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Exponential Stability of the Numerical Solution of a Hyperbolic System with Nonlocal Characteristic Velocities

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Abstract: In this paper, we investigate the problem of the exponential stability of a stationary solution for a hyperbolic system with nonlocal characteristic velocities and measurement error. The formulation of the initial boundary value problem of boundary control for the specified hyperbolic system is given. A difference scheme is constructed for the numerical solution of the considered initial boundary value problem. The definition of the exponential stability of the numerical solution in ℓ^2-norm with respect to a discrete perturbation of the equilibrium state of the initial boundary value difference problem is given. A discrete Lyapunov function for a numerical solution is constructed, and a theorem on the exponential stability of a stationary solution of the initial boundary value difference problem in ℓ^2-norm with respect to a discrete perturbation is proved.

Keywords: hyperbolic system; nonlocal characteristic velocity; stability; explicit difference scheme

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1. Introduction

The basic concepts of partial differential equation theory were formed during the study of classical problems of mathematical physics and have now been well studied. However, modern problems of natural science lead to the need to formulate and research qualitatively new problems, with a prominent example being the class of nonlocal problems.

Nonlocal differential problems typically refer to those that involve conditions dependent on the values of the solution over a broader range or even over the entire time space, such as conditions involving integral relationships linking the value of the desired solution and possibly its derivatives, at all internal spatial points of the domain [1–3].

In recent decades, nonlocal problems for partial differential equations have been actively studied by many scientists. The investigation of nonlocal problems is caused by both theoretical interest and practical necessity. This is due to the fact that the mathematical models of various physical, chemical, biological, and environmental processes often involve problems in which, instead of classical relations, a certain connection is set between the values of the desired function (or its derivatives) within the considered domain. Problems of this type may arise in the study of phenomena related to plasma physics [4], heat propagation [5], the process of moisture transfer in capillary-porous media [6], issues of demography and mathematical biology [7], and some technological processes [8].

Problems with nonlocal relations arise in the mathematical modeling of various physical processes in cases where the boundary of a real process flow is not available for direct measurements, but information about its flow in internal points of the domain can be obtained.

Among the nonlocal problems, a class of problems for hyperbolic systems with integral relations in characteristic velocities can be distinguished.

Note that the problem of exponential stability around a stationary solution for hyperbolic equations with a nonlocal characteristic velocity was investigated in [9–20]. A detailed review of these works can be found in [13].

However, refs. [1–3] should be singled out separately, where the problem of exponential stability for hyperbolic equations with integral relations in characteristic velocities was studied.

In [1], the issue on the exponential stability of a solution was investigated for a scalar conservation law with a positive nonlocal characteristic velocity. Based on Lyapunov stability theory, the exponential stability of both a linearized and, in some cases, even a nonlinear control system was studied.

In [2], in using the results of [1], the global stability of the solution for a class of nonlinear transport equations with a positive nonlocal characteristic velocity was investigated. The results of the exponential stability were transferred to the case of a discrete problem based on an upwind difference scheme.

In [3], the exponential stability of the solution was proven for both a scalar conservation law with a positive nonlocal characteristic velocity and measurement error, and for a discrete analog of this problem.

The main purposes of this paper were to formulate and study qualitatively new nonlocal problems for one-dimensional hyperbolic systems with nonlocal characteristic velocities, as well as to develop effective ways to prove the exponential stability of the numerical solution to the initial boundary value difference problems for them.

One of the main requirements that a management system must meet is ensuring its sustainability. The conditions under which the system is stable essentially determine the operating conditions of the system. Therefore, during the design process of automatic control systems, stability analyses of the systems are necessarily conducted.

The theoretical foundation for analyzing the stability of nonlinear control systems is currently provided through the method of Lyapunov functions. A central aspect within this method is the construction of a Lyapunov function. For a specific nonlinear hyperbolic system with nonlocal characteristic speeds, the corresponding Lyapunov function enables the resolution of a complex set of problems that hold significant practical importance.

The construction of Lyapunov functions in a given domain of the phase space of a nonlinear hyperbolic system with nonlocal characteristic velocities is a difficult problem. There are currently no general constructive methods for solving this problem applicable to sufficiently wide classes of nonlinear hyperbolic systems with nonlocal characteristic velocities. Most methods for constructing the Lyapunov function are based on using the specifics of the system under study, which in some cases, allow specifying the class of functions to which the desired Lyapunov function belongs. A classic example of such a situation is mechanical systems, in which the total energy of the system can often be chosen as the Lyapunov function. However, in general, the class of Lyapunov function is usually not known in advance.

Therefore, the development of constructive methods for constructing Lyapunov functions suitable for a wide class of nonlinear hyperbolic systems with nonlocal characteristic velocities is of great theoretical and practical interest.

In this paper, we propose and justify a method for the numerical construction of the Lyapunov function and its use for solving a number of stability problems of nonlinear hyperbolic systems with nonlocal characteristic velocities.

The main purpose of the paper is to develop and justify a numerical method for constructing Lyapunov functions for analyzing the stability of a nonlinear hyperbolic system with nonlocal characteristic velocities. To achieve this goal, the Lyapunov direct method is used, a difference method for solving a mixed problem for a nonlinear hyperbolic system with nonlocal characteristic velocities.

The main result of this work is an approach to solving the problem of constructing numerical Lyapunov functions in the problem of analyzing the stability of the zero solution of a nonlinear hyperbolic system with nonlocal characteristic velocities. New sufficient conditions for the exponential stability of a trivial solution to a nonlinear hyperbolic system with nonlocal characteristic velocities are obtained.

Algorithms for the numerical method of constructing the Lyapunov function on a grid based on reducing the problem of constructing the Lyapunov function in the neighborhood of zero to a linear problem have been developed. Algorithms for checking the properties of the Lyapunov function constructed on a throughout, the entire considered domain, are provided.

The Lyapunov function method is widely used in many control problems. Analytical methods typically play the primary role in constructing Lyapunov functions. However, these methods do not provide a constructive solution to the problem of constructing the Lyapunov function for a nonlinear hyperbolic system with nonlocal characteristic velocities. Therefore, the further development and complication of problems arising in control theory, mechanics, and computational mathematics currently necessitate algorithmizing and implementing the second Lyapunov method.

