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A New Nonlinear Integral Inequality with a Tempered Ψ–Hilfer Fractional Integral and Its Application to a Class of Tempered Ψ–Caputo Fractional Differential Equations

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Abstract: In this paper, the tempered Ψ–Riemann–Liouville fractional derivative and the tempered Ψ–Caputo fractional derivative of order $n-1<\alpha<n\in\mathbb{N}$ are introduced for C^{n-1}–functions. A nonlinear version of the second Henry–Gronwall inequality for integral inequalities with the tempered Ψ–Hilfer fractional integral is derived. By using this inequality, an existence and uniqueness result and a sufficient condition for the non-existence of blow-up solutions of nonlinear tempered Ψ–Caputo fractional differential equations are proved. Illustrative examples are given.

Keywords: tempered Ψ–Hilfer fractional integral inequality; tempered Ψ–Riemann–Liouville fractional derivative; tempered Ψ–Caputo fractional derivative; generalized Henry–Gronwall inequality; blow-up solution

MSC: 34A40; 34A08; 26A33; 26D15; 26D10

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1. Introduction

The classical linear Gronwall integral inequality has played a fundamental role in the theory of ordinary and partial differential equations.

Many linear and nonlinear versions of this inequality can be found in the monographs [1–3]. All of such integral inequalities contained in these monographs have regular kernels. The most known and very often quoted nonlinear one is the Bihari inequality, originally proved in the paper [4]. Many other nonlinear integral inequalities are, in some sense, modifications of this inequality. The first result on linear integral inequality with the weakly singular kernel

$$(t-s)^{\alpha-1},\quad \alpha>0, \tag{1}$$

frequently called the Henry lemma or Henry inequality, is proved in the famous monograph by D. Henry ([5], Lemma 7.1.1). In the book, it plays a fundamental role in the theory of semilinear parabolic equations. Another result also proved in this monograph ([5], Lemma 7.1.2) concerns the linear integral inequality with the weakly singular kernel

$$(t-s)^{\alpha-1}s^{\gamma-1},\quad \alpha,\gamma>0.$$

These two results are proved by an iteration argument. Unfortunately, this method is not applicable in nonlinear cases. A new approach (so-called desingularization method), presented in the papers [6,7], is suitable also for the investigation of nonlinear integral inequalities with various types of weakly singular kernels. This method is helpful in the theory of fractional differential equations (see, e.g., [8]), abstract evolution differential equations (see, e.g., [7,9–11]), and parabolic partial differential equations. The results from the papers [6,7] are included altogether with their proofs in monographs ([12], Theorem 1.2.17) and ([13], Theorems 1.4.7–1.4.9).A generalization of ([13], Theorem 1.4.9) to nonlinear fractional iterative integral inequalities is proved in [14]. It is also applied in the proof of a sufficient condition for the nonexistence of blow-up solutions in a class of nonlinear integral equations with several integrals possessing weakly singular kernels of the form of (1), as in ([5], Lemma 7.1.1).

The desingularization method was successfully applied in many papers on integral inequalities with weakly singular kernels (see, e.g., recently published papers [15–18]) and in the study of asymptotic properties of fractional differential equations. The first result of this type was published in the paper [19]. Later, other papers followed (e.g., [9,20–26]). It is worth to mention that the Henry lemmas were generalized to weakly singular nonlinear integral inequalities with a delay [27], stochastic inequalities with singular kernels [28], integral inequalities with doubly singular kernels [29], etc.

In the present paper, we apply the desingularization method to nonlinear integral inequalities with the weakly singular kernel

$$\mathcal{K}_{\Psi}^{\alpha,\gamma,\lambda}(t,s) := (\Psi(t)-\Psi(s))^{\alpha-1}\Psi(s)^{\gamma-1}\mathrm{e}^{-\lambda(\Psi(t)-\Psi(s))}\Psi'(s), \tag{2}$$

where Ψ is a C^1–function with a positive derivative. The case $\gamma = 1$ was studied in the paper [30] and its linear form in [31], where the definition of the tempered Ψ–Caputo fractional derivative was introduced.

It is obvious that the second Henry inequality is obtained if $\Psi(t) \equiv 1$, $\lambda = 0$, and the first one if, in addition, $\gamma = 1$. Clearly, the new integral inequality can be applied to some modifications of the above-mentioned fractional problems in the framework of tempered Ψ–fractional differential equations.

The structure of this paper is as follows: In the next part, we introduce the tempered Ψ–Riemann–Liouville fractional derivative and the tempered Ψ–Caputo fractional derivative. Moreover, some of their properties are derived. In Section 3, we prove the Henry–Gronwall inequality for integrals with kernel $\mathcal{K}_{\Psi}^{\alpha,\gamma,\lambda}$. In Section 4, we apply the integral inequality to obtain results for fractional differential equations involving the tempered Ψ–Caputo derivative. Here, we also provide examples of initial value problems with explicit solutions. Finally, Section 5 summarizes the results and outlines possible future research directions.

In the whole paper, we denote by $\mathbb{N}$ and $\mathbb{N}_0$ the set of all positive and nonnegative integers, respectively. Next, for $a,b \in \mathbb{Z}$, $a < b$, we use $\mathbb{Z}_a^b$ for a discrete interval $[a,b] \cap \mathbb{Z} = \{a, a+1, \dots, b\}$.

2. Preliminaries

In this section, we recall known definitions and prove auxiliary results. Here, we also define the tempered Ψ–Riemann–Liouville fractional derivative.

Definition 1 ([31])**.** *Let $\alpha > 0$, $\lambda \geq 0$, and $\Psi \in C^1[a,b]$ satisfy $\Psi'(t) > 0$ for all $t \in [a,b]$. The tempered Ψ–Hilfer fractional integral of order $\alpha > 0$ of a function $x \in C[a,b]$ is defined by*

$$I_a^{\alpha,\lambda,\Psi}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \mathcal{K}_{\Psi}^{\alpha,\lambda}(t,s)\,x(s)\,ds$$

for $t \in [a,b]$, where $\mathcal{K}_{\Psi}^{\alpha,\lambda}(t,s) = \mathcal{K}_{\Psi}^{\alpha,1,\lambda}(t,s)$ for $\mathcal{K}_{\Psi}^{\alpha,\gamma,\lambda}(t,s)$ given by (2) and $\Gamma(\cdot)$ is the Euler gamma function.

Note that

$$I_a^{\alpha,\lambda,\Psi}x(t) = \mathrm{e}^{-\lambda\Psi(t)}I_a^{\alpha,\Psi}(\mathrm{e}^{\lambda\Psi(t)}x(t)), \tag{3}$$

where $I_a^{\alpha,\Psi} = I_a^{\alpha,0,\Psi}$ is the Ψ–Riemann–Liouville fractional integral [32,33], sometimes referred to as the fractional integral with respect to function Ψ. It is worth to mention that if $x \in C[a,b]$, then $I_a^{\alpha,\lambda,\Psi} x \in C^{\lfloor \alpha \rfloor}$, where $\lfloor \cdot \rfloor$ is the floor function.

