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The Enumeration of $(\odot, \vee)$-Multiderivations on a Finite MV-Chain

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Abstract: In this paper, $(\odot, \vee)$-multiderivations on an MV-algebra A are introduced, the relations between $(\odot, \vee)$-multiderivations and $(\odot, \vee)$-derivations are discussed. The set $\mathrm{MD}(A)$ of $(\odot, \vee)$-multiderivations on A can be equipped with a preorder, and $(\mathrm{MD}(A)/\sim, \preccurlyeq)$ can be made into a partially ordered set with respect to some equivalence relation $\sim$. In particular, for any finite MV-chain L_n, $(\mathrm{MD}(L_n)/\sim, \preccurlyeq)$ becomes a complete lattice. Finally, a counting principle is built to obtain the enumeration of $\mathrm{MD}(L_n)$.

Keywords: MV-algebra; $(\odot, \vee)$-multiderivation; complete lattice; enumeration; cardinality

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1. Introduction

The concept of derivation originating from analysis has been delineated for a variety of algebraic structures which come in analogy with the Leibniz rule

$$\frac{d}{dx}(fg) = \frac{d}{dx}(f)g + f\frac{d}{dx}(g).$$

Posner [1] introduced the derivation on prime rings $(R, +, \cdot)$ as a mapping d from R to R such that for all $x, y \in R$:

$$(1)\ d(x \cdot y) = d(x) \cdot y + x \cdot d(y), \quad (2)\ d(x+y) = d(x) + d(y).$$

It implies that

$$(3)\ \ d(1) = 0, \quad (4)\ \ d(0) = 0,$$

which are the 0-ary version of (1) and (2), respectively.

The derivations on lattices $(L, \vee, \wedge)$ were defined in [2] by Szász and were developed in [3] by Ferrari as a map d from L to L such that for all elements x, y in L:

$$(i)\ d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)), \quad (ii)\ d(x \vee y) = d(x) \vee d(y).$$

Xin et al. [4,5] investigated the derivations on a lattice satisfying only condition (i). In fact, a derivation d on L with both the Leibniz rule (i) and the linearity (ii) implies that $d(x) = x \wedge u$ for some $u \in L$ [6] (Proposition 2.5). If u is the maximum of a lattice, then such a derivation is actually the identity. It seems that this is an important reason for the derivations on, for instance, BCI-algebra [7], residuated lattices [8], basic algebra [9], L-algebra [10], and differential lattices [6], which are defined with the unique requirement of the Leibniz rule (i) (for the discussion in detail, cf. Section 2).

The derivation on an MV-algebra $(A, \oplus, ^*, 0)$ was firstly introduced by Alshehri [11] as a mapping d from A to A satisfying an $(\odot, \oplus)$-condition: $\forall x, y \in A$,

$$d(x \odot y) = (d(x) \odot y) \oplus (x \odot d(y)),$$

where $x \odot y$ is defined to be $(x^* \oplus y^*)^*$. Then, several derivations on MV-algebras have been considered in [12–15]. However, the interplay of the ring operations $\cdot$ and $+$ is more similar to the interplay between the MV-operations $\odot$ and $\vee$ rather than that between the MV-operations $\odot$ and $\oplus$. In fact, the main interplay between $\cdot$ and $+$ in rings is the distributivity of $\cdot$ over $+$. In MV-algebras, $\odot$ distributes over $\vee$, as in rings, while it is not true that $\odot$ distributes over $\oplus$. It is also true that $\odot$ distributes over $\wedge$, but $\vee$ is preferable because the identity element of $\vee$ is absorbing for $\odot$, that is, $0 \odot x = 0$ for any element x in an MV-algebra A, as in rings, while the same is not true for $\wedge$. Therefore, the $(\odot, \vee)$-derivation on MV-algebras [16] is a nature improvement of Alshehri's celebrated work [11] of the $(\odot, \oplus)$-derivation (cf. Section 2 for more discussion).

Let E and F be nonempty sets. A multifunction $f \colon E \to \Delta(F)$ is a map (or function) from E into $\Delta(F)$, the collection of nonempty subsets of F. The multifunction [17] is also known as set-valued function [18]. Significantly, multifunctions have many diverse and interesting applications in control problems [19,20] and mathematical economics [21,22]. Motivated by the role played by derivations on MV-algebras and the work of multiderivations on lattices [23], it is imperative to undertake a systematic study of the corresponding algebraic structure for derivations on MV-algebras.

This article is a continuation of work on $(\odot, \vee)$-multiderivations based on the nature $(\odot, \vee)$-derivation on MV-algebras [16], that is, a set-valued generalization of point-valued $(\odot, \vee)$-derivations. Section 2 starts with a review of the $(\odot, \vee)$-derivations on an MV-algebra A. In Section 3, we first define a natural preorder on $\Delta(A)$ that $M \preceq N$ iff for every $m \in M$ there exists $n \in N$ such that $m \leq n$. Then, we introduce $(\odot, \vee)$-multiderivations on MV-algebras. The relations between $(\odot, \vee)$-derivations and $(\odot, \vee)$-multiderivations on an MV-algebra are given (Propositions 5–7). In Section 4, we investigate the set of $(\odot, \vee)$-multiderivations $\mathrm{MD}(A)$ on an MV-algebra A. Let $\sigma, \sigma' \in \mathrm{MD}(A)$. Define $\sigma \preccurlyeq \sigma'$ if $\sigma(x) \preceq \sigma'(x)$ for any $x \in A$, and an equivalence relation $\sim$ on $\mathrm{MD}(A)$ by $\sigma \sim \sigma'$ iff $\sigma \preccurlyeq \sigma'$ and $\sigma' \preccurlyeq \sigma$. Then, $(\mathrm{MD}(A)/\!\sim, \preccurlyeq)$ is a poset. For an n-element MV-chain L_n, we show that $(\mathrm{MD}(L_n)/\!\sim, \preccurlyeq)$ is isomorphic to the complete lattice $\mathrm{Der}(L_n)$, the underlying set of $(\odot, \vee)$-derivations on L_n (Theorem 1), so we deduce that $|\,\mathrm{MD}(L_n)/\!\sim| = |\,\mathrm{Der}(L_n)|$, then [16] (Theorem 3.11) can be applied. Moreover, we define an equivalence relation $\sim$ on $\Delta(A)$, and present the fact that the poset $\Delta(L_n \times L_2)/\!\sim$ is isomorphic to the complete lattice $\mathrm{Der}(L_{n+1})$ (Proposition 11). However, the cardinalities of different equivalence classes with respect to the equivalence relation $\sim$ are different in general (Example 5). In Section 5, by building a counting principle (Theorem 3) for $(\odot, \vee)$-multiderivations on an n-element MV-chain L_n, we finally obtain the enumeration of $\mathrm{MD}(L_n)$: $(7 \cdot 3^{n-1} - 2^{n+2} + 1)/2$.

Notation. Throughout this paper, A denotes an MV-algebra; $|X|$ denotes the cardinality of a set X; $\Delta(X)$ denotes the set of nonempty subsets of a set X; $\sqcup$ means disjoint union; $\mathbb{N}$ denotes the set of natural numbers; "iff" is the abbreviation for "if and only if".

2. Preliminaries

Definition 1 ([24])**.** *An algebra $(A, \oplus, ^*, 0)$ is an* MV-algebra *if the following axioms are satisfied:*

(MV1) (associativity) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.
(MV2) (commutativity) $x \oplus y = y \oplus x$.
(MV3) (existence of the unit 0) $x \oplus 0 = x$.
(MV4) (involution) $x^{**} = x$.
(MV5) (maximal element 0^*) $x \oplus 0^* = 0^*$.
(MV6) (Łukasiewicz axiom) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Define $1 = 0^*$ and **the natural order** on A as follows: $y \geq x$ iff $x \odot y^* = 0$. Then, the interval $[a, b] = \{r \in A \mid a \leq r \leq b\}$ for any $a, b \in A$ and $a \leq b$. Note that A is a bounded distributive lattice with respect to the natural order [24] (Proposition 1.5.1) with 0, 1, and

$$x \vee y = (x \odot y^*) \oplus y \ , \ x \wedge y = x \odot (x^* \oplus y). \tag{1}$$

An MV-chain is an MV-algebra which is linearly ordered with respect to the natural order.

Example 1 ([24])**.** *Let $L = [0,1]$ be the real unit interval. Define*

$$x \oplus y = \min\{1, x+y\} \text{ and } x^* = 1 - x \text{ for any } x, y \in L.$$

Then $(L, \oplus, {}^*, 0)$ is an MV*-chain. Note that $x \odot y = \max\{0, x+y-1\}$.*

Example 2. *For every $2 \leq n \in \mathbb{N}_+$, let*

$$L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \cdots, \frac{n-2}{n-1}, 1\right\}.$$

Then the n-element subset L_n is an MV*-subalgebra of L.*

Lemma 1 ([24,25])**.** *If A is an* MV*-algebra, then the following statements are true $\forall x, y, z \in A$:*

1. $x \oplus y \geq x \vee y \geq x \geq x \wedge y \geq x \odot y$.
2. $x \oplus y = 0$ *iff* $x = y = 0$. $x \odot y = 1$ *iff* $x = y = 1$.
3. *If* $y \geq x$, *then* $y \vee z \geq x \vee z$, $y \wedge z \geq x \wedge z$.
4. *If* $y \geq x$, *then* $y \oplus z \geq x \oplus z$, $y \odot z \geq x \odot z$.
5. $y \geq x$ *iff* $x^* \geq y^*$.
6. $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$.
7. $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.
8. $x \odot y \leq z$ *iff* $x \leq y^* \oplus z$.

Let Ω be an index set. The **direct product** $\prod_{i \in \Omega} A_i$ [24] of a family of MV-algebras $\{A_i\}_{i \in \Omega}$ is the MV-algebra with cartesian product of the family and pointwise MV-operations. We denote $A_1 \times A_2 \times \cdots \times A_n$ when Ω is a positive integer n. We call $a \in A$ **idempotent** if $a \oplus a = a$. Let $\mathbf{B}(A)$ be the set of idempotent elements of A and B_{2^n} be the 2^n-element Boolean algebra. Note that B_4 is actually $L_2 \times L_2$ [24].

Lemma 2 ([24], Proposition 3.5.3)**.** *Let A be a subalgebra of $[0,1]$. Let $A^+ = \{x \in A \mid x > 0\}$ and $a = \inf A^+$ be the infimum of A^+. If $a = 0$, then A is a dense subchain of $[0,1]$. If $a > 0$, then $A = L_n$ for some $n \geq 2$.*

Definition 2 ([16])**.** *If A is an* MV*-algebra, then a map d from A to A is an $(\odot, \vee)$-**derivation** on A if $\forall x, y \in A$,*

$$d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y)). \tag{2}$$

Let $\mathrm{Der}(A)$ *be the set of $(\odot, \vee)$-derivations on A. For $X = \{x_1, x_2, \cdots, x_n\}$ and a map $d : X \to X$, we shall write d as*

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ d(x_1) & d(x_2) & \cdots & d(x_n) \end{pmatrix}.$$

The mappings Id_A and $\mathbf{0}_A$, defined by $\mathrm{Id}_A(x) = x$ and $\mathbf{0}_A(x) = 0$ $(\forall x \in A)$, respectively, are $(\odot, \vee)$-derivations on A. For $u \in A$, the operator $\chi^{(u)}(x) := \begin{cases} u, & \text{if } x = 1 \\ x. & \text{otherwise} \end{cases} \in \mathrm{Der}(A)$. More examples are given in [16].

Proposition 1 ([16])**.** *If A is an* MV*-algebra and $d \in \mathrm{Der}(A)$, then the followings hold for all $x, y \in A$:*

1. $0 = d(0)$.
2. $x \geq d(x)$.
3. *If* $d(x) = x$, *then* $d(y) = y$ *for* $y \leq x$.

Remark 1. *Now let us give some explanations of the naturality of an $(\odot, \vee)$-derivation in Definition 2. The interplay of the ring operations $\cdot$ and $+$ is more similar to the interplay between the MV-operations $\odot$ and $\vee$ rather than that between the MV-operations $\odot$ and $\oplus$.*

Next we discuss why we include only Equation (2). Recall that $d(0) = 0$ is the 0-ary version of $d(x + y) = d(x) + d(y)$ in derivations on a ring. For MV-algebras, $d(0) = 0$ is the 0-ary version of (a); see Proposition 1 (1). $d(1) = 0$ is the 0-ary version of $d(x \cdot y) = d(x) \cdot d(y)$ in derivations on a ring. Hence, it seems that the most faithful and natural derivation notion on A as a translation of the ring-theoretic notion of derivation (cf. Introduction) would include:

(a) $d(x \odot y) = (d(x) \odot y) \vee (x \odot d(y))$,
(b) $d(1) = 0$,
(c) $d(x \vee y) = d(x) \vee d(y)$,
(d) $d(0) = 0$.

However, (b) and (c) imply that d is trivial (note that (a) is automatically assumed).

Lemma 3. *If A is an* MV*-algebra and $d : A \to A$ is a map satisfying* (a), (b) *and* (c) *for any $x, y \in A$. Then, $d = \mathbf{0}_A$.*

Proof. Assume $x \le y$, it follows from (c) that $d(y) = d(x \vee y) = d(x) \vee d(y)$ and thus $d(x) \le d(y)$. Together with (b) $d(1) = 0$, we have $d(x) = 0$ for any $x \in A$ since $x \le 1$. Hence, $d = \mathbf{0}_A$. □

Next, we consider what will happen if the condition (b′) $d(1) = 1$ replaces (b) $d(1) = 0$.