In this regard, the method of numerical construction of Lyapunov functions for a wide class of nonlinear systems developed in the article has theoretical and practical value.

Papers [20–25] are devoted to the problems of constructing and investigating the exponential stability of the numerical solution to mixed problems for hyperbolic systems. They proposed systematic approaches to the construction and study of the computational models' adequacy for a mixed dissipative problem posed for symmetric t-hyperbolic systems. One-dimensional and two-dimensional hyperbolic systems with variable coefficients and minor terms, as well as with standard dissipative boundary conditions, have been considered. Difference schemes for the numerical calculation of stable solutions to the tasks have been constructed. Discrete analogues of Lyapunov function have been constructed for the numerical verification of the solutions' stability in the problems. A priori estimates of the discrete analogue of the Lyapunov function have been obtained. These estimates allow us to assert the exponential stability of a numerical solution. Theorems on the exponential stability of the solution of the boundary value problem for a hyperbolic system and on the stability of the difference scheme in Sobolev spaces have been proved. These stability theorems enable us to prove the convergence of the numerical solution.

Note that in all papers [1–3], the case of a positive nonlocal characteristic velocity for the scalar case is considered. However, in works [20–25], nonlocal characteristic velocities are not considered (they are limited to local characteristic velocities).

The main difficulty of this study was related precisely to the nonlocal characteristic velocities of the difference scheme.

In this study, the results of the works [1–3,20–25] were transferred to the case of a hyperbolic system with nonlocal characteristic velocities.

Thus, in this article, we study the issue of the equilibrium state's numerical controllability of a hyperbolic system's difference scheme of conservation laws with nonlocal characteristic velocities. We prove the result of the numerical controllability of the local and global equilibrium states, i.e. there is a control that transfers the numerical solution from any given initial data to any desired final data within a certain period of time, provided that the initial and final data are both close to a given equilibrium, and also when there are no restrictions on the distance between the initial and final data.

So the main result of this paper is Theorem 1, which states that the stationary state of the proposed initial boundary value difference problem is stable in ℓ^2-norm with respect to any bounded discrete perturbation function.

2. Materials and Methods

Statement of a Mixed Problem

Consider the following symmetric t-hyperbolic system:

$$\frac{\partial U}{\partial t} + \mathrm{M}(A(t))\frac{\partial U}{\partial x} = 0, \quad t \in [0, +\infty), \ x \in [0,1], \tag{1}$$

where

$$U = \begin{pmatrix} \bar{U} \\ \bar{\bar{U}} \end{pmatrix}, \ \mathrm{M}(A(t)) = \begin{pmatrix} \bar{\mathrm{M}}(\bar{A}(t)) & 0 \\ 0 & -\bar{\bar{\mathrm{M}}}\left(\bar{\bar{A}}(t)\right) \end{pmatrix},$$

$$\bar{U} \triangleq ({}_1u, \ {}_2u, \cdots, {}_mu)^T, \ \bar{\bar{U}} \triangleq ({}_{m+1}u, \ {}_{m+2}u, \cdots, {}_nu)^T,$$

$$\bar{A}(t) \triangleq ({}_1a(t), {}_2a(t), \cdots, {}_ma(t))^T, \ \bar{\bar{A}}(t) \triangleq ({}_{m+1}a(t), {}_{m+2}a(t), \cdots, {}_na(t))^T,$$

$$\bar{\mathrm{M}}(\bar{A}(t)) \triangleq diag({}_1\mu({}_1a(t)), {}_2\mu({}_2a(t)), \cdots, {}_m\mu({}_ma(t))),$$

$$\bar{\bar{\mathrm{M}}}\left(\bar{\bar{A}}(t)\right) \triangleq diag({}_{m+1}\mu({}_{m+1}a(t)), {}_{m+2}\mu({}_{m+2}a(t)), \cdots, {}_n\mu({}_na(t))),$$

and ${}_i\mu(s), \ i = \overline{1,n}$ are some given functions.

Here, $\mathrm{M}(A(t))$ is a matrix of characteristic velocities depending on the integral of unknown vector function $U(t,x)$ over the whole domain $[0,1]$:

$$A(t) = \int_0^1 U(t,x)\,dx, \ \ t \in (0, +\infty) \tag{2}$$

Initial conditions for system (1) are as follows:

$$U(0,x) = \Phi(x), \quad x \in [0,1]. \tag{3}$$

Here, $\Phi(x) \triangleq ({}_1\varphi(x), {}_2\varphi(x), \cdots, {}_n\varphi(x))^T$ is a given initial vector function.

According to the theory of hyperbolic systems, the boundary conditions for system (1) are set as follows:

$$\begin{cases} for \ \ x = 0: \ \ \bar{\mathrm{M}}(\bar{A}(t))\bar{U}(t,0) = \bar{\mathbf{V}}(t), \\ for \ \ x = 1: \ \ -\bar{\bar{\mathrm{M}}}\left(\bar{\bar{A}}(t)\right)\bar{\bar{U}}(t,1) = \bar{\bar{\mathbf{V}}}(t), \end{cases} \quad 0 < t < +\infty. \tag{4}$$

where $\bar{\mathbf{V}}(t) \triangleq ({}_1V(t), {}_2V(t), \cdots, {}_mV(t))^T, \bar{\bar{\mathbf{V}}}(t) \triangleq ({}_{m+1}V(t), {}_{m+2}V(t), \cdots, {}_nV(t))^T$ are controllers of vector functions.

It follows from [1,2] that with the appropriate choices of $\mathrm{M}(A(t)), \ \Phi(x)$, and $\mathbf{V}(t) \triangleq \left(\bar{\mathbf{V}}(t), \ \bar{\bar{\mathbf{V}}}(t)\right)^T$, it is possible to prove the correctness of the formulation to the mixed problem (1)–(4).