Definition 2. *Let $n \in \mathbb{N}$, $n-1 < \alpha < n$, $\lambda \geq 0$, and $\Psi \in C^n[a,b]$ satisfy $\Psi'(t) > 0$ for all $t \in [a,b]$. The tempered Ψ–Riemann–Liouville fractional derivative of order α of a function $x \in C^{n-1}[a,b]$ is defined by*

$$ {}^{RL}D_a^{\alpha,\lambda,\Psi} x(t) = \mathrm{e}^{-\lambda\Psi(t)} \left(I_a^{n-\alpha,\lambda,\Psi} x(t) \right)_{\lambda,\Psi}^{[n]} $$

where

$$ x_{\lambda,\Psi}^{[n]}(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n \left(\mathrm{e}^{\lambda\Psi(t)} x(t) \right), $$

i.e.,

$$ {}^{RL}D_a^{\alpha,\lambda,\Psi} x(t) = \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(n-\alpha)} \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n \int_a^t (\Psi(t) - \Psi(s))^{n-\alpha-1} \Psi'(s) \mathrm{e}^{\lambda\Psi(s)} x(s)\, ds. $$

Note that

$$ {}^{RL}D_a^{\alpha,\lambda,\Psi} x(t) = \mathrm{e}^{-\lambda\Psi(t)}\, {}^{RL}D_a^{\alpha,\Psi} (\mathrm{e}^{\lambda\Psi(t)} x(t)) \tag{4} $$

for the Ψ–Riemann–Liouville fractional derivative, ${}^{RL}D_a^{\alpha,\Psi} = {}^{RL}D_a^{\alpha,0,\Psi}$, as introduced in [34].

Definition 3. *Let $n \in \mathbb{N}$, $n-1 < \alpha < n$, $\lambda \geq 0$, and $\Psi \in C^n[a,b]$ satisfy $\Psi'(t) > 0$ for all $t \in [a,b]$. The tempered Ψ–Caputo fractional derivative of order α of a function $x \in C^{n-1}[a,b]$ is defined by*

$$ {}^{C}D_a^{\alpha,\lambda,\Psi} x(t) = {}^{RL}D_a^{\alpha,\lambda,\Psi} \left[x(t) - \mathrm{e}^{-\lambda\Psi(t)} \sum_{k=0}^{n-1} \frac{(\Psi(t) - \Psi(a))^k}{k!}\, x_{\lambda,\Psi}^{[k]}(a) \right] \tag{5} $$

for $t \in [a,b]$.

By denoting $y(t) = \mathrm{e}^{\lambda\Psi(t)} x(t)$, we obtain

$$ {}^{C}D_a^{\alpha,\lambda,\Psi} x(t) = \mathrm{e}^{-\lambda\Psi(t)}\, {}^{RL}D_a^{\alpha,\Psi} \left[y(t) - \sum_{k=0}^{n-1} \frac{(\Psi(t) - \Psi(a))^k}{k!}\, y_{\Psi}^{[k]}(a) \right], $$

where $y_\Psi^{[n]}(t) = y_{0,\Psi}^{[n]}(t)$. Now, if $x \in C^n[a,b]$, by using the definition of the Ψ–Caputo fractional derivative, ${}^{C}D_a^{\alpha,\Psi}$, from ([34], Definition 1), by ([35], Theorem 3), ${}^{C}D_a^{\alpha,\lambda,\Psi} x(t)$ reads

$$ \begin{aligned} {}^{C}D_a^{\alpha,\lambda,\Psi} x(t) &= \mathrm{e}^{-\lambda\Psi(t)}\, {}^{C}D_a^{\alpha,\Psi} y(t) \\ &= \mathrm{e}^{-\lambda\Psi(t)} I_a^{n-\alpha,\Psi} \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n y(t) \\ &= \mathrm{e}^{-\lambda\Psi(t)} I_a^{n-\alpha,\Psi} x_{\lambda,\Psi}^{[n]}(t) \\ &= \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(n-\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{n-\alpha-1} \Psi'(s)\, x_{\lambda,\Psi}^{[n]}(s)\, ds, \end{aligned} \tag{6} $$

which agrees with the definition of the tempered Ψ–Caputo fractional derivative from ([31], Definition 6).

Next, we summarize several properties of the above-defined fractional operators in auxiliary lemmas.

Lemma 1. *Let $\lambda \geq 0$ and $\Psi \in C^1[a,b]$ satisfy $\Psi'(t) > 0$ for all $t \in [a,b]$. A continuous function x defined on $[a,b]$ fulfills the following:*

1. $I_a^{\alpha,\lambda,\Psi} I_a^{\beta,\lambda,\Psi} x(t) = I_a^{\alpha+\beta,\lambda,\Psi} x(t)$ *for* $\alpha, \beta > 0$;
2. $\left(I_a^{\alpha,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[n]} = \mathrm{e}^{\lambda\Psi(t)} I_a^{\alpha-n,\lambda,\Psi} x(t)$ *for* $\alpha > n \in \mathbb{N}_0$;
3. $\left(I_a^{n,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[n]} = \mathrm{e}^{\lambda\Psi(t)} x(t)$ *for* $n \in \mathbb{N}$.

Proof. By making use of fractional integral composition (see [34,36]), one derives

$$\begin{aligned} I_a^{\alpha,\lambda,\Psi} I_a^{\beta,\lambda,\Psi} x(t) &= \mathrm{e}^{-\lambda\Psi(t)} I_a^{\alpha,\Psi} I_a^{\beta,\Psi} (\mathrm{e}^{\lambda\Psi(t)} x(t)) \\ &= \mathrm{e}^{-\lambda\Psi(t)} I_a^{\alpha+\beta,\Psi} (\mathrm{e}^{\lambda\Psi(t)} x(t)) \\ &= I_a^{\alpha+\beta,\lambda,\Psi} x(t). \end{aligned}$$

Statement 1 is proved.

If $n = 0$, then $\left(I_a^{\alpha,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[n]} = \mathrm{e}^{\lambda\Psi(t)} I_a^{\alpha,\lambda,\Psi} x(t)$. This confirms Statement 2 for $n = 0$. Let $n > 0$. Then, $I_a^{\alpha,\lambda,\Psi} x \in C^{\lfloor\alpha\rfloor}[a,b]$ with $\lfloor\alpha\rfloor \geq n$. By subsequently differentiating, one obtains

$$\begin{aligned} \left(I_a^{\alpha,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[n]} &= \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \left(\mathrm{e}^{\lambda\Psi(t)} I_a^{\alpha,\lambda,\Psi} x(t)\right) \\ &= \frac{1}{\Gamma(\alpha)}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \int_a^t (\Psi(t)-\Psi(s))^{\alpha-1}\Psi'(s)\mathrm{e}^{\lambda\Psi(s)} x(s)\, ds \\ &= \frac{1}{\Gamma(\alpha-1)}\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{n-1} \int_a^t (\Psi(t)-\Psi(s))^{\alpha-2}\Psi'(s)\mathrm{e}^{\lambda\Psi(s)} x(s)\, ds \\ &= \left(I_a^{\alpha-1,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[n-1]} = \cdots = \left(I_a^{\alpha-n+1,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[1]} \\ &= \frac{1}{\Gamma(\alpha-n+1)\Psi'(t)}\frac{d}{dt}\int_a^t (\Psi(t)-\Psi(s))^{\alpha-n}\Psi'(s)\mathrm{e}^{\lambda\Psi(s)} x(s)\, ds. \end{aligned}$$