Lemma 4. *If $d : A \to A$ is a mapping from an* MV*-algebra A to A with* (a) *and* (b′) *for any $x, y \in A$, then, $d = \mathrm{Id}_A$.*

Proof. Assume d satisfies (a) and (b′). We obtain that d satisfies Proposition 1 (3) since d satisfies (a). Both with (b′) $d(1) = 1$, we obtain $d(x) = x$ for any $x \in A$. Therefore, $d = \mathrm{Id}_A$. □

Recall that for a given $a \in A$, a **principal** $(\odot, \vee)$-derivation d_a on A [16] is defined by $d_a(x) := a \odot x$ for all $x \in A$. An $(\odot, \vee)$-derivation d is **isotone** [16] if $\forall x, y \in A$, $y \ge x$ implies that $d(y) \ge d(x)$. Note that $\mathbf{0}_A$ and Id_A are both principal and isotone. More generally, we obtain the following.

Proposition 2 ([16] (Proposition 3.19))**.** *Let A be an MV-algebra and d be a map satisfying* (a) *and* (b″)*. Then, the followings are equivalent:*

1. *d is isotone;*
2. *$d(1) \odot x = d(x)$ for all $x \in A$;*
3. *$d(x) \vee d(y) = d(x \vee y)$.*

If d satisfies (b), then the principal derivations on MV-algebra A will not be included, expect $\mathbf{0}_A$. Even identity derivations Id_A will not be within our scope of consideration. Hence, the scope of the study will be significantly narrowed.

Remark 2. *Note that d is isotone if d satisfies* (c)*. In fact, if $x \le y$, then $d(y) = d(x \vee y) = d(x) \vee d(y)$ and thus $d(x) \le d(y)$. The isotone case is a special case of d, thus the scope of research will be narrowed. This case has been partially studied in [16], Section 3.3.*

Therefore, we use the derivation meaning from Definition 2 in our series papers since [16] on.

3. $(\odot, \vee)$-Multiderivations on an MV-Algebra

Let X and Y be two nonempty sets. Recall that a **set-valued function** or **multivalued function** (for short, **multifunction**) F between X and Y is a map $F : X \to \Delta(Y)$. The set $F(x)$ is called the image of x under F (cf. [26], Appendix A).

Definition 3. *Let A be an* MV*-algebra and $M, N \in \Delta(A)$. We define four binary operations $\oplus, \odot, \vee, \wedge$ and an unary operation $*$ on $\Delta(A)$ by:*

$$M \star N = \{m \star n \mid m \in M, n \in N\} \text{ and } M^* = \{m^* \mid m \in M\}$$

where $\star \in \{\oplus, \odot, \vee, \wedge\}$.

Remark 3.

1. *Note that $M \vee N$ means the pointwise $m \vee n$ operation from Equation (1) of sets, which is different from the supremum of M and N. $M \wedge N$ has a similar meaning.*
2. *We abbreviate $M \star \{x\}$ and $\{x\}^*$ by $M \star x$ and x^*, respectively. But if $\{x\}$ appears by itself such as $M \preceq \{x\}$, we still use $\{x\}$.*

We define a binary relation $M \preceq N$ iff for every $m \in M$ there exists $n \in N$ such that $m \leq n$. Denote $M \prec N$ if $M \preceq N$ and $M \neq N$.

Then, $\preceq$ is a preorder on $\Delta(A)$. In fact, the reflexivity and transitivity of $\preceq$ are clear. However, $\preceq$ does not satisfy antisymmetry in general. In fact, $\preceq$ satisfies antisymmetry iff the MV-algebra A is trivial: If A is trivial, we have $\Delta(A) = \{\{0\}\}$ and $\{0\} \preceq \{0\}$. Hence, $\preceq$ satisfies antisymmetry. Conversely, suppose A is nontrivial, we have $A \neq \{1\}$, but $\{1\} \preceq A$ and $A \preceq \{1\}$, a contradiction.

Lemma 5. *Let A be an* MV*-algebra and $x, a, b, c, e, f \in A$. Then, the followings hold:*

1. *If $x \leq b \odot c$, then there exists $t \in A$ such that $t \leq b$ and $x = t \odot c$.*
2. *If $x \leq b \vee c$, then there exist $t, s \in A$ such that $t \leq b, s \leq c$ and $x = t \vee s$.*
3. *$[a, b] \odot c = [a \odot c, b \odot c]$.*
4. *$[a, b] \vee [e, f] = [a \vee e, b \vee f]$.*

Proof. (1) Assume $x \leq b \odot c$, then

$$x = (b \odot c) \wedge x = (b \odot c) \odot ((b \odot c)^* \oplus x) = b \odot ((b \odot c)^* \oplus x) \odot c.$$

Thus, we may choose $t = b \odot ((b \odot c)^* \oplus x)$.

(2) Assume $x \leq b \vee c$. Recall that A is a distributive lattice. So

$$x = (b \vee c) \wedge x = (b \wedge x) \vee (c \wedge x).$$

Hence, we can obtain $x = t \vee s$ by taking $t = b \wedge x, s = c \wedge x$.

(3) For each $x \in [a, b]$, we obtain $a \odot c \leq x \odot c \leq b \odot c$ by Lemma 1 (4). Thus, $[a, b] \odot c \subseteq [a \odot c, b \odot c]$. It suffices to prove that $[a \odot c, b \odot c] \subseteq [a, b] \odot c$. For any $a \odot c \leq x \leq b \odot c$, by (1) there is $t = b \odot ((b \odot c)^* \oplus x) \leq b$ such that $x = t \odot c$. If we can prove $a \leq t$, then the result follows immediately. Note that

$$t = b \odot ((b \odot c)^* \oplus x) = b \odot (b^* \oplus c^* \oplus x) = b \wedge (c^* \oplus x).$$

Since $a \odot c \leq x$, we have $a \leq c^* \oplus x$ by Lemma 1 (8). Together with $a \leq b$, we obtain $a \leq b \wedge (c^* \oplus x) = t$. Thus, we conclude that $[a, b] \odot c = [a \odot c, b \odot c]$.

(4) For any $t \in [a, b], s \in [e, f]$, we have $a \vee e \leq t \vee s \leq b \vee f$ by Lemma 1 (3). Thus, $[a, b] \vee [e, f] \subseteq [a \vee e, b \vee f]$. It is enough to prove that $[a \vee e, b \vee f] \subseteq [a, b] \vee [e, f]$. For any $a \vee e \leq x \leq b \vee f$, there exist $t, s \in A$ such that

$$t = b \wedge x \leq b, \quad s = f \wedge x \leq f \quad \text{and} \quad x = t \vee s$$

by (2). If we can prove $a \leq t$ and $e \leq s$, then the result follows. Note that since $a \leq b$ and $a \leq a \vee e \leq x$, we have $a \leq b \wedge x = t$. Similarly, $e \leq s$. Therefore, $[a \vee e, b \vee f] = [a, b] \vee [e, f]$. □

The following result holds for any MV-algebra A since it is a distributive lattice under the natural order.

Lemma 6 ([23] (Lemma 2.1))**.** *Let L be a lattice and $M, N, P, Q \in \Delta(L)$. Then, the following statements hold:*

1. $M \wedge N \preceq M \preceq M \vee N$.
2. *If $M \preceq N$ and $P \preceq Q$, then $M \wedge P \preceq N \wedge Q$ and $M \vee P \preceq N \vee Q$. In particular, $M \preceq N$ implies $M \wedge P \preceq N \wedge P$.*
3. $M \subseteq M \wedge M, M \subseteq M \vee M$. *If M is a sublattice of L, then $M = M \vee M$.*
4. $M \vee N = N \vee M$.
5. $(M \vee N) \vee P = M \vee (N \vee P)$.
6. *If $M \vee N \subseteq M$, then $N \preceq M$.*
7. *If L is distributive, then $(M \vee N) \wedge P \subseteq (M \wedge P) \vee (N \wedge P)$.*

Remark 4.

1. *Note that the converse inclusion of Lemma 6 (3), i.e., $M \wedge M \subseteq M$ and $M \vee M \subseteq M$, does not hold in general. For example, consider the Boolean lattice $B_4 = \{0, a, b, 1\}$ (see Figure 1), $M = \{a, b\} \subseteq B_4$, then $0 = a \wedge b \in M \wedge M$ and $1 = a \vee b \in M \vee M$, but $0, 1 \notin M$.*
2. *The converse of Lemma 6 (6), i.e., $N \preceq M$ implies $M \vee N \subseteq M$ may not hold. For example, in L_3, let $N = \{0, \frac{1}{2}\}, M = \{0, 1\}$. We have $N \preceq M$ but $M \vee N = \{0, \frac{1}{2}, 1\} \nsubseteq M$.*
3. *The converse inclusion of Lemma 6 (7) holds if P is a singleton but need not hold in general. This is slightly different from [23]. For example, let $B_8 = \{0, a, b, c, u, v, w, 1\}$ be the 8-element Boolean lattice as Figure 2, $M = \{u\}, N = \{w\}$ and $P = \{a, b, c\}$. We can check that $u = a \vee b = (u \wedge a) \vee (w \wedge b) \in (M \wedge P) \vee (N \wedge P)$ but $u \notin P = (M \vee N) \wedge P$.*

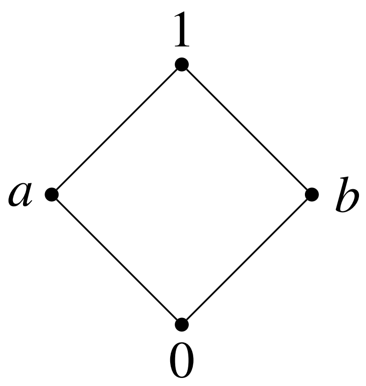

Figure 1. Hasse diagram of B_4.

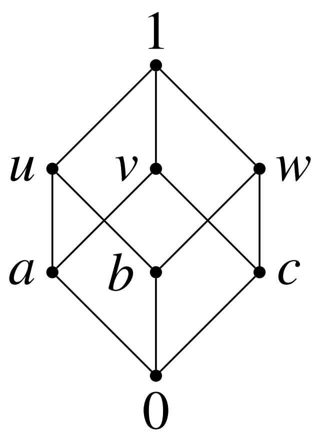

Figure 2. Hasse diagram of B_8.

According to Lemma 1, one obtains

Lemma 7. *Assume that A is an* MV*-algebra, $M, N, P, Q \in \Delta(A)$, and $m \in M$. Then, the following statements hold:*

1. *If $M \preceq N$ and $P \preceq Q$, then $M \oplus P \preceq N \oplus Q$ and $M \odot P \preceq N \odot Q$. In particular, $M \preceq N$ implies $M \oplus P \preceq N \oplus P$ and $M \odot P \preceq N \odot P$.*
2. *$m \odot (P \vee Q) = (m \odot P) \vee (m \odot Q)$.*
3. *$m \odot (P \cup Q) = (m \odot P) \cup (m \odot Q)$.*
4. *$M \odot N \preceq M \wedge N \preceq M \preceq M \vee N \preceq M \oplus N$.*
5. *If $M \oplus N \subseteq M$, then $N \preceq M$.*

Proof. (1) Suppose $M \preceq N$ and $P \preceq Q$. For any $x = m \oplus p \in M \oplus P$, there are $n \in N$ and $q \in Q$ such that $m \leq n$ and $p \leq q$. It follows from Lemma 1 (4) that $m \oplus p \leq m \oplus q \leq n \oplus q$, where $n \oplus q \in N \oplus Q$. Thus, $M \oplus P \preceq N \oplus Q$. Similarly, we have $M \odot P \preceq N \odot Q$. In particular, we obtain $M \oplus P \preceq N \oplus P$ and $M \odot P \preceq N \odot P$.

(2) For any $p \in P$ and $q \in Q$, we have $m \odot (p \vee q) = (m \odot p) \vee (m \odot q) \in (m \odot P) \vee (m \odot Q)$ by Lemma 1 (7). Thus, $m \odot (P \vee Q) \subseteq (m \odot P) \vee (m \odot Q)$. The reverse inclusion can be verified similarly. Therefore, $m \odot (P \vee Q) = (m \odot P) \vee (m \odot Q)$.

(3) We have $x \in m \odot (P \cup Q)$, iff there is $y \in P \cup Q$ such that $x = m \odot y$, iff there is $y \in P$ or $y \in Q$ such that $x = m \odot y$, iff $x \in m \odot P$ or $x \in m \odot Q$, iff $x \in (m \odot P) \cup (m \odot Q)$. Hence, $m \odot (P \cup Q) = (m \odot P) \cup (m \odot Q)$.

(4) For any $m \in M$ and $n \in N$, we know $m \odot n \leq m \wedge n \leq m \leq m \vee n \leq m \oplus n$ by Lemma 1 (1). The result follows immediately.

