}\end{gathered} \quad (1)$$

The fuzzy concentration, denoted as $\widetilde{w}(x,\tau,\xi)$, represents a measure of a specific quantity that is associated with the crisp variables τ, x, and ξ, where ξ represents a fractional order between 0 and 1. The term $\frac{\partial^{\xi}\widetilde{w}(x,\tau,\xi)}{\partial^{\xi}\tau}$ of order ξ corresponds to the fuzzy derivative of the time fractional. The velocity of the quantity is represented by $\widetilde{v}(x)$, while $\widetilde{d}(x)$ represents the diffusion coefficient or diffusivity, and $\widetilde{u}(x)$ denotes fuzzy functions related to the crisp variable x. The fuzzy initial condition is represented by $\widetilde{w}(x,0)$, and the fuzzy boundary conditions are given by $\widetilde{w}(x,0)$ and $\widetilde{w}(l,\tau)$, where $\widetilde{f}$ and $\widetilde{z}$ represent fuzzy convex numbers. Finally, in Equation (1), the fuzzy functions $\widetilde{v}(x)$, $\widetilde{d}(x)$, $\widetilde{u}(x)$, and $\widetilde{g}(x)$ are defined as follows:

$$\begin{cases} \tilde{d}(x) = \tilde{\Theta}_1 c_1(x) \\ \tilde{u}(x) = \tilde{\Theta}_2 c_2(x) \\ \tilde{g}(x) = \tilde{\Theta}_3 c_3(x) \\ \tilde{v}(x) = \tilde{\Theta}_4 c_4(x) \end{cases} \tag{2}$$

where $c_1(x)$, $c_2(x)$, $c_3(x)$, and $c_4(x)$ are the crisp functions, with $\tilde{\Theta}_1$, $\tilde{\Theta}_2$, $\tilde{\Theta}_3$, and $\tilde{\Theta}_4$ being the fuzzy convex numbers.

According to the r-cut technique discussed in [30], the defuzzification of Equation (1) can be presented for all $r \in [0,1]$ as follows:

$$\left[\tilde{w}(x,\tau)\right]_r = \underline{w}(x,\tau;r), \overline{w}(x,\tau;r) \tag{3}$$

$$\left[\frac{\partial^{\xi}\tilde{w}(x,\tau,\xi)}{\partial^{\xi}\tau}\right]_r = \frac{\partial^{\xi}\underline{w}(x,\tau,\xi;r)}{\partial^{\xi}\tau}, \frac{\partial^{\xi}\overline{w}(x,\tau,\xi;r)}{\partial^{\xi}\tau} \tag{4}$$

$$\left[\frac{\partial\tilde{w}(x,\tau)}{\partial x}\right]_r = \frac{\partial\underline{w}(x,\tau;r)}{\partial x}, \frac{\partial\overline{w}(x,\tau;r)}{\partial x} \tag{5}$$

$$\left[\frac{\partial^2\tilde{w}(x,\tau)}{\partial x^2}\right]_r = \frac{\partial^2\underline{w}(x,\tau;r)}{\partial x^2}, \frac{\partial^2\overline{w}(x,\tau;r)}{\partial x^2} \tag{6}$$

$$\left[\tilde{v}(x)\right]_r = \underline{v}(x;r), \overline{v}(x;r) \tag{7}$$

$$\left[\tilde{d}(x)\right]_r = \underline{d}(x;r), \overline{d}(x;r) \tag{8}$$

$$\left[\tilde{u}(x)\right]_r = \underline{u}(x;r), \overline{u}(x;r) \tag{9}$$

$$\left[\tilde{w}(x,0)\right]_r = \underline{w}(x,0;r), \overline{w}(x,0;r) \tag{10}$$

$$\left[\tilde{w}(0,\tau)\right]_r = \underline{w}(0,\tau;r), \overline{w}(0,\tau;r) \tag{11}$$

$$\left[\tilde{w}(l,\tau)\right]_r = \underline{w}(l,\tau;r), \overline{w}(l,\tau;r) \tag{12}$$

$$\left[\tilde{g}(x)\right]_r = \underline{g}(x;r), \overline{g}(x;r) \tag{13}$$

$$\begin{cases} \left[\tilde{f}\right]_r = \underline{f}(r), \overline{f}(r) \\ \left[\tilde{z}\right]_r = \underline{z}(r), \overline{z}(r) \end{cases} \tag{14}$$

where

$$\begin{cases} [\tilde{d}(x)]_r = [\underline{\Theta}_1(r), \overline{\Theta}_1(r)]c_1(x) \\ [\tilde{u}(x)]_r = [\underline{\Theta}_2(r), \overline{\Theta}_2(r)]c_2(x) \\ [\tilde{g}(x)]_r = [\underline{\Theta}_3(r), \overline{\Theta}_3(r)]c_3(x) \\ [v(x)]_r = [\underline{\Theta}_4(r), \overline{\Theta}_4(r)]c_4(x) \end{cases} \tag{15}$$

The Zadeh extension principle defined in [29] is employed to define the membership function:

$$\begin{cases} \underline{w}(x,\tau;r) = min\left\{\tilde{w}\left(\tilde{\mu}(r),\tau\right)\middle|\tilde{\mu}(r) \in \tilde{w}(x,\tau;r)\right\} \\ \overline{w}(x,\tau;r) = max\left\{\tilde{w}\left(\tilde{\mu}(r),\tau\right)\middle|\tilde{\mu}(r) \in \tilde{w}(x,\tau;r)\right\} \end{cases} \tag{16}$$

Now, Equation (1) is rewritten for $0 < x < l, \tau > 0$ and $r \in [0,1]$ to obtain the general formula of FTFCDE, as follows:

$$\begin{cases} \frac{\partial^\xi \underline{w}(x,\tau,\xi)}{\partial^\xi \tau} = -[\underline{\Theta}_4(r)]c_4(x)\frac{\partial \underline{w}(x,\tau;r)}{\partial x} + [\underline{\Theta}_1(r)]c_1(x)\frac{\partial^2 \underline{w}(x,\tau;r)}{\partial x^2} + [\underline{\Theta}_2(r)]c_2(x) \\ \underline{w}(x,0,\xi;r) = \underline{\Theta}_3(r)c_3(x) \\ \underline{w}(0,\tau,\xi;r) = \underline{f}(x,r), \underline{w}(l,\tau,\xi;r) = \underline{z}(l,\tau,\xi;r) \end{cases} \tag{17}$$

$$\begin{cases} \frac{\partial^\xi \overline{w}(x,\tau,\xi)}{\partial^\xi \tau} = -[\overline{\Theta}_4(r)]c_4(x)\frac{\partial \overline{w}(x,\tau;r)}{\partial x} + [\overline{\Theta}_1(r)]c_1(x)\frac{\partial^2 \overline{w}(x,\tau;r)}{\partial x^2} + [\overline{\Theta}_2(r)]c_2(x) \\ \overline{w}(x,0,\xi;r) = \overline{\Theta}_3(r)c_3(x) \\ \overline{w}(0,\tau,\xi;r) = \overline{f}(x,r), \overline{w}(l,\tau,\xi;r) = \overline{z}(l,\tau,\xi;r) \end{cases} \tag{18}$$

Equations (17) and (18) present the lower form and upper form, respectively, of the FTFCDE.