Now, if $\alpha = n$, this is equal to

$$\frac{1}{\Psi'(t)}\frac{d}{dt}\int_a^t \mathrm{e}^{\lambda\Psi(s)} x(s)\Psi'(s)\, ds = \mathrm{e}^{\lambda\Psi(t)} x(t)$$

proving Statement 3; on the other hand, for $\alpha > n$, one can differentiate once more to obtain

$$\frac{1}{\Gamma(\alpha-n)}\int_a^t (\Psi(t)-\Psi(s))^{\alpha-n-1}\Psi'(s)\mathrm{e}^{\lambda\Psi(s)} x(s)\, ds = \mathrm{e}^{\lambda\Psi(t)} I_a^{\alpha-n,\lambda,\Psi} x(t),$$

which proves Statement 2 for $n > 0$. □

From now on, we refer to the statements of the latter lemma by adding the corresponding number, e.g., Lemma 1(1). The same holds for the next lemma.

Lemma 2. *Let $n \in \mathbb{N}$, $n-1 < \alpha < n$, $\lambda \geq 0$, and $\Psi \in C^n[a,b]$ satisfy $\Psi'(t) > 0$ for all $t \in [a,b]$. Then, the following holds:*

1. *for* $k \in \mathbb{N}_0$,

$$^C D_a^{\alpha,\lambda,\Psi}\left(\mathrm{e}^{-\lambda\Psi(t)}(\Psi(t)-\Psi(a))^k\right) = \begin{cases} 0, & n > k, \\ \frac{\mathrm{e}^{-\lambda\Psi(t)}k!}{\Gamma(k+1-\alpha)}(\Psi(t)-\Psi(a))^{k-\alpha}, & n \leq k. \end{cases}$$

2. $^C D_a^{\alpha,\lambda,\Psi} I_a^{\beta,\lambda,\Psi} x(t) = I_a^{\beta-\alpha,\lambda,\Psi} x(t)$ *for* $\alpha < \beta$ *and* $x \in C[a,b]$.

3. ${}^{C}D_a^{\alpha,\lambda,\Psi} I_a^{\alpha,\lambda,\Psi} x(t) = x(t)$ *for* $x \in C[a,b]$.
4. *for* $x \in C^{n-1}[a,b]$.

$$I_a^{\alpha,\lambda,\Psi}\, {}^{C}D_a^{\alpha,\lambda,\Psi} x(t) = x(t) - \mathrm{e}^{-\lambda\Psi(t)} \sum_{k=0}^{n-1} \frac{(\Psi(t)-\Psi(a))^k}{k!} x_{\lambda,\Psi}^{[k]}(a).$$

Proof. First, observe that

$$\begin{aligned}\left[\mathrm{e}^{-\lambda\Psi(t)}(\Psi(t)-\Psi(a))^k\right]_{\lambda,\Psi}^{[n]} &= \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^n \left[(\Psi(t)-\Psi(a))^k\right] \\ &= \begin{cases} 0, & n > k, \\ \frac{k!}{(k-n)!}(\Psi(t)-\Psi(a))^{k-n}, & n \le k. \end{cases}\end{aligned}$$

Then, Formula (6) is applied to immediately see that Statement 1 holds whenever $n > k$. If $n \le k$, the substitution $\Psi(s) = \sigma(\Psi(t)-\Psi(a)) + \Psi(a)$ results in

$$\begin{aligned}&{}^{C}D_a^{\alpha,\lambda,\Psi}\left(\mathrm{e}^{-\lambda\Psi(t)}(\Psi(t)-\Psi(a))^k\right) \\ &= \frac{\mathrm{e}^{-\lambda\Psi(t)}}{\Gamma(n-\alpha)} \int_a^t (\Psi(t)-\Psi(s))^{n-\alpha-1}\Psi'(s)\left[\mathrm{e}^{-\lambda\Psi(s)}(\Psi(s)-\Psi(a))^k\right]_{\lambda,\Psi}^{[n]} ds \\ &= \frac{\mathrm{e}^{-\lambda\Psi(t)}k!}{\Gamma(n-\alpha)(k-n)!} \int_a^t (\Psi(t)-\Psi(s))^{n-\alpha-1}(\Psi(s)-\Psi(a))^{k-n}\,\Psi'(s)\,ds \\ &= \frac{\mathrm{e}^{-\lambda\Psi(t)}(\Psi(t)-\Psi(a))^{k-\alpha}k!}{\Gamma(n-\alpha)(k-n)!} \int_0^1 (1-\sigma)^{n-\alpha-1}\sigma^{k-n}\,ds \\ &= \frac{\mathrm{e}^{-\lambda\Psi(t)}k!}{\Gamma(n-\alpha)(k-n)!} B(n-\alpha,k-n+1)(\Psi(t)-\Psi(a))^{k-\alpha} \\ &= \frac{\mathrm{e}^{-\lambda\Psi(t)}k!}{\Gamma(k+1-\alpha)}(\Psi(t)-\Psi(a))^{k-\alpha},\end{aligned}$$

where $B(\cdot,\cdot)$ is the Euler beta function. So, Statement 1 is proved.

Now, $I_a^{\beta,\lambda,\Psi} x \in C^{\lfloor\beta\rfloor}[a,b]$ with $\lfloor\beta\rfloor \ge n-1$. So, in general, Formula (6) cannot be used. Instead, we use Definition 3. Under the assumptions of Statement 2, we obtain, as a consequence of Lemma 1(2),

$$\left(I_a^{\beta,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[k]}\Bigg|_{t=a} = 0, \quad k = 0,1,\ldots,n-1 = \lfloor\alpha\rfloor < \beta.$$

Consequently, by using (5), Definition 2, and Lemma 1, we have

$$\begin{aligned}{}^{C}D_a^{\alpha,\lambda,\Psi} I_a^{\beta,\lambda,\Psi} x(t) &= {}^{RL}D_a^{\alpha,\lambda,\Psi} I_a^{\beta,\lambda,\Psi} x(t) \\ &= \mathrm{e}^{-\lambda\Psi(t)}\left(I_a^{n-\alpha,\lambda,\Psi} I_a^{\beta,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[n]} \\ &= \mathrm{e}^{-\lambda\Psi(t)}\left(I_a^{n+\beta-\alpha,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[n]} \\ &= \mathrm{e}^{-\lambda\Psi(t)}\mathrm{e}^{\lambda\Psi(t)} I_a^{\beta-\alpha,\lambda,\Psi} x(t) \\ &= I_a^{\beta-\alpha,\lambda,\Psi} x(t).\end{aligned}$$

This proves Statement 2.