(5) Assume $M \oplus N \subseteq M$, then for any $n \in N$, there exists $m \in M$ such that $m \oplus n \in M$. So by Lemma 1 (1) we obtain $n \leq m \oplus n$. Therefore, $N \preceq M$. □

To study whether $(\Delta(A), \oplus, ^*, \{0\})$ is an MV-algebra, we first give

Lemma 8. *If A is an* MV*-algebra, then, for any $M, N, P \in \Delta(A)$, the followings hold:*

1. *$(M \oplus N) \oplus P = M \oplus (N \oplus P)$.*
2. *$M \oplus N = N \oplus M$.*
3. *$M \oplus 0 = M$.*
4. *$M^{**} = M$.*
5. *$M \oplus 0^* = \{0^*\}$.*

Proof. (1)–(5) follow from (MV1)–(MV5), respectively. □

Remark 5. *Since (MV1)–(MV5) are satisfied on $\Delta(A)$, it is natural to consider whether (MV6) $(M^* \oplus N)^* \oplus N = (N^* \oplus M)^* \oplus M$ holds on $\Delta(A)$. The answer is no. For example, let $M = \{\frac{1}{2}\}$ and $N = \{0, 1\}$ on three-element* MV*-chain L_3. It is easy to see that $(\frac{1}{2}^* \oplus \{0, 1\})^* \oplus \{0, 1\} = \{0, \frac{1}{2}\} \oplus \{0, 1\} = \{0, \frac{1}{2}, 1\} \neq \{\frac{1}{2}, 1\} = (\{0, 1\}^* \oplus \frac{1}{2})^* \oplus \frac{1}{2}$. That is, $(M^* \oplus N)^* \oplus N \neq (N^* \oplus M)^* \oplus M$.*

If A is a nontrivial MV-algebra, and $\varphi : A \to \Delta(A)$ is a multifunction on A. φ is called additive and negative, if $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y)$ and $\varphi(x^*) = (\varphi(x))^*$ for all $x, y \in A$, respectively.

Proposition 3. *Let A be an* MV*-algebra and $\varphi : A \to \Delta(A)$ be a multifunction on A. If φ is additive and negative, then $(\varphi(A), \oplus, ^*, \varphi(0))$ is an* MV*-algebra, where $\varphi(A) = \{\varphi(x) \mid x \in A\}$.*

Proof. It is sufficient to prove (MV3), (MV5) and (MV6), since we know that $(\varphi(A), \oplus, ^*, \varphi(0))$ satisfies (MV1), (MV2) and (MV4) by Lemma 8. Since φ is additive and negative, it follows that $\varphi(x) \oplus \varphi(0) = \varphi(x \oplus 0) = \varphi(x)$ and $\varphi(x) \oplus \varphi(0)^* = \varphi(x \oplus 0^*) = \varphi(0^*) = \varphi(0)^*$. Furthermore, $(\varphi(x)^* \oplus \varphi(y))^* \oplus \varphi(y) = \varphi(x^* \oplus y)^* \oplus \varphi(y) = \varphi((x^* \oplus y)^* \oplus y) = \varphi((y^* \oplus x)^* \oplus x) = \varphi(y^* \oplus x)^* \oplus \varphi(x) = (\varphi(y)^* \oplus \varphi(x))^* \oplus \varphi(x)$ for any $x, y \in A$. Thus, $(\varphi(A), \oplus, ^*, \varphi(0))$ is an MV-algebra. □

Now let us define the $(\odot, \vee)$-multiderivation.

Definition 4. If A is an MV-algebra, a multifunction $\sigma : A \to \Delta(A)$ is called an **$(\odot, \vee)$-multiderivation on A** if

$$\sigma(x \odot y) = (\sigma(x) \odot y) \vee (x \odot \sigma(y)) \tag{3}$$

for all $x, y \in A$. Denote the set of $(\odot, \vee)$-multiderivations on A by $\mathrm{MD}(A)$.

Example 3. *(i) Consider the* MV*-chain L_4. We define a multifunction σ on L_4 by $\sigma(0) = \{0\}$, $\sigma(\frac{1}{3}) = \{0, \frac{1}{3}\}, \sigma(\frac{2}{3}) = \{0, \frac{2}{3}\}, \sigma(1) = \{0, 1\}$. Then, we can check σ is an $(\odot, \vee)$-multiderivation on L_4. In fact, $\sigma = \beta_1$ (see Corollary 1).*

(ii) Consider the standard MV*-algebra $L = [0, 1]$. We define a multifunction $\sigma : L \to \Delta(L)$ by $\sigma(x) = [0, x]$ for all $x \in L$. Then, we can verify that σ is an $(\odot, \vee)$-multiderivation on L (see Proposition 6).*

(iii) Let A be an MV*-algebra and $S \subseteq A$ be a subalgebra of A. Define a multifunction σ_S on A by $\sigma_S(x) = x \odot S$, $\forall x \in A$, then $\sigma_S \in \mathrm{MD}(A)$, which is called a **principal** $(\odot, \vee)$-**multiderivation**. In fact, for any $x, y \in A$, since the subalgebra S must be a sublattice of A, it follows that $S = S \vee S$ by Lemma 6 (3). According to Lemma 7 (2), we immediately have $\sigma_S(x \odot y) = x \odot y \odot S = x \odot y \odot (S \vee S) = (x \odot y \odot S) \vee (x \odot y \odot S) = (\sigma_S(x) \odot y) \vee (x \odot \sigma_S(y))$.*

Proposition 4. *If A is an* MV*-algebra and $\sigma \in \mathrm{MD}(A)$. Then, the followings hold for all $x, y \in A$,*

1. *$\sigma(0) = \{0\}$.*
2. *$\sigma(x) \preceq \{x\}$.*
3. *$\sigma(x) \odot \sigma(y) \preceq \sigma(x \odot y) \preceq \sigma(x) \vee \sigma(y)$.*
4. *$x \odot \sigma(1) \preceq \sigma(x)$.*
5. *If I is a lower set of A, then $\sigma(x) \subseteq I$ holds for any $x \in I$.*
6. *Let $1 \in \sigma(1)$. Then, $x \in \sigma(x)$.*

Proof. (1) Taking $x = y = 0$ in Equation (3), we obtain $\sigma(0) = \sigma(0 \odot 0) = (\sigma(0) \odot 0) \vee (0 \odot \sigma(0)) = \{0\}$.

(2) Since $x \odot x^* = 0$, we know that $\{0\} = \sigma(0) = \sigma(x \odot x^*) = (\sigma(x) \odot x^*) \vee (x \odot \sigma(x^*))$ by (1). So $\sigma(x) \odot x^* = \{0\}$ and we obtain $\sigma(x) \preceq \{x\}$.

(3) By Lemma 6 (3), we have $\sigma(x) \odot \sigma(y) \subseteq (\sigma(x) \odot \sigma(y)) \vee (\sigma(x) \odot \sigma(y))$. Moreover, $\sigma(x) \odot \sigma(y) \preceq \sigma(x) \odot y$ and $\sigma(x) \odot \sigma(y) \preceq x \odot \sigma(y)$ by (2) and Lemma 7 (1). Thus,

$$\sigma(x) \odot \sigma(y) \subseteq (\sigma(x) \odot \sigma(y)) \vee (\sigma(x) \odot \sigma(y)) \preceq (\sigma(x) \odot y) \vee (x \odot \sigma(y)) = \sigma(x \odot y)$$

by Lemma 6 (2). Moreover, by Lemma 7 (1) and Lemma 6 (2) we have

$$\sigma(x \odot y) = (\sigma(x) \odot y) \vee (x \odot \sigma(y)) \preceq \sigma(x) \vee \sigma(y).$$

(4) Since $x = 1 \odot x$, it follows that $\sigma(x) = \sigma(1 \odot x) = \sigma(x) \vee (x \odot \sigma(1))$ by Equation (3). Then, we can obtain $x \odot \sigma(1) \preceq \sigma(x)$ by Lemma 6 (6).

(5) For any $x \in I$, we know $\sigma(x) \preceq \{x\}$ by (2). It induces that $y \leq x$ holds for any $y \in \sigma(x)$. Then, $y \in I$ since I is a lower set. Thus, $\sigma(x) \subseteq I$.

(6) Since $1 \in \sigma(1)$, there must exist $y \in \sigma(x)$ such that $x = x \odot 1 \leq y$ by (4). Moreover, by (2) we know $y \leq x$ always holds for y. Hence, we obtain $x = y$ and $x \in \sigma(x)$. $\square$

Now, let us explore the relations between $(\odot, \vee)$-derivation d and $(\odot, \vee)$-multiderivation σ on A.

On the one hand, given an $(\odot, \vee)$-derivation d on A, how can we construct an $(\odot, \vee)$-multiderivation on A? We get started with a direct construction. Assume $d \in \mathrm{Der}(A)$. Define a multifunction $\alpha : A \to \Delta(A)$ as follows:

$$\alpha(x) = \{d(x)\} \quad \text{for any} \quad x \in A.$$

Then, $\alpha \in \mathrm{MD}(A)$.

Proposition 5. *If A is an* MV*-algebra and $d \in \mathrm{Der}(A)$, define a multifunction $\beta : A \to \Delta(A)$ on A as follows*

$$\beta(x) := \{0, d(x)\}.$$

Then, $\beta \in \mathrm{MD}(A)$ iff $d(x) \odot y = x \odot d(y)$ holds for any $x, y \in A$ with $d(x) \odot y > 0$ and $x \odot d(y) > 0$.

Proof. Assuming $\beta \in \mathrm{MD}(A)$, it follows that

$$\begin{aligned}
\{0, d(x \odot y)\} &= \beta(x \odot y) \\
&= (\beta(x) \odot y) \vee (x \odot \beta(y)) \\
&= (\{0, d(x)\} \odot y) \vee (x \odot \{0, d(y)\}) \\
&= \{0, d(x) \odot y\} \vee \{0, x \odot d(y)\} \\
&= \{0, d(x) \odot y, x \odot d(y), d(x \odot y)\}
\end{aligned}$$

for any $x, y \in A$. From the chain of equalities, we know that $d(x) \odot y, x \odot d(y) \in \{0, d(x \odot y)\}$. If both $d(x) \odot y > 0$ and $x \odot d(y) > 0$, then $d(x) \odot y = d(x \odot y) = x \odot d(y)$.

Conversely, let $x, y \in A$.

Then,

$$\beta(x \odot y) = \{0, d(x \odot y)\}$$

and

$$(\beta(x) \odot y) \vee (x \odot \beta(y)) = \{0, d(x) \odot y, x \odot d(y), d(x \odot y)\}.$$

There are only two cases:

If $d(x) \odot y = 0$ or $x \odot d(y) = 0$, without loss of generality, assume that $d(x) \odot y = 0$. Then,

$$d(x \odot y) = 0 \vee (x \odot d(y)) = x \odot d(y).$$

Thus, $(\beta(x) \odot y) \vee (x \odot \beta(y)) = \{0, d(x \odot y)\} = \beta(x \odot y)$.

If $d(x) \odot y = x \odot d(y)$, then

$$d(x \odot y) = d(x) \odot y = x \odot d(y).$$

Thus, $(\beta(x) \odot y) \vee (x \odot \beta(y)) = \{0, d(x \odot y)\} = \beta(x \odot y)$.

Consequently, we infer $\beta \in \mathrm{MD}(A)$. $\square$

Corollary 1. *If A is an* MV*-algebra, and $a \in A$, a multifunction $\beta_a : A \to \Delta(A)$ on A is defined as follows*

$$\beta_a(x) := \{0, d_a(x)\}.$$

Then $\beta_a \in \mathrm{MD}(A)$.

Proof. If $d = d_a$ in Proposition 5, then for any $x, y \in A$, we know $d(x) \odot y = a \odot x \odot y = x \odot d(y)$. Hence, we infer that $\beta_a \in \mathrm{MD}(A)$ by Proposition 5. $\square$

Remark 6. *The conclusion is not necessarily true for general $(\odot, \vee)$-derivations. For example, $d = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$ is an $(\odot, \vee)$-derivation on L_4. But $\beta(\frac{2}{3} \odot 1) = \{0, \frac{2}{3}\} \neq \{0, \frac{1}{3}, \frac{2}{3}\} = \{0, \frac{2}{3}\} \vee \{0, \frac{1}{3}\} = (\{0, \frac{2}{3}\} \odot 1) \vee (\frac{2}{3} \odot \{0, \frac{2}{3}\}) = (\beta(\frac{2}{3}) \odot 1) \vee (\frac{2}{3} \odot \beta(1))$.*

Proposition 6. *Let A be an* MV*-algebra and $d \in \mathrm{Der}(A)$. Define a multifunction $\gamma : A \to \Delta(A)$ on A as follows*

$$\gamma(x) := [0, d(x)].$$

Then $\gamma \in \mathrm{MD}(A)$.

Proof. Since $d \in \mathrm{Der}(A)$, we obtain $\gamma(x \odot y) = [0, d(x \odot y)] = [0, (d(x) \odot y) \vee (x \odot d(y))]$. Moreover, we have

$$\begin{aligned}(\gamma(x) \odot y) \vee (x \odot \gamma(y)) &= ([0, d(x)] \odot y) \vee (x \odot [0, d(y)]) && \text{(Definition 3)} \\ &= [0, d(x) \odot y] \vee [0, x \odot d(y)] && \text{(Lemma 5 (3))} \\ &= [0, (d(x) \odot y) \vee (x \odot d(y))]. && \text{(Lemma 5 (4))}\end{aligned}$$

Hence, we conclude that $\gamma \in \mathrm{MD}(A)$. □

On the other hand, if there is a given $(\odot, \vee)$-multiderivation σ on A, then we can construct a corresponding $(\odot, \vee)$-derivation d from σ. We need the following lemma to prepare.