We will now proceed to defuzzify Equation (1) using the double parametric form. However, if we consider the singular parametric form, we can express Equation (1) in the following manner:

$$\begin{aligned} &\left[\frac{\partial^\xi \underline{w}(x,\tau,\xi;r)}{\partial^\xi \tau}, \frac{\partial^\xi \overline{w}(x,\tau,\xi;r)}{\partial^\xi \tau}\right] \\ &\quad = -[\underline{v}(x,r), \overline{v}(x,r)]\left[\frac{\partial \underline{w}_{i,n}(x,\tau;r)}{\partial x}, \frac{\partial \overline{w}_{i,n}(x,\tau;r)}{\partial x}\right] \\ &\quad + \left[\underline{d}(x,r), \overline{d}(x,r)\right]\left[\frac{\partial^2 \underline{w}_{i,n}(x,\tau;r)}{\partial x^2}, \frac{\partial^2 \overline{w}_{i,n}(x,\tau;r)}{\partial x^2}\right] + [\underline{u}(x,\tau;r), \overline{u}(x,\tau;r)] \end{aligned} \tag{19}$$

subjected to the imprecise initial and boundary conditions:

$$[\underline{w}(x,0;r), \overline{w}(x,0;r)] = \left[\underline{g}(x,\tau;r), \overline{g}(x,\tau;r)\right]$$

$$[\underline{w}(0,\tau;r), \overline{w}(0,\tau;r)] = \left[\underline{f}(0,\tau;r), \overline{f}(0,\tau;r)\right]$$

$$[\underline{w}(l,\tau;r), \overline{w}(l,\tau;r)] = [\underline{z}(l,\tau;r), \overline{z}(l,\tau;r)]$$

Now, by utilizing the double parametric form techniques in [28], we rewrite Equation (19) as follows:

$$\begin{aligned} &\left\{\psi\left(\frac{\partial^\xi \underline{w}(x,\tau,\xi;r)}{\partial^\xi \tau} - \frac{\partial^\xi \overline{w}(x,\tau,\xi;r)}{\partial^\xi \tau}\right) + \frac{\partial^\xi \underline{w}(x,\tau,\xi;r)}{\partial^\xi \tau}\right\} \\ &\quad = -\{\psi\,(\underline{v}(x,r) - \overline{v}(x,r)) + \underline{v}(x,r)\}\left\{\psi\left(\frac{\partial \underline{w}_{i,n}(x,\tau;r)}{\partial x} - \frac{\partial \overline{w}_{i,n}(x,\tau;r)}{\partial x}\right) + \frac{\partial \underline{w}_{i,n}(x,\tau;r)}{\partial x}\right\} \\ &\quad + \left\{\psi\left(\underline{d}(x,r) - \overline{d}(x,r)\right) + \underline{d}(x,r)\right\}\left\{\psi\left(\frac{\partial^2 \underline{w}_{i,n}(x,\tau;r)}{\partial x^2} - \frac{\partial^2 \overline{w}_{i,n}(x,\tau;r)}{\partial x^2}\right) + \frac{\partial^2 \underline{w}_{i,n}(x,\tau;r)}{\partial x^2}\right\} \\ &\quad + \{\psi\,(\underline{u}(x,\tau;r) - \overline{u}(x,\tau;r)) + \underline{u}(x,\tau;r)\} \end{aligned} \tag{20}$$

subjected to fuzzy initial and boundary conditions:

$$\{\psi\,(\underline{w}(x,0;r) - \overline{w}(x,0;r)) + \underline{w}(x,0;r)\} = \left\{\psi\left(\underline{g}(x;r) - \overline{g}(x;r)\right) + \underline{g}(x;r)\right\}$$

$$\{\psi\,(\underline{w}(0,\tau;r) - \overline{w}(0,\tau;r)) + \underline{w}(0,\tau;r)\} = \left\{\psi\left(\underline{f}(x;r) - \overline{f}(x;r)\right) + \underline{f}(x;r)\right\}$$

$$\{\psi\,(\underline{w}(l,\tau;r) - \overline{w}(l,\tau;r)) + \underline{w}(l,\tau;r)\} = \{\psi\,(\underline{z}(x;r) - \overline{z}(x;r)) + \underline{z}(x;r)\}$$

where $\psi \in [0,\ 1]$.

Now, denote

$$\frac{\partial^{\xi}\widetilde{w}(x,\tau,\psi)}{\partial^{\xi}\tau} = \left\{ \psi\left(\frac{\partial^{\xi}\underline{w}(x,\tau,\xi;r)}{\partial^{\xi}\tau} - \frac{\partial^{\xi}\overline{w}(x,\tau,\xi;r)}{\partial^{\xi}\tau}\right) + \frac{\partial^{\xi}\underline{w}(x,\tau,\xi;r)}{\partial^{\xi}\tau}\right\}$$
$$\widetilde{v}(x)\frac{\partial\widetilde{w}(x,\tau,\psi)}{\partial x} = \{ \psi\,(\underline{v}(x,r) - \overline{v}(x,r)) + \underline{v}(x,r)\}\left\{ \psi\left(\frac{\partial\underline{w}_{i,n}(x,\tau;r)}{\partial x} - \frac{\partial\overline{w}_{i,n}(x,\tau;r)}{\partial x}\right) + \frac{\partial\underline{w}_{i,n}(x,\tau;r)}{\partial x}\right\}$$
$$\widetilde{d}(x)\frac{\partial^2\widetilde{w}(x,\tau,\psi)}{\partial x^2} = \left\{ \psi\left(\underline{d}(x,r) - \overline{d}(x,r)\right) + \underline{d}(x,r)\right\}\left\{ \psi\left(\frac{\partial^2\underline{w}_{i,n}(x,\tau;r)}{\partial x^2} - \frac{\partial^2\overline{w}_{i,n}(x,\tau;r)}{\partial x^2}\right) + \frac{\partial^2\underline{w}_{i,n}(x,\tau;r)}{\partial x^2}\right\}$$
$$\widetilde{u}(x,\tau;r,\psi) = \{ \psi\,(\underline{u}(x,\tau;r) - \overline{u}(x,\tau;r)) + \underline{u}(x,\tau;r)\}$$
$$\widetilde{w}(x,0,r,\psi) = \{ \psi\,(\underline{w}(x,0;r) - \overline{w}(x,0;r)) + \underline{w}(x,0;r)\}$$
$$\widetilde{g}(x,r,\psi) = \left\{ \psi\left(\underline{g}(x;r) - \overline{g}(x;r)\right) + \underline{g}(x;r)\right\}$$
$$\widetilde{w}(0,\tau,r,\psi) = \{ \psi\,(\underline{w}(0,\tau;r) - \overline{w}(0,\tau;r)) + \underline{w}(0,\tau;r)\}$$
$$\widetilde{f}(x,r,\psi) = \left\{\psi\left(\underline{f}(x;r) - \overline{f}(x;r)\right) + \underline{f}(x;r)\right\}$$
$$\widetilde{w}(l,\tau,r,\psi) = \{ \psi\,(\underline{w}(l,\tau;r) - \overline{w}(l,\tau;r)) + \underline{w}(l,\tau;r)\}$$
$$\widetilde{z}(x,r,\psi) = \{ \psi\,(\underline{z}(x;r) - \overline{z}(x;r)) + \underline{z}(x;r)\}$$