In this work, we confine ourselves to the family of characteristic speeds of the type

$$\begin{cases} \bar{\mathbf{V}}(t) - \bar{\mathrm{M}}^*\bar{U}^* = \bar{R}\{\bar{\mathrm{M}}(\bar{A}(t))[\bar{U}(t,1) + \bar{\Delta}(t)] - \bar{\mathrm{M}}^*\bar{U}^*\}, \\ -\bar{\bar{\mathbf{V}}}(t) + \bar{\bar{\mathrm{M}}}^*\bar{\bar{U}}^* = \bar{\bar{R}}\left\{-\bar{\bar{\mathrm{M}}}\left(\bar{\bar{A}}(t)\right)\left[\bar{\bar{U}}(t,0) + \bar{\bar{\Delta}}(t)\right] + \bar{\bar{\mathrm{M}}}^*\bar{\bar{U}}^*\right\}, \end{cases} \quad t \in (0, +\infty)$$

where

$$\begin{cases} \bar{\mathrm{M}}^* \triangleq \bar{\mathrm{M}}(\bar{U}^*) = diag({}_1\mu({}_1u^*), {}_2\mu({}_2u^*), \cdots, {}_m\mu({}_mu^*)), \\ \bar{\bar{\mathrm{M}}}^* \triangleq \bar{\bar{\mathrm{M}}}\left(\bar{\bar{U}}^*\right) = diag({}_{m+1}\mu({}_{m+1}u^*), {}_{m+2}\mu({}_{m+2}u^*), \cdots, {}_n\mu({}_nu^*)), \end{cases}$$

$$\begin{cases} \bar{U}^* \triangleq ({}_1u^*, {}_2u^*, \cdots, {}_mu^*)^T, \\ \bar{\bar{U}}^* \triangleq ({}_{m+1}u^*, {}_{m+2}u^*, \cdots, {}_nu^*)^T, \end{cases} \quad U^* \triangleq \begin{pmatrix} \bar{U}^* \\ \bar{\bar{U}}^* \end{pmatrix}, \quad \mathrm{M}^* \triangleq \begin{pmatrix} \bar{\mathrm{M}}^* & 0 \\ 0 & \bar{\bar{\mathrm{M}}}^* \end{pmatrix},$$
$$\begin{cases} \bar{R} \triangleq diag({}_1r, {}_2r, \cdots, {}_mr), \\ \bar{\bar{R}} \triangleq diag({}_{m+1}r, {}_{m+2}r, \cdots, {}_nr), \end{cases} \quad R \triangleq \begin{pmatrix} \bar{R} & 0 \\ 0 & \bar{\bar{R}} \end{pmatrix},$$
$$\begin{cases} \bar{\Delta}(t) \triangleq ({}_1\delta(t), {}_2\delta(t), \cdots, {}_m\delta(t))^T, \\ \bar{\bar{\Delta}}(t) \triangleq ({}_{m+1}\delta(t), {}_{m+2}\delta(t), \cdots, {}_n\delta(t))^T, \end{cases} \quad \Delta(t) \triangleq \begin{pmatrix} \bar{\Delta}(t) \\ \bar{\bar{\Delta}}(t) \end{pmatrix}.$$

and ${}_ir \in [0,1), \ \ i = \overline{1,n}$ are given coefficients, and U^*, where ${}_iu^* > 0, \ \ i = \overline{1,n}$ is a given state of equilibrium, and $\Delta(t)$ is limited disturbance. Note that for a given equilibrium state U^*, the value of the characteristic vector function is calculated as follows:

$$\mathrm{M}(A(t))|_{U=U^*} = \mathrm{M}(U^*).$$

In this paper, we limit ourselves to the family of characteristic velocities of the type

$${}_i\mu(s) = \frac{{}_iP}{{}_iQ + s}, \quad s \in [0, +\infty), \quad i = \overline{1,n} \tag{5}$$

with ${}_iP > 0, \ \ {}_iQ > 0$, and $\ \forall i \in \{m+1, m+2, \cdots, n\}$.

Thus, consider the following mixed control problem

$$\begin{cases} \frac{\partial U}{\partial t} + \mathrm{M}(A(t))\frac{\partial U}{\partial x} = 0, \quad t \in (0,+\infty), x \in (0,1), \\ U(0,x) = \Phi(x), \quad x \in (0,1), \\ \bar{\mathbf{V}}(t) - \bar{\mathrm{M}}^*\bar{U}^* = \bar{R}\{\bar{\mathrm{M}}(\bar{A}(t))[\bar{U}(t,1) + \bar{\Delta}(t)] - \bar{\mathrm{M}}^*\bar{U}^*\}, \quad t \in (0,+\infty), \\ -\bar{\bar{\mathbf{V}}}(t) + \bar{\bar{\mathrm{M}}}^*\bar{\bar{U}}^* = \bar{\bar{R}}\Big\{-\bar{\bar{\mathrm{M}}}\Big(\bar{\bar{A}}(t)\Big)\Big[\bar{\bar{U}}(t,0) + \bar{\bar{\Delta}}(t)\Big] + \bar{\bar{\mathrm{M}}}^*\bar{\bar{U}}^*\Big\}, \quad t \in (0,+\infty), \\ \bar{\mathrm{M}}(\bar{A}(t))\bar{U}(t,0) = \bar{\mathbf{V}}(t), \quad -\bar{\bar{\mathrm{M}}}\Big(\bar{\bar{A}}(t)\Big)\bar{\bar{U}}(t,1) = \bar{\bar{\mathbf{V}}}(t), \ t \in [0,+\infty), \\ A(t) = \int_0^1 U(t,x)\,dx, \quad t \in (0,+\infty). \end{cases} \tag{6}$$

where U is the vector function to be defined.

Consider transformations with respect to equilibrium U^*:

$$\tilde{U}(t,x) = U(t,x) - U^*, \quad \tilde{A}(t) = A(t) - U^*, \quad \tilde{\Phi}(x) = \Phi(x) - U^*, \tilde{\mathrm{M}}_{\tilde{A}}(t) = \mathrm{M}\big(U^* + \tilde{A}(t)\big).$$

