axioms

MDPI

Article

Iteration with Bisection to Approximate the Solution of a Boundary Value Problem

Richard Avery [1,†], Douglas R. Anderson [2,*,†] and Jeffrey Lyons [3,†]

[1] College of Arts and Sciences, Dakota State University, Madison, SD 57042, USA; rich.avery@dsu.edu
[2] Department of Mathematics, Concordia College, Moorhead, MN 56562, USA
[3] Department of Mathematical Sciences, The Citadel, Charleston, SC 29409, USA; jlyons3@citadel.edu
* Correspondence: andersod@cord.edu
† These authors contributed equally to this work.

Abstract: Due to the restrictive growth and/or monotonicity requirements inherent in their employment, classical iterative fixed-point theorems are rarely used to approximate solutions to an integral operator with Green's function kernel whose fixed points are solutions of a boundary value problem. In this paper, we show how one can decompose a fixed-point problem into multiple fixed-point problems that one can easily iterate to approximate a solution of a differential equation satisfying one boundary condition, then apply a bisection method in an intermediate value theorem argument to meet a second boundary condition. Error estimates on the iterates are also established. The technique will be illustrated on a second-order right focal boundary value problem, with an example provided showing how to apply the results.

Keywords: boundary value problems; nonlinear analysis; fixed-point theorems; alternative inversion; iteration; Green's function; positive solutions

MSC: 47H10; 34B18

check for updates

Citation: Avery, R.; Anderson, D.R.; Lyons, J. Iteration with Bisection to Approximate the Solution of a Boundary Value Problem. *Axioms* **2024**, *13*, 222. https://doi.org/10.3390/axioms13040222

Academic Editors: Ricardo Roque Enguiça and Clemente Cesarano

Received: 26 February 2024
Revised: 21 March 2024
Accepted: 21 March 2024
Published: 27 March 2024

1. Introduction

Iteration is a powerful tool to find a solution to a boundary value problem when the solutions of the boundary value problem are fixed points of an operator that has monotonic or contractive properties, e.g., Abushammala, Khuri, and Sayfy [1], Bello, Alkali, and Roko [2], or Dang and Luan [3], but those are very restrictive properties that are rarely satisfied by an operator whose fixed points are solutions of the boundary value problem. See Zeidler [4] or Granas and Dugundji [5] for a thorough treatment of the classical techniques. For papers using Green's function approaches, see Kafri and Khuri [6], Kafri, Khuri, and Sayfy [7], Khuri and Louhichi [8], or Khuri and Sayfy [9]. See also Duffy [10] for a review of Green's function techniques. Many techniques have provided alternatives to standard Green function techniques to convert a boundary value problem to a fixed-point problem. For instance, see Haq and Ali [11] or Hossain and Islam [12] for numerical solutions to boundary value problems using Haar wavelets and the Galerkin method, respectively. Another approach is an S-iteration process for quasi-contractive mappings to find a solution to a nonlinear boundary value problem; see Kumar, Latif, Rafiq, and Hussain [13] or Thenmozhi and Marudai [14]. Additionally, see the techniques that correspond to bringing the operator inside the nonlinear term by Burton [15] and Avery and Peterson [16], the mean value method of Mann [17], and the bisection arguments coupled with splitting an interval into two components recently by Avery, Anderson, and Henderson [18,19]. In this paper, we will apply a new inversion technique, and the key to the method we present is the upper limit of integration being t instead of one. That is, we can iterate on a restricted domain with meaningful fixed points. We will take advantage of the fact that if x^* is a fixed point of

$$Ax(t) = mt - \int_0^t (t-s)f(x(s))\, ds$$

and

$$m = \int_0^1 f(x^*(s))ds,$$

then x^* is a solution of the second-order boundary value problem (1), (2) given below. It is worth noting that the fixed points of A correspond to solutions of the nonlinear initial value problem (1) with initial conditions $x(0) = 0$ and $x'(0) = m$.

We will employ a bisection method argument to iterate to find such a constant m by applying Banach's Theorem [20] on a collection of intervals where components of A are contractive, i.e., the requirements for an operator of the form

$$H_{m,i}x(t) = z_{m,i}(t) - \int_{\frac{i-1}{n}}^t (t-s)f(x(s))\, ds$$

to be contractive on the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ are not nearly as restrictive as they are employing other inversion techniques to find solutions of the boundary value problem. Moreover, stability will not be a consequence of the convergence of the iterative scheme like it is for the monotonic and contractive techniques. The technique we present is very much dependent on the structure of the boundary value problem (note that the choice of m is how we obtain the second boundary condition satisfied, and so, the boundary conditions drive this technique). The boundary conditions of focus in this manuscript are of the right focal variety, as given below. We refer the reader to the book by Agarwal [21] for more details on focal boundary value problems. The arguments we present focus on the structure of the second-order right focal boundary value problem given by

$$x''(t) + f(x(t)) = 0, \qquad t \in (0,1), \tag{1}$$
$$x(0) = x'(1) = 0, \tag{2}$$

where $f : \mathbb{R} \to [0, \infty)$ is differentiable. The Green's function for (1), (2) is given by

$$G(t,s) = \min\{t,s\}.$$

Every solution of (1), (2) is a fixed point of the operator $H : C[0,1] \to C^2[0,1]$ defined by

$$Hx(t) = \int_0^1 G(t,s)f(x(s))\, ds, \tag{3}$$

where the norm $||\cdot||$ is the usual supremum norm on the Banach space $B = C[0,1]$.

2. Preliminaries

For a positive real number R, an integer n with $n \geq 2$, let

$$B_{R,1} = \left\{x \in C\left[0, \frac{1}{n}\right] \; : \; |x(t)| \leq R \text{ for } t \in \left[0, \frac{1}{n}\right]\right\}$$

which is bounded, closed, and convex subset of the Banach Space $C\left[0, \frac{1}{n}\right]$ with the supnorm $\|\cdot\|_1$. Let m be a non-negative real number, $x \in B_{R,1}$, $t \in \left[0, \frac{1}{n}\right]$, and define

$$H_{m,1}x(t) = mt - \int_0^t (t-s)f(x(s))\, ds. \tag{4}$$

Lemma 1. *Let n be an integer with $n \geq 2$, $R \in (0, \infty)$, $m \in \mathbb{R}$ with $m \leq Rn$, and*

$$x_{m,1,0} \equiv 0 \in B_{R,1}.$$

Recursively define the sequence

$$x_{m,1,k+1} = H_{m,1}x_{m,1,k}$$

for integers $k \geq 1$. *If* $\frac{R}{n} < 1$,
$(A1)$ $f : [-Rn, Rn] \to [0, 2Rn^2]$ *is differentiable, and*
$(A2)$ $|f'(t)| \leq 2Rn$ *for all* $t \in [-Rn, Rn]$,
then $\{x_{m,1,k}\}_{k=0}^{\infty} \subset B_{R,1}$, *there exists an* $x^*_{m,1} \in B_{R,1}$ *such that*

$$x_{m,1,k} \to x^*_{m,1}$$

as $k \to \infty$, *and for* $\kappa = \frac{R}{n}$,

$$\|x_{m,1,k} - x^*_{m,1}\|_1 \leq \left(\frac{\kappa^k}{1-\kappa}\right)\|x_{m,1,1} - x_{m,1,0}\|_i \leq \left(\frac{R\kappa^k}{1-\kappa}\right).$$