Similarly, we have

$$\left(I_a^{\alpha,\lambda,\Psi} x(t)\right)_{\lambda,\Psi}^{[k]}\Bigg|_{t=a} = 0, \quad k = 0,1,\ldots,n-1 = \lfloor\alpha\rfloor,$$

and

$$ {}^{C}D_a^{\alpha,\lambda,\Psi} I_a^{\alpha,\lambda,\Psi} x(t) = \mathrm{e}^{-\lambda\Psi(t)} \left(I_a^{n,\lambda,\Psi} x(t) \right)_{\lambda,\Psi}^{[n]} = x(t) $$

due to Lemma 1(3), which proves Statement 3.

To show Statement 4, we apply an analogous result for $\lambda = 0$ proved in ([34], Theorem 1):

$$ \begin{aligned} I_a^{\alpha,\lambda,\Psi}\, {}^{C}D_a^{\alpha,\lambda,\Psi} x(t) &= \mathrm{e}^{-\lambda\Psi(t)} I_a^{\alpha,\Psi}\, {}^{C}D_a^{\alpha,\Psi} (\mathrm{e}^{\lambda\Psi(t)} x(t)) \\ &= \mathrm{e}^{-\lambda\Psi(t)} \left[\mathrm{e}^{\lambda\Psi(t)} x(t) - \sum_{k=0}^{n-1} \frac{(\Psi(t)-\Psi(a))^k}{k!} \left(\mathrm{e}^{\lambda\Psi(t)} x(t) \right)_{\Psi}^{[k]} \Big|_{t=a} \right] \\ &= x(t) - \mathrm{e}^{-\lambda\Psi(t)} \sum_{k=0}^{n-1} \frac{(\Psi(t)-\Psi(a))^k}{k!} x_{\lambda,\Psi}^{[k]}(a). \end{aligned} $$

This completes the proof. □

Let us consider the following initial value problem:

$$ {}^{C}D_a^{\alpha,\lambda,\Psi} x(t) = f(t, x(t)), \quad t \geq a, \tag{7} $$

$$ x_{\lambda,\Psi}^{[k]}(a) = x_a^k, \quad k \in \mathbb{Z}_0^{n-1} \tag{8} $$

for some $x_a^k \in \mathbb{R}^N$, $k \in \mathbb{Z}_0^{n-1}$, where $n-1 < \alpha < n \in \mathbb{N}$, $\lambda \geq 0$, $\Psi \in C^n[a,\infty)$ satisfies $\Psi'(t) > 0$ for all $t \geq a$, and $f \in C([a,\infty) \times \mathbb{R}^N, \mathbb{R}^N)$. Here, the differential operator is to be understood component-wise. In accordance with ([37], Definition 4) and Definition 3, function $x \in C^{n-1}[a, a+h)$ for some $0 < h \leq \infty$ is a solution of initial value problem (7) and (8) if $D_a^{\alpha,\lambda,\Psi} x(t)$ exists and is continuous on $[a, a+h)$, and x fulfills Equation (7) for all $t \in [a, a+h)$ and initial conditions (8).

The following theorem extends ([37], Theorem 2) to C^{n-1}–functions.

Theorem 1. *Function x is a solution of initial value problem (7) and (8) if and only if it satisfies*

$$ x(t) = \mathrm{e}^{-\lambda\Psi(t)} \sum_{k=0}^{n-1} \frac{(\Psi(t)-\Psi(a))^k}{k!} x_a^k + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{K}_{\Psi}^{\alpha,\lambda}(t,s)\, f(s, x(s))\, ds, \quad t \geq a. \tag{9} $$

Proof. If x solves (7) and (8), the application of operator $I_a^{\alpha,\lambda,\Psi}$ to Equation (7) yields (9), due to Lemma 2(4) and conditions (8).

Now, assume that x fulfills integral Equation (9). Applying ${}^{C}D_a^{\alpha,\lambda,\Psi}$ results in

$$ \begin{aligned} {}^{C}D_a^{\alpha,\lambda,\Psi} x(t) &= {}^{C}D_a^{\alpha,\lambda,\Psi} \left(\mathrm{e}^{-\lambda\Psi(t)} \sum_{k=0}^{n-1} \frac{(\Psi(t)-\Psi(a))^k}{k!} x_a^k \right) \\ &\quad + {}^{C}D_a^{\alpha,\lambda,\Psi} I_a^{\alpha,\lambda,\Psi} f(t, x(t)) \\ &= f(t, x(t)), \end{aligned} $$

where Statements 1 and 3 of Lemma 2 were applied. It only remains to verify initial conditions (8). For each $j \in \mathbb{Z}_0^{n-1}$, we have

$$ \begin{aligned} \left(\mathrm{e}^{-\lambda\Psi(t)} \sum_{k=0}^{n-1} \frac{(\Psi(t)-\Psi(a))^k}{k!} x_a^k \right)_{\lambda,\Psi}^{[j]} &= \left(\frac{1}{\Psi'(t)} \frac{d}{dt} \right)^j \sum_{k=0}^{n-1} \frac{(\Psi(t)-\Psi(a))^k}{k!} x_a^k \\ &= \sum_{k=j}^{n-1} \frac{(\Psi(t)-\Psi(a))^{k-j}}{(k-j)!} x_a^k, \end{aligned} $$

that is equal to x_a^j at $t = a$. Moreover,

$$\left(I_a^{\alpha,\lambda,\Psi} f(t, x(t))\right)_{\lambda,\Psi}^{[j]} = \mathrm{e}^{\lambda\Psi(t)} I_a^{\alpha-j,\lambda,\Psi} f(t, x(t))$$

by Lemma 1(2). Here, we used $0 \le j \le \lfloor \alpha \rfloor < \alpha$. One can easily see that

$$\left(I_a^{\alpha,\lambda,\Psi} f(t, x(t))\right)_{\lambda,\Psi}^{[j]}\bigg|_{t=a} = 0 \quad \text{for each } j \in \mathbb{Z}_0^{n-1}.$$

This verifies the initial conditions and completes the proof. □

3. Integral Inequalities

Here, we investigate the integral inequality

$$u(t) \le a(t) + b(t) \int_a^t \mathcal{K}_\Psi^{\alpha,\gamma,\lambda}(t,s)\, F(s)\, \omega(u(s))\, ds, \quad a \le t < T. \tag{10}$$

In [33], an inequality of the Henry–Gronwall type (see ([33], Theorem 3)) is proved for inequality (10) with $\lambda = 0$, $\omega(u) \equiv u$, and $F(t) \equiv 1$. In the papers [6,7], inequality (10) with $\lambda = 0$ and $\Psi(t) \equiv t$ is studied.

First, we recall a generalized Hölder inequality.