Lemma 9. *If A is an* MV*-algebra, and $M, N \in \Delta(A)$, if both* $\sup(M)$ *and* $\sup(N)$ *exist, then*

1. $\sup(M \odot N)$ *exists and* $\sup(M \odot N) = \sup(M) \odot \sup(N)$.
2. $\sup(M \vee N)$ *exists and* $\sup(M \vee N) = \sup(M) \vee \sup(N)$.

Proof. Denote $m_0 = \sup(M)$ and $n_0 = \sup(N)$.

(1) Firstly, we prove that $m_0 \odot n_0$ is an upper bound of $M \odot N$. For any $m \in M$ and $n \in N$, we immediately have $m \odot n \leq m_0 \odot n_0$ by Lemma 1 (4). Hence, it is enough to show that $m_0 \odot n_0$ is the least upper bound. Assume that $m \odot n \leq x$ for all $m \in M, n \in N$. It tells us that $m \leq n^* \oplus x$ and so $m_0 \leq n^* \oplus x$ by Lemma 1 (8) and the definition of least upper bound. Then, we have $m_0 \odot n \leq x$. Similarly, we obtain $n \leq m_0^* \oplus x$ and $n_0 \leq m_0^* \oplus x$. Thus, we can prove that $m_0 \odot n_0 \leq x$. Finally, $\sup(M \odot N) = \sup(M) \odot \sup(N)$ holds.

(2) For any $m \in M$ and $n \in N$, we have $m \leq m_0$ and $n \leq n_0$. So, $m \vee n \leq m_0 \vee n_0$ and $\sup(M \vee N) \leq \sup(M) \vee \sup(N)$. Conversely, since $M \vee N \succeq M, N$, it implies that $\sup(M \vee N) \geq \sup(M), \sup(N)$ and thus $\sup(M \vee N) \geq \sup(M) \vee \sup(N)$. Therefore, $\sup(M \vee N) = \sup(M) \vee \sup(N)$. □

Proposition 7. *If A is an* MV*-algebra, $\sigma \in \mathrm{MD}(A)$, and $\sup(\sigma(x))$ exists for any $x \in A$, define $\sup \sigma : A \to A$ by $(\sup \sigma)(x) = \sup(\sigma(x))$. Then, $\sup \sigma \in \mathrm{Der}(A)$.*

Proof. For any $x, y \in A$, we have

$$\begin{aligned}(\sup \sigma)(x \odot y) &= \sup(\sigma(x \odot y)) && \text{(Definition of } \sup \sigma) \\ &= \sup((\sigma(x) \odot y) \vee (x \odot \sigma(y))) && \text{(Equation (3))} \\ &= \sup(\sigma(x) \odot y) \vee \sup(x \odot \sigma(y)) && \text{(Lemma 9 (2))} \\ &= (\sup(\sigma(x)) \odot \sup\{y\}) \vee (\sup\{x\} \odot \sup(\sigma(y))) && \text{(Lemma 9 (1))} \\ &= ((\sup \sigma)(x) \odot y) \vee (x \odot (\sup \sigma)(y)). && \text{(Definition of } \sup \sigma)\end{aligned}$$

Hence, $\sup \sigma \in \mathrm{Der}(A)$. □

Remark 7. (1) *If* MV*-algebra A is complete, then $\sup \sigma$ is always an $(\odot, \vee)$-derivation on A for an arbitrary $(\odot, \vee)$-multiderivation σ on A.*

(2) *If $\sigma \in \mathrm{MD}(A)$ and the image $\sigma(x)$ is finite for any $x \in A$, then $\sup \sigma$ is always an $(\odot, \vee)$-derivation on A.*

Next, we construct $(\odot, \vee)$-multiderivations on subalgebras and direct products of MV-algebras from a given $(\odot, \vee)$-multiderivation.

Proposition 8. *Let A be an* MV*-algebra and $\sigma \in \mathrm{MD}(A)$. If S is a subalgebra of A and $\sigma(x) \subseteq S$ for any $x \in S$, then $\sigma|_S \in \mathrm{MD}(S)$.*

Proof. For any $x, y \in S$, we know that $\sigma(x), \sigma(y) \subseteq S$ and so $\sigma(x) \odot y, x \odot \sigma(y) \subseteq S$. Then,

$$\sigma|_S(x \odot y) = (\sigma(x) \odot y) \vee (x \odot \sigma(y)) = (\sigma|_S(x) \odot y) \vee (x \odot \sigma|_S(y)) \subseteq S \vee S = S$$

by Lemma 6 (3). Thus, $\sigma|_S \in \mathrm{MD}(S)$. □

Definition 5. If Ω is a nonempty set, for each $i \in \Omega$, let σ_i be a multifunction on A_i. The **direct product of** $\{\sigma_i\}_{i\in\Omega}$ $\prod_{i\in\Omega} \sigma_i : \prod_{i\in\Omega} A_i \to \Delta(\prod_{i\in\Omega} A_i)$ is defined by

$$\left(\prod_{i\in\Omega} \sigma_i\right)(g) = \prod_{i\in\Omega} \sigma_i(g(i)) = \{(x_i)_{i\in\Omega} \mid x_i \in \sigma_i(g(i))\}$$

for all $g \in \prod_{i\in\Omega} A_i$.

Lemma 10. *Let Ω be a nonempty set, $\{A_i\}_{i\in\Omega}$ be a family of* MV*-algebras, and $M_i, N_i \in \Delta(A_i)$. Then, $\prod_{i\in\Omega}(M_i \vee N_i) = \prod_{i\in\Omega} M_i \vee \prod_{i\in\Omega} N_i$.*

Proof. We first show that $\prod_{i\in\Omega}(M_i \vee N_i) \subseteq \prod_{i\in\Omega} M_i \vee \prod_{i\in\Omega} N_i$. For any $x \in \prod_{i\in\Omega}(M_i \vee N_i)$, there are $m_i \in M_i, n_i \in N_i$ for any $i \in \Omega$ such that $x = (m_i \vee n_i)_{i\in\Omega}$. Denote $m = (m_i)_{i\in\Omega}, n = (n_i)_{i\in\Omega}$, we have $x = (m_i \vee n_i)_{i\in\Omega} = (m_i)_{i\in\Omega} \vee (n_i)_{i\in\Omega} = m \vee n \in \prod_{i\in\Omega} M_i \vee \prod_{i\in\Omega} N_i$. And vice versa. Therefore, $\prod_{i\in\Omega}(M_i \vee N_i) = \prod_{i\in\Omega} M_i \vee \prod_{i\in\Omega} N_i$. □

Proposition 9. *Assume that Ω is a nonempty set and $\{A_i\}_{i\in\Omega}$ is a family of* MV*-algebras. Then, $\sigma_i \in \mathrm{MD}(A_i)$ for any $i \in \Omega$ iff $\prod_{i\in\Omega} \sigma_i \in \mathrm{MD}(\prod_{i\in\Omega} A_i)$.*

Proof. Denote $A = \prod_{i\in\Omega} A_i$ and $\sigma = \prod_{i\in\Omega} \sigma_i$. For all $x = (x_i)_{i\in\Omega}, y = (y_i)_{i\in\Omega} \in A$, we have

$$\sigma(x \odot y) = \sigma((x_i)_{i\in\Omega} \odot (y_i)_{i\in\Omega}) = \prod_{i\in\Omega} \sigma_i(x_i \odot y_i),$$

$$\begin{aligned}(\sigma(x) \odot y) \vee (x \odot \sigma(y)) &= \left(\prod_{i\in\Omega} \sigma_i(x_i) \odot (y_i)_{i\in\Omega}\right) \vee \left((x_i)_{i\in\Omega} \odot \prod_{i\in\Omega} \sigma_i(y_i)\right) \\ &= \prod_{i\in\Omega}(\sigma_i(x_i) \odot y_i) \vee \prod_{i\in\Omega}(x_i \odot \sigma_i(y_i)) \\ &= \prod_{i\in\Omega}((\sigma_i(x_i) \odot y_i) \vee (x_i \odot \sigma_i(y_i))). \qquad \text{(Lemma 10)}\end{aligned}$$

We can immediately obtain $\sigma_i \in \mathrm{MD}(A_i)$ for all $i \in \Omega$ iff $\sigma(x \odot y) = (\sigma(x) \odot y) \vee (x \odot \sigma(y))$ by Equation (3). □

Finally, we investigate the condition when an $(\odot, \vee)$-multiderivation σ is isotone.

Definition 6. *If A is an* MV*-algebra, and $\sigma \in \mathrm{MD}(A)$, we say σ is* ***isotone*** *if $\sigma(x) \preceq \sigma(y)$ whenever $x \leq y$.*

Proposition 10. *If A is an* MV*-algebra, and $\sigma \in \mathrm{MD}(A)$, then σ is isotone iff $\sigma(x \wedge y) \preceq \sigma(x) \wedge y$ for all $x, y \in A$.*

Proof. Assume σ is isotone, then,

$$\sigma(x \wedge y) \subseteq \sigma(x \wedge y) \wedge \sigma(x \wedge y) \preceq \sigma(x) \wedge \sigma(y) \preceq \sigma(x) \wedge y$$

by Lemma 6 (3) and (2). Conversely, assume that $\sigma(x \wedge y) \preceq \sigma(y) \wedge x$ for all $x, y \in A$. Let $x, y \in A$ with $x \leq y$. Then, $\sigma(x) = \sigma(y \wedge x) \preceq \sigma(y) \wedge x$. Thus, for every $a \in \sigma(x)$ there is $b \in \sigma(y)$ such that $a \leq b \wedge x$. Hence, $a \leq b$ and so $\sigma(x) \preceq \sigma(y)$. □

Corollary 2. *If A is an* MV*-algebra, and $S \subseteq A$ is a subalgebra of A, then the principal $(\odot, \vee)$-multiderivation σ_S is isotone.*

Proof. Method 1: Let $x, y \in A$ and $x \leq y$. For any $s \in S$, Lemma 1 (4) implies $x \odot s \leq y \odot s$. Thus, $\sigma_S(x) \preceq \sigma_S(y)$.

Method 2: It is enough to verify that $\sigma_S(x \wedge y) \preceq \sigma_S(x) \wedge y$ for all $x, y \in A$ by Proposition 10. For any $s \in S$, Lemma 1 (6) implies

$$(x \wedge y) \odot s = (x \odot s) \wedge (y \odot s) \leq (x \odot s) \wedge y.$$

Thus, $\sigma_S(x \wedge y) = (x \wedge y) \odot S \preceq (x \odot S) \wedge y = \sigma_S(x) \wedge y$. $\square$

4. The Order Structure of $(\odot, \vee)$-Multiderivations on a Finite MV-Chain

Let $\mathrm{MF}(A)$ be the set of multifunctions on an MV-algebra A. Define $\preccurlyeq$ on $\mathrm{MF}(A)$ by:

$$(\forall\, \sigma, \sigma' \in \mathrm{MF}(A)) \quad \sigma \preccurlyeq \sigma' \text{ if } \sigma(x) \preceq \sigma'(x), \forall x \in A.$$

Then, $\preccurlyeq$ is a preorder on $\mathrm{MF}(A)$ and $\mathbf{0}_{\mathrm{MF}(A)} \preccurlyeq \sigma \preccurlyeq \mathbf{1}_{\mathrm{MF}(A)}$ for any $\sigma \in \mathrm{MF}(A)$, where $\mathbf{0}_{\mathrm{MF}(A)}$ and $\mathbf{1}_{\mathrm{MF}(A)}$ are defined by $\mathbf{0}_{\mathrm{MF}(A)}(x) := \{0\}$ and $\mathbf{1}_{\mathrm{MF}(A)}(x) := \{1\}$ for any $x \in A$, respectively. For any $\sigma \in \mathrm{MD}(A)$, we have $\mathbf{0}_{\mathrm{MF}(A)} \preccurlyeq \sigma \preccurlyeq \mathrm{Id}_{\mathrm{MF}(A)}$, where $\mathrm{Id}_{\mathrm{MF}(A)}(x) = \{x\}$, and it is plain that $\{0\} \preceq \sigma(x) \preceq \{x\}$, $\forall x \in A$.

For $\sigma, \sigma' \in \mathrm{MF}(A)$, set

$$(\sigma \boxtimes \sigma')(x) := \sigma(x) \boxtimes \sigma'(x), \tag{4}$$

for any $x \in A$ and $\boxtimes \in \{\vee, \wedge, \cup, \cap\}$.

Remark 8.