Then, system (6) with (5) for $t \in (0,+\infty)$ can be rewritten as follows:

$$\begin{cases} \frac{\partial \tilde{U}}{\partial t} + \tilde{\mathrm{M}}_{\tilde{A}}(t)\frac{\partial \tilde{U}}{\partial x} = 0, \quad x \in (0,1), \\ \tilde{U}(0,x) = \tilde{\Phi}(x), \quad x \in (0,1), \\ \tilde{\bar{\mathbf{V}}}(t) = \bar{R}\tilde{\bar{\mathrm{M}}}_{\tilde{\bar{A}}}(t)\Big[\tilde{\bar{U}}(t,1) + \bar{\Delta}(t)\Big] + (\bar{E} - \bar{R})\Big\{\bar{\mathrm{M}}^* - \tilde{\bar{\mathrm{M}}}_{\tilde{\bar{A}}}(t)\Big\}\bar{U}^*, \\ \tilde{\bar{\bar{\mathbf{V}}}}(t) = \bar{\bar{R}}\tilde{\bar{\bar{\mathrm{M}}}}_{\tilde{\bar{\bar{A}}}}(t)\Big[\tilde{\bar{\bar{U}}}(t,0) + \bar{\bar{\Delta}}(t)\Big] + \Big(\bar{\bar{E}} - \bar{\bar{R}}\Big)\Big\{\bar{\bar{\mathrm{M}}}^* - \tilde{\bar{\bar{\mathrm{M}}}}_{\tilde{\bar{\bar{A}}}}(t)\Big\}\bar{\bar{U}}^*, \\ \tilde{\bar{\mathrm{M}}}_{\tilde{\bar{A}}}(t)\tilde{\bar{U}}(t,0) = \tilde{\bar{\mathbf{V}}}(t), \quad \tilde{\bar{\bar{\mathrm{M}}}}_{\tilde{\bar{\bar{A}}}}(t)\tilde{\bar{\bar{U}}}(t,1) = \tilde{\bar{\bar{\mathbf{V}}}}(t). \end{cases} \tag{7}$$

Here,

$$\tilde{U} = \begin{pmatrix} \tilde{\bar{U}} \\ \tilde{\bar{\bar{U}}} \end{pmatrix} = \begin{pmatrix} \bar{U}(t,x) - \bar{U}^* \\ \bar{\bar{U}}(t,x) - \bar{\bar{U}}^* \end{pmatrix}, \quad \tilde{\mathrm{M}}_{\tilde{A}}(t) = \begin{pmatrix} \tilde{\bar{\mathrm{M}}}_{\tilde{\bar{A}}}(t) & 0 \\ 0 & -\tilde{\bar{\bar{\mathrm{M}}}}_{\tilde{\bar{\bar{A}}}}(t) \end{pmatrix},$$

$${}_i\mu(s) = \frac{{}_iP}{{}_iQ + s}, \quad {}_iP > 0, \quad {}_iQ > 0, \ s \in [0,+\infty), \ \ i = \overline{1,n}.$$

$$\tilde{A}(t) = \int_0^1 \tilde{U}(t,x)dx \quad where \quad \int_0^1 {}_i\tilde{u}(t,x)dx \geq -{}_iu^*, \ i = \overline{1,n},$$

$$\tilde{\bar{\mathrm{M}}}_{\tilde{\bar{A}}}(t) = \bar{\mathrm{M}}\left(\bar{U}^* + \tilde{\bar{A}}(t)\right),\ \tilde{\bar{\bar{\mathrm{M}}}}_{\tilde{\bar{\bar{A}}}}(t) = \bar{\bar{\mathrm{M}}}\left(\bar{\bar{U}}^* + \tilde{\bar{\bar{A}}}(t)\right),$$
$$\bar{E} = diag\left(\underbrace{1,1,\cdots,1}_{m}\right),\ \ \bar{\bar{E}} = diag\left(\underbrace{1,1,\cdots,1}_{n-m}\right).$$

Using expressions specified for functions ${}_i\mu,\ \ i = \overline{1,n}$ of the characteristic velocities (5) in Equation (7), we have

$$\begin{gathered}\left\{\mathrm{M}^* - \tilde{\mathrm{M}}_{\tilde{A}}(t)\right\}U^* = \left[\ diag\left(\tfrac{{}_1P}{{}_1Q+{}_1u^*},\cdots \tfrac{{}_nP}{{}_nQ+{}_nu^*}\right) - diag\left(\tfrac{{}_1P}{{}_1Q+{}_1u^*+{}_1\tilde{a}(t)},\cdots \tfrac{{}_nP}{{}_nQ+{}_nu^*+{}_n\tilde{a}(t)}\right)\ \right]U^* = \\ = diag\left(\ \tfrac{{}_1P_1\tilde{a}(t)}{({}_1Q+{}_1u^*)({}_1Q+{}_1u^*+{}_1\tilde{a}(t))},\cdots,\tfrac{P_n\tilde{a}_n(t)}{({}_nQ+{}_nu^*)[{}_nQ+{}_nu^*+{}_n\tilde{a}(t)]}\ \right)U^* = \Omega\tilde{\mathrm{M}}_{\tilde{A}}(t)\tilde{A}(t),\end{gathered} \tag{8}$$

where

$$\Omega \triangleq diag({}_1\varpi, {}_2\varpi,\cdots,{}_n\varpi), \qquad {}_i\varpi = \frac{{}_iu^*}{{}_iQ + {}_iu^*},\ \ i = \overline{1,n}.$$

Note that the matrix inequality is valid for $\Omega < E$.

For the convenience of recording, we omit the «∼» symbol. Then, for $t \in (0,+\infty)$, the system of Equation (7) with Equation (8) can be rewritten in the following form:

$$\begin{cases} \frac{\partial U}{\partial t} + \mathrm{M}_A(t)\frac{\partial U}{\partial x} = 0, \quad x \in (0,1), \\ U(0,x) = \Phi(x), \quad x \in (0,1), \\ \bar{\mathbf{V}}(t) = \bar{R}\bar{\mathrm{M}}_{\bar{A}}(t)[\bar{U}(t,1) + \bar{\Delta}(t)] + (\bar{E} - \bar{R})\{\bar{\mathrm{M}}^* - \bar{\mathrm{M}}_{\bar{A}}(t)\}\bar{U}^*, \\ \bar{\bar{\mathbf{V}}}(t) = \bar{\bar{R}}\bar{\bar{\mathrm{M}}}_{\bar{\bar{A}}}(t)\left[\bar{\bar{U}}(t,0) + \bar{\bar{\Delta}}(t)\right] + \left(\bar{\bar{E}} - \bar{\bar{R}}\right)\left\{\bar{\bar{\mathrm{M}}}^* - \bar{\bar{\mathrm{M}}}_{\bar{\bar{A}}}(t)\right\}\bar{\bar{U}}^*, \\ \bar{\mathrm{M}}_{\bar{A}}(t)\bar{U}(t,0) = \bar{\mathbf{V}}(t),\ \ \bar{\bar{\mathrm{M}}}_{\bar{\bar{A}}}(t)\bar{\bar{U}}(t,1) = \bar{\bar{\mathbf{V}}}(t). \end{cases} \tag{9}$$