Lemma 3. *Let $n \in \mathbb{N}$ and $p_j > 1$ for $j \in \mathbb{Z}_1^n$ satisfy*

$$\sum_{j=1}^{n} \frac{1}{p_j} = 1. \tag{11}$$

Then,

$$\int_a^b \prod_{j=1}^{n} u_j(s)\, ds \le \prod_{j=1}^{n} \left(\int_a^b |u_j(s)|^{p_j} ds \right)^{\frac{1}{p_j}}. \tag{12}$$

This lemma was proved by A. Kufner, O. John, and S. Fučík in ([38], p. 67) (see also ([39], 5.9c, pp. 355–356)).

Theorem 2. *Let $a \in \mathbb{R}$; $\alpha, \gamma \in (0,1)$; $\lambda > 0$; $p, q, r > 1$; $p < (1-\alpha)^{-1}$; $q < (1-\gamma)^{-1}$ satisfy $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$; $a, b \in C[a,T)$ be nonnegative functions; $b(\cdot)$ be nondecreasing, where $a < T \le \infty$; and $\Psi \in C^1[a,\infty)$ satisfy $\Psi'(t) > 0$ for all $t \in (a,\infty)$. Let $\omega \in C[0,\infty)$ be a positive, nondecreasing function; $F, u \in C[a,T)$ be nonnegative functions; and u satisfy inequality (10). Then,*

$$u(t) \le \left[\Xi^{-1}\left(\Xi(A(t)) + B(t) \int_a^t \mathrm{e}^{r\Psi(s)} \Psi'(s) F(s)^r ds \right) \right]^{1/r} \tag{13}$$

for all $a \le t < T$ for which the right side makes sense, where

$$\begin{gathered} A(t) = 2^{r-1} \sup_{a \le s \le t} a(s)^r, \qquad B(t) = 2^{r-1} K^r \sup_{a \le s \le t} b(s)^r, \\ K = K(\alpha,\gamma,\lambda,p,q) = K_1(\alpha,\lambda,p)\, K_2(\gamma,q), \\ K_1(\alpha,\lambda,p) = \left[\frac{\Gamma(p(\alpha-1)+1)}{(p\lambda)^{p(\alpha-1)+1}} \right]^{1/p}, \\ K_2(\gamma,q) = \left[\frac{\Gamma(q(\gamma-1)+1)}{q^{q(\gamma-1)+1}} \right]^{1/q}, \\ \Xi(z) = \int_{z_0}^{z} \frac{d\sigma}{\omega(\sigma^{1/r})^r} \quad \textit{for } z_0, z \ge 0, \end{gathered} \tag{14}$$

and Ξ^{-1} is the inverse of Ξ.

Proof. Since $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$, we rewrite $\Psi'(t)$ as $\Psi'(t)^{1/p}\,\Psi'(t)^{1/q}\,\Psi'(t)^{1/r}$. By using this equality, the identity $\mathrm{e}^{\Psi(t)}\,\mathrm{e}^{-\Psi(t)}=1$, and Lemma 3, we obtain

$$
\begin{aligned}
&\int_a^t \mathcal{K}_\Psi^{\alpha,\lambda,\Psi}(t,s)\,F(s)\,\omega(u(s))\,ds \\
&\quad\le \left(\int_a^t \Psi'(s)(\Psi(t)-\Psi(s))^{p(\alpha-1)}\mathrm{e}^{-p\lambda(\Psi(t)-\Psi(s))}ds\right)^{1/p} \\
&\qquad\times\left(\int_a^t \Psi'(s)\mathrm{e}^{-q\Psi(s)}\Psi(s)^{q(\gamma-1)}ds\right)^{1/q}\left(\int_a^t \Psi'(s)\,\mathrm{e}^{r\Psi(s)}F(s)^r\omega(u(s))^r ds\right)^{1/r} \\
&\quad=\left(\int_0^{\Psi(t)-\Psi(a)} \sigma^{p(\alpha-1)}\mathrm{e}^{-p\lambda\sigma}\,d\sigma\right)^{1/p}\left(\int_{\Psi(a)}^{\Psi(t)} \mathrm{e}^{-q\tau}\tau^{q(\gamma-1)}\,d\tau\right)^{1/q} \\
&\qquad\times\left(\int_a^t \mathrm{e}^{r\Psi(s)}F(s)^r\omega(u(s))^r\Psi'(s)\,ds\right)^{1/r} \\
&\quad\le\left(\frac{\Gamma(p(\alpha-1)+1)}{(p\lambda)^{p(\alpha-1)+1}}\right)^{1/p}\left(\frac{\Gamma(q(\gamma-1)+1)}{q^{q(\gamma-1)+1}}\right)^{1/q} \\
&\qquad\times\left(\int_a^t \mathrm{e}^{r\Psi(s)}F(s)^r\omega(u(s))^r\,\Psi'(s)\,ds\right)^{1/r}.
\end{aligned}
\tag{15}
$$

By (10), this yields the inequality

$$u(t)\le a(t)+Kb(t)\left(\int_a^t \mathrm{e}^{r\Psi(s)}F(s)^r\omega(u(s))^r\,\Psi'(s)\,ds\right)^{1/r}, \tag{16}$$

where the constant K is defined by (14). By using the estimation $(\xi+\zeta)^r\le 2^{r-1}(\xi^r+\zeta^r)$ valid for any $\xi,\zeta\ge 0$, we obtain, from (16),

$$u(t)^r\le \bar{a}(t)+\bar{b}(t)\int_a^t \mathrm{e}^{r\Psi(s)}F(s)^r\omega(u(s))^r\,\Psi'(s)\,ds, \tag{17}$$

where

$$\bar{a}(t)=2^{r-1}a(t)^r,\qquad \bar{b}(t)=2^{r-1}K^r b(t)^r. \tag{18}$$

If $v(t)=u(t)^r$, we rewrite inequality (17) as

$$v(t)\le \bar{a}(t)+\bar{b}(t)\int_a^t \mathrm{e}^{r\Psi(s)}\Psi'(s)F(s)^r[\omega(v(s)^{1/r})]^r\,ds. \tag{19}$$

A theorem of Butler and Rogers ([40], Theorem, p. 78) implies

$$v(t)\le \Xi^{-1}\left(\Xi(A(t))+B(t)\int_a^t \mathrm{e}^{r\Psi(s)}\Psi'(s)F(s)^r\,ds\right). \tag{20}$$

Thus, inequality (13) is verified. □

For our purpose, it is worth to explicitly state the following corollary.

Corollary 1. *Let $a\in\mathbb{R}$; $\alpha,\gamma\in(0,1)$; $\lambda>0$; $p,q,r>1$; $p<(1-\alpha)^{-1}$; $q<(1-\gamma)^{-1}$ satisfy $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$; $b\in C[a,T)$ be a nonnegative, nondecreasing function, where $a<T\le\infty$; and $\Psi\in C^1[a,\infty)$ satisfy $\Psi'(t)>0$ for all $t\in(a,\infty)$. Let $u\in C[a,T)$ be a nonnegative function satisfying*

$$u(t)\le b(t)\int_a^t \mathcal{K}_\Psi^{\alpha,\gamma,\lambda}(t,s)\,ds,\quad a\le t<T.$$

Then,

$$u(t) \leq 2^{\frac{r-1}{r}} K \left(\frac{e^{r\Psi(t)} - e^{r\Psi(a)}}{r} \right)^{\frac{1}{r}} \sup_{a \leq s \leq t} b(s) \tag{21}$$

for all $a \leq t < T$, where K is given by (14).