1. *Note that $\sigma(x) \vee \sigma'(x)$ is meant in the sense of Definition 3, rather than the supremum of $\sigma(x)$ and $\sigma'(x)$.*
2. *Note that $\sigma \vee \sigma'$ is an upper bound of σ and σ' by Lemma 6 (1) but is not necessarily a least upper bound. For example, define $\sigma \in \mathrm{MF}(B_4)$ by $\sigma(a) = \sigma(b) = \{a, b\}$, $\sigma(0) = \{0\}, \sigma(1) = \{1\}$. Then,*

   $$(\sigma \vee \sigma)(a) = (\sigma \vee \sigma)(b) = \{a, b, 1\}.$$

   It is clear that both σ and $\sigma \vee \sigma$ are upper bounds of σ and σ, but $\sigma \prec \sigma \vee \sigma$. In a word, $\sigma \vee \sigma$ is not a least upper bound of σ and σ.
   More generally, let A be an MV*-algebra which is not an* MV*-chain with two incomparable elements a, b. Define $\sigma \in \mathrm{MF}(A)$ as $\sigma(a) = \sigma(b) = \{a, b\}$, $\sigma(x) = \{x\}$ for $x \in A \backslash \{a, b\}$. $\sigma \vee \sigma$ is not a least upper bound of σ and σ.*

In the sense of category theory, a preordered set P is called **complete** [27] (Section 8.5) if for every subset S of P both $\sup S$ and $\inf S$ exist (in P). Note that $\sup S$ and $\inf S$ need not be unique. For example, let $P = \{a, b\}$ and define a preorder $\preceq$ as follows: $a \preceq b$, $b \preceq a$. Take $S = \{a, b\}$. Then, both a and b are $\sup S$, also $\inf S$. Therefore, we use "a" rather than "the" concerning $\sup S$ and $\inf S$ in the following.

Let $\{\sigma_i\}_{i \in \Omega}$ be a nonempty family of multifunctions on an MV-algebra A. Define a multifunction $\bigcup_{i \in \Omega} \sigma_i$ on A, by

$$\left(\bigcup_{i \in \Omega} \sigma_i\right)(x) := \bigcup_{i \in \Omega} \sigma_i(x),$$

for any $x \in A$.

Analogue to [28] (Theorem I.4.2), we have the following.

Lemma 11. *If A is an* MV*-algebra, then* $(\mathrm{MF}(A), \preccurlyeq, \mathbf{0}_{\mathrm{MF}(A)}, \mathbf{1}_{\mathrm{MF}(A)})$ *is a complete bounded preordered set, where* $\bigcup_{i\in\Omega}\sigma_i$ *is a least upper bound of* $\{\sigma_i\}_{i\in\Omega}$*, and* $\sigma \wedge \sigma'$ *is a greatest lower bound of* σ *and* σ'*, respectively.*

Proof. Note that $\mathbf{0}_{\mathrm{MF}(A)} \preccurlyeq \sigma \preccurlyeq \mathbf{1}_{\mathrm{MF}(A)}$ for any $\sigma \in \mathrm{MF}(A)$.

Let $\{\sigma_i\}_{i\in\Omega}$ be a nonempty family of $\mathrm{MF}(A)$. Then, $\sigma_i \preccurlyeq \bigcup_{i\in\Omega}\sigma_i$. Now we will prove that $\bigcup_{i\in\Omega}\sigma_i$ is a least upper bound of $\{\sigma_i\}_{i\in\Omega}$. Assume that $\sigma_i \preccurlyeq \eta$ for every $i \in \Omega$. For any $y \in (\bigcup_{i\in\Omega}\sigma_i)(x)$ where $x \in A$, there exists $k \in \Omega$ such that $y \in \sigma_k(x)$. Since $\sigma_k(x) \preceq \eta(x)$, there is $z \in \eta(x)$ such that $y \leq z$, which shows $\bigcup_{i\in\Omega}\sigma_i \preccurlyeq \eta$. Therefore, $\bigcup_{i\in\Omega}\sigma_i$ is a least upper bound of $\{\sigma_i\}_{i\in\Omega}$.

Let
$$X^\ell = \{\lambda \in \mathrm{MF}(A) \mid \lambda \preccurlyeq \sigma_i, \forall i \in \Omega\}$$
be the set of lower bounds of $\{\sigma_i\}_{i\in\Omega}$ in $\mathrm{MF}(A)$. Next, we verify that $\bigcup_{\lambda\in X^\ell}\lambda$ is indeed a greatest lower bound of $\{\sigma_i\}_{i\in\Omega}$. For any $i \in \Omega$ and $\lambda \in X^\ell$, we have $\lambda \preccurlyeq \sigma_i$. Thus, $\bigcup_{\lambda\in X^\ell}\lambda \preccurlyeq \sigma_i$ and $\bigcup_{\lambda\in X^\ell}\lambda \in X^\ell$. Hence, $\bigcup_{\lambda\in X^\ell}\lambda$ is a greatest lower bound of $\{\sigma_i\}_{i\in\Omega}$. Therefore, $\mathrm{MF}(A)$ is complete.

For any $\sigma, \sigma' \in \mathrm{MF}(A)$, since $\sigma \wedge \sigma' \preccurlyeq \sigma, \sigma'$, it follows that $\sigma \wedge \sigma'$ is a lower bound of σ and σ'. To verify that $\sigma \wedge \sigma'$ is a greatest lower bound, let $\eta \preccurlyeq \sigma, \sigma'$. Then, for any $y \in \eta(x)$ $(x \in A)$, there are $z \in \sigma(x)$ and $w \in \sigma'(x)$ such that $y \leq z$ and $y \leq w$ by $\eta(x) \preceq \sigma(x), \sigma'(x)$. Hence,
$$y \leq z \wedge w \in \sigma(x) \wedge \sigma'(x).$$
Therefore, $\eta(x) \preceq \sigma(x) \wedge \sigma'(x)$. Thus, $\eta \preccurlyeq \sigma \wedge \sigma'$. □

As already mentioned, $\preceq$ is not always a partial order on $\Delta(A)$, where $M \preceq N$ iff for each $m \in M$ there exists $n \in N$ such that $m \leq n$. The binary relation $\sim$ on $\Delta(A)$ defined by $M \sim N$ iff $M \preceq N$ and $N \preceq M$ is an equivalence relation. Given $M \in \Delta(A)$, the equivalence class of M with respect to $\sim$ will be denoted by $\overline{M}$. If $M = \{x\}$ is a singleton, then we abbreviate $\overline{\{x\}}$ by $\overline{x}$. Thus, we can obtain a partial order $\preceq$ on $\Delta(A)/\!\sim$ defined by $\overline{M} \preceq \overline{N}$ iff $M \preceq N$. We claim that $\preceq$ is well defined. In fact, if $M \sim M', N \sim N'$ and $M \preceq N$, then $M' \preceq M \preceq N \preceq N'$.

Recall that for a subset M of A, the **lower set generated by *M*** [29] is the set
$$\downarrow M = \{x \in A \mid \text{there exists } m \in M \text{ such that } x \leq m\}.$$

Lemma 12. *Let* $M, N \in \Delta(A)$*. Then,* $\overline{M} = \overline{N}$ *iff* $\downarrow M = \downarrow N$*.*

Proof. It is sufficient to show that $M \preceq N$ iff $\downarrow M \subseteq \downarrow N$.

Let $M \preceq N$. For every $x \in \downarrow M$, there is $m \in M$ such that $x \leq m$. Then, $M \preceq N$ gives $m \leq n$ for some $n \in N$. Hence, $x \leq n$ and $x \in \downarrow N$. Therefore, $\downarrow M \subseteq \downarrow N$.

Conversely, assume that $\downarrow M \subseteq \downarrow N$. For any $m \in M$, we have $m \in \downarrow M \subseteq \downarrow N$. Thus, there exists $n \in N$ such that $m \leq n$. Hence, $M \preceq N$.

Similarly, $N \preceq M$ iff $\downarrow N \subseteq \downarrow M$. □

Corollary 3. *In general, let A be an* MV*-algebra,* $M \in \Delta(A)$*, and* $a \in A$*. Then,* $\overline{M} = \overline{a}$ *iff* $\sup M$ *exists and* $\sup M = a \in M$*.*

Assume $\overline{M} = \overline{a}$*. Then a is an upper bound of M since* $M \preceq \{a\}$*. To prove a is a least upper bound of M, let b be an upper bound of M. Since* $\{a\} \preceq M$*, there exists* $m \in M$ *such that* $a \leq m$*. Hence,* $a \leq m \leq b$*, which shows* $\sup M = a \in M$*.*

Conversely, let $\sup M = a \in M$*. It suffices to verify that* $\downarrow M = \downarrow a$ *by Lemma 12. If* $x \in \downarrow M$*, then there is* $m \in M$ *such that* $x \leq m \leq a$*. It follows that* $x \in \downarrow a$ *and* $\downarrow M \subseteq \downarrow a$*. If* $x \in \downarrow a$*, then* $x \leq a \in M$*. Thus,* $x \in \downarrow M$ *and* $\downarrow a \subseteq \downarrow M$*. Therefore,* $\downarrow M = \downarrow a$*.*

Corollary 4. *Let* L_n *with* $n \geq 2$ *and* $M \in \Delta(L_n)$*. Then,* $\overline{M} = \overline{\sup M}$*.*

Proof. Observe that $\sup M$ is exactly $\frac{i}{n-1}$ for a certain $0 \le i \le n-1$. It suffices to verify that $\downarrow M = \downarrow \sup M$ by Lemma 12. Suppose $x \in \downarrow M$, there is $m \in M$ such that $x \le m$. Since $m \le \sup M$, it follows that $x \le \sup M$. Hence, $x \in \downarrow \sup M$. Conversely, assume $x \in \downarrow \sup M$, which means $x \le \sup M = \frac{i}{n-1}$. Since $\sup M \in M$, it follows that $x \in \downarrow M$. Therefore, $\downarrow M = \downarrow \sup M$ and $\overline{M} = \overline{\sup M}$. □

Note that the family of all lower sets of a poset A is a complete lattice by [30] (Example O-2.8). We will prove that the family of all nonempty lower sets of A is also a complete lattice, denoted by $(L_0(A), \subseteq)$.

Corollary 5. *Let A be an* MV*-algebra, then $\Delta(A)/\sim$ is isomorphic to the complete lattice $(L_0(A), \subseteq)$.*

Proof. Since A has a least element 0, the intersection of a family of nonempty lower sets of A is still a nonempty lower set. Therefore, $L_0(A)$ is a complete lattice.

Define $\varphi : \Delta(A)/\sim \to L_0(A)$ by $\overline{M} \mapsto \downarrow M$. Lemma 12 shows that φ is well defined and injective, and φ is also surjective since $M = \downarrow M$ if $M \in L_0(A)$. As discussed in the proof of Lemma 12, $\overline{M} \preceq \overline{N}$ iff $\downarrow M \subseteq \downarrow N$ for all $M, N \in \Delta(A)$, which gives both φ and φ^{-1} are order preserving. Hence, φ is an isomorphism. □

Next, we study the order structure on $\Delta(L_n)/\sim$. First, we need

Lemma 13. *Let A be an* MV*-chain, $M, N \in \Delta(A)$, and $\sup M, \sup N$ exist.*

1. *If $\overline{M} \preceq \overline{N}$, then $\sup M \le \sup N$.*
2. *If $\sup M < \sup N$, then $\overline{M} \preceq \overline{N}$.*
3. *$\overline{M} = \overline{N}$ iff the following conditions hold:*
   - *(a)* $\sup M = \sup N$.
   - *(b)* $\sup M \in M \Leftrightarrow \sup N \in N$.

   In particular, if A is a finite MV*-chain, then $\overline{M} = \overline{N}$ iff (a) holds.*

Proof. (1) Suppose $\overline{M} \preceq \overline{N}$, then $M \preceq N$. For any $m \in M$ there is $n \in N$ such that $m \le n \le \sup N$. According to the definition of $\sup M$, we have $\sup M \le \sup N$.

(2) Let $\sup M < \sup N$. Assume on the contrary $M \npreceq N$. Then, there is $m \in M$ such that $m > n$ for any $n \in N$. The definition of $\sup N$ implies $m \ge \sup N$. Thus, $\sup N \le m \le \sup M$, which contradicts the fact that $\sup M < \sup N$.

(3) Assume that $\overline{M} = \overline{N}$. (a) follows from (1).

To prove that $\sup M \in M \Leftrightarrow \sup N \in N$, we assume $\sup M \in M$. Then, there exists $n_0 \in N$ such that $\sup M \le n_0$ by $M \preceq N$. Since $N \preceq M$, we have $n_0 \le \sup M$. Hence, $n_0 = \sup M$. Therefore, $\sup N = \sup M = n_0 \in N$ by (a). Symmetrically, $\sup N \in N \Rightarrow \sup M \in M$.

Conversely, assume that (a) and (b) hold, it suffices to show that $\downarrow M = \downarrow N$ by Lemma 12. Assume that $\downarrow M \ne \downarrow N$; without loss of generality, there is $y \in \downarrow M$ but $y \notin \downarrow N$. That is to say, for arbitrary $n \in N$ we have $n < y$. So, $\sup N \in N$ implies $\sup N < y$. Since $y \in \downarrow M$, there is $m \in M$ such that $y \le m$. It follows $\sup N < y \le m < \sup M$ by the definition of $\sup N$, which is contrary to $\sup M = \sup N$. Thus, $\overline{M} = \overline{N}$.