Here,

$$U = \begin{pmatrix} \bar{U}(t,x) - \bar{U}^* \\ \bar{\bar{U}}(t,x) - \bar{\bar{U}}^* \end{pmatrix},\ \ \mathrm{M}_A(t) = \begin{pmatrix} \bar{\mathrm{M}}_{\bar{A}}(t) & 0 \\ 0 & -\bar{\bar{\mathrm{M}}}_{\bar{\bar{A}}}(t) \end{pmatrix}.$$
$$\bar{\mathrm{M}}_{\bar{A}}(t) = \bar{\mathrm{M}}(\bar{U}^* + \bar{A}(t)),\ \ \bar{\bar{\mathrm{M}}}_{\bar{\bar{A}}}(t) = \bar{\bar{\mathrm{M}}}\left(\bar{\bar{U}}^* + \bar{\bar{A}}(t)\right),$$

3. Exponential Stability of the Numerical Solution

In this section, we establish the exponential stability of the numerical solution to the initial boundary value difference problem.

To obtain the initial boundary value difference problem, we will apply an upwind difference scheme for the numerical calculation of system (6).

To do this, we cover the spatial domain [0,1] with a uniform grid $\Omega_h = \{x_j = ih,\ j = \overline{0,J}\}$, where h is a step on x. With the integral $A(t)$ for each value on $t^k \triangleq k\tau$ (τ is a step by time), $k \in \{0,1,2,\cdots\}$, we use the quadrature formula to calculate the following:

$$A^k \triangleq \left({}_1a^k, {}_2a^k,\cdots,{}_na^k\right)^T,\ \ {}_ia^k = h\sum_{j=1}^{J}{}_iu_j^k,\ i = \overline{1,m}, {}_ia^k = h\sum_{j=0}^{J-1}{}_iu_j^k,\ i = \overline{m+1,n},\ \ k \in \{0,1,2,\cdots\}.$$

Next, we define a discrete value M^k:

$$\mathrm{M}^k \triangleq \mathrm{M}\left(A^k\right) \equiv diag\left({}_1\mu^k, {}_2\mu^k,\cdots,{}_n\mu^k\right), {}_i\mu^k \triangleq \mu\left({}_ia^k\right) = \frac{{}_iP}{{}_iQ + {}_ia^k},\ {}_iP > 0,\ \ {}_iQ > 0,\ \ i = \overline{1,n};\ \ k \in \{0,1,2,\cdots\}.$$

Assume that the Courant–Friedrichs–Levy (CFL) condition is satisfied:

$$0 < \Lambda^k \triangleq \frac{\tau}{h}\mathrm{M}^k < E, \quad k \in \{0,1,2,\cdots\}, \tag{10}$$

where $\Lambda^k = diag\left({}_1\lambda^k, {}_2\lambda^k,\cdots,{}_n\lambda^k\right),\ \ {}_i\lambda^k = \frac{\tau}{h}{}_i\mu^k,\ \ i = \overline{1,n}$, and $\ k \in \{1,2,\cdots K\}$.

For the numerical solution of system (6), we propose an upwind difference scheme:

$$\begin{cases} \bar{U}_j^{k+1} = \left(\bar{E} - \bar{\Lambda}^k\right)\bar{U}_j^k + \bar{\Lambda}^k \bar{U}_{j-1}^k, \quad j = \overline{1, J}; \; k \in \{0, 1, \cdots\}; \\ \bar{\bar{U}}_j^{k+1} = \left(\bar{\bar{E}} - \bar{\bar{\Lambda}}^k\right)\bar{\bar{U}}_j^k + \bar{\bar{\Lambda}}^k \bar{\bar{U}}_{j+1}^k, \quad j = \overline{0, J-1}; \; k \in \{0, 1, \cdots\}; \\ \bar{U}_0^{k+1} = \bar{R}\bar{U}_J^{k+1} + (\bar{E} - \bar{R})\left(\bar{\mathrm{M}}^k\right)^{-1}\bar{\mathrm{M}}^* \bar{U}^* + \bar{R}\bar{\Delta}^{k+1}, \quad k \in \{0, 1, \cdots\}; \\ \bar{\bar{U}}_J^{k+1} = \bar{\bar{R}}\bar{\bar{U}}_0^{k+1} + \left(\bar{\bar{E}} - \bar{\bar{R}}\right)\left(\bar{\bar{\mathrm{M}}}^k\right)^{-1}\bar{\bar{\mathrm{M}}}^* \bar{\bar{U}}^* + \bar{\bar{R}}\bar{\bar{\Delta}}^{k+1}, \quad k \in \{0, 1, \cdots\}; \\ U_j^0 = \Phi(x_j), \quad j = \overline{0, J}. \end{cases} \tag{11}$$

Here,

$$\bar{\Lambda}^k = diag\left({}_1\lambda^k, {}_2\lambda^k, \cdots, {}_m\lambda^k\right), \; \bar{\bar{\Lambda}}^k = diag\left({}_{m+1}\lambda^k, {}_{m+2}\lambda^k, \cdots, {}_n\lambda^k\right),$$

$$\bar{R} = diag({}_1r, {}_2r, \cdots, {}_mr), \; \bar{\mathrm{M}}^* = diag({}_1\mu({}_1u^*), {}_2\mu({}_2u^*), \cdots, {}_m\mu({}_mu^*)),$$
$$\bar{U}^* = ({}_1u^*, {}_2u^*, \cdots, {}_mu^*)^T, \; \bar{U}_j^k = \left({}_1u_j^k, {}_2u_j^k, \cdots, {}_mu_j^k\right)^T, \; \bar{\Delta}^k \triangleq \left({}_1\delta^k, {}_2\delta^k, \cdots, {}_m\delta^k\right)^T$$

$$\bar{\bar{\mathrm{M}}}^* = diag({}_{m+1}\mu({}_{m+1}u^*), {}_{m+2}\mu({}_{m+2}u^*), \cdots, {}_n\mu({}_nu^*)),$$
$$\bar{\bar{U}}^* = ({}_{m+1}u^*, {}_{m+2}u^*, \cdots, {}_nu^*)^T, \; \bar{\bar{U}}_j^k = \left({}_{m+1}u_j^k, {}_{m+2}u_j^k, \cdots, {}_nu_j^k\right)^T,$$
$$\bar{\bar{\Delta}}^k \triangleq \left({}_{m+1}\delta^k, {}_{m+2}\delta^k, \cdots, {}_n\delta^k\right)^T, \; \bar{\bar{R}} = diag({}_{m+1}r, {}_{m+2}r, \cdots, {}_nr).$$