In particular,

$$\int_a^t \mathcal{K}_\Psi^{\alpha,\gamma,\lambda}(t,s)\, ds \leq 2^{\frac{r-1}{r}} K \left(\frac{e^{r\Psi(t)} - e^{r\Psi(a)}}{r} \right)^{\frac{1}{r}}$$

for all $a \leq t < T$.

Proof. The direct use of Theorem 2 with $a(t) \equiv 0$ and $F(t) \equiv 1 \equiv \omega(t)$ yields (21). Notice that this time, $\Xi(z) = \ln \frac{z}{z_0}$, so the inverse in (13) is defined for any t. $\square$

4. Applications to Initial Value Problems Involving the Tempered Ψ–Caputo Derivative

This section applies the results proved in Section 3 to initial value problems corresponding to fractional differential equations with the tempered Ψ–Caputo fractional derivative. We consider the following assumptions:

H1 There is $\gamma \in (0,1)$ such that

$$t^{1-\gamma}\|f(t,x) - f(t,y)\| \leq L\|x-y\| \quad \text{for all } (t,x), (t,y) \in [a,\infty) \times \mathbb{R}^N.$$

H2 $\Psi(t) \leq t$ for all $t \geq a$.

The type of "Lipschitz condition" assumed in **H1** was introduced in the paper [41].

First, we state a result on the existence of a unique solution of the initial value problem.

Theorem 3. *Let $\alpha \in (0,1)$, $\lambda > 0$, and $\Psi \in C^1[a,\infty)$ satisfy $\Psi'(t) > 0$ for all $t \in (a,\infty)$. Moreover, let conditions **H1** and **H2** be fulfilled and*

$$\alpha + \gamma > 1. \tag{22}$$

Then, there exists $h > 0$ such that there is a unique solution x of initial value problem (7) and (8) on the interval $I_h = [a, a+h)$.

Proof. Let us fix $p, q, r > 1$, $p < (1-\alpha)^{-1}$, $q < (1-\gamma)^{-1}$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Note that by (22), such p, q, and r exist. Indeed, one can set

$$p = \frac{1}{1-\alpha+\varepsilon}, \quad q = \frac{1}{1-\gamma+\varepsilon}, \quad r = \frac{1}{\varepsilon}, \quad \varepsilon = \frac{\alpha+\gamma-1}{3}.$$

Then, it is easy to see that $0 < \varepsilon < \frac{1}{3}$, $r > 3$, $p < (1-\alpha)^{-1}$, $q < (1-\gamma)^{-1}$, and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2 - \alpha - \gamma + 3\varepsilon = 1.$$

Let $h > 0$ be arbitrary and fixed. By Theorem 1, it suffices to prove the existence of a unique solution of integral Equation (9) on I_h. Let $X_h = C[a, a+h)$ be the Banach space equipped with the norm $|u| = \sup_{t \in I_h} \|u(t)\|$. Define the operator $\mathcal{F}\colon X_h \to X_h$ by

$$\mathcal{F}(x)(t) = e^{-\lambda\Psi(t)} x_a^0 + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{K}_\Psi^{\alpha,\lambda}(t,s) f(s,x(s))\, ds, \quad t \in I_h. \tag{23}$$

If $x, y \in X_h$, then by using conditions H1 and H2, we obtain

$$\begin{aligned}\|\mathcal{F}(x)(t) - \mathcal{F}(y)(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{K}_{\Psi}^{\alpha,\gamma,\lambda}(t,s) \underbrace{\Psi(s)^{1-\gamma}\|f(s,x(s)) - f(s,y(s))\|}_{\leq s^{1-\gamma}\|f(s,x(s))-f(s,y(s))\|} ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{K}_{\Psi}^{\alpha,\gamma,\lambda}(t,s) \underbrace{s^{1-\gamma}\|f(s,x(s)) - f(s,y(s))\|}_{\leq L\|x(s)-y(s)\|} ds \\ &\leq L\left(\frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{K}_{\Psi}^{\alpha,\gamma,\lambda}(t,s)\, ds\right)|x-y|.\end{aligned} \tag{24}$$

By applying Corollary 1, we obtain

$$\|\mathcal{F}(x)(t) - \mathcal{F}(y)(t)\| \leq \frac{2^{\frac{r-1}{r}} LK}{\Gamma(\alpha)} \left(\frac{e^{r\Psi(t)} - e^{r\Psi(a)}}{r}\right)^{\frac{1}{r}} |x-y| \tag{25}$$

for all $t \in [a, a+h)$. Hence, we obtain

$$|\mathcal{F}(x) - \mathcal{F}(y)| \leq \frac{2^{\frac{r-1}{r}} LK}{\Gamma(\alpha)} \left(\frac{e^{r\Psi(a+h)} - e^{r\Psi(a)}}{r}\right)^{\frac{1}{r}} |x-y|. \tag{26}$$

Since the coefficient on the right side of the above inequality tends to 0 as $h \to 0^+$, for $h > 0$ sufficiently small, operator $\mathcal{F}$ is contractive. The Banach fixed point theorem yields the existence of a unique fixed point. □

Remark 1. *If condition* ***H1*** *holds only for all* $(t,x),(t,y) \in [a, a+H) \times \mathbb{R}^N$ *for some* $H > 0$, *Theorem 3 remains valid with* $0 < h < H$.

Next, we present a version of Theorem 3 under the local "Lipschitz condition"

H1' There is $\gamma \in (0,1)$ such that

$$t^{1-\gamma}\|f(t,x) - f(t,y)\| \leq L\|x-y\| \quad \text{for all } (t,x),(t,y) \in [a,\infty) \times B_R$$

for some $R > 0$, where $B_R = \{z \in \mathbb{R}^N \mid \|z - e^{-\lambda\Psi(a)}x_a^0\| \leq R\}$.

Theorem 4. *Theorem 3 remains valid if* ***H1*** *is replaced by* ***H1'***.