Moreover,

$$x^*_{m,1}(t) = H_{m,1}x^*_{m,1}(t) = mt - \int_0^t (t-s)f(x^*_{m,1}(s))\, ds.$$

Proof. Let $x \in B_{R,1}$. Since for all $t \in \left[0, \frac{1}{n}\right]$ we have that $mt \geq 0$ and $\int_0^t (t-s)f(x(s))\, ds \geq 0$, it follows that

$$\begin{aligned} \|H_{m,1}x\|_1 &= \max_{t \in \left[0, \frac{1}{n}\right]} \left| mt - \int_0^t (t-s)f(x(s))\, ds \right| \\ &\leq \max\left\{ \frac{m}{n}, \int_0^{\frac{1}{n}} \left(\frac{1}{n} - s\right) f(x(s))\, ds \right\} \\ &\leq \max\left\{ \frac{Rn}{n}, (2Rn^2) \int_0^{\frac{1}{n}} \left(\frac{1}{n} - s\right) ds \right\} \\ &= R. \end{aligned}$$

Thus,

$$H_{m,1} : B_{R,1} \to B_{R,1}$$

and

$$\{x_{m,1,k}\}_{k=0}^{\infty} \subset B_{R,1}.$$

Below, we will demonstrate that the operator is contractive on the appropriate set. For each positive integer k and for each $s \in \left[0, \frac{1}{n}\right]$, let $w_k(s)$ be between $x_{m,1,k-1}(s)$ and $x_{m,1,k}(s)$ such that

$$f(x_{m,1,k}(s)) - f(x_{m,1,k-1}(s)) = f'(w_k(s))(x_{m,1,k}(s) - x_{m,1,k-1}(s))$$

by the mean value theorem. Hence, for each positive integer k,

$$\begin{aligned}
\|x_{m,1,k+1} - x_{m,1,k}\|_1 &= \|H_{m,1}x_{m,1,k} - H_{m,1}x_{m,1,k-1}\|_1 \\
&= \max_{t\in[0,\frac{1}{n}]} \left| \int_0^t (t-s) f(x_{m,1,k}(s))\, ds - \int_0^t (t-s) f(x_{m,1,k-1}(s))\, ds \right| \\
&\leq \max_{t\in[0,\frac{1}{n}]} \int_0^t (t-s) \left| f(x_{l,1,m}(s)) - f(x_{l,1,m-1}(s)) \right| ds \\
&\leq \int_0^{\frac{1}{n}} \left(\frac{1}{n} - s\right) \left| f'(w_k(s))(x_{m,1,k}(s) - x_{m,1,k-1}(s)) \right| ds \\
&\leq \int_0^{\frac{1}{n}} 2Rn \left(\frac{1}{n} - s\right) \|x_{m,1,k} - x_{m,1,k-1}\|_1\, ds \\
&\leq 2Rn\|x_{m,1,k} - x_{m,1,k-1}\|_1 \left(\int_0^{\frac{1}{n}} \frac{1}{n} - s\, ds \right) \\
&\leq 2Rn\|x_{m,1,k} - x_{m,1,k-1}\|_1 \left(\frac{1}{2n^2} \right) \\
&\leq \left(\frac{R}{n}\right) \|x_{m,1,k} - x_{m,1,k-1}\|_1,
\end{aligned}$$

where we have assumed that $\kappa = \frac{R}{n} < 1$. Hence by the Banach Fixed-Point Theorem [20] there is an $x^*_{m,1} \in B_{R,1}$ such that $x_{m,1,k} \to x^*_{m,1}$. Moreover,

$$x^*_{m,1} = H_{m,1}x^*_{m,1}.$$

For any positive integers k and r, by mathematical induction we have

$$\begin{aligned}
\|x_{m,1,k+r+1} - x_{m,1,k+r}\|_1 &\leq \kappa\|x_{m,1,k+r} - x_{m,1,k+r-1}\|_1 \\
&\leq \cdots \\
&\leq \kappa^r \|x_{m,1,k+1} - x_{m,1,k}\|_1
\end{aligned}$$

hence, for any natural numbers k and p, applying the triangle inequality, we have

$$\begin{aligned}
\|x_{m,1,k+p} - x_{m,1,k}\|_1 &\leq \sum_{j=0}^{p-1} \|x_{m,1,k+j+1} - x_{m,1,k+j}\|_1 \\
&\leq \sum_{j=0}^{p-1} \kappa^j \|x_{m,1,k+1} - x_{m,1,k}\|_1 \\
&\leq \sum_{j=0}^{\infty} \kappa^j \|x_{m,1,k+1} - x_{m,1,k}\|_1 \\
&= \left(\frac{1}{1-\kappa}\right) \|x_{m,1,k+1} - x_{m,1,k}\|_1 \\
&\leq \left(\frac{\kappa^k}{1-\kappa}\right) \|x_{m,1,1} - x_{m,1,0}\|_1 \\
&\leq \left(\frac{\kappa^k}{1-\kappa}\right) \|x_{m,1,1}\|_1 \\
&\leq \left(\frac{R\kappa^k}{1-\kappa}\right).
\end{aligned}$$

Thus, letting $p \to \infty$ from the inequality above, we arrive at the error estimate

$$\|x^*_{m,1} - x_{m,1,k}\|_1 \leq \left(\frac{R\kappa^k}{1-\kappa}\right).$$

This ends the proof. □

With the existence of $x^*_{m,1}$ given the hypotheses in Lemma 1 we can recursively define the balls $B_{R,m,i}$ in the Banach Space $B_i = C\left[\frac{i-1}{n}, \frac{i}{n}\right]$ with the supnorm $\|\cdot\|_i$ and the operator

$$H_{m,i} : B_{R,m,i} \to B_{R,m,i}$$

with the necessary hypotheses to have a unique fixed point $x^*_{m,i} \in B_{R,m,i}$ for each integer i with $1 < i \leq n$ applying Lemma 2 which follows. For a positive real number R, an integer i with $2 \leq i \leq n$, and non-negative real number m define $z_{m,i} \in B_i$ by

$$z_{m,i}(t) \equiv mt - \sum_{j=1}^{i-1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} (t-s) f(x^*_{m,j}(s))\, ds$$

and

$$B_{R,m,i} = \left\{ x \in C\left[\frac{i-1}{n}, \frac{i}{n}\right] \; : \; \|x - z_{m,i}\|_i \leq R \right\},$$

respectively. For $x \in B_{R,m,i}$, let

$$H_{m,i}x(t) = z_{m,i}(t) - \int_{\frac{i-1}{n}}^{t} (t-s) f(x(s))\, ds. \tag{5}$$

Lemma 2. *Let n be an integer with $n \geq 2$, i be an integer with $1 < i \leq n$, $R \in (0, \infty)$, $m \in \mathbb{R}$ with $m \leq Rn$, and $x_{m,i,0} \equiv z_{m,i} \in B_{R,m,i}$. Recursively define the sequence*

$$x_{m,i,k+1} = H_{m,i} x_{m,i,k}$$

for integers $k \geq 1$. If $\frac{R}{n} < 1$,
$(A1)$ $f : [-Rn, Rn] \to [0, 2Rn^2]$ is differentiable; and
$(A2)$ $|f'(t)| \leq 2Rn$ for all $t \in [-Rn, Rn]$;
then $\{x_{m,i,k}\}_{k=1}^{\infty} \subset B_{R,m,i}$, there exists an $x^*_{m,i} \in B_{R,m,i}$ such that

$$x_{m,i,k} \to x^*_{m,i}$$

as $k \to \infty$, and for $\kappa = \frac{R}{n}$,

$$\|x_{m,i,k} - x^*_{m,i}\|_i \leq \left(\frac{\kappa^k}{1-\kappa}\right) \|x_{m,i,1} - x_{m,i,0}\|_i \leq \left(\frac{R\kappa^k}{1-\kappa}\right).$$