Assume A is a finite MV-chain, and (b) always holds. Hence, $\overline{M} = \overline{N}$ iff (a) holds. □

Remark 9. *Note that $\sup M = \sup N$ may not imply $\overline{M} \preceq \overline{N}$. For example, let $A = [0,1]$ be the standard* MV*-algebra and $\frac{1}{2} \in A$. Define $M = \downarrow \frac{1}{2}$ and $N = \{a \in A \mid 0 \le a < \frac{1}{2}\}$. Then, $\sup M = \sup N = \frac{1}{2}$, but $\overline{M} \npreceq \overline{N}$, since $\frac{1}{2} \in M$, there is no $y \in N$ such that $\frac{1}{2} \le y$.*

Example 4. *Consider the MV-chain L_n with $n \ge 2$. Then, $\Delta(L_n)/\sim$ is order isomorphic to L_n.*

Proof. Define $f : L_n \to \Delta(L_n)/\sim$ by $f(x) = \overline{x}$ for any $x \in L_n$. If $\overline{x} = \overline{y}$, then $x = \sup\{x\} = \sup\{y\} = y$ by Lemma 13 (3). Thus, f is injective. To prove f is surjective, assume $\overline{M} \in \Delta(L_n)/\sim$, then $f(\sup M) = \overline{\sup M} = \overline{M}$ by Corollary 4.

It is enough to verify that f and f^{-1} are order preserving. If $x \leq y$, then $f(x) = \overline{x} \preceq \overline{y} = f(y)$ since $\{x\} \preceq \{y\}$ and Corollary 4. Conversely, suppose $\overline{x} \preceq \overline{y}$, we have $x = \sup\{x\} \leq \sup\{y\} = y$ by Lemma 13 (1). Therefore, f is an isomorphism. $\square$

We next investigate the preorder on the set of $(\odot, \vee)$-multiderivations.

Similar to $\Delta(A)$, we can define an equivalence relation on $\mathrm{MD}(A)$ by $\sigma \sim \sigma'$ iff $\sigma \preccurlyeq \sigma'$ and $\sigma' \preccurlyeq \sigma$, and define $\overline{\sigma} \preccurlyeq \overline{\sigma'}$ in $\mathrm{MD}(A)/\sim$ iff $\sigma \preccurlyeq \sigma'$. Observe that $\preccurlyeq$ in $\mathrm{MD}(A)/\sim$ is a well-defined partial order by the hereditary order of $\preceq$. Clearly, $(\mathrm{MD}(A)/\sim, \preccurlyeq)$ is a poset. By the definition of $\preceq$, we know $\overline{\sigma} = \overline{\sigma'}$ iff $\overline{\sigma(x)} = \overline{\sigma'(x)}$ for any $x \in A$.

For any $\sigma \in \mathrm{MD}(A)$, $\downarrow\sigma\colon A \to \Delta(A)$ is defined as $(\downarrow\sigma)(x) = \downarrow\sigma(x)$. We claim that $\overline{\sigma} = \overline{\downarrow\sigma}$. In fact, $\sigma \preccurlyeq \downarrow\sigma$ is trivial. For any $y \in \downarrow\sigma(x)$, there exists $z \in \sigma(x)$ such that $y \leq z$ by the definition of $\downarrow\sigma(x)$. Therefore, $\downarrow\sigma(x) \preceq \sigma(x)$ for any $x \in A$ and $\downarrow\sigma \preccurlyeq \sigma$.

Lemma 14. *If A is an* MV*-algebra, then:*

1. $\sigma \vee \sigma' \in \mathrm{MD}(A)$ *for all* $\sigma, \sigma' \in \mathrm{MD}(A)$.
2. $\downarrow\sigma \in \mathrm{MD}(A)$ *for any* $\sigma \in \mathrm{MD}(A)$.

Proof. (1) Let $\sigma, \sigma' \in \mathrm{MD}(A)$ and $x, y \in A$. Then, we have

$$\begin{aligned}
(\sigma \vee \sigma')(x \odot y) &= \sigma(x \odot y) \vee \sigma'(x \odot y) && \text{(Definition of } \sigma \vee \sigma'\text{)}\\
&= ((\sigma(x) \odot y) \vee (x \odot \sigma(y))) \vee ((\sigma'(x) \odot y) \vee (x \odot \sigma'(y))) && (\sigma, \sigma' \in \mathrm{MD}(A))\\
&= ((\sigma(x) \odot y) \vee (\sigma'(x) \odot y)) \vee ((x \odot \sigma(y)) \vee (x \odot \sigma'(y))) && \text{(Lemma 6 (4) and (5))}\\
&= ((\sigma(x) \vee \sigma'(x)) \odot y) \vee (x \odot (\sigma(y) \vee \sigma'(y))) && \text{(Lemma 7 (2))}\\
&= ((\sigma \vee \sigma')(x) \odot y) \vee (x \odot (\sigma \vee \sigma')(y)) && \text{(Definition of } \sigma \vee \sigma'\text{)}
\end{aligned}$$

and so $\sigma \vee \sigma' \in \mathrm{MD}(A)$.

(2) Assume $\sigma \in \mathrm{MD}(A)$. Let $a \in (\downarrow\sigma)(x \odot y) = \downarrow\sigma(x \odot y) = \downarrow((\sigma(x) \odot y) \vee (x \odot \sigma(y)))$. There exist $x_1 \in \sigma(x)$ and $y_1 \in \sigma(y)$ such that $a \leq (x_1 \odot y) \vee (x \odot y_1)$. It follows that

$$\begin{aligned}
a &= a \wedge ((x_1 \odot y) \vee (x \odot y_1))\\
&= (a \wedge (x_1 \odot y)) \vee (a \wedge (x \odot y_1)) && \text{(Distributivity of } A\text{)}\\
&= (b \odot y) \vee (x \odot c), && \text{(Lemma 5 (1))}
\end{aligned}$$

where $b \leq x_1$ and $c \leq y_1$. Hence, $a \in ((\downarrow\sigma)(x) \odot y) \vee (x \odot (\downarrow\sigma)(y))$.

Conversely, let $a \in ((\downarrow\sigma)(x) \odot y) \vee (x \odot (\downarrow\sigma)(y))$. There exist $x_1 \in \sigma(x)$ and $y_1 \in \sigma(y)$ such that

$$a = (b \odot y) \vee (x \odot c) \leq (x_1 \odot y) \vee (x \odot y_1),$$

where $b \leq x_1$ and $c \leq y_1$. Thus, $a \in (\downarrow\sigma)(x \odot y)$.

Therefore, $\downarrow\sigma \in \mathrm{MD}(A)$. $\square$

Remark 10. *When A is an* MV*-chain, $\sigma \vee \sigma' \in \mathrm{MD}(A)$ is a least upper bound of σ and σ' in $\mathrm{MD}(A)$. We know $\sigma \cup \sigma'$ is a least upper bound of σ and σ' in $\mathrm{MF}(A)$. Note that $\mathrm{MD}(A) \subseteq \mathrm{MF}(A)$ and the preordered on $\mathrm{MF}(A)$. It suffices to verify that $\sigma \vee \sigma' \sim \sigma \cup \sigma'$. For all $x \in A$, $(\sigma \cup \sigma')(x) \preceq (\sigma \vee \sigma')(x)$ is trivial. For any $y \in (\sigma \vee \sigma')(x)$, there exist $z \in \sigma(x)$ and $z' \in \sigma'(x)$ such that $y = z \vee z'$. Since A is an* MV*-chain, $y = z$ or $y = z'$. Hence, $y \in (\sigma \cup \sigma')(x)$, which implies $(\sigma \vee \sigma')(x) \preceq (\sigma \cup \sigma')(x)$. Therefore, $(\sigma \cup \sigma')(x) \sim (\sigma \vee \sigma')(x)$ for all $x \in A$, and hence, $\sigma \vee \sigma' \in \mathrm{MD}(A)$ is a least upper bound of σ and σ' in $\mathrm{MD}(A)$.*

At the end of this section, we characterize the lattice $\mathrm{MD}(L_n)/\sim$ $(n \geq 2)$.

Theorem 1. *If L_n is the n-element* MV*-chain with $n \geq 2$, then the lattices* $\mathrm{MD}(L_n)/\sim$ *and* $\mathrm{Der}(L_n)$ *are isomorphic.*

Proof. Define a map $f : \mathrm{MD}(L_n)/\sim \to \mathrm{Der}(L_n)$ by

$$f(\overline{\sigma}) = \sup \sigma.$$

By Proposition 7 we know $\sup \sigma \in \mathrm{Der}(L_n)$. The order $\leqq$ on $\mathrm{Der}(L_n)$ is defined as $d \leqq d'$ iff $d(x) \leq d'(x), \forall x \in L_n$.

Firstly, we prove that f is well defined. Suppose $\overline{\sigma} = \overline{\sigma'}$, that is, $\overline{\sigma(x)} = \overline{\sigma'(x)}$ for any $x \in L_n$. We get

$$(\sup \sigma)(x) = \sup(\sigma(x)) = \sup(\sigma'(x)) = (\sup \sigma')(x)$$

for any $x \in L_n$ by Lemma 13 (3). Thus, $f(\overline{\sigma}) = \sup(\sigma) = \sup(\sigma') = f(\overline{\sigma'})$.

If $f(\overline{\sigma}) = f(\overline{\sigma'})$, that is, $\sup(\sigma) = \sup(\sigma')$, then $\sup(\sigma(x)) = \sup(\sigma'(x))$ for any $x \in L_n$. Lemma 13 (3) implies $\overline{\sigma(x)} = \overline{\sigma'(x)}$ for any $x \in L_n$ and thus $\overline{\sigma} = \overline{\sigma'}$. Hence, f is injective. For any $d \in \mathrm{Der}(L_n)$, there is $\gamma_d \in \mathrm{MD}(L_n)$ where $\gamma_d(x) := [0, d(x)]$ such that

$$f(\overline{\gamma_d})(x) = (\sup \gamma_d)(x) = \sup(\gamma_d(x)) = \sup[0, d(x)] = d(x)$$

for all $x \in L_n$ by Propositions 6 and 7. Thus, $f(\overline{\gamma_d}) = d$ and f is surjective.

To prove that f is an order-isomorphism, let $\overline{\sigma} \preccurlyeq \overline{\sigma'}$, that is, for any $x \in L_n$, $\overline{\sigma(x)} \preceq \overline{\sigma'(x)}$. Corollary 4 implies that $\overline{\sigma(x)} = \overline{\sup(\sigma(x))}$ for any $x \in L_n$. It follows that

$$\overline{(\sup \sigma)(x)} = \overline{\sup(\sigma(x))} \preceq \overline{\sup(\sigma'(x))} = \overline{(\sup \sigma')(x)}$$

and thus $(\sup \sigma)(x) \leq (\sup \sigma')(x)$ for any $x \in L_n$ since $(\sup \sigma)(x)$ is a singleton. Hence, $f(\overline{\sigma}) = \sup \sigma \leqq \sup \sigma' = f(\overline{\sigma'})$. Conversely, assume $d, d' \in \mathrm{Der}(L_n)$ and $d \leqq d'$, which means $d(x) \leq d'(x)$ for all $x \in L_n$. Now the construction in Proposition 6 gives $\gamma_d = f^{-1} : A \to \Delta(A)$, where $\gamma_d(x) = [0, d(x)]$. Furthermore, we have

$$\gamma_d(x) = [0, d(x)] \preceq [0, d'(x)] = \gamma_{d'}(x)$$

for any $x \in L_n$ by the definition of $\preceq$. Thus, $\gamma_d \preccurlyeq \gamma_{d'}$ and $f^{-1}(d) = \overline{\gamma_d} \preccurlyeq \overline{\gamma_{d'}} = f^{-1}(d')$. $\square$

Proposition 11. *If L_n is the n-element* MV*-chain with $n \geq 2$, then the lattices $\Delta(L_n \times L_2)/\sim$ and* $\mathrm{Der}(L_{n+1})$ *are isomorphic.*

Proof. Recall that $\mathrm{Der}(L_{n+1})$ is isomorphic to the lattice $(\mathcal{A}(L_{n+1}), \leqq)$ where $\mathcal{A}(L_{n+1}) = \{(x, y) \in L_{n+1} \times L_{n+1} \mid y \leq x\} \backslash \{(0, 0)\}$ [16, Theorem 5.6] and $\leqq$ is defined by: for any $(x_1, y_1), (x_2, y_2) \in L_{n+1} \times L_{n+1}$, $(x_1, y_1) \leqq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$. Moreover, $\Delta(L_n \times L_2)/\sim$ is isomorphic to the lattice $L_0(L_n \times L_2)$ by Corollary 5.

Define a map $f : \mathcal{A}(L_{n+1}) \to L_0(L_n \times L_2)$ by:

$$f\left(\tfrac{k}{n}, \tfrac{\ell}{n}\right) = \begin{cases} \downarrow(\frac{k-1}{n-1}, 0), & \text{if } \ell = 0; \\ \downarrow(\frac{k-1}{n-1}, 0) \cup \downarrow(\frac{\ell-1}{n-1}, 1), & \text{if } \ell > 0, \end{cases}$$

where $0 \leq k, \ell \leq n-1$. It is easy to see that f is injective. Now we show that f is surjective. For any $M \in L_0(L_n \times L_2)$, we claim M has at most two maximal elements. By way of contradiction, assume M has three different maximal elements denoted by (a_n, b_n), $n = 1, 2, 3$; then, there exist $1 \leq i < j \leq 3$ such that $b_i = b_j$ since $b_n \in L_2$. Thus, (a_i, b_i) and (a_j, b_j) are comparable, which contradicts the fact that (a_i, b_i) and (a_j, b_j) are different maximal elements. If M has only one maximal element denoted by $(\frac{k}{n-1}, a)$, then

$$M = \downarrow(\tfrac{k}{n-1}, a) = \begin{cases} f\left(\frac{k+1}{n}, 0\right), & \text{if } a = 0; \\ f\left(\frac{k+1}{n}, \frac{k+1}{n}\right), & \text{if } a = 1. \end{cases}$$

If M has exactly two maximal elements denoted by $(\frac{k}{n-1}, 0)$ and $(\frac{\ell}{n-1}, 1)$, then

$$M = \downarrow(\tfrac{k}{n-1}, 0) \cup \downarrow(\tfrac{\ell}{n-1}, 1) = f\left(\tfrac{k+1}{n}, \tfrac{\ell+1}{n}\right).$$

Therefore, f is surjective.