Proof. Let p, q, and r be as in the proof of Theorem 3. Let us fix $h > 0$ such that

$$\|e^{-\lambda\Psi(a+h)}x_a^0 - e^{-\lambda\Psi(a)}x_a^0\| \leq \frac{R}{2}. \tag{27}$$

Let us consider the set

$$X_h = \left\{ u \in C[a, a+h) \,\middle|\, \sup_{t \in I_h} \|u(t) - e^{-\lambda\Psi(t)}x_a^0\| \leq \frac{R}{2} \right\}$$

equipped with the supremum norm $|\cdot|$ and define the operator $\mathcal{F}\colon X_h \to C[a, a+h)$ by (23). Clearly, $(X_h, |\cdot|)$ is a Banach space. Let us denote $M := \sup_{(s,x) \in I_h \times B_R} \|f(s,x)\|$. Then, for $(t,x) \in I_h \times X_h$,

$$\|\mathcal{F}(x)(t) - e^{-\lambda\Psi(t)}x_a^0\| \leq \frac{M}{\Gamma(\alpha)} \int_a^t \mathcal{K}_{\Psi}^{\alpha,\lambda}(t,s)\, ds.$$

The substitution $\lambda(\Psi(t) - \Psi(s)) = \sigma$ yields

$$\begin{aligned}\|\mathcal{F}(x)(t) - \mathrm{e}^{-\lambda\Psi(t)}x_a^0\| &\leq \frac{M}{\Gamma(\alpha)\lambda^\alpha}\int_0^{\lambda(\Psi(t)-\Psi(a))}\sigma^{\alpha-1}\mathrm{e}^{-\sigma}d\sigma \\ &= \frac{M\,\gamma(\alpha,\lambda(\Psi(t)-\Psi(a)))}{\Gamma(\alpha)\lambda^\alpha},\end{aligned}$$

where $\gamma(\cdot,\cdot)$ is the incomplete gamma function (see, e.g., [42]). Note that $\gamma(\alpha,\lambda(\Psi(t) - \Psi(a))) \to 0^+$ as $t \to a^+$. So, if $h > 0$ is sufficiently small, then $\mathcal{F}\colon X_h \to X_h$. Now, if $x \in X_h$, then

$$\begin{aligned}\sup_{t\in I_h}\|x(t) - \mathrm{e}^{-\lambda\Psi(a)}x_a^0\| &\leq \sup_{t\in I_h}\|x(t) - \mathrm{e}^{-\lambda\Psi(t)}x_a^0\| \\ &\quad + \sup_{t\in I_h}\|\mathrm{e}^{-\lambda\Psi(t)}x_a^0 - \mathrm{e}^{-\lambda\Psi(a)}x_a^0\| \\ &\leq \frac{R}{2} + \frac{R}{2} = R\end{aligned}$$

by the property of X_h and (27). Hence, $x(t) \in B_R$ for all $t \in I_h$. As a consequence, one can show exactly as in the proof of Theorem 3 that $\mathcal{F}$ is a contraction by assuming that $h > 0$ is small enough. The use of the Banach fixed point theorem completes the proof. □

The following examples illustrate the above existence results.

Example 1. *Let us consider the initial value problem*

$$\begin{aligned}{}^{C}D_a^{\alpha,\lambda,\Psi}x(t) &= c, \quad t \geq a, \\ x_{\lambda,\Psi}^{[0]}(a) &= x_a^0\end{aligned} \tag{28}$$

for some $c, x_a^0 \in \mathbb{R}$; $\alpha \in (0,1)$; $\lambda > 0$; and $\Psi \in C^1[a,\infty)$ satisfying condition **H2** *and $\Psi'(t) > 0$ for all $t \geq a$.*

It is obvious that condition **H1** is fulfilled for any $\gamma \in (0,1)$ and $L \geq 0$. Theorem 3 gives the existence of a unique solution to (28). In this case, it can be evaluated from the integral equation

$$\begin{aligned}x(t) &= \mathrm{e}^{-\lambda\Psi(t)}x_a^0 + \frac{c}{\Gamma(\alpha)}\int_a^t \mathcal{K}_\Psi^{\alpha,\lambda}(t,s)\,ds \\ &= \mathrm{e}^{-\lambda\Psi(t)}x_a^0 + \frac{c\,\gamma(\alpha,\lambda(\Psi(t)-\Psi(a)))}{\Gamma(\alpha)}.\end{aligned}$$

Example 2. *Let us consider the initial value problem*

$$\begin{aligned}{}^{C}D_a^{\alpha,\lambda,\Psi}x(t) &= \frac{\mathrm{e}^{-\lambda\Psi(t)}\Gamma(1-\beta)}{\Gamma(1-\alpha-\beta)}\left(\mathrm{e}^{\lambda\Psi(t)}x(t) - x_a^0\right)^{\frac{\alpha+\beta}{\beta}}, \quad t \geq a, \\ x_{\lambda,\Psi}^{[0]}(a) &= x_a^0\end{aligned} \tag{29}$$

for some $x_a^0 \in \mathbb{R}$; $\alpha, \beta \in (0,1)$; $\alpha + \beta < 1$; $\lambda > 0$; and $\Psi \in C^1[a,\infty)$ satisfying condition **H2** *and $\Psi'(t) > 0$ for all $t \geq a$.*

Let us estimate

$$|f(t,x)-f(t,y)| = \frac{e^{-\lambda\Psi(t)}\Gamma(1-\beta)}{\Gamma(1-\alpha-\beta)}\left|\left(e^{\lambda\Psi(t)}x - x_a^0\right)^{\frac{\alpha+\beta}{\beta}} - \left(e^{\lambda\Psi(t)}y - x_a^0\right)^{\frac{\alpha+\beta}{\beta}}\right|$$
$$= \frac{(\alpha+\beta)e^{-\lambda\Psi(t)}\Gamma(1-\beta)}{\beta\,\Gamma(1-\alpha-\beta)}\left|e^{\lambda\Psi(t)}\theta - x_a^0\right|^{\frac{\alpha}{\beta}}|x-y|$$

for some θ between x and y. If $x, y \in B_R$ for some $R > 0$, then $\theta \in B_R$, and one obtains

$$\begin{aligned}\left|e^{\lambda\Psi(t)}\theta - x_a^0\right|^{\frac{\alpha}{\beta}} &= e^{\frac{\alpha\lambda\Psi(t)}{\beta}}\left|\theta - e^{-\lambda\Psi(t)}x_a^0\right|^{\frac{\alpha}{\beta}} \\ &\le e^{\frac{\alpha\lambda\Psi(t)}{\beta}}\left(\left|\theta - e^{-\lambda\Psi(a)}x_a^0\right| + \left|e^{-\lambda\Psi(a)}x_a^0 - e^{-\lambda\Psi(t)}x_a^0\right|\right)^{\frac{\alpha}{\beta}} \\ &\le e^{\frac{\alpha\lambda\Psi(t)}{\beta}}(2R)^{\frac{\alpha}{\beta}}\end{aligned}$$

for all t sufficiently close to a, let us say that $t \in [a, a+H)$ for some $H > 0$. Consequently,

$$|f(t,x)-f(t,y)| \le t^{\gamma-1}\left[\frac{(\alpha+\beta)\,\Gamma(1-\beta)\,(2R)^{\frac{\alpha}{\beta}}}{\beta\,\Gamma(1-\alpha-\beta)}\,e^{\lambda\Psi(t)\left(1-\frac{\alpha}{\beta}\right)}t^{1-\gamma}\right]|x-y|.$$

Now, one can take the supremum of the bracket over all $t \in [a, a+H)$ as the constant L. It results that condition **H1′** is fulfilled for all $t \in [a, a+H)$. The use of Theorem 4 along with a remark analogous to Remark 1 proves the existence of a unique solution to (29). It can be easily verified that the solution is given by

$$x(t) = e^{-\lambda\Psi(t)}\left(x_a^0 + (\Psi(t)-\Psi(a))^{-\beta}\right).$$

Next, we give a result on the nonexistence of a blow-up solution. This means that under certain conditions, every solution of the initial value problem is bounded. We need one more assumption.