We will now introduce the following matrices to be considered:

$$\bar{\mathbf{U}}^k \triangleq diag\left({}_1u_1^k, {}_2u_1^k, \cdots, {}_mu_1^k, {}_1u_2^k, {}_2u_2^k, \cdots, {}_mu_2^k, \cdots\cdots, {}_1u_J^k, {}_2u_J^k, \cdots, {}_mu_J^k\right),$$
$$\bar{\mathbf{U}}^0 \triangleq diag\left({}_1\varphi_1, {}_2\varphi_1, \cdots, {}_m\varphi_1, {}_1\varphi_2, {}_2\varphi_2, \cdots, {}_m\varphi_2, \cdots\cdots, {}_1\varphi_J, {}_2\varphi_J, \cdots, {}_m\varphi_J\right)$$
$$\bar{\mathbf{U}}^* \triangleq diag\left(\overbrace{{}_1u^*, {}_2u^*, \cdots, {}_mu^*, {}_1u^*, {}_2u^*, \cdots, {}_mu^*, \cdots\cdots, {}_1u^*, {}_2u^*, \cdots, {}_mu^*}^{m \times J}\right),$$
$$\bar{\mathbf{\Delta}}^k \triangleq diag\left({}_1\delta^k, {}_2\delta^k, \cdots, {}_m\delta^k\right).$$

$$\bar{\bar{\mathbf{U}}}^k \triangleq diag\begin{pmatrix} {}_{m+1}u_0^k, {}_{m+2}u_0^k, \cdots, {}_nu_0^k, {}_{m+1}u_1^k, {}_{m+2}u_1^k, \cdots, {}_nu_1^k, \cdots\cdots, \\ {}_{m+1}u_{J-1}^k, {}_{m+2}u_{J-1}^k, \cdots, {}_nu_{J-1}^k \end{pmatrix},$$
$$\bar{\bar{\mathbf{U}}}^0 \triangleq diag\begin{pmatrix} {}_{m+1}\varphi_0, {}_{m+2}\varphi_0, \cdots, {}_n\varphi_0, {}_{m+1}\varphi_1, {}_{m+2}\varphi_1, \cdots, {}_n\varphi_1, \cdots\cdots, \\ {}_{m+1}\varphi_{J-1}, {}_{m+2}\varphi_{J-1}, \cdots, {}_n\varphi_{J-1} \end{pmatrix}$$
$$\bar{\bar{\mathbf{U}}}^* \triangleq diag\left(\overbrace{{}_{m+1}u^*, {}_{m+2}u^*, \cdots, {}_nu^*, {}_{m+1}u^*, {}_{m+2}u^*, \cdots, {}_nu^*, \cdots\cdots, {}_{m+1}u^*, {}_{m+2}u^*, \cdots, {}_nu^*}^{(n-m) \times J}\right),$$
$$\bar{\mathbf{\Delta}}^k \triangleq diag\left({}_{m+1}\delta^k, {}_{m+2}\delta^k, \cdots, {}_n\delta^k\right).$$

$$\mathbf{U}^k \triangleq diag\begin{pmatrix} {}_{m+1}u_0^k, {}_{m+2}u_0^k, \cdots, {}_nu_0^k, {}_1u_1^k, {}_2u_1^k, \cdots, {}_nu_1^k, \cdots\cdots, \\ {}_1u_{J-1}^k, {}_2u_{J-1}^k, \cdots, {}_nu_{J-1}^k, {}_1u_J^k, {}_2u_J^k, \cdots, {}_mu_J^k \end{pmatrix},$$
$$\mathbf{U}^0 \triangleq diag\begin{pmatrix} {}_{m+1}\varphi_0, {}_{m+2}\varphi_0, \cdots, {}_n\varphi_0, {}_1\varphi_1, {}_2\varphi_1, \cdots, {}_n\varphi_1, \cdots\cdots, \\ {}_1\varphi_{J-1}, {}_2\varphi_{J-1}, \cdots, {}_n\varphi_{J-1}, {}_1\varphi_J, {}_2\varphi_J, \cdots, {}_m\varphi_J \end{pmatrix},$$
$$\mathbf{U}^* \triangleq diag\begin{pmatrix} {}_{m+1}u^*, {}_{m+2}u^*, \cdots, {}_nu^*, \overbrace{{}_1u^*, {}_2u^*, \cdots, {}_nu^*, \cdots\cdots, {}_1u^*, {}_2u^*, \cdots, {}_nu^*}^{(n-1)J}, \\ {}_1u^*, {}_2u^*, \cdots, {}_mu^* \end{pmatrix},$$
$$\bar{\mathbf{\Delta}}^k \triangleq diag\left({}_1\delta^k, {}_2\delta^k, \cdots, {}_n\delta^k\right).$$

Definition 1. *Let* $\Xi > 0$*. The equilibrium state* U^* *of the initial boundary value difference problem (11) is called stable in* l^2*-norm with respect to discrete perturbations that satisfy matrix inequalities* $\mathbf{\Delta}^k \leq \Xi$, $k \in \{1, 2, \cdots\}$ *if there are positive real constants* $\zeta_1 > 0$, $\zeta_2 > 0$, *and* $\zeta_3 > 0$ *such that for any initial condition* $\Phi(x_j)$, $j = \overline{0, J}$, *the solution* U_j^k, *where* $k \in \{1, 2, \cdots\}$ *and* $j = \overline{0, J}$, *of the initial boundary value difference problem* (11) *satisfies the inequality*

$$\left\|\mathbf{U}^k - \mathbf{U}^*\right\|_{l^2} \leq \zeta_2 e^{-\zeta_1 t^k} \|\mathbf{\Phi} - \mathbf{U}^*\|_{l^2} + \zeta_3 \max_{0 \leq s < k} (|\Delta^s|), \quad k \in \{1, 2, \cdots\}, \tag{12}$$

where

$$\mathbf{U}^k \triangleq \begin{pmatrix} \bar{U}_0^k \\ U_1^k \\ \cdots \\ U_{J-1}^k \\ \bar{U}_J^k \end{pmatrix}, \quad \mathbf{\Phi} = \begin{pmatrix} \bar{\bar{\Phi}}(x_0) \\ \Phi(x_1) \\ \cdots \\ \Phi(x_{J-1}) \\ \bar{\Phi}(x_J) \end{pmatrix}, \quad \mathbf{U}^* \triangleq \left.\begin{pmatrix} \bar{U}^* \\ U^* \\ \cdots \\ U^* \\ \bar{U}^* \end{pmatrix}\right\} n \times J, \quad |\Delta^s| = \max_{1 \leq i \leq n} |_i\delta^s|.$$

and

$$\left\|\mathbf{U}^k - \mathbf{U}^*\right\|_{l^2}^2 \triangleq h\left(\left[\bar{U}_0^k - \bar{U}^*\right], \left[\bar{U}_0^k - \bar{U}^*\right]\right) + h\left(\left[\bar{U}_J^k - \bar{U}^*\right], \left[\bar{U}_J^k - \bar{U}^*\right]\right) + h\sum_{j=1}^{J-1}\left(\left[U_j^k - U^*\right], \left[U_j^k - U^*\right]\right), \; k \in \{0, 1, \cdots\}.$$

Definition 2 (The discrete Lyapunov function). *It is said that the function* $\mathbf{L} : \mathbb{R}^{n \times J} \to \mathbb{R}_0^+$ *is called a discrete Lyapunov function for the initial boundary value difference problem* (11) *if the following hold:*

1. *There exist positive constants* $\chi_1 > 0$ *and* $\chi_2 > 0$ *such that for all* $k \in \{0, 1, \cdots\}$,

$$\chi_1 \left\|\mathbf{U}^k - \mathbf{U}^*\right\|_{l^2}^2 \leq \mathbf{L}\left(\mathbf{U}^k\right) \leq \chi_2 \left\|\mathbf{U}^k - \mathbf{U}^*\right\|_{l^2}^2,$$

2. *There exist positive constants* $\eta > 0$ *and* $\nu > 0$ *such that for all* $k \in \{0, 1, \cdots\}$,

$$\frac{\mathbf{L}\left(\mathbf{U}^{k+1}\right) - \mathbf{L}\left(\mathbf{U}^k\right)}{\Delta t} \leq -\eta \mathbf{L}\left(\mathbf{U}^k\right) + \nu\left(\Delta^k, \Delta^k\right).$$

To simplify the notation, in the following, we define a sequence of discrete values L^k *as*

$$\mathrm{L}^k = \mathbf{L}\left(\mathbf{U}^k\right), \quad k \in \{0, 1, \cdots\}$$

where $\mathbf{U}^k$ *is the given solution to the initial boundary value difference problem* (11).

It should be noted that the presence of a discrete Lyapunov function ensures the stability of the equilibrium state U^* of the initial boundary value difference problem (11) in l^2-norm with respect to discrete perturbations.

Theorem 1 (Discrete stability for the case $\mathbf{U}^* \geq 0$). *Let the conditin CFL* (10) *be fulfilled and* $\Xi \geq 0$. *Then, for every* $\mathbf{U}^*$ *and* R *satisfying, respectively, the matrix inequality* $\mathbf{U}^* \geq 0$ *and* $0 \leq R < E$, *every* $\mathrm{U} > 0$ *and for any initial vector of the function* $\mathbf{\Phi}$ *satisfying the matrix inequality with* $\mathbf{U}^0 \geq 0$, *and*

$$\|\mathbf{\Phi} - \mathbf{U}^*\|_{l^2} < \mathrm{U}, \tag{13}$$

the solution $\mathbf{U}^k$ *to the initial boundary value difference problem* (11) *satisfies the matrix inequality* $\mathbf{U}^k \geq 0$, $k \in \{0, 1, \cdots\}$, *and the stationary state* $\mathbf{U}^*$ *of the initial boundary value difference problem* (11) *is stable in* l^2*-norm with respect to any discrete perturbation function* Δ^k, $k \in \{0, 1, \cdots\}$, *such that the matrix inequality* $\mathbf{\Delta^k} \leq \mathbf{\Xi}$ *is valid.*

To analyze the stability of the initial boundary value difference problem (11) using the Lyapunov discrete method, we use the following transformations:

$$\begin{gathered}\tilde{\bar{U}}_j^k=\bar{U}_j^k-\bar{U}^*,\ \ \tilde{\bar{A}}^k=h\textstyle\sum_{j=1}^{J}\tilde{\bar{U}}_j^k,\ \ \tilde{\bar{\mathrm{M}}}_{\tilde{\bar{A}}^k}^k=\bar{\mathrm{M}}\left(\bar{U}^*+\tilde{\bar{A}}^k\right),\ \ \tilde{\bar{\Lambda}}^k=\frac{\tau}{h}\tilde{\bar{\mathrm{M}}}_{\tilde{\bar{A}}^k}^k,\ \ k=\{0,1,\cdots\}.\\ \bar{\mathrm{M}}\left(\bar{U}^*+\tilde{\bar{A}}^k\right)\equiv diag\left(\frac{{}_1P}{{}_1Q+{}_1u^*+{}_1\tilde{a}^k},\frac{{}_2P}{{}_2Q+{}_2u^*+{}_2\tilde{a}^k},\cdots,\frac{{}_mP}{{}_mQ+{}_mu^*+{}_m\tilde{a}^k}\right),\\ \tilde{\bar{A}}^k\triangleq diag\left({}_1\tilde{a}^k,{}_2\tilde{a}^k,\cdots,{}_m\tilde{a}^k\right),\ \ {}_i\tilde{a}^k=h\textstyle\sum_{j=1}^{J}\left({}_iu_j^k-{}_iu^*\right),\ \ i=\overline{1,m},\ \ \tilde{\Phi}_j=\Phi_j-U^*,\\ \tilde{\bar{\bar{U}}}_j^k=\bar{\bar{U}}_j^k-\bar{\bar{U}}^*,\ \ \tilde{\bar{\bar{A}}}^k=h\textstyle\sum_{j=0}^{J-1}\tilde{\bar{\bar{U}}}_j^k,\ \ \tilde{\bar{\bar{\mathrm{M}}}}_{\tilde{\bar{\bar{A}}}^k}^k=\bar{\bar{\mathrm{M}}}\left(\bar{\bar{U}}^*+\tilde{\bar{\bar{A}}}^k\right),\ \ \tilde{\bar{\bar{\Lambda}}}^k=\frac{\tau}{h}\tilde{\bar{\bar{\mathrm{M}}}}_{\tilde{\bar{\bar{A}}}^k}^k,\ \ k=\{0,1,\cdots\}.\\ \bar{\bar{\mathrm{M}}}\left(\bar{\bar{U}}^*+\tilde{\bar{\bar{A}}}^k\right)\equiv diag\left(\frac{{}_{m+1}P}{{}_{m+1}Q+{}_{m+1}u^*+{}_{m+1}\tilde{a}^k},\frac{{}_{m+2}P}{{}_{m+2}Q+{}_{m+2}u^*+{}_{m+2}\tilde{a}^k},\cdots,\frac{{}_nP}{{}_nQ+{}_nu^*+{}_n\tilde{a}^k}\right),\\ \tilde{\bar{\bar{A}}}^k\triangleq diag\left({}_{m+1}\tilde{a}^k,{}_{m+2}\tilde{a}^k,\cdots,{}_n\tilde{a}^k\right),\ \ {}_i\tilde{a}^k=h\sum_{j=0}^{J-1}\left({}_iu_j^k-{}_iu^*\right),\ \ i=\overline{m+1,n},\end{gathered}\tag{14}$$

For simplicity, we omit the «∼» symbol in the notation (14) and discretize system (9) as follows:

$$\begin{cases}\bar{U}_j^{k+1}=\left(1-\bar{\Lambda}^k\right)\bar{U}_j^k+\bar{\Lambda}^k\bar{U}_{j-1}^k,\quad j=\overline{1,J};\ \ k\in\{0,1,\cdots\};\\ \bar{\bar{U}}_j^{k+1}=\left(1-\bar{\bar{\Lambda}}^k\right)\bar{\bar{U}}_j^k+\bar{\bar{\Lambda}}^k\bar{\bar{U}}_{j+1}^k,\quad j=\overline{0,J-1};\ \ k\in\{0,1,\cdots\};\\ \bar{U}_0^{k+1}=\bar{R}\bar{U}_J^{k+1}+(\bar{E}-\bar{R})\bar{\Theta}\bar{A}^{k+1}+\bar{R}\bar{\Delta}^{k+1},\quad k\in\{0,1,\cdots\};\\ \bar{\bar{U}}_J^{k+1}=\bar{\bar{R}}\bar{\bar{U}}_0^{k+1}+\left(\bar{\bar{E}}-\bar{\bar{R}}\right)\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}+\bar{\bar{R}}\bar{\bar{\Delta}}^{k+1},\quad k\in\{0,1,\cdots\};\\ U_j^0=\Phi_j.\end{cases}\tag{15}$$

Here,

$$\bar{\Theta}=diag\left(\frac{{}_1u^*}{({}_1Q+{}_1u^*)},\frac{{}_2u^*}{({}_2Q+{}_2u^*)},\cdots,\frac{{}_mu^*}{({}_mQ+{}_mu^*)}\right);$$

$$\bar{A}^k=h\sum_{j=1}^{J}\bar{U}_j^k,\ \ \bar{\mathrm{M}}_{\bar{A}^k}^k=\bar{\mathrm{M}}\left(\bar{U}^*+\bar{A}^k\right),\ \ \bar{\Lambda}^k=\frac{\tau}{h}\bar{\mathrm{M}}_{\bar{A}^k}^k,\ \ k=\{0,1,\cdots\};$$

$$\bar{\mathrm{M}}\left(\bar{U}^*+\bar{A}^k\right)\equiv diag\left(\frac{{}_1P}{{}_1Q+{}_1u^*+{}_1a^k},\frac{{}_2P}{{}_2Q+{}_2u^*+{}_2a^k},\cdots,\frac{{}_mP}{{}_mQ+{}_mu^*+{}_ma^k}\right);$$

$${}_ia^k=h\sum_{j=1}^{J}{}_iu_j^k\geq-{}_iu^*,\quad i=\overline{1,m};\,{}_ia^k=h\sum_{j=0}^{J-1}{}_iu_j^k\geq-{}_iu^*,\ \ i=\overline{m+1,n};\quad k=0,1,\cdots;$$

$$\bar{\bar{\Theta}}=diag\left(\frac{{}_{m+1}u^*}{{}_{m+1}Q+{}_{m+1}u^*},\frac{{}_{m+2}u^*}{{}_{m+2}Q+{}_{m+2}u^*},\cdots,\frac{{}_nu^*}{{}_nQ+{}_nu^*}\right);$$

$$\bar{\bar{A}}^k=h\sum_{j=0}^{J-1}\bar{\bar{U}}_j^k,\ \ \bar{\bar{\mathrm{M}}}_{\bar{\bar{A}}^k}^k=\bar{\bar{\mathrm{M}}}\left(\bar{\bar{U}}^*+\bar{\bar{A}}^k\right),\ \ \bar{\bar{\Lambda}}^k=\frac{\tau}{h}\bar{\bar{\mathrm{M}}}_{\bar{\bar{A}}^k}^k,\ \ k=\{0,1,\cdots\};$$

$$\bar{\bar{\mathrm{M}}}\left(\bar{\bar{U}}^*+\bar{\bar{A}}^k\right)\equiv diag\left(\frac{{}_{m+1}P}{{}_{m+1}Q+{}_{m+1}u^*+{}_{m+1}a^k},\frac{{}_{m+2}P}{{}_{m+2}Q+{}_{m+2}u^*+{}_{m+2}a^k},\cdots,\frac{{}