Since a bijection with supremum preserving is an order isomorphism, it suffices to verify that f preserves the supremum, that is,

$$f\left((\tfrac{k}{n}, \tfrac{\ell}{n}) \vee (\tfrac{p}{n}, \tfrac{q}{n})\right) = f\left(\tfrac{k}{n}, \tfrac{\ell}{n}\right) \cup f(\tfrac{p}{n}, \tfrac{q}{n})$$

for all $(\frac{k}{n}, \frac{\ell}{n}), (\frac{p}{n}, \frac{q}{n}) \in \mathcal{A}(L_{n+1})$.

Case 1. If $\ell = q = 0$, then

$$\begin{aligned} f\left(\tfrac{k}{n}, 0\right) \cup f(\tfrac{p}{n}, 0) &= \downarrow(\tfrac{k-1}{n-1}, 0) \cup \downarrow(\tfrac{p-1}{n-1}, 0) \\ &= \downarrow(\max\{\tfrac{k-1}{n-1}, \tfrac{p-1}{n-1}\}, 0) \\ &= \downarrow(\tfrac{\max\{k,p\}-1}{n-1}, 0) \\ &= f\left((\tfrac{k}{n}, 0) \vee (\tfrac{p}{n}, 0)\right). \end{aligned}$$

Case 2. If $\ell = 0, q > 0$, then

$$\begin{aligned} f\left(\tfrac{k}{n}, 0\right) \cup f(\tfrac{p}{n}, \tfrac{q}{n}) &= \downarrow(\tfrac{k-1}{n-1}, 0) \cup \left(\downarrow(\tfrac{p-1}{n-1}, 0) \cup \downarrow(\tfrac{q-1}{n-1}, 1)\right) \\ &= \downarrow(\tfrac{\max\{k,p\}-1}{n-1}, 0) \cup \downarrow(\tfrac{q-1}{n-1}, 1) \\ &= f\left((\tfrac{k}{n}, 0) \vee (\tfrac{p}{n}, \tfrac{q}{n})\right). \end{aligned}$$

The case $\ell > 0, q = 0$ is similar.

Case 3. If $\ell > 0, q > 0$, then

$$\begin{aligned} f\left(\tfrac{k}{n}, \tfrac{\ell}{n}\right) \cup f(\tfrac{p}{n}, \tfrac{q}{n}) &= \left(\downarrow(\tfrac{k-1}{n-1}, 0) \cup \downarrow(\tfrac{\ell-1}{n-1}, 1)\right) \cup \left(\downarrow(\tfrac{p-1}{n-1}, 0) \cup \downarrow(\tfrac{q-1}{n-1}, 1)\right) \\ &= \downarrow(\tfrac{\max\{k,p\}-1}{n-1}, 0) \cup \downarrow(\tfrac{\max\{\ell,q\}-1}{n-1}, 1) \\ &= f\left(\tfrac{\max\{k,p\}}{n}, \tfrac{\max\{\ell,q\}}{n}\right) \\ &= f\left((\tfrac{k}{n}, \tfrac{\ell}{n}) \vee (\tfrac{p}{n}, \tfrac{q}{n})\right). \end{aligned}$$

Now we verify that f is an isomorphism of posets and hence an isomorphism of lattices. For all $x, y \in \mathcal{A}(L_{n+1})$,

$$x \leqq y \Leftrightarrow x \vee y = y \Leftrightarrow f(x) \cup f(y) = f(x \vee y) = f(y) \Leftrightarrow f(x) \subseteq f(y).$$

Hence, f is an isomorphism of lattices.

Therefore, $\mathcal{A}(L_{n+1}) \cong L_0(L_n \times L_2)$ and then $\Delta(L_n \times L_2)/\sim\, \cong \mathrm{Der}(L_{n+1})$. $\square$

Corollary 6. *If L_n is the n-element* MV*-chain with $n \geq 2$, then* $\mathrm{MD}(L_{n+1})/\sim$ *is isomorphic to the lattice* $\Delta(L_n \times L_2)/\sim$.

Proof. It follows from Theorem 1 and Proposition 11. $\square$

Note that according to the isomorphism in Theorem 1, $|\,\mathrm{MD}(L_n)/\sim| = |\,\mathrm{Der}(L_n)\,| = \frac{(n-1)(n+2)}{2}$ by [16] (Theorem 3.11). However, the following Example 5 shows that the cardinalities of different equivalence classes with respect to the equivalence relation $\sim$ are different in general.

Example 5. *Let $n = 2$ and define $\delta \in \mathrm{MF}(L_2)$ by $\delta(0) = \{0\}, \delta(1) = \{0, 1\}$. Then, it is easy to check that*

$$\mathrm{MD}(L_2) = \{\mathbf{0}_{\mathrm{MF}(L_2)}, \mathrm{Id}_{\mathrm{MF}(L_2)}, \delta\},$$
$$\mathrm{MD}(L_2)/\sim = \{\{\mathbf{0}_{\mathrm{MF}(L_2)}\}, \{\mathrm{Id}_{\mathrm{MF}(L_2)}, \delta\}\}.$$

It is clear that $|\,\overline{\mathbf{0}_{\mathrm{MF}(L_2)}}\,| = 1$ but $|\,\overline{\mathrm{Id}_{\mathrm{MF}(L_2)}}\,| = 2$. Hence, $2 = |\,\mathrm{MD}(L_2)/\sim| \nmid |\,\mathrm{MD}(L_2)\,| = 3$.

So, the cardinality of $\mathrm{MD}(L_n)$ is not easy to deduce from Theorem 1. In the next section, we will investigate the enumeration of the set of $(\odot, \vee)$-multiderivations on L_n by constructing a counting principle (Theorem 3).

5. The Enumeration of $(\odot, \vee)$-Multiderivations on a Finite MV-Chain

In this section, we determine the cardinality of $\mathrm{MD}(L_n)$. For small values of n, this can be performed with calculations using Python (see the Appendix A Figure A1) in Table 1:

Table 1. The cardinality of $\mathrm{MD}(L_n)$.

n	2	3	4	5	6
$\lvert\mathrm{MD}(L_n)\rvert$	3	16	63	220	723

The result cannot be obtained after $n \geq 6$ due to the limitation of computing resources. But we have shown the following general formula.

Theorem 2. *Let $n \geq 2$ be a positive integer. Then, $|\,\mathrm{MD}(L_n)| = \dfrac{7 \cdot 3^{n-1} - 2^{n+2} + 1}{2}$.*

In order to prove Theorem 2, we need the following Lemmas.

Lemma 15. *Assume that A is an MV-chain and $\sigma \in \mathrm{MD}(A)$; then, the following results hold:*

1. *If $M \subseteq A$, then $M = M \vee M$.*
2. *For any $x \in A, n \in \mathbb{N}_+$, we have $\sigma(x^n) = x^{n-1} \odot \sigma(x)$, where $x^0 = 1, x^n = \underbrace{x \odot x \odot \cdots \odot x}_{n}$.*

Proof. (1) It follows immediately from Lemma 6 (3), as M is a sublattice.

(2) We prove $\sigma(x^n) = x^{n-1} \odot \sigma(x)$ by induction on n. Obviously, $\sigma(x^1) = \sigma(x) = 1 \odot \sigma(x) = x^{1-1} \odot \sigma(x)$.

Now, assume that $\sigma(x^n) = x^{n-1} \odot \sigma(x)$. By Equation (3), we have

$$\begin{aligned}
\sigma(x^{n+1}) &= \sigma(x^n \odot x) \\
&= (\sigma(x^n) \odot x) \vee (x^n \odot \sigma(x)) \\
&= (x^{n-1} \odot \sigma(x) \odot x) \vee (x^n \odot \sigma(x)) \\
&= x^n \odot \sigma(x),
\end{aligned}$$

so (2) holds. $\square$

Note that an MV-chain can be completely characterized by (1). That is, if A is an MV-algebra, then A is an MV-chain iff $M = M \vee M$ for every $M \subseteq A$. In fact, by way of contraposition, assume that $x, y \in A$ and x, y are incomparable, denote $z = x \vee y$. Let $M = \{x, y\}$. Then, $z = x \vee y \in M \vee M$ but $z \notin M$. This leads to a contradiction.

Let $n \in \mathbb{N}_+$ and $n \geq 2$. In L_n, we know $\frac{n-m-1}{n-1} = (\frac{n-2}{n-1})^m$ for every $m \in \{1, 2, \cdots, n-1\}$. So, any $x \in L_n \backslash \{1\}$ has a representation as a power of $\frac{n-2}{n-1}$.

Next, we give a counting principle for $(\odot, \vee)$-multiderivations on a finite MV-chain L_n.

Theorem 3. *Let σ be a multifunction on L_n and $v = \frac{n-2}{n-1}$. Then, $\sigma \in \mathrm{MD}(L_n)$ iff σ satisfies the following conditions:*

1. $\sigma(v^m) = v^{m-1} \odot \sigma(v), \forall m \in \{1, 2, \cdots, n-1\}$.
2. $\sigma(v) = \sigma(v) \vee (v \odot \sigma(1))$.
3. $\sigma(v) \preceq \{v\}$.

Proof. Assume $\sigma \in \mathrm{MD}(L_n)$; then, for each $m \in \{1, 2, \cdots, n-1\}$, we have $\sigma(v^m) = v^{m-1} \odot \sigma(v)$ by Lemma 15 (2), and $\sigma(v) = \sigma(v \odot 1) = \sigma(v) \vee (v \odot \sigma(1))$ by Equation (3). Thus, σ satisfies (1) and (2). Furthermore, (3) holds by Proposition 4 (2).

Conversely, suppose that σ satisfies (1), (2) and (3). Let $x, y \in L_n$. There are four cases:

If $x = y = 1$, then it is easy to see that $\sigma(1 \odot 1) = \sigma(1) = \sigma(1) \vee \sigma(1)$ by Lemma 15 (1).

If $x = 1$ or $y = 1$, and $x \neq y$. With out loss of generality, suppose that $x \neq 1$ and $y = 1$, then $x = v^k$ for some $k \in \{1, 2, \cdots, n-1\}$. By (1), we have $\sigma(x \odot 1) = \sigma(x) = \sigma(v^k) = v^{k-1} \odot \sigma(v)$. Also, we have

$$
\begin{aligned}
\sigma(x) \vee (x \odot \sigma(1)) &= (v^{k-1} \odot \sigma(v)) \vee (v^k \odot \sigma(1)) \\
&= (v^{k-1} \odot \sigma(v)) \vee (v^{k-1} \odot (v \odot \sigma(1))) \\
&= v^{k-1} \odot (\sigma(v) \vee (v \odot \sigma(1)) && \text{(Lemma 7 (2))} \\
&= v^{k-1} \odot \sigma(v). && \text{((2) of Theorem 3)}
\end{aligned}
$$

Hence, $\sigma(x \odot 1) = \sigma(x) = (\sigma(x) \odot 1) \vee (x \odot \sigma(1))$.

If $x \neq 1$ and $y \neq 1$, then assume that $x = v^k$ and $y = v^\ell$ for some $k, \ell \in \{1, 2, \cdots, n-1\}$. We have

$$\sigma(x \odot y) = \sigma(v^k \odot v^\ell) = \sigma(v^{k+\ell})$$

and

$$(\sigma(x) \odot y) \vee (x \odot \sigma(y)) = ((v^{k-1} \odot \sigma(v)) \odot v^\ell) \vee (v^k \odot (v^{\ell-1} \odot \sigma(v))) = v^{k+\ell-1} \odot \sigma(v)$$

by Lemma 15 (1). Then, there are three cases:

For $k + \ell < n - 1$, by (1) we obtain $\sigma(v^{k+\ell}) = v^{k+\ell-1} \odot \sigma(v)$.

For $k + \ell = n - 1$, by (3) we have $\sigma(v^{k+\ell}) = \sigma(v^{n-1}) = \sigma(0) \preceq \{0\}$ and so $\sigma(0) = \{0\}$. And $v^{k+\ell-1} \odot \sigma(v) = v^{n-2} \odot \sigma(v) = v^* \odot \sigma(v) = \{0\}$. Thus, $\sigma(x \odot y) = (\sigma(x) \odot y) \vee (x \odot \sigma(y))$.

For $n - 1 < k + \ell \leq 2n - 2$, we have $\sigma(v^{k+\ell}) = \sigma(0) = \{0\} = 0 \odot \sigma(v) = v^{k+\ell-1} \odot \sigma(v)$ by (3) and thus Equation (3) holds.