Moreover,

$$x^*_{m,i} = H_{m,i} x^*_{m,i}.$$

Proof. Letting $x \in B_{R,m,i}$ we see that

$$\begin{aligned} \|H_{m,i}x - z_{m,i}\|_i &= \max_{t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]} \left| \int_{\frac{i-1}{n}}^{t} (t-s) f(x(s))\, ds \right| \\ &\leq 2Rn^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}} \frac{i}{n} - s\, ds \\ &= 2Rn^2 \left(\frac{1}{2n^2}\right) \\ &\leq R. \end{aligned}$$

Hence, we have verified that $H_{m,i}x \in B_{R,m,i}$; therefore,

$$H_{m,i} : B_{R,m,i} \to B_{R,m,i}.$$

As a result,

$$\{x_{m,i,k}\}_{k=1}^{\infty} \subset B_{R,m,i}.$$

For each positive integer k and for each $s \in \left[\frac{i-1}{n}, \frac{1}{n}\right]$, let $z_k(s)$ be between $x_{m,i,k-1}(s)$ and $x_{m,i,k}(s)$ such that

$$f(x_{m,i,k}(s)) - f(x_{m,i,k-1}(s)) = f'(z_k(s))(x_{m,i,k}(s) - x_{m,i,k-1}(s))$$

by the mean value theorem. Hence, for each positive integer k,

$$\begin{aligned}
\|x_{m,i,k+1} - x_{m,i,k}\|_i &= \|H_{m,i}x_{m,i,k} - H_{m,i}x_{m,i,k-1}\|_i \\
&= \max_{t\in\left[\frac{i-1}{n},\frac{i}{n}\right]} \left| \int_{\frac{i-1}{n}}^{t} (t-s)(f(x_{m,i,k}(s)) - f(x_{m,i,k-1}(s)))\, ds \right| \\
&= \max_{t\in\left[\frac{i-1}{n},\frac{i}{n}\right]} \left| \int_{\frac{i-1}{n}}^{t} (t-s)f'(z_k(s))(x_{m,i,k}(s) - x_{m,i,k-1}(s))\, ds \right| \\
&\le \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\frac{i}{n} - s\right) |f'(z_k(s))(x_{m,i,k}(s) - x_{m,i,k-1}(s))|\, ds \\
&\le 2Rn\|x_{m,i,k} - x_{m,i,k-1}\|_i \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\frac{i}{n} - s\right) ds \\
&= 2Rn\|x_{m,i,k} - x_{m,i,k-1}\|_i \left(\frac{1}{2n^2}\right) \\
&\le \frac{R}{n}\|x_{m,i,k} - x_{m,i,k-1}\|_i,
\end{aligned}$$

where we have assumed that $\kappa = \frac{R}{n} < 1$. Hence by the Banach Fixed-Point Theorem [20] there is an $x^*_{m,i} \in B_{R,m,i}$ such that $x_{m,i,k} \to x^*_{m,i}$. The error estimate

$$\|x^*_{m,i} - x_{m,i,k}\|_i \le \left(\frac{R\kappa^k}{1-\kappa}\right)$$

follows from the identical Banach argument in the proof of Lemma 1. □

3. Criteria for Existence of Solutions

In this section, we establish certain criteria for a solution of boundary value problem (1), (2) to exist. Subsequently, we will show how to apply the bisection method to approximate a solution to (1), (2). First, a remark about the notation employed.

Remark 1. *The following notation is to be used in the sequel below. For an integer n with $n \ge 2$, $R \in (0, \infty)$, and $m \in \mathbb{R}$ with $0 \le m \le Rn$ define x^*_m on $[0, 1]$ by*

$$x^*_m(t) = x^*_{m,i}(t)$$

for $t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$.

In the following Theorem 1, we provide criteria for the existence of a solution to the boundary value problem (1), (2).

Theorem 1. *Let n be a natural number with $n \geq 2$, $R \in (0, \infty)$, and $m \in \mathbb{R}$ with $0 \leq m \leq Rn$. If*

$$m = \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x_m^*(s))\, ds = \int_0^1 f(x_m^*(s))\, ds$$

then x_m^* is a solution of boundary value problem (1), (2).

Proof. Letting $t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$ for some $i \in 1, 2, \cdots, n-1, n$ we have

$$x_{m,i}^*(t) = H_{m,i} x_{m,i}^*(t),$$

$$(x_{m,i}^*)'(t) = m - \sum_{j=1}^{i-1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x_{m,j}^*(s))\, ds - \int_{\frac{i-1}{n}}^{t} f(x_{m,i}^*(s))\, ds,$$

and

$$(x_{m,i}^*)''(t) = -f(x_{m,i}^*(t)).$$

Also, for $i \in 1, 2, \cdots, n-1$ we have

$$\begin{aligned} x_{m,i+1}^*\left(\frac{i}{n}\right) &= H_{m,i+1} x_{m,i+1}^*\left(\frac{i}{n}\right) \\ &= z_{m,i+1}\left(\frac{i}{n}\right) - \int_{\frac{i}{n}}^{\frac{i}{n}} \left(\frac{i}{n} - s\right) f(x_{m,i+1}^*(s))\, ds \\ &= z_{m,i+1}\left(\frac{i}{n}\right) \\ &= \frac{mi}{n} - \sum_{j=1}^{i} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left(\frac{i}{n} - s\right) f(x_{m,j}^*(s))\, ds \\ &= z_{m,i}\left(\frac{i}{n}\right) - \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\frac{i}{n} - s\right) f(x_{m,i}^*(s))\, ds \\ &= H_{m,i} x_{m,i}^*\left(\frac{i}{n}\right) \\ &= x_{m,i}^*\left(\frac{i}{n}\right) \end{aligned}$$

and

$$\begin{aligned} (x_{m,i+1}^*)'\left(\frac{i}{n}\right) &= m - \sum_{j=1}^{i} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x_{m,j}^*(s))\, ds - \int_{\frac{i}{n}}^{\frac{i}{n}} f(x_{m,i+1}^*(s))\, ds \\ &= m - \sum_{j=1}^{i-1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x_{m,j}^*(s))\, ds - \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(x_{m,i}^*(s))\, ds \\ &= (x_{m,i}^*)'\left(\frac{i}{n}\right) \end{aligned}$$

thus $x_m^* \in C^2[0, 1]$. We also have that

$$x_m^*(0) = x_{m,1}^*(0) = 0(t) - \int_0^0 (0 - s) f(x_{m,1}^*(s))\, ds = 0$$

and by hypothesis $m = \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x_m^*(s))\, ds$. Therefore

$$(x_m^*)'(1) = (x_{m,n}^*)'(1) = m - \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x_m^*(s))\, ds = 0,$$

and for $t \in (0,1)$

$$(x_m^*)''(t) = -f(x_m^*(t)).$$

This ends the proof. □

For a real number m with $0 \le m \le Rn$ define the function

$$g(m) = m - \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x_m^*(s))\, ds = m - \int_0^1 f(x_m^*(s))\, ds.$$

In the following theorem, we verify that g is continuous.