Substituting these into Equation (20), we obtain

$$\begin{array}{c}\frac{\partial^{\xi}\widetilde{w}(x,\tau,\xi,\psi)}{\partial^{\xi}\tau} = -\widetilde{v}(x)\frac{\partial\widetilde{w}(x,\tau,\psi)}{\partial x} + \widetilde{d}(x)\frac{\partial^2\widetilde{w}(x,\tau,\psi)}{\partial x^2} + \widetilde{b}(x,\tau,\psi)\ ,\ \ 0 < x < l, 0 < \psi < l,\ \tau > 0 \\ \widetilde{w}(x,0,\psi) = \widetilde{g}(x,r,\psi)\ ,\ \widetilde{w}(0,\tau,\psi) = \widetilde{f}\ ,\ \widetilde{w}(l,\tau,\psi) = \widetilde{z}\end{array} \tag{21}$$

We can derive the lower and upper bounds of the solutions in the single parametric form by assuming $\psi = 0$ and $\psi = 1$, respectively. This can be expressed as

$$\widetilde{w}(x,\tau;r,0) = \underline{w}(x,\tau;r)\ and\ \widetilde{w}(x,\tau;r,1) = \overline{w}(x,\tau;r)$$

3. The FTCS Method for Solving the FTFCDE

The time fractional derivative is discretized based on the Caputo formula, as follows:

$$\frac{\partial^{\xi}\widetilde{w}(x,\tau,\xi,\psi)}{\partial^{\xi}\tau} = \frac{\Delta\tau^{-\xi}}{\Gamma(2-\xi)}[\widetilde{w}_{i,n+1}(x,\tau;r,\psi) - \widetilde{w}_{i,n}(x,\tau;r,\psi) + \sum_{j=1}^{n} u_j\ (\widetilde{w}_{i,n+1-j}(x,\tau;r,\psi) - \widetilde{w}_{i,n-j}(x,\tau;r,\psi))] \tag{22}$$

The first- and second-order partial derivatives are discretized based on forward and central difference approximations, respectively, as follows:

$$\begin{array}{c}\frac{\partial\widetilde{w}(x,\tau;r,\psi)}{\partial x} = \frac{\widetilde{w}_{i+1,n}(x,\tau;r,\psi) - \widetilde{w}_{i-1,n}(x,\tau;r,\psi)}{2h} \\ \frac{\partial^2\widetilde{w}(x,\tau;r,\psi)}{\partial x^2} = \frac{\widetilde{w}_{i+1,n}(x,\tau;r,\psi) - 2\widetilde{w}_{i,n}(x,\tau;r,\psi) + \widetilde{w}_{i-1,n}(x,\tau;r,\psi)}{h^2}\end{array} \tag{23}$$

Substitute Equations (22) and (23) into Equation (21) to obtain

$$\begin{array}{l}\frac{\Delta\tau^{-\xi}}{\Gamma(2-\xi)}[\widetilde{w}_{i,n+1}(x,\tau;r,\psi) - \widetilde{w}_{i,n}(x,\tau;r,\psi) + \sum\limits_{j=1}^{n} u_j\ (\widetilde{w}_{i,n+1-j}(x,\tau;r,\psi) - \widetilde{w}_{i,n-j}(x,\tau;r,\psi))]. \\ = -\widetilde{v}(x,r)\frac{\widetilde{w}_{i+1,n}(x,\tau;r,\psi) - \widetilde{w}_{i-1,n}(x,\tau;r,\psi)}{2h} \\ +\widetilde{d}(x,r)\frac{\widetilde{w}_{i+1,n}(x,\tau;r,\psi) - 2\widetilde{w}_{i,n}(x,\tau;r,\psi) + \widetilde{w}_{i-1,n}(x,\tau;r,\psi)}{h^2} + \widetilde{u}(x;r,\psi)\end{array} \tag{24}$$

$$\begin{array}{l}[2\widetilde{w}_{i,n+1}(x,\tau;r,\psi) - 2\widetilde{w}_{i,n}(x,\tau;r,\psi) + 2\sum\limits_{j=1}^{n} u_j\ \left(\widetilde{w}_{i,n+1-j}(x,\tau;r,\psi) - \widetilde{w}_{i,n-j}(x,\tau;r,\psi)\right)] \\ = \frac{\widetilde{v}(x,r)\Delta\tau^{\xi}\Gamma(2-\xi)}{h}\left[\widetilde{w}_{i-1,n}(x,\tau;r,\psi) - \widetilde{w}_{i+1,n}(x,\tau;r,\psi)\right] \\ +\frac{\widetilde{d}(x,r)\Delta\tau^{\xi}\Gamma(2-\xi)}{h^2}\left[2\widetilde{w}_{i+1,n}(x,\tau;r,\psi) - 4\widetilde{w}_{i,n}(x,\tau;r,\psi) + 2\widetilde{w}_{i-1,n}(x,\tau;r,\psi)\right] \\ +2\Delta\tau^{\xi}\Gamma(2-\xi)\widetilde{u}(x,r)\end{array} \tag{25}$$

Now, let $\widetilde{p}_1(r) = \frac{\widetilde{v}(x,r)\Delta\tau^{\xi}\Gamma(2-\xi)}{h}$, $\widetilde{p}_2(r) = \frac{\widetilde{d}(x,r)\Delta\tau^{\xi}\Gamma(2-\xi)}{h^2}$, and from Equation (25), we obtain

$$2\widetilde{w}_{i,n+1}(x,\tau;r,\psi) = \left(2\widetilde{p}_2-\widetilde{p}_1\right)\widetilde{w}_{i+1,n}(x,\tau;r,\psi)+\left(2-4\widetilde{p}_2\right)\widetilde{w}_{i,n}(x,\tau;r,\psi)+\left(\widetilde{p}_1+2\widetilde{p}_2\right)\widetilde{w}_{i-1,n}(x,\tau;r,\psi) -2\sum_{j=1}^{n} b_j\left(\widetilde{w}_i^{n+1-j}(x,\tau;r,\psi)-\widetilde{w}_i^{n-j}(x,\tau;r,\psi)\right)+2\Delta\tau^{\xi}\Gamma(2-\xi)\widetilde{u}(x,r) \tag{26}$$

Linear equations are derived by evaluating Equation (26) for every spatial grid point. At the completion of each time level, a system of linear equations is obtained. Solving this system yields the values of $\widetilde{w}(x,\tau,\xi,\psi)$ for that specific time level.

4. The Saulyev Method for Solving the FTFCDE

The Saulyev scheme utilizes two time levels to approximate the first and second space derivative. Although the scheme seems implicit, the solution is obtained explicitly, making it an explicit method by nature [31].

The first- and second-order partial derivatives are discretized by employing two-time-level approximation. The discretization process can be outlined as follows:

$$\frac{\partial\widetilde{w}(x,\tau,\psi)}{\partial x}=\frac{\widetilde{w}_{i+1,n}(x,\tau;r,\psi)-\widetilde{w}_{i-1,n+1}(x,\tau;r,\psi)}{2h} \tag{27}$$

$$\frac{\partial^2\widetilde{w}(x,\tau,\psi)}{\partial x^2}=\frac{\widetilde{w}_{i+1,n}(x,\tau;r,\psi)-\widetilde{w}_{i,n}(x,\tau;r,\psi)-\widetilde{w}_{i,n+1}(x,\tau;r,\psi)+\widetilde{w}_{i-1,n+1}(x,\tau;r,\psi)}{h^2} \tag{28}$$

Now, let $\widetilde{p}_1(r)=\frac{\widetilde{v}(x,r)\Delta\tau^{\xi}\Gamma(2-\xi)}{h}$, $\widetilde{p}_2(r)=\frac{\widetilde{d}(x,r)\Delta\tau^{\xi}\Gamma(2-\xi)}{h^2}$, and from Equation (28), obtain

$$\left(1+\widetilde{p}_2\right)\widetilde{w}_{i,n+1}(x,\tau;r,\psi)-\left(\frac{\widetilde{p}_1}{2}+\widetilde{p}_2\right)\widetilde{w}_{i-1,n+1}(x,\tau;r,\psi) =\left(1-\widetilde{p}_2\right)\widetilde{w}_{i,n}(x,\tau;r,\psi)+\left(\widetilde{p}_2-\frac{\widetilde{p}_1}{2}\right)\widetilde{w}_{i+1,n}(x,\tau;r,\psi)-\sum_{j=1}^{n}u_j\left(\widetilde{w}_i^{n+1-j}(x,\tau;r,\psi)-\widetilde{w}_i^{n-j}(x,\tau;r,\psi)\right) +\Delta\tau^{\xi}\Gamma(2-\xi)\,\widetilde{u}(x,r) \tag{29}$$

5. The Stability Analysis

In the following section, we will discuss the stability of the FTCS for FTFCDE without a source term using the von Neumann method [32,33]. To begin, we assume that the discretization of the initial condition introduces a fuzzy error, represented as $\widetilde{\varepsilon}_i^{0}$.

Let $\widetilde{w}_i^{0}=\acute{\widetilde{w}}_i^{0}-\widetilde{\varepsilon}_i^{0}$, $\widetilde{w}_i^{n}$, and $\acute{\widetilde{w}}_i^{n}$ be the fuzzy numerical solutions of the scheme in Equation (13) with respect to the initial data $\widetilde{g}_i^{0}$ and $\acute{\widetilde{g}}_i^{0}$, respectively.

Let $[\widetilde{w}._{i+1}^{n}(x,\tau;\xi)]_r=\psi[\overline{w}(r)-\underline{w}(r)]+\underline{w}(r)]$, where $\psi,r\in[0,1]$.

The fuzzy error bound is defined as

$$\left[\widetilde{\varepsilon}_i^{n}\right]_r=\left[\acute{\widetilde{w}}_i^{n}-\widetilde{w}_i^{n}\right]_r,\quad n=1,2,\ \ldots\ldots X\times M,\ i=1,2,\ldots,X-1 \tag{30}$$

Then, the following is obtained:

$$\left\|\widetilde{\varepsilon}^{n}\right\|_2^2=\sum_{i=-\infty}^{\infty}\left|\widetilde{\lambda}^{n}\right|^2 \tag{31}$$

Suppose that $\widetilde{\varepsilon}_i^{n}$ can be expressed in the form

$$\widetilde{\varepsilon}_i^{n}=\widetilde{\lambda}^{n}e^{\sqrt{-\theta_i}},\ where\ \widetilde{\theta}_i=qih \tag{32}$$

The Equation (26) can be rewritten as follows:

$$\begin{aligned} 2\widetilde{w}_{i,n+1} = \left(2\widetilde{p}_2 - \widetilde{p}_1\right)\widetilde{w}_{i+1,n}(x,\tau;r,\psi) + \left(2 - 4\widetilde{p}_2 - 2u_1\right)\widetilde{w}_{i,n}(x,\tau;r,\psi) + \left(\widetilde{p}_1 + 2\,\widetilde{p}_2\right)\widetilde{w}_{i-1,n}(x,\tau;r,\psi) \\ -2\sum_{j=1}^{n-1}\left(u_{j+1} - u_j\right)\widetilde{w}_i^{n-j}(x,\tau;r,\psi) + u_n\widetilde{w}_i^{0}(x,\tau;r,\psi) \end{aligned} \tag{33}$$

Next, let us express the fuzzy round-off error for Equation (33) in the following manner:

$$2\widetilde{\varepsilon}_{i,n+1} = \left(2\widetilde{p}_2 - \widetilde{p}_1\right)\widetilde{\varepsilon}_{i+1,n} + \left(2 - 4\widetilde{p}_2 - 2u_1\right)\widetilde{\varepsilon}_{i,n} + \left(\widetilde{p}_1 + 2\,\widetilde{p}_2\right)\widetilde{\varepsilon}_{i-1,n} - 2\sum_{j=1}^{n-1}\left(u_{j+1} - u_j\right)\widetilde{\varepsilon}_i^{n-j} + u_n\widetilde{\varepsilon}_i^{0} \tag{34}$$

Substituting Equation (32) into Equation (34), obtain

$$\begin{aligned} 2\widetilde{\lambda}^{n+1} e^{\sqrt{-\theta_i}} = \left(2\widetilde{p}_2 - \widetilde{p}_1\right)\widetilde{\lambda}^{n} e^{\sqrt{-\theta_{i+1}}} + \left(2 - 4\widetilde{p}_2 - 2u_1\right)\widetilde{\lambda}^{n} e^{\sqrt{-\theta_i}} + \left(\widetilde{p}_1 + 2\,\widetilde{p}_2\right)\widetilde{\lambda}^{n} e^{\sqrt{-\theta_{i-1}}} \\ -2\sum_{j=1}^{n-1}\left(u_{j+1} - u_j\right)\widetilde{\lambda}^{n-j} e^{\sqrt{-\theta_i}} + b_n\widetilde{\lambda}^{0} e^{\sqrt{-\theta_i}} \end{aligned} \tag{35}$$

Divide Equation (35) by $e^{\sqrt{-\theta i}}$ to obtain

$$2\widetilde{\lambda}^{n+1} = [\left(2 - 4\widetilde{p}_2 - 2u_1\right) + 2\widetilde{p}_2\left(e^{\sqrt{-\theta_i}} + e^{-\sqrt{-\theta_i}}\right) - \widetilde{p}_1\left(e^{\sqrt{-\theta_i}} - e^{-\sqrt{-\theta_i}}\right)]\widetilde{\lambda}^{n} - 2\sum_{j=1}^{n-1}\left(u_{j+1} - u_j\right)\widetilde{\lambda}^{n-j} + u_n\widetilde{\lambda}^{0} \tag{36}$$

$$2\widetilde{\lambda}^{n+1} = [\left(2 - 4\widetilde{p}_2 - 2u_1\right) + 2\widetilde{p}_2\left(2 - 4\sin^2\left(\frac{\theta}{2}\right)\right) - \widetilde{p}_1\left(2\sqrt{-1}\,sin\,\theta\right)]\widetilde{\lambda}^{n} - 2\sum_{j=1}^{n-1}\left(u_{j+1} - u_j\right)\widetilde{\lambda}^{n-j} + u_n\widetilde{\lambda}^{0} \tag{37}$$

By simplifying Equation (37), we obtain

$$\widetilde{\lambda}^{n+1} = \left[1 - u_1 - 4\,\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right) - \sqrt{-1}\,\widetilde{p}_1 sin\,\theta\right]\widetilde{\lambda}^{n} - \sum_{j=1}^{n-1}\left(u_{j+1} - u_j\right)\widetilde{\lambda}^{n-j} + u_n\widetilde{\lambda}^{0} \tag{38}$$

Proposition 1. *If $\widetilde{\lambda}^{n}$ is the fuzzy solution of Equation (38) and $2\widetilde{p}_2 - \widetilde{p}_1^{\,2} \le \frac{1}{4}(1 - u_1)$, then $\left|\widetilde{\lambda}^{n}\right| \le \left|\widetilde{\lambda}^{0}\right|$.*

Proof. From Equation (38) and when $n = 0$, we obtain

$$\widetilde{\lambda}^{1} = \left[\left(1 - 4\,\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)\right) - \sqrt{-1}\,\widetilde{p}_1 sin\,\theta\right]\widetilde{\lambda}^{0}$$

Using the property

$$\left|\left(1 - 4\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)\right) - \sqrt{-1}\left(\widetilde{p}_1 sin\theta\right)\right| \le \left|\left(1 - 4\widetilde{p}_2\sin^2\left(\frac{\theta}{2}\right)\right)^2\right| + \left|\left(\widetilde{p}_1 sin\theta\right)^2\right|$$

we obtain

$$\begin{aligned} \left|\widetilde{\lambda}^{1}\right| &\le \left|\left(1 - 4\,\widetilde{p}_2\sin^2\left(\tfrac{\theta}{2}\right)\right)^2 + \widetilde{p}_1^{\,2}\sin^2(\theta)\right|\widetilde{\lambda}^{0} \\ \left|\widetilde{\lambda}^{1}\right| &\le \left|1 - 8\,\widetilde{p}_2\sin^2\left(\tfrac{\theta}{2}\right) + \widetilde{p}_1^{\,2}\left(4\sin^2\left(\tfrac{\theta}{2}\right)\cos^2\left(\tfrac{\theta}{2}\right)\right)\right|\widetilde{\lambda}^{0} \\ \left|\widetilde{\lambda}^{1}\right| &\le \left|1 - \left(8\,\widetilde{p}_2 - 4\widetilde{p}_1^{\,2}\right)\right|\widetilde{\lambda}^{0} \end{aligned}$$

Since $2\widetilde{p}_2 - \widetilde{p}_1^{\,2} \leq \frac{1}{4}$ *andmax* $\sin^2\left(\frac{\theta}{2}\right) = 1$, we have

$$\left|\overset{\sim}{\lambda}^{1}\right| \leq \left|\overset{\sim}{\lambda}^{0}\right|$$

Now, suppose that

$$\left|\overset{\sim}{\lambda}^{m}\right| \leq \left|\overset{\sim}{\lambda}^{0}\right|,\ m = 1,\ 2\ ,\ 3\ ,\ \ldots, n-1$$

From Proposition 1 and Equation (38), we obtain

$$\left|\overset{\sim}{\lambda}^{n+1}\right| < \left(1 - u_1 - 4\,\widetilde{p}_2 \sin^2\left(\frac{\theta}{2}\right) - \sqrt{-1}\,\widetilde{p}_1 sin\ \theta\right)\left|\overset{\sim}{\lambda}^{n}\right| - \sum_{j=1}^{n-1}\left(u_{j+1} - u_j\right)\left|\overset{\sim}{\lambda}^{n-j}\right| + u_n\left|\overset{\sim}{\lambda}^{0}\right| \ \leq\ \left|\overset{\sim}{\lambda}^{0}\right|$$

□

Theorem 1. *The stability of the FTCS Scheme in Equation (38) is*

$$2\widetilde{p}_2 - \widetilde{p}_1^{\,2} \leq \frac{1}{4}$$

Proof. By combining the formula in Equation (38) with proposition 1, we can deduce that

$$\left\|\overset{\sim}{\varepsilon}^{n}\right\|_2 \leq \left\|\overset{\sim}{\varepsilon}^{0}\right\|_2,\quad n = 1, 2, \ldots, N-1$$

So, it means that the FTCS scheme in Equation (38) is stable under the condition

$$2\widetilde{p}_2 - \widetilde{p}_1^{\,2} \leq \frac{1}{4}$$

Using the same procedure, we can show that the Saulyev's scheme in Equation (29) is unconditionally stable [34]. □

6. Numerical Example

Consider the fuzzy time fractional advection diffusion equations [31]:

$$\frac{\partial^{\xi}\widetilde{w}(x,\tau)}{\partial\tau^{\xi}} = -\frac{\partial\widetilde{w}(x,\tau,\psi)}{\partial x} + \frac{\partial^2\widetilde{w}(x,\tau)}{\partial x^2},\qquad 0 < x < l,\ \tau > 0,\quad 0 < \xi \leq 1 \tag{39}$$

Given the boundary conditions $\widetilde{w}(0,\tau) = \widetilde{w}(1,\tau) = 0$ and the initial condition:

$$\widetilde{w}(x,0) = \widetilde{k}\rceil^{-x},\ \ 0 < x < 1 \tag{40}$$

The fuzzy number will remain the same as follows when represented in double parametric form:

$$\widetilde{k}(r) = ((\psi(0.2 - 0.2r)) + 0.1r - 0.1)$$

The exact solution to Equation (39) was provided precisely in [31]:

$$\widetilde{w}(x,\tau,\xi;r,\psi) = \sum_{n=0}^{\infty}\frac{2^n\ \tau^{n\xi}}{\Gamma(n\xi+1)}\widetilde{k}(r,\psi)\rceil^{-x} \tag{41}$$

Tables 1 and 2 and Figures 1 and 2 demonstrate that the proposed methods exhibit a strong agreement with the exact solution at τ = 0.005, ξ = 0.5, and for all r, $\psi \in$ [0, 1]. Additionally, they fulfill the requirements of the double parametric form, leading to the shape of a triangular fuzzy number. Comparatively, the FTCS scheme provides more accurate results than Saulyev's scheme; however, Saulyev's scheme boasts the advantage of unconditional stability, setting it apart from other explicit methods. Figure 3 also illustrates that the numerical results of the proposed methods yield more accurate solutions near the inflection point ($\psi = 0.5$). Figure 4 also illustrates the exact FTCS scheme and Saulyev methods for the fuzzy solution of Equation (39) at $\xi = 0.5$ for all $r \in [0, 1]$.

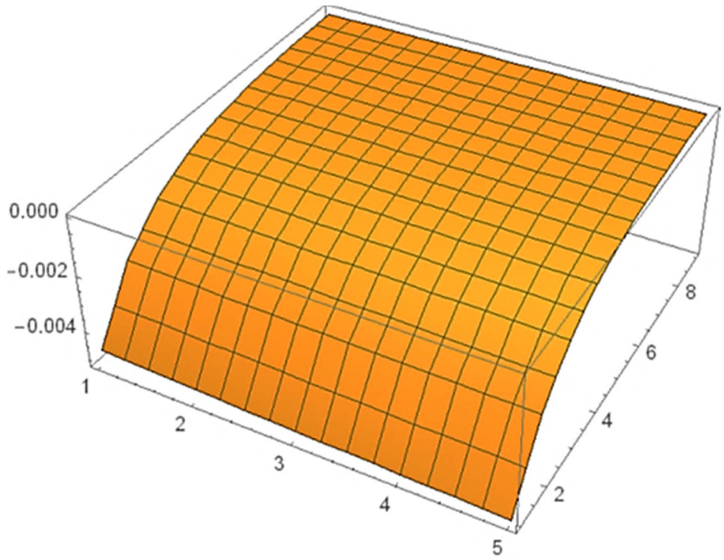


Figure 1. The fuzzy exact solution for Equation (39) at $\tau = 0.005$, $x = 5.4$, $r = 0.6$, and $\psi = 0.4$.

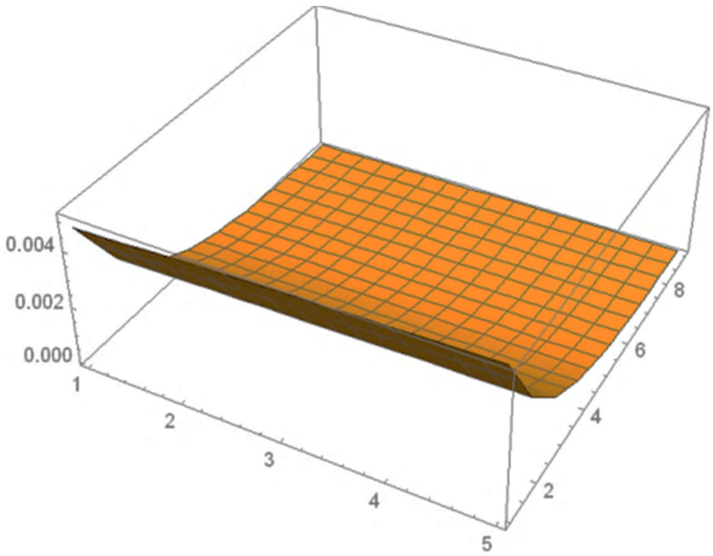


Figure 2. The fuzzy exact solution of Equation (39) at $\tau = 0.005$, $x = 5.4$, $r = 0.6$, and $\psi = 0.6$.

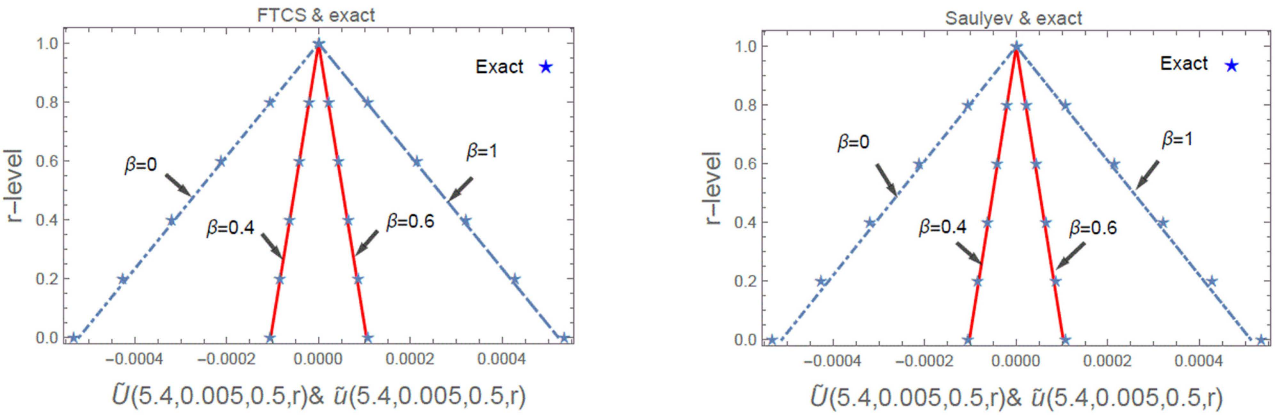
At $h = \Delta x = 0.6$ and $\mathrm{k} = \Delta\tau^{\xi} = 0.01$ to obtain $p(r, \psi) = \frac{\Delta\tau^{\xi}}{h^2} = 0.2777$, we obtain the following numerical results:

Table 1. Numerical solution of Equation (39) via FTCS and Saulyev at $\tau = 0.5$ and $x = 5.4$ for all r, $\psi \in [0, 1]$.

		FTCS		Saulyev	
ψ	r	$\widetilde{w}(5.4,0.5;r,\psi)$	$\widetilde{E}(4.5,0.5;r,\psi)$	$\widetilde{w}(5.4,0.5;r,\psi)$	$\widetilde{E}(5.4,0.5;r,\psi)$
Lower Solution When $\psi = 0$	0	−0.0004929227	4.09024×10^{-5}	−0.0004478379	8.59872×10^{-5}
	0.2	−0.0003943382	3.27219×10^{-5}	−0.0003582703	6.87898×10^{-5}
	0.4	−0.0002957536	2.45414×10^{-5}	−0.0002687027	5.15923×10^{-5}
	0.6	−0.0001971691	1.63610×10^{-5}	−0.0001791352	3.43949×10^{-5}
	0.8	−0.0000985845	8.18048×10^{-6}	−0.0000895676	1.71974×10^{-5}
	1	0	0	0	0
Upper Solution When $\psi = 1$	0	0.0004929227	4.09024×10^{-5}	0.0004478379	8.59872×10^{-5}
	0.2	0.0003943382	3.27219×10^{-5}	0.0003582703	6.87898×10^{-5}
	0.4	0.0002957536	2.45414×10^{-5}	0.0002687027	5.15923×10^{-5}
	0.6	0.0001971691	1.63610×10^{-5}	0.0001791352	3.43949×10^{-5}
	0.8	0.0000985845	8.18048×10^{-6}	0.0000895676	1.71974×10^{-5}
	1	0	0	0	0

Table 2. Numerical solution for Equation (39) via FTCS and Saulyev at $\tau = 0.5$ and $x = 5.4$ for all r, $\psi \in [0, 1]$.

		FTCS		Saulyev	
ψ	r	$\widetilde{w}(5.4,0.5;r,\psi)$	$\widetilde{E}(4.5,0.5;r,\psi)$	$\widetilde{w}(5.4,0.5;r,\psi)$	$\widetilde{E}(5.4,0.5;r,\psi)$
$\psi = 0.4$	0	−0.0000985845	8.18048×10^{-6}	−0.0000895676	1.71974×10^{-5}
	0.2	−0.0000788676	6.54438×10^{-6}	−0.0000716541	1.3758×10^{-5}
	0.4	−0.0000591507	4.90829×10^{-6}	−0.0000537405	1.03185×10^{-5}
	0.6	−0.0000394338	3.27219×10^{-6}	−0.0000358270	6.87898×10^{-6}
	0.8	−0.0000197169	1.6361×10^{-6}	−0.0000179135	3.43949×10^{-6}
	1	0	0	0	0
$\psi = 0.6$	0	0.0000985845	8.18048×10^{-6}	0.0000895676	1.71974×10^{-5}
	0.2	0.0000788676	6.54438×10^{-6}	0.0000716541	1.3758×10^{-5}
	0.4	0.0000591507	4.90829×10^{-6}	0.0000537405	1.03185×10^{-5}
	0.6	0.0000394338	3.27219×10^{-6}	0.0000358270	6.87898×10^{-6}
	0.8	0.0000197169	1.6361×10^{-6}	0.0000179135	3.43949×10^{-6}
	1	0	0	0	0



(**a**) (**b**)

Figure 3. Numerical and exact solution for Equation (39) via (**a**) FTCS scheme and (**b**) Saulyev scheme at $\tau = 0.5$ and $x = 4.5$ for all $r, \psi \in [0, 1]$.

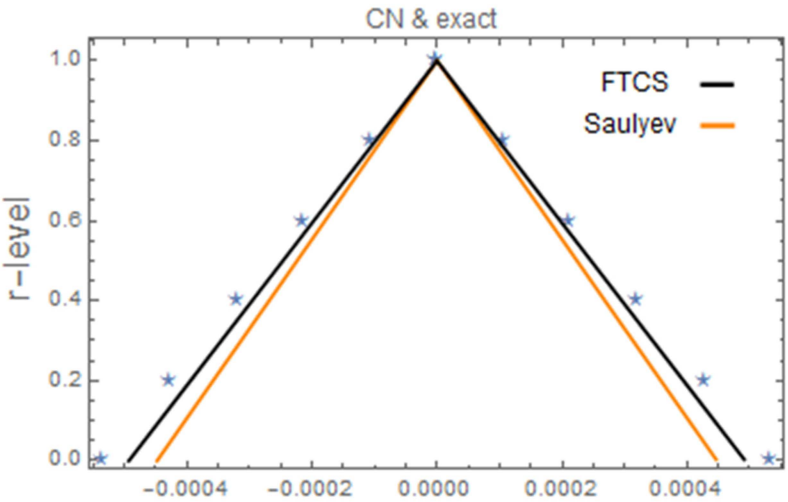


Figure 4. Exact FTCS scheme and Saulyev methods for the fuzzy solution of Equation (39) at $\xi = 0.5$, $x = 5.4$, $\tau = 0.005$, and for all $r \in [0, 1]$.

7. Summary

This paper employs the approach of a double parametric form to derive numerical solutions for FTFCDE using two explicit finite difference schemes, namely, FTCS and Saulyev. The double parametric form technique is found to be straightforward and computationally efficient, effectively converting the uncertain governing equation into a crisp one. The results obtained through both schemes align with the properties of the double parametric form and demonstrate prefect agreement with the fuzzy exact solution. Additionally, both explicit schemes are found to be stable. A comparative analysis of the numerical and exact solutions for various values of β indicates that the FTCS scheme yields slightly more accurate results than the Saulyev scheme.

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References

1. Al-Smadi, M.; Arqub, O.A. Computational algorithm for solving fredholsm time-fractional partial integrodifferential equations of dirichlet functions type with error estimates. *Appl. Math. Comput.* **2019**, *342*, 280–294.
2. Bira, B.; Raja Sekhar, T.; Zeidan, D. Exact solutions for some time-fractional evolution equations using Lie group theory. *Math. Methods Appl. Sci.* **2018**, *41*, 6717–6725. [CrossRef]
3. Zureigat, H.; Al-khateeb, A.; Abuteen, E.; Abu-Ghurra, S. Numerical Solution of Fuzzy Heat Equation with Complex Dirichlet Conditions. *Int. J. Fuzzy Log. Intell. Syst.* **2023**, *23*, 11–19. [CrossRef]
4. Momani, S.; Odibat, Z. Numerical solutions of the space-time fractional advection-dispersion equation. *Numer. Meth. Partial. Differ. Equ. Int. J.* **2008**, *24*, 1416–1429. [CrossRef]
5. Al-Smadi, M. Simplified iterative reproducing kernel method for handling time-fractional BVPs with error estimation. *Ain Shams Eng. J.* **2017**, *9*, 2517–2525. [CrossRef]
6. Zureigat, H.; Al-Omari, S.; Al-Smadi, M.; Momani, S. A Solution of Complex Fuzzy Time-Fractional Heat Equation by an Explicit Scheme. In Proceedings of the 2023 International Conference on Fractional Differentiation and Its Applications (ICFDA), Ajman, United Arab Emirates, 14–16 March 2023; IEEE: Piscataway, NJ, USA, 2023; pp. 1–5.
7. Batiha, B.; Ghanim, F.; Alayed, O.; Hatamleh, R.E.; Heilat, A.S.; Zureigat, H.; Bazighifan, O. Solving Multispecies Lotka–Volterra Equations by the Daftardar-Gejji and Jafari Method. *Int. J. Math. Math. Sci.* **2022**, *2022*, 1839796. [CrossRef]
8. Tasbozan, O.; Senol, M.; Kurt, A.; Ozkan, O. New solutions of fractional Drinfeld-Sokolov-Wilson system in shallow water waves. *Ocean. Eng.* **2018**, *161*, 62–68. [CrossRef]
9. Zureigat, H.; Al-Smadi, M.; Al-Khateeb, A.; Al-Omari, S.; Alhazmi, S. Numerical Solution for Fuzzy Time-Fractional Cancer Tumor Model with a Time-Dependent Net Killing Rate of Cancer Cells. *Int. J. Environ. Res. Public. Health* **2023**, *20*, 3766. [CrossRef]
10. Zeidan, D.; Sekhar, T.R. On the wave interactions in the drift-flux equations of two-phase flows. *Appl. Math. Comput.* **2018**, *327*, 117–131.
11. Almutairi, M.; Zureigat, H.; Izani Ismail, A.; Fareed Jameel, A. Fuzzy numerical solution via finite difference scheme of wave equation in double parametrical fuzzy number form. *Mathematics* **2021**, *9*, 667. [CrossRef]
12. Abu Arqub, O.; El-Ajou, A.; Momani, S.; Shawagfeh, N. Analytical solutions of fuzzy initial value problems by HAM. *Appl. Math. Info Sci.* **2013**, *7*, 1903–1919. [CrossRef]
13. Zureigat, H.H.; Ismail AI, M. Numerical solution of fuzzy heat equation with two different fuzzifications. In Proceedings of the 2016 SAI Computing Conference (SAI), London, UK, 13–15 July 2016; IEEE: Piscataway, NJ, USA, 2016; pp. 85–90.
14. Agarwal, R.P.; Lakshmikantham, V.; Nieto, J.J. On the concept of solution for fractional differential equations with uncertainty. *Nonlinear Anal. Theory Methods Appl.* **2010**, *726*, 2859–2862. [CrossRef]
15. Arshad, S.; Lupulescu, V. Fractional differential equation with the fuzzy initial condition. *Electron. J. Differ. Equ.* **2011**, *2011*, 1–8.
16. Salahshour, S.; Ahmadian, A.; Chan, C.S.; Baleanu, D. Toward the existence of solutions of fractional sequential differential equations with uncertainty. In Proceedings of the 2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Istanbul, Turkey, 2–5 August 2015; IEEE: Piscataway, NJ, USA, 2015; pp. 1–6.
17. Long, H.V.; Son, N.T.K.; Hoa, N.V. Fuzzy fractional partial differential equations in partially ordered metric spaces. *Iran. J. Fuzzy Syst.* **2017**, *14*, 107–126.
18. Souahi, A.; Guezane-Lakoud, A.; Hitta, A. On the existence and uniqueness for high order fuzzy fractional differential equations with uncertainty. *Adv. Fuzzy Syst.* **2016**, *2016*, 1–9. [CrossRef]
19. Salahshour, S.; Allahviranloo, T.; Abbasbandy, S. Solving fuzzy fractional differential equations by fuzzy Laplace transforms. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 1372–1381. [CrossRef]
20. Allahviranloo, T.; Salahshour, S.; Abbasbandy, S. Explicit solutions of fractional differential equations with uncertainty. *Soft Comput.* **2012**, *16*, 297–302. [CrossRef]
21. Mazandarani, M.; Kamyad, A.V. Modified fractional Euler method for solving fuzzy fractional initial value problem. *Commun. Nonlinear Sci. Numer. Simul.* **2013**, *18*, 12–21. [CrossRef]
22. Ahmadian, A.; Suleiman, M.; Salahshour, S.; Baleanu, D. A Jacobi operational matrix for solving a fuzzy linear fractional differential equation. *Adv. Differ. Equ.* **2013**, *2013*, 104. [CrossRef]
23. Jafarian, A.; Golmankhaneh, A.K.; Baleanu, D. On fuzzy fractional Laplace transformation. *Adv. Math. Phys.* **2014**, *2014*, 1–9. [CrossRef]
24. Raj, S.R.; Saradha, M. Solving hybrid fuzzy fractional differential equations by Adam-Bash forth method. *Appl. Math. Sci.* **2015**, *9*, 1429–1432. [CrossRef]
25. Ahmadian, A.; Salahshour, S.; Chan, C.S. Fractional differential systems: A fuzzy solution based on operational matrix of shifted Chebyshev polynomials and its applications. *IEEE Trans. Fuzzy Syst.* **2017**, *25*, 218–236. [CrossRef]
26. Salah, A.; Khan, M.; Gondal, M.A. A novel solution procedure for fuzzy fractional heat equations by homotopy analysis transform method. *Neural Comput. Appl.* **2013**, *23*, 269–271. [CrossRef]
27. Shah, N.A.; El-Zahar, E.R.; Dutt, H.M.; Arefin, M.A. Novel Evaluation of Fuzzy Fractional Cauchy Reaction-Diffusion Equation. *J. Funct. Spaces* **2022**, *2022*, 6499384. [CrossRef]
28. Saadeh, R. Numerical solutions of fractional convection-diffusion equation using finite-difference and finite-volume schemes. *J. Math. Comput. Sci.* **2021**, *11*, 7872–7891.

29. Zureigat, H.; Al-Smadi, M.; Al-Khateeb, A.; Al-Omari, S.; Alhazmi, S.E. Fourth-order numerical solutions for a fuzzy time-fractional convection–diffusion equation under Caputo generalized Hukuhara derivative. *Fractal Fract.* **2022**, *7*, 47. [CrossRef]
30. Chakraverty, S.; Tapaswini, S. Non-probabilistic solutions of imprecisely defined fractional-order diffusion equations. *Chin. Phys. B.* **2014**, *23*, 120–202. [CrossRef]
31. Al-Shibani, F.S.; Ismail, A.M.; Abdullah, F.A. Explicit finite difference methods for the solution of the one dimensional time fractional advection-diffusion equation. In Proceedings of the AIP Conference Proceedings 2014, Boulder, CO, USA, 20–25 July 2014; American Institute of Physics: College Park, PA, USA, 2014; Volume 1605, pp. 380–385.
32. Batiha, B.; Al-khateeb, A.; Zureigat, H. Improving Numerical Solutions for the Generalized Huxley Equation: The New Iterative Method (NIM). *Appl. Math* **2023**, *1*, 427–493.
33. Zureigat, H.; Ismail, A.I.; Sathasivam, S. A compact Crank–Nicholson scheme for the numerical solution of fuzzy time fractional diffusion equations. *Neural Comput. Appl.* **2020**, *32*, 6405–6412. [CrossRef]
34. Zureigat, H.; Ismail, A.I.; Sathasivam, S. Numerical solutions of fuzzy time fractional advection-diffusion equations in double parametric form of fuzzy number. *Math. Methods Appl. Sci.* **2021**, *44*, 7956–7968. [CrossRef]