Therefore, we conclude that $\sigma \in \mathrm{MD}(L_n)$. $\square$

Lemma 16. *Let $P, Q \in \Delta(L_n)$. Then, the following results hold:*

1. $P \subseteq P \vee Q$ *iff* $\min Q \leq \min P$.
2. $P \vee Q \subseteq P$ *iff* $[\min P, 1] \cap Q \subseteq P$.

Proof. Denote $p_0 = \min P$, $q_0 = \min Q$.

(1) Assume $P \subseteq P \vee Q$, then there exist $p \in P, q \in Q$ such that $p_0 = p \vee q \geq q$. Thus, $q_0 \leq q \leq p_0$.

Conversely, suppose $q_0 \leq p_0$, then $p = p \vee q_0$ for any $p \in P$ since $p_0 \leq p$. Hence, $P \subseteq P \vee Q$.

(2) Assume $P \vee Q \subseteq P$; then, for all $q \in [p_0, 1] \cap Q$, we have $q = p_0 \vee q \in P \vee Q \subseteq P$. Thus, $[p_0, 1] \cap Q \subseteq P$.

Conversely, assume $[p_0, 1] \cap Q \subseteq P$ and $p \in P$, $q \in Q$. If $q \le p$, then $p \vee q = p \in P$. If $q > p$, then $p \vee q = q \in [p_0, 1] \cap Q \subseteq P$. In either case, $p \vee q \in P$ and so $P \vee Q \subseteq P$. $\square$

Lemma 17. *Let $Q, Q' \in \Delta(L_n)$ and $1 \notin Q$. Denote $v = \frac{n-2}{n-1}$. Then, the following results hold:*

1. *If $0 \notin Q$, then $Q = v \odot Q'$ iff $Q' = Q \oplus v^*$.*
2. *If $0 \in Q$, denote $Q_1 = Q \setminus \{0\}$. Then, $Q = v \odot Q'$ iff $Q' = \{0\} \sqcup (Q_1 \oplus v^*)$, $\{v^*\} \sqcup (Q_1 \oplus v^*)$ or $\{0, v^*\} \sqcup (Q_1 \oplus v^*)$.*

Proof. (1) Let $0 \notin Q$ and $Q = v \odot Q'$. Then, $0 \notin Q'$, otherwise, $0 = v \odot 0 \in v \odot Q' = Q$, a contradiction. Thus, $0 \notin Q'$, which implies $\{v^*\} \preceq Q'$. Hence, we have

$$\begin{aligned} Q' = Q' \vee v^* &= \{q' \vee v^* \mid q' \in Q'\} \\ &= \{(q' \odot v) \oplus v^* \mid q' \in Q'\} \\ &= (Q' \odot v) \oplus v^* \\ &= Q \oplus v^*. \end{aligned}$$

Conversely, assume $Q' = Q \oplus v^*$. Since $1 \notin Q$, we have $Q \preceq \{v\}$. Hence,

$$\begin{aligned} Q = Q \wedge v &= \{q \wedge v \mid n \in Q\} \\ &= \{v \odot (q \oplus v^*) \mid n \in Q\} \\ &= v \odot (Q \oplus v^*) \\ &= v \odot Q'. \end{aligned}$$

(2) Assume $0 \in Q$ and $Q = v \odot Q'$; then, $0 = v \odot q'$ for some $q' \in Q'$. Thus, $0 \in Q'$ or $v^* \in Q'$. Denote $Q'_0 = \{0, v^*\} \cap Q'$ and $Q'_1 = Q' \setminus Q'_0$. By $v \odot Q'_0 = \{0\}$ and $\{v^*\} \preceq v \odot Q'_1$, we have

$$\begin{aligned} Q_1 = Q \setminus \{0\} &= (v \odot Q') \setminus \{0\} \\ &= (v \odot (Q'_0 \cup Q'_1)) \setminus \{0\} \\ &= ((v \odot Q'_0) \cup (v \odot Q'_1)) \setminus \{0\} \quad \text{(Lemma 7 (3))} \\ &= (\{0\} \cup (v \odot Q'_1)) \setminus \{0\} \\ &= v \odot Q'_1. \end{aligned}$$

Since $0 \notin Q_1$, we obtain $Q'_1 = Q_1 \oplus v^*$ by (1). Therefore,

$$Q' = Q'_0 \sqcup Q'_1 = Q'_0 \sqcup (Q_1 \oplus v^*),$$

where $Q'_0 = \{0\}$, $\{v^*\}$ or $\{0, v^*\}$.

Conversely, assume $0 \in Q$ and $Q' = Q'_0 \sqcup (Q_1 \oplus v^*)$, where $Q'_0 = \{0\}$, $\{v^*\}$ or $\{0, v^*\}$. From $1 \notin Q_1$, it follows that $Q_1 \preceq \{v\}$ and

$$\begin{aligned} v \odot Q' &= v \odot (Q'_0 \cup (Q_1 \oplus v^*)) \\ &= (v \odot Q'_0) \cup (v \odot (Q_1 \oplus v^*)) \qquad \text{(Lemma 7 (3))} \\ &= \{0\} \cup (Q_1 \wedge v) = \{0\} \cup Q_1 = Q. \end{aligned}$$

Hence, we complete the proof. $\square$

We are now in a position to prove Theorem 2:

Proof of Theorem 2. Assume that σ is a multifunction on L_n and denote $\frac{n-2}{n-1}$ by v. According to Theorem 3, σ is uniquely determined by $\sigma(v)$ and $\sigma(1)$ if $\sigma \in \mathrm{MD}(L_n)$. Hence, it is enough to consider the values of $\sigma(v)$ and $\sigma(1)$. By Theorem 3, $\sigma \in \mathrm{MD}(L_n)$ iff

$$\sigma(v) \preceq \{v\}, \tag{5}$$

and

$$\sigma(v) = \sigma(v) \vee (v \odot \sigma(1)). \tag{6}$$

For convenience, we denote $P = \sigma(v)$, $Q' = \sigma(1)$, $Q = v \odot \sigma(1)$, $p_0 = \min P$ and $q_0 = \min Q$. Equation (5) implies $1 \notin P$. By Lemma 16, we know Equation (6) implies that $q_0 \leq p_0$ and $[p_0, 1] \cap Q \subseteq P$. Assume that $p_0 = \frac{k}{n-1}$ and $|P| = \ell$, where $0 \leq k \leq n-2$ and $1 \leq \ell \leq n-k-1$. Then, $P \setminus \{p_0\} \subseteq \left[\frac{k+1}{n-1}, \frac{n-2}{n-1}\right]$. Thus, P has $C_{n-k-2}^{\ell-1}$ choices with respect to k and ℓ. Now, we will determine all choices of Q and Q'.

Case 1. If $q_0 = p_0$, then $Q = [q_0, 1] \cap Q = [p_0, 1] \cap Q \subseteq P$. Hence, $Q \setminus \{q_0\}$ can take any subset of $P \setminus \{p_0\}$ and so Q has $2^{\ell-1}$ choices.

If $q_0 > 0$, then $0 \notin Q$, and by Lemma 17 (1) and $Q = v \odot Q'$ we know $Q' = Q \oplus v^*$. Hence, Q' has $2^{\ell-1}$ choices.

If $q_0 = 0$, then $0 \in Q$, by Lemma 17 (2) and $Q = v \odot Q'$ we have $Q' = \{0\} \sqcup (Q_1 \oplus v^*)$, $\{v^*\} \sqcup (Q_1 \oplus v^*)$ or $\{0, v^*\} \sqcup (Q_1 \oplus v^*)$. Thus, Q' has $3 \cdot 2^{\ell-1}$ choices.

Case 2. If $0 < q_0 < p_0$, denote $Q_1 = (0, p_0) \cap Q$ and $Q_2 = [p_0, 1] \cap Q$. Since $0 \notin Q$, we have $Q = Q_1 \sqcup Q_2$. Notice that $Q_1 \neq \varnothing$, so there are $2^{k-1} - 1$ choices of Q_1. Furthermore, since $Q_2 = [p_0, 1] \cap Q \subseteq P$, Q_2 can take any subset of P and so has 2^ℓ choices. Thus, there are $(2^{k-1} - 1) \cdot 2^\ell$ choices of Q in this case. Since $0 \notin Q$, it follows that Q' has also $(2^{k-1} - 1) \cdot 2^\ell$ choices by Lemma 17 (1).

Case 3. If $0 = q_0 < p_0$, denote $Q_1 = (0, p_0) \cap Q$ and $Q_2 = [p_0, 1] \cap Q$, so we have $Q = \{0\} \sqcup Q_1 \sqcup Q_2$. Since $Q_1 \subset (0, p_0)$, there are 2^{k-1} choices of Q_1. Moreover, Q_2 has 2^ℓ choices as in Case 2. Thus, there are $2^{k+\ell-1}$ choices of Q in this case. Since $0 \in Q$, it follows that Q' has $3 \cdot 2^{k+\ell-1}$ choices by Lemma 17 (2).

According to Theorem 3, we can determine the unique $(\odot, \vee)$-multiderivation for each choices of $\sigma(1)$ and $\sigma(v)$.

Therefore, it follows

$$\begin{aligned}
|\operatorname{MD}(L_n)| &= \sum_{k=1}^{n-2} \sum_{\ell=1}^{n-k-1} \binom{n-k-2}{\ell-1} (2^{\ell-1} + (2^{k-1}-1) \cdot 2^\ell + 3 \cdot 2^{k-1} \cdot 2^\ell) + \sum_{\ell=1}^{n-1} \binom{n-2}{\ell-1} (3 \cdot 2^{\ell-1}) \\
&= \sum_{k=0}^{n-2} \sum_{\ell=1}^{n-k-1} \binom{n-k-2}{\ell-1} (2^{k+\ell+1} - 2^{\ell-1}) \\
&= \sum_{k=0}^{n-2} \left((2^{k+2} - 1) \sum_{\ell=1}^{n-k-1} \binom{n-k-2}{\ell-1} \cdot 2^{\ell-1} \right) \\
&= \sum_{k=0}^{n-2} (2^{k+2} - 1)(2+1)^{n-k-2} \\
&= 3^n \sum_{k=0}^{n-2} \left(\left(\frac{2}{3}\right)^{k+2} - \left(\frac{1}{3}\right)^{k+2} \right) \\
&= \frac{7 \cdot 3^{n-1} - 2^{n+2} + 1}{2}. \quad \square
\end{aligned}$$

6. Conclusions and Questions

In this paper, the point-to-point $(\odot, \vee)$-derivations on MV-algebras have been extended to point-to-set $(\odot, \vee)$-multiderivations. We show that $(\operatorname{MD}(L_n)/\sim, \preccurlyeq)$ is isomorphic to the complete lattice $\operatorname{Der}(L_n)$, the underlying set of $(\odot, \vee)$-derivations on L_n. This unveils a certain relevance between $(\odot, \vee)$-multiderivations and $(\odot, \vee)$-derivations. Moreover, by building a counting principle, we obtain the enumeration of $\operatorname{MD}(L_n)$.

This general study of $(\odot, \vee)$-multiderivations has the advantage of developing into a system theory of sets and has potential wide applications: other logical algebras, control theory, interval analysis, and artificial intelligence.

We list three questions to be considered in the future:

(1) We have found two ways to construct $(\odot, \vee)$-multiderivations from $(\odot, \vee)$-derivations in Propositions 5 and 6. Are there other ways?

(2) We ask whether the equivalent characterization and enumeration of $(\odot, \vee)$- multiderivations on finite MV-chains can be extended to finite MV-algebras.

(3) We ask whether MV-algebras A and A' are isomorphic if $(\mathrm{MD}(A), \preccurlyeq)$ and $(\mathrm{MD}(A'), \preccurlyeq)$ are order isomorphic.

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Appendix A. Calculation Program by Python in Table 1

```
from itertools import product

#the set of MV-chain Ln
n = 6  # Adjust n as needed
L = list(range(n))

# operators on Ln
def omul(a, b):
    return max(a + b + 1 - n, 0)

def join(a, b):
    return max(a, b)

# operators on Delta(Ln)
def Omul(A, B):
    C = []
    for i in A:
        for j in B:
            k = omul(i, j)
            if k not in C:
                C.append(k)
    return C

def Join(A, B):
    C = []
    for i in A:
        for j in B:
            k = join(i, j)
            if k not in C:
```

Figure A1. *Cont.*

```
                C.append(k)
        return C

    # judge whether F is a multiderivation
    def IsMulDer(F):
        for i in range(n):
            for j in range(n):
                if set(F[omul(i, j)]) != set(Join(Omul(F[i], [j
                    ]), Omul([i], F[j]))):
                    return False
        return True

    # get the list of all multifunctions on Ln
    def powerset(s):
        for i in range(1 << len(s)):
            yield [s[j] for j in range(len(s)) if (i & (1 << j))
                ]

    def generate_PLn(n):
        elements = []
        for i in range(1, n+1):
            a = list(powerset(range(i)))
            if [] in a:
                a.remove([])
            elements.append(a)
        return list(product(*elements))

    def find_MulDer():
        MulDer = 0
        for F in generate_PLn(n):
            if IsMulDer(F):
                MulDer += 1
                print(F)
        return MulDer

    MulDer_count = find_MulDer()
    print(MulDer_count)
```

Figure A1. MD(L_n).py.

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