Theorem 2. *Let M, R be positive real numbers, n be a natural number with $2n > M$, and $m \in \mathbb{R}$ with $0 \le m \le Rn$. If $\frac{R}{n} < 1$,*
$(A1)$ $f : [-Rn, Rn] \to [0, 2Rn^2]$ is differentiable, and
$(A2)$ $|f'(t)| \le MR$ for all $t \in [-Rn, Rn]$,
then g is continuous on $[0, Rn]$.

Please note that since $2n > M$, we also have that $2nR > MR > |f'(t)|$ for all $t \in [-Rn, Rn]$ hence the hypotheses of Lemmas 1 and 2 are satisfied.

Proof. Let $w, z \in [0, Rn]$ and for each $s \in [0,1]$, let $\mu(s)$ be between $x_w^*(s)$ and $x_z^*(s)$ such that

$$f(x_w^*(s)) - f(x_z^*(s)) = f'(\mu(s))(x_w^*(s) - x_z^*(s))$$

by the mean value theorem. Thus,

$$\begin{aligned} |g(w) - g(z)| &= \left| w - \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x_w^*(s))\, ds - z + \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x_z^*(s))\, ds \right| \\ &\le |w - z| + \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} |f(x_w^*(s)) - f(x_z^*(s))|\, ds \\ &\le |w - z| + \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} |f'(\mu(s))(x_w^*(s) - x_z^*(s))|\, ds \\ &\le |w - z| + \sum_{j=1}^{n} \left(\frac{1}{n}\right)(MR)\|x_w^* - x_z^*\| \\ &= |w - z| + (MR)\|x_w^* - x_z^*\|. \end{aligned}$$

By finite induction on i, we will show that

$$\|x_w^* - x_z^*\|_i \le \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2 MR}{2n^2 - MR} + 1\right)^{i-1} |w - z|$$

for $1 \le i \le n$. For $i = 1$ we have that:

$$\begin{aligned} \|x_w^* - x_z^*\|_1 &= \|H_{m,1}x_w^* - H_{m,1}x_z^*\|_1 \\ &= \max_{t \in [0, \frac{1}{n}]} \left| (wt - zt) + \int_0^t (t - s)(f(x_z^*(s)) - f(x_w^*(s))\, ds \right| \\ &\le |w - z| + \int_0^{\frac{1}{n}} \left(\frac{1}{n} - s\right) |f'(\mu(s))(x_w^*(s) - x_z^*(s))|\, ds \\ &\le |w - z| + \left(\frac{1}{2n^2}\right)(MR)\|x_w^* - x_z^*\|_1 \\ &= |w - z| + \left(\frac{MR}{2n^2}\right)\|x_w^* - x_z^*\|_1 \end{aligned}$$

so

$$\|x_w^* - x_z^*\|_1 \leq \left(\frac{2n^2}{2n^2 - MR}\right)|w - z|.$$

Suppose the statement is true for all $j \leq i - 1$ for some $i \geq 2$ with $i \leq n$, thus we have

$$\begin{aligned}
\|x_w^* - x_z^*\|_i &= \|H_{m,i}x_w^* - H_{m,i}x_z^*\|_i \\
&= \max_{t\in\left[\frac{i-1}{n},\frac{i}{n}\right]}\left|(wt - zt) + \int_{\frac{i-1}{n}}^{t}(t-s)(f(x_z^*(s)) - f(x_w^*(s))\,ds\right. \\
&\quad \left. + \sum_{j=1}^{i-1}\int_{\frac{j-1}{n}}^{\frac{j}{n}}(t-s)(f(x_z^*(s)) - f(x_w^*(s)))\,ds\right| \\
&\leq |w-z| + \int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(\frac{i}{n} - s\right)|f'(\mu(s))(x_w^*(s) - x_z^*(s))| \\
&\quad + \sum_{j=1}^{i-1}\int_{\frac{j-1}{n}}^{\frac{j}{n}}\left(\frac{i}{n} - s\right)|f'(\mu(s))(x_w^*(s) - x_z^*(s))|ds \\
&\leq |w-z| + \int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(\frac{i}{n} - s\right)(MR)\|x_w^* - x_z^*\|_i ds \\
&\quad + \sum_{j=1}^{i-1}\int_{\frac{j-1}{n}}^{\frac{j}{n}}(1)(MR)\|x_w^* - x_z^*\|_j ds \\
&\leq |w-z| + \left(\frac{1}{2n^2}\right)(MR)\|x_w^* - x_z^*\|_i \\
&\quad + \sum_{j=1}^{i-1}\left(\frac{MR}{n}\right)\left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2MR}{2n^2 - MR} + 1\right)^{j-1}|w-z| \\
&\leq |w-z| + \left(\frac{MR}{2n^2}\right)\|x_w^* - x_z^*\|_i \\
&\quad + (MR)\left(\frac{2n^2}{2n^2 - MR}\right)|w-z|\left(\frac{\left(\frac{2n^2MR}{2n^2-MR} + 1\right)^{i-1} - 1}{\left(\frac{2n^2MR}{2n^2-MR} + 1\right) - 1}\right) \\
&\leq |w-z| + \left(\frac{MR}{2n^2}\right)\|x_w^* - x_z^*\|_i \\
&\quad + |w-z|\left(\frac{2n^2MR}{2n^2 - MR}\right)\left(\frac{\left(\frac{2n^2MR}{2n^2-MR} + 1\right)^{i-1} - 1}{\left(\frac{2n^2MR}{2n^2-MR}\right)}\right) \\
&= |w-z| + \left(\frac{MR}{2n^2}\right)\|x_w^* - x_z^*\|_i + \left[\left(\frac{2n^2MR}{2n^2 - MR} + 1\right)^{i-1} - 1\right]|w-z| \\
&= \left(\frac{MR}{2n^2}\right)\|x_w^* - x_z^*\|_i + \left(\frac{2n^2MR}{2n^2 - MR} + 1\right)^{i-1}|w-z|.
\end{aligned}$$

Hence,

$$\left(1 - \frac{MR}{2n^2}\right)\|x_w^* - x_z^*\|_i \leq \left(\frac{2n^2MR}{2n^2 - MR} + 1\right)^{i-1}|w-z|.$$

Thus,

$$\|x_w^* - x_z^*\|_i \leq \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2MR}{2n^2 - MR} + 1\right)^{i-1}|w-z|.$$

Therefore, we have verified that for all integers i with $1 \le i \le n$ that

$$\|x_w^* - x_z^*\|_i \le \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2 MR}{2n^2 - MR} + 1\right)^{i-1} |w - z|.$$

For some $i' \in [1, n]$ we have

$$\|x_w^* - x_z^*\| = \|x_w^* - x_z^*\|_{i'} \le \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2 MR}{2n^2 - MR} + 1\right)^{n-1} |w - z|.$$

Therefore, we have that

$$\begin{aligned} |g(w) - g(z)| &\le |w - z| + (MR)\|x_w^* - x_z^*\| \\ &\le |w - z| + (MR)\left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2 MR}{2n^2 - MR} + 1\right)^{n-1} |w - z| \\ &\le \left[1 + \left(\frac{2n^2 MR}{2n^2 - MR}\right)\left(\frac{2n^2 MR}{2n^2 - MR} + 1\right)^{n-1}\right] |w - z| \end{aligned}$$

hence g is uniformly continuous on $[0, Rn]$. □

The following Theorem 3 shows how to apply the bisection method to approximate a solution of the boundary value problem (1), (2) now that we have that g is continuous.

Theorem 3. *Let M, R be positive real numbers, n be a natural number with $2n > M$, and $m \in \mathbb{R}$ with $0 \le m \le Rn$. Assume $\frac{R}{n} < 1$,*
$(A1)$ $f : [-Rn, Rn] \to [0, 2Rn^2]$ is differentiable, and
$(A2)$ $|f'(t)| \le MR$ for all $t \in [-Rn, Rn]$.
If $g(0)g(Rn) < 0$, then there exists an $\omega \in [0, Rn]$ such that $g(\omega) = 0$, and thus x_ω^* is a solution of (1), (2). Moreover, there is a sequence $\{\omega_j\}_{j=0}^{\infty} \subseteq [0, Rn]$ such that

$$\omega_j \to \omega$$

with

$$|\omega - \omega_j| \le \frac{Rn}{2^{j+1}}$$

and

$$\|x_\omega^* - x_{\omega_j}^*\| \le \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2 MR}{2n^2 - MR} + 1\right)^{n-1} \left(\frac{Rn}{2^{j+1}}\right).$$

Proof. In Theorem 2, we verified that $g : [0, Rn] \to \mathbb{R}$ is a continuous real-valued function. By assumption $g(0)g(Rn) < 0$, thus by the intermediate value theorem there exists an $\omega \in [0, Rn]$ such that $g(\omega) = 0$ and by Theorem 1 x_ω^* is a solution of (1), (2).

Let

$$a_0 = 0,\ b_0 = Rn \text{ and } \omega_0 = \frac{a_0 + b_0}{2}$$

then recursively define the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ by

$$a_{n+1} = \omega_n, b_{n+1} = b_n \text{ and } \omega_{n+1} = \frac{a_{n+1} + b_{n+1}}{2}$$

if $g(\omega_n)g(b_n) \le 0$ and

$$a_{n+1} = a_n, b_{n+1} = \omega_n \text{ and } \omega_{n+1} = \frac{a_{n+1} + b_{n+1}}{2}$$

if $g(\omega_n)g(a_n) < 0$. Observe that for each whole number n that

$$g(a_n)g(b_n) < 0$$

thus, by the intermediate value theorem there is an $\omega \in [a_n, b_n]$ such that $g(\omega) = 0$. By induction, we have that

$$b_j - a_j = \frac{b_{j-1} - a_{j-1}}{2} = \frac{b_0 - a_0}{2^j} = \frac{Rn}{2^j}$$

and since ω_j is the midpoint of the interval $[a_j, b_j]$ and $\omega \in [a_j, b_j]$, we have that

$$\omega_j - a_j = \frac{Rn}{2^{j+1}} \text{ and } b_j - \omega_j = \frac{Rn}{2^{j+1}}$$

and

$$\omega \in [a_j, \omega_j] \text{ or } \omega \in [\omega_j, b_j].$$

Hence,

$$|\omega - \omega_j| \leq \frac{Rn}{2^{j+1}}.$$

Also, from the proof of Theorem 2, we have proven that

$$\|x^*_\omega - x^*_{\omega_j}\| \leq \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2 MR}{2n^2 - MR} + 1\right)^{n-1} |\omega - \omega_j|.$$

Hence, we have that

$$\|x^*_\omega - x^*_{\omega_j}\| \leq \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2 MR}{2n^2 - MR} + 1\right)^{n-1} \left(\frac{Rn}{2^{j+1}}\right).$$

This ends the proof. □

4. Error Estimates

Iterative techniques are used to approximate solutions and at this stage in the manuscript, both x^*_ω and $x^*_{\omega_n}$ are functions derived through a limiting process. Although we have proven that x^*_ω and $x^*_{\omega_n}$ exist, the best we can actually hope to do is approximate these functions. In the following, we provide error estimates for an approximation of x^*_m, which will be the foundation of our error estimates.

For a non-negative real number m and natural number k from Lemma 1, define

$$y_{m,k,1} = x_{m,1,k}.$$

Let $D_i = C\left[\frac{i-1}{n}, \frac{i}{n}\right]$ be the Banach space with the supnorm $\| \cdot \|_i$. For each integer i with $2 < i \leq n$ and positive real number R, define $w_{m,i}$ by

$$w_{m,i}(t) \equiv mt - \sum_{j=1}^{i-1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} (t-s) f(y_{m,k,j}(s))\, ds.$$

Please note that $w_{m,i}$ is an approximation of $z_{m,i}$ and does not have any functions defined by a limiting process in its definition. Each of the functions in the definition of $w_{m,i}$ is the result of k iterations. Let

$$D_{R,m,i} = \{y \in C\left[\frac{i-1}{n}, \frac{i}{n}\right] \ : \ \|y - w_{m,i}\|_i \leq R\}$$

and for $y \in D_{R,m,i}$, let

$$J_{m,i}y(t) = w_{m,i}(t) - \int_{\frac{i-1}{n}}^{t} (t-s)f(y(s))\, ds. \tag{6}$$

Define the sequence $\{y_{m,j,i}\}_{j=0}^{k}$ recursively by

$$y_{m,0,i} \equiv w_{m,i}$$

and

$$y_{m,j+1,i} = J_{m,i}y_{m,j,i}$$

for some integer $k \geq 1$ and define $y^*_{m,k}$ on $[0,1]$ by

$$y^*_{m,k}(t) = y_{m,k,i}(t)$$

when $t \in \left[\frac{i-1}{n}, \frac{i}{n}\right]$.

Remark 2. *Please note that once one has found $y_{m,k,i}$ where $1 \leq i < n$, it takes another k iterations to find $y_{m,k,i+1}$ and a total of nk iterations to find $y^*_{m,k}$ which is an approximation of x^*_m. Below, we have given an upper bound for the error of this estimate.*

Theorem 4. *Let M, R be positive real numbers with $MR \geq 1$, n be a natural number with $2n > M$, and $m \in \mathbb{R}$ with $0 \leq m \leq Rn$. Let $\tau = \frac{4R+1}{2}$. If*
$(A1)$ $f : [-Rn, Rn] \to [0, 2Rn^2]$ is differentiable,
$(A2)$ $|f'(t)| \leq MR$ for all $t \in [-Rn, Rn]$,
and $\kappa = \frac{R}{n} < 1$, then

$$\|y^*_{m,k} - x^*_m\| \leq \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{n-1}.$$

Proof. We will verify that for integers i with $1 \leq i \leq n$ that

$$\|y^*_{m,k} - x^*_m\|_i \leq \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{i-1}$$

by finite induction. By Lemma 1, we have

$$\|y^*_{m,k} - x^*_m\|_1 = \|x_{m,1,k} - x^*_{m,1}\|_1 \leq \left(\frac{R\kappa^k}{1-\kappa}\right)$$

and following the same arguments as in Lemma 2, we have

$$\|y_{m,k,i} - y_{m,k-1,i}\|_i \leq \left(\frac{R\kappa^k}{1-\kappa}\right)\|y_{m,1,i} - y_{m,0,i}\|_i.$$

Now, for an integer i with $2 \leq i \leq n-1$, suppose for each integer j with $1 \leq j \leq i-1$ that

$$\|y^*_{m,k} - x^*_m\|_j \leq \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{j-1}.$$

Thus,

$$\begin{aligned}
\|y^*_{m,k} - x^*_m\|_i &= \|y_{m,k,i} - x^*_m\|_i \\
&= \|J_{m,i} y_{m,k-1,i} - H_{m,i} x^*_m\|_i \\
&= \max_{t\in\left[\frac{i-1}{n},\frac{i}{n}\right]} \bigg| (mt - mt) + \int_{\frac{i-1}{n}}^{t} (t-s)(f(x^*_m(s)) - f(y_{m,k-1,i}(s))\, ds \\
&\quad + \sum_{j=1}^{i-1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} (t-s)(f(x^*_m(s)) - f(y_{m,k,j}(s))\, ds \bigg| \\
&\leq \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\frac{i}{n} - s\right) |f'(\mu(s))(x^*_m(s) - y_{m,k-1,i}(s))|\, ds \\
&\quad + \sum_{j=1}^{i-1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left(\frac{i}{n} - s\right) |f'(\mu(s))(x^*_m(s) - y_{m,k,j}(s))|\, ds \\
&\leq \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left(\frac{i}{n} - s\right) MR\|x^*_m - y_{m,k-1,i}\|_i\, ds \\
&\quad + \sum_{j=1}^{i-1} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left(\frac{i}{n} - s\right) MR\|x^*_m - y_{m,k,j}\|_j\, ds \\
&\leq \left(\frac{MR}{2n^2}\right) \|x^*_m - y_{m,k-1,i}\|_i \\
&\quad + \sum_{j=1}^{i-1} \left(\frac{MR}{2}\right) \left(\frac{(i-(j-1))^2 - (i-j)^2}{n^2}\right) \|x^*_m - y_{m,k,j}\|_j \\
&\leq \left(\frac{MR}{2n^2}\right) \|x^*_m - y_{m,k,i}\|_i + \left(\frac{MR}{2n^2}\right) \|y_{m,k,i} - y_{m,k-1,i}\|_i \\
&\quad + \sum_{j=1}^{i-1} \left(\frac{MR}{2}\right) \left(\frac{2(i-j)+1}{n^2}\right) \|x^*_m - y^*_{m,k}\|_j \\
&\leq \left(\frac{MR}{2n^2}\right) \|x^*_m - y_{m,k,i}\|_i + \left(\frac{MR}{2n^2}\right) \left(\frac{R\kappa^k}{1-\kappa}\right) \|y_{m,1,i} - y_{m,0,i}\|_i \\
&\quad + \sum_{j=1}^{i-1} \left(\frac{MRi}{n^2}\right) \left(\frac{R\kappa^k}{1-\kappa}\right) \left(\frac{4\tau n^2}{2n^2 - MR}\right)^{j-1} \\
&\leq \left(\frac{MR}{2n^2}\right) \|x^*_m - y_{m,k,i}\|_i + \left(\frac{MR^2}{2n^2}\right) \left(\frac{R\kappa^k}{1-\kappa}\right) \\
&\quad + \left(\frac{MR}{n}\right) \left(\frac{2n^2 - MR}{4\tau n^2 - 2n^2 + MR}\right) \left(\frac{R\kappa^k}{1-\kappa}\right) \left(\frac{4\tau n^2}{2n^2 - MR}\right)^{i-1}.
\end{aligned}$$

Therefore, we have,

$$\begin{aligned}
\left(1 - \frac{MR}{2n^2}\right) \|y^*_{m,k} - x^*_m\|_i &\leq \left(\frac{MR^2}{2n^2}\right) \left(\frac{R\kappa^k}{1-\kappa}\right) \\
&\quad + \left(\frac{MR}{n}\right) \left(\frac{2n^2 - MR}{4\tau n^2 - 2n^2 + MR}\right) \left(\frac{R\kappa^k}{1-\kappa}\right) \left(\frac{4\tau n^2}{2n^2 - MR}\right)^{i-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|y^*_{m,k} - x^*_m\|_i &\leq \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{MR}{n}\right)\left(\frac{2n^2 - MR}{4\tau n^2 - 2n^2 + MR}\right)\left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{i-1} \\
&\quad + \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{MR^2}{2n^2}\right)\left(\frac{R\kappa^k}{1-\kappa}\right) \\
&\leq \left(\frac{2nMR}{2n^2(2\tau - 1)}\right)\left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{i-1} \\
&\quad + \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{MR^2}{2n^2}\right)\left(\frac{R\kappa^k}{1-\kappa}\right) \\
&\leq \left(\frac{2n(2n)R}{2n^2(4R)}\right)\left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{i-1} \\
&\quad + \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{MR^2}{2n^2}\right)\left(\frac{R\kappa^k}{1-\kappa}\right) \\
&\leq \left(\frac{\left(\frac{R\kappa^k}{1-\kappa}\right)}{2}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{i-1} + \left(\frac{\left(\frac{R\kappa^k}{1-\kappa}\right)}{2}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right) \\
&\leq \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{i-1}.
\end{aligned}$$

It follows that

$$\|y^*_{m,k} - x^*_m\| \leq \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{n-1},$$

and thus

$$\lim_{k\to\infty} y^*_{m,k} = x^*_m.$$

This ends the proof. □

In the following theorem, we apply the previous two results to prove that when the nonlinearity f satisfies sufficient criteria, one can obtain as close to a solution of boundary value problem (1), (2) as desired. The key criterion is a positive real number m such that $g(0)g(m) < 0$.

Theorem 5. *Let M, R be positive real numbers with $MR \geq 1$, n be a natural number with $2n > M$, $m \in \mathbb{R}$ with $0 \leq m \leq Rn$. Let $\tau = \frac{4R+1}{2}$. If*
$(A1)$ $f : [-Rn, Rn] \to [0, 2Rn^2]$ is differentiable,
$(A2)$ $|f'(t)| \leq MR$ for all $t \in [-Rn, Rn]$,
and $\kappa = \frac{R}{n} < 1$, then there exists an $\omega \in [0, Rn]$ such that x^*_ω is a solution of (1), (2) and a sequence $\{\omega_k\}_{k=1}^{\infty} \subset [0, Rn]$ with

$$\omega_k \to \omega \text{ and } y^*_{\omega_k,k} \to x^*_\omega.$$

Proof. From the proof of Theorem 3, we have that there is an $\omega \in [0, Rn]$ such that x^*_ω is a solution of (1), (2) and for an integer k, there is an $x^*_{\omega,k}$ such that

$$\|x^*_\omega - x^*_{\omega_k}\| \leq \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2MR}{2n^2 - MR} + 1\right)^{n-1}\left(\frac{Rn}{2^{k+1}}\right).$$

From the proof of Theorem 4, we have that

$$\|y^*_{\omega_k,k} - x^*_{\omega_k}\| \le \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{n-1}.$$

Hence,

$$\begin{aligned} \|x^*_\omega - y^*_{\omega_k,k}\| &\le \|x^*_\omega - x^*_{\omega_k}\| + \|x^*_{\omega_k} - y^*_{\omega_k,k}\| \\ &\le \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2 MR}{2n^2 - MR} + 1\right)^{n-1}\left(\frac{Rn}{2^{k+1}}\right) \\ &\quad + \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{n-1}, \end{aligned}$$

and therefore,

$$\lim_{k\to\infty} y^*_{\omega_k,k} = x^*_\omega.$$

This ends the proof. □

5. Existence Application

A simple Banach contraction principle argument applied to the operator H given in (3) involves bounding the derivative. The simplest of applications requires that $|f'(x)| < 2$ for all x in an interval containing the range of the solution. Notice that the criterion in $(A2)$ is much less restrictive with

$$|f'(t)| < MR < 2nR$$

for all $t \in [-Rn, Rn]$ which contains the range of a solution. The simplest of applications of monotonic fixed-point results requires that f' be of one sign on an interval containing the range of a solution and for a sequence of iterates to be bounded. Often, neither of these requirements are satisfied.

The following existence of a solution argument follows directly from Theorem 3 with much less restrictive hypotheses than are needed in Banach or monotonicity arguments.

Theorem 6. *Let M, R be positive real numbers, n be a natural number with $2n > M$, and $m \in \mathbb{R}$ with $0 \le m \le Rn$. If $\frac{R}{n} < 1$,*
$(A1)$ $f : [-Rn, Rn] \to (0, Rn)$ is differentiable, and
$(A2)$ $|f'(t)| \le MR$ for all $t \in [-Rn, Rn]$,
then there exists an $\omega \in [0, Rn]$ such that $g(\omega) = 0$, and thus x^*_ω is a solution to boundary value problem (1), (2).

Proof. Since $f : [-Rn, Rn] \to (0, Rn)$, we have that

$$g(0) = 0 - \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x^*_0(s))\, ds < 0$$

and

$$\begin{aligned} g(Rn) &= Rn - \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} f(x^*_{Rn}(s))\, ds \\ &> Rn - \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} Rn\, ds \\ &= Rn - \sum_{j=1}^{n} R = 0. \end{aligned}$$

Therefore,

$$g(0)g(Rn) < 0.$$

Hence, by Theorem 3, there exists an $\omega \in [0, Rn]$ such that $g(\omega) = 0$, and thus, by Theorem 1, we have that x^*_ω is a solution of (1), (2). □

Consider the following example applying Theorem 6,

$$x''(t) + 4\cos^2(x(t)) + 1 = 0, \qquad t \in (0,1), \tag{7}$$
$$x(0) = x'(1) = 0. \tag{8}$$

Thus,

$$f(x) = 4\cos^2(x) + 1 \text{ and } f'(x) = -8\cos(x)\sin(x) = -4\sin(2x).$$

We can apply Theorems 5 and 6 with constants

$$n = 3,\ R = \frac{5}{3},\ M = \frac{12}{5},\ \kappa = \frac{5}{9},\ \text{and } \tau = \frac{23}{6}.$$

For convenience, let

$$\delta_k = \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{n-1}.$$

Thus, from Theorem 4, we know that

$$\|y_{m,k} - x^*_m\| \le \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{n-1} = \delta_k.$$

Also, define the function h_k by

$$h_k(m) = m - \int_0^1 f(y_{m,k}(s))\, ds.$$

By an application of the mean value theorem and since $\max_{x\in[0,5]} |f'(x)| = 4$, for each $s \in [0,1]$, we have that

$$|f(y_{\omega_k,k}(s)) - f(x^*_\omega(s)| \le 4|y_{\omega_k,k}(s)) - f(x^*_\omega(s))|.$$

Therefore,

$$\begin{aligned} |g(m) - h_k(m)| &= \left| m - \int_0^1 f(x^*_m(s))\, ds - m + \int_0^1 f(y_{m,k}(s))\, ds \right| \\ &\le \int_0^1 \left| f(y_{m,k}(s)) - f(x^*_m(s)) \right| ds \\ &\le \int_0^1 4\left| y_{m,k}(s) - x^*_m(s) \right| ds \\ &\le \int_0^1 4\|y_{m,k} - x^*_m\|\, ds \le 4\delta_k. \end{aligned}$$

Let

$$\epsilon_k = 4\delta_k.$$

In particular, we have that

$$h_k(m) - \epsilon_k \le g(m) \le h_k(m) + \epsilon_k.$$

Hence, we have that if

$$h_k(m) > \epsilon_k, \ \text{ then } \ g(m) > 0,$$

and if

$$h_k(m) < -\epsilon_k, \ \text{ then } \ g(m) < 0.$$

We will use this to determine the sign of $g(\omega_k)$ in our application of Theorem 3 to find a sequence ω_k which will lead to an approximation of ω. Applying the error estimate that is in the proof of Theorem 5 provides that

$$\|x^*_\omega - y^*_{\omega_k,k}\| \leq \left(\frac{2n^2}{2n^2 - MR}\right)\left(\frac{2n^2MR}{2n^2 - MR} + 1\right)^{n-1}\left(\frac{Rn}{2^{k+1}}\right) + \left(\frac{R\kappa^k}{1-\kappa}\right)\left(\frac{4\tau n^2}{2n^2 - MR}\right)^{n-1}.$$

If we want to find ω_k such that $\|x^*_\omega - y^*_{\omega_k,k}\| < \frac{10^{-3}}{2}$, we need $k \geq 20$.

Below, we show how to find ω_{20} for the nonlinear second-order right focal boundary value problem (7), (8).

Since we are applying Theorem 6 and $g(0) < 0$ and $g(5) > 0$, we know there is an $\omega \in [0,5]$ with $g(\omega) = 0$, and thus, x^*_ω is a solution of (7), (8) by Theorem 1, so we begin with $a_0 = 0$ and $b_0 = 5$. Following the bisection argument of Theorem 3, let

$$\omega_0 = \frac{a_0 + b_0}{2} = \frac{5}{2}.$$

Since

$$h_{20}(\omega_0) < -\epsilon_{20}$$

which was determined using Mathematica, we have that $g(\omega_0) < 0$, and we know that $g(5) > 0$. Hence, by the intermediate value theorem, $\omega \in [\omega_0, 5]$, and we let $a_1 = \omega_0$ and $b_1 = 5$. We continue in this manner until we arrive at ω_{20}. We summarize this information in the following Table 1.

Table 1. Iteration Values Resulting from the Bisection Method

n	a_n	b_n	ω_n	$g(\omega_n)$
0	0	5	$\frac{5}{2}$	negative
1	$\frac{5}{2}$	5	$\frac{15}{4}$	positive
2	$\frac{5}{2}$	$\frac{15}{4}$	$\frac{25}{8}$	positive
3	$\frac{5}{2}$	$\frac{25}{8}$	$\frac{45}{16}$	positive
4	$\frac{5}{2}$	$\frac{45}{16}$	$\frac{85}{32}$	negative
5	$\frac{85}{32}$	$\frac{45}{16}$	$\frac{175}{64}$	positive
6	$\frac{85}{32}$	$\frac{175}{64}$	$\frac{345}{128}$	negative
7	$\frac{345}{128}$	$\frac{175}{64}$	$\frac{695}{256}$	negative
8	$\frac{695}{256}$	$\frac{175}{64}$	$\frac{1395}{512}$	negative
9	$\frac{1395}{512}$	$\frac{175}{64}$	$\frac{2795}{1024}$	negative
10	$\frac{2795}{1024}$	$\frac{175}{64}$	$\frac{5595}{2048}$	negative
11	$\frac{5595}{2048}$	$\frac{175}{64}$	$\frac{11195}{4096}$	negative
12	$\frac{11195}{4096}$	$\frac{175}{64}$	$\frac{22395}{8192}$	positive
13	$\frac{11195}{4096}$	$\frac{22395}{8192}$	$\frac{44785}{16384}$	positive
14	$\frac{11195}{4096}$	$\frac{44785}{16384}$	$\frac{89565}{32768}$	negative
15	$\frac{89565}{32768}$	$\frac{44785}{16384}$	$\frac{179135}{65536}$	negative
16	$\frac{179135}{65536}$	$\frac{44785}{16384}$	$\frac{358275}{131072}$	negative
17	$\frac{358275}{131072}$	$\frac{44785}{16384}$	$\frac{716555}{262144}$	negative
18	$\frac{716555}{262144}$	$\frac{44785}{16384}$	$\frac{1433115}{524288}$	positive
19	$\frac{716555}{262144}$	$\frac{1433115}{524288}$	$\frac{2866225}{1048576}$	positive

Therefore,

$$a_{20} = \frac{716555}{262144} \quad \text{and} \quad b_{20} = \frac{2866225}{1048576}.$$

Hence,

$$\omega \in \left[\frac{716555}{262144}, \frac{2866225}{1048576}\right]$$

and

$$\omega_{20} = \frac{a_{20} + b_{20}}{2} = \frac{5732445}{2097152}$$

with

$$\|x^*_\omega - y_{\omega_{20},20}\| \leq \frac{10^{-3}}{2}.$$

Please note that $y_{\omega_{20},20}$ satisfies the boundary condition $y_{\omega_{20},20}(0) = 0$. However, it does not satisfy the boundary condition at $t = 1$. If it is important that the approximation also satisfies the right boundary condition, then one can use $Hy_{\omega_{20},20}$ as the approximation. It will satisfy both boundary conditions, and the error bound with this approximation utilizing the bound on the derivative given by $\max_{x\in[0,5]} |f'(x)] = 4$ is

$$\begin{aligned}
\|x^*_\omega - Hy_{\omega_{20},20}\| &= \|Hx^*_\omega - Hy_{\omega_{20},20}\| \\
&= \left\| \int_0^1 G(t,s)(f(x^*_\omega(s) - f(y_{\omega_{20},20}(s))ds \right\| \\
&\leq \max_{t\in[0,1]} \int_0^1 G(t,s)4\|x^*_\omega - y_{\omega_{20},20}\|ds \\
&= 2\|x^*_\omega - y_{\omega_{20},20}\| \leq 2\left(\frac{10^{-3}}{2}\right) = 10^{-3}.
\end{aligned}$$

6. Conclusions and Next Steps

While it is very clear that Theorem 6 had a solution given the hypotheses since an application of the Schauder fixed-point theorem [22] gives us existence immediately, that we can iterate to get as close to the solution x^*_ω as we want is this paper's contribution. In particular, we have shown how one can decompose a fixed-point problem into multiple fixed-point problems that one can easily iterate to approximate a solution of a differential equation satisfying one boundary condition, then applying a bisection method in an intermediate value theorem argument to meet a second boundary condition. We also established error estimates on the iterates generated by our technique. This approach is illustrated on a second-order right focal boundary value problem, with the example above showing how to apply the results. Open to the research community is the creation of a computer program to efficiently carry out the iterative scheme developed in this paper for the second-order right focal boundary value problem. There are also open questions related to iterative schemes using the decomposition strategy developed in this paper for other types of boundary value problems.

Author Contributions: Conceptualization, R.A.; methodology, D.R.A.; software, D.R.A. and J.L.; validation, D.R.A., R.A. and J.L.; formal analysis, D.R.A., R.A. and J.L.; investigation, D.R.A.; writing—original draft preparation, D.R.A.; writing—review and editing, R.A. and J.L.; supervision, D.R.A.; project administration, D.R.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Abushammala, M.; Khuri, S.A.; Sayfy, A. A novel fixed point iteration method for the solution of third order boundary value problems. *Appl. Math. Comput.* **2015**, *271*, 131–141. [CrossRef]
2. Bello, N.; Alkali, A.J.; Roko, A. A fixed point iterative method for the solution of two-point boundary value problems for a second order differential equations. *Alexandria Eng. J.* **2017**, *57*, 2515–2520. [CrossRef]
3. Dang, Q.A.; Luan, V.T. Iterative method for solving a nonlinear fourth order boundary value problem. *Comput. Math. Appl.* **2010**, *60*, 112–121. [CrossRef]
4. Zeidler, E. *Nonlinear Functional Analysis and Its Applications I, Fixed Point Theorems*; Springer: New York, NY, USA, 1986.
5. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA,2003.
6. Kafri, H.Q.; Khuri, S.A. Bratu's problem: A novel approach using fixed-point iterations and Green's functions. *Comput. Phys. Commun.* **2015**, *198*, 97–104. [CrossRef]
7. Kafri, H.Q.; Khuri, S.A.; Sayfy, A. A new approach based on embedding Green's functions into fixed-point iterations for highly accurate solution to Troesch's problem. *Int. J. Comput. Methods Eng. Sci. Mech.* **2016**, *17*, 93–105. [CrossRef]
8. Khuri, S.A.; Louhichi, I. A novel Ishikawa-Green's fixed point scheme for the solution of BVPs. *Appl. Math. Lett.* **2018**, *82*, 50–57.
9. Khuri, S.A.; Sayfy, A. A fixed point iteration method using Green's functions for the solution of nonlinear boundary value problems over semi-infinite intervals. *Int. J. Comput. Math.* **2020**, *97*, 1303–1319. [CrossRef]
10. Duffy, D. *Green's Functions with Applications*; CRC Press: New York, NY, USA, 2015.
11. Haq, F.I.; Ali, A. Numerical solution of fourth order boundary-value problems using Haar wavelets. *Appl. Math. Sci.* **2011**, *5*, 3131–3146.
12. Hossain, M.B.; Islam, M.S. Numerical solutions of general fourth order two point boundary value problems by Galerkin method with Legendre polynomials. *Dhaka Univ. J. Sci.* **2014**, *62*, 103–108. [CrossRef]
13. Kumar, V.; Latif, A.; Rafiq, A.; Hussain, N. *S*-iteration process for quasi-contractive mappings. *J. Inequalities Appl.* **2013**, 206. [CrossRef]
14. Thenmozhi, S.; Marudai, M. Solution of nonlinear boundary value problem by *S*-iteration. *J. Appl. Math. Comput.* **2022**, *68*, 1047–1068. [CrossRef]
15. Burton, T.A. Fixed points, differential equations, and proper mappings. *Semin. Fixed Point Theory-Cluj-Napoca* **2002**, *3*, 19–32.
16. Avery, R.I.; Peterson, A.C. Multiple positive solutions of a discrete second order conjugate problem. *Panam. Math. J.* **1998**, *8*, 1–12.
17. Mann, W. Mean Value Methods in Iteration. *Proc. Am. Math. Soc.* **1953**, *4*, 506–510. [CrossRef]
18. Avery, R.; Anderson, D.; Henderson, J. Decomposing a Conjugate Fixed-Point Problem into Multiple Fixed-Point Problems. *Dyn. Syst. Appl.* **2023**, *32*, 255–274. [CrossRef]
19. Avery, R.; Anderson, D.; Henderson, J. Decomposing a Fixed-Point Problem into Multiple Fixed-Point Problems. *Rocky Mt. J. Math.* **2023**; accepted. [CrossRef]
20. Banach, S. Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
21. Agarwal, R.P. *Focal Boundary Value Problems for Differential and Difference Equations*; Mathematics and Its Applications 436; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1998. . [CrossRef]
22. Schauder, J. Der Fixpunktsatz in Funktionalraumen. *Stud. Math.* **1930**, *2*, 171–180. [CrossRef]