H3 There is $\gamma \in (0,1)$ such that

$$t^{1-\gamma}\|f(t,x)\| \le F(t)\omega(\|x\|) \quad \text{for all } (t,x) \in [a,\infty)\times\mathbb{R}^N.$$

Theorem 5. *Let $\alpha \in (0,1)$, $\lambda > 0$, and $\Psi \in C^1[a,\infty)$ satisfy condition* **H2** *and $\Psi'(t) > 0$ for all $t \in (a,\infty)$. Let us assume that condition* **H3** *is fulfilled for some positive function $F \in C[a,\infty)$. Moreover, let (22) hold and*

$$\int_{v_0}^{\infty}\frac{\sigma^{r-1}d\sigma}{\omega(\sigma)^r} = \infty \tag{30}$$

for some $r > (\alpha+\gamma-1)^{-1}$ and $v_0 \ge 0$. Then, there is no blow-up solution to initial value problem (7) and (8).

Proof. Let us set

$$\varepsilon = \frac{1}{2}\left(\alpha+\gamma-1-\frac{1}{r}\right), \quad p = \frac{1}{1-\alpha+\varepsilon}, \quad q = \frac{1}{1-\gamma+\varepsilon}.$$

Then, $0 < \varepsilon < \frac{1}{2}\min\{\alpha,\gamma\}$, $1 < p < (1-\alpha)^{-1}$, $1 < q < (1-\gamma)^{-1}$, $1 < r$, and

$$\frac{1}{p}+\frac{1}{q} = 2-\alpha-\gamma+2\varepsilon = 1-\frac{1}{r}.$$

Let $x\colon [a,T) \to \mathbb{R}^N$ be a continuous solution of integral Equation (9) with $a < T < \infty$, and $\lim_{t\to T^-} \|x(t)\| = \infty$. Analogously to estimates (24), by using **H3**, we derive, for $t \in [a,T)$,

$$\begin{aligned}\|x(t)\| &\le \|x_a^0\| + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{K}_\Psi^{\alpha,\lambda}(t,s)\|f(s,x(s))\|\,ds \\ &\le \|x_a^0\| + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{K}_\Psi^{\alpha,\gamma,\lambda}(t,s)s^{1-\gamma}\|f(s,x(s))\|\,ds \\ &\le \|x_a^0\| + \frac{1}{\Gamma(\alpha)} \int_a^t \mathcal{K}_\Psi^{\alpha,\gamma,\lambda}(t,s)F(s)\omega(\|x(s)\|)\,ds.\end{aligned}$$

From Theorem 2, it follows that

$$\begin{aligned}\Xi(\|x(t)\|^r) &= \int_{v_0}^{\|x(t)\|^r} \frac{dz}{\omega(z^{1/r})^r} \\ &= r\int_{v_0^{1/r}}^{\|x(t)\|} \frac{\sigma^{r-1}d\sigma}{\omega(\sigma)^r} \\ &\le \Xi(2^{r-1}\|x_a^0\|^r) + \frac{2^{r-1}K^r}{\Gamma(\alpha)^r}\int_a^t \mathrm{e}^{r\Psi(s)}\Psi'(s)F(s)^r ds \quad \text{for all } t \in [a,T),\end{aligned} \tag{31}$$

where K is given by (14). Since

$$\lim_{t\to T^-} \int_{v_0^{1/r}}^{\|x(t)\|} \frac{\sigma^{r-1}d\sigma}{\omega(\sigma)^r} = \infty \tag{32}$$

and the limit of the right-hand side of inequality (31) is finite as $t \to T^-$, we have a contradiction. This completes the proof. □

Example 3. *Let us consider the initial value problem*

$$\begin{aligned}{}^{C}D_1^{\frac{1}{4},\lambda,\Psi}x(t) &= \frac{x(t)\,\Gamma(\frac{3}{4})}{\sqrt{\pi}\left(\sqrt{\sqrt{t}-1} + (\sqrt{t}-1)^{\frac{1}{4}}\right)}, \quad t \ge 1, \\ x_{\lambda,\Psi}^{[0]}(1) &= 1\end{aligned} \tag{33}$$

with $\Psi(t) = \sqrt{t}$ *for* $t \ge 1$ *and* $\lambda > 0$.

For $\gamma = \frac{7}{8}$, condition **H3** has the form

$$t^{1-\gamma}|f(t,x)| \le \frac{t^{\frac{7}{8}}\,\Gamma(\frac{3}{4})}{\sqrt{\pi}\left(\sqrt{\sqrt{t}-1} + (\sqrt{t}-1)^{\frac{1}{4}}\right)}\,|x|.$$

Since

$$\int_{v_0}^{\infty} \frac{\sigma^{r-1}\,d\sigma}{\sigma^r} = \int_{v_0}^{\infty} \frac{d\sigma}{\sigma} = \infty$$

for any fixed $v_0 > 0$ and arbitrary $r > 8$, all assumptions of Theorem 5 are fulfilled. Therefore, initial value problem (33) does not possess a blow-up solution. It can be verified that its solution is given by

$$x(t) = \mathrm{e}^{-\lambda\sqrt{t}}\left(1 + (\sqrt{t}-1)^{-\frac{1}{4}}\right).$$

Remark 2. *In Example 3, the majorant function (ω) had the form $\omega(z) = z$. Another often-used majorant satisfying (30) is $\omega(z) = [\ln(z^r + c)]^{\frac{1}{r}}$ for $c > 0$. Then,*

$$\int_{v_0}^{\infty} \frac{\sigma^{r-1} d\sigma}{\omega(\sigma)^r} = \int_{v_0}^{\infty} \frac{\sigma^{r-1} d\sigma}{\ln(\sigma^r + c)} = \frac{1}{r} \int_{v_0^r}^{\infty} \frac{d\tau}{\ln(\tau + c)} = \infty$$

for any fixed $v_0 \geq 0$ and appropriate r.

5. Conclusions

In this paper, new definitions of fractional derivatives of order α for $n-1 < \alpha < n \in \mathbb{N}$ were presented, namely, the tempered Ψ–Riemann–Liouville fractional derivative and the tempered Ψ–Caputo fractional derivative. Both definitions were given for only C^{n-1}–functions, unlike the recent definition of the Caputo one from [31], which required a C^n–function. Next, a new Henry-type nonlinear integral inequality with a weakly singular kernel was derived. It was applied to prove the existence of a unique solution of an initial value problem corresponding to fractional differential equations with the tempered Ψ–Caputo derivative. A result on nonexistence of a blow-up solution was also proved. Illustrative examples of initial value problems were given.

Additional potential uses of the new nonlinear integral inequality include investigating the stability, asymptotics, and controllability of solutions to initial value problems; the study of boundedness and other asymptotic properties of nonoscillatory solutions like in [43,44]; or generalization to retarded or stochastic integral inequalities as mentioned in Section 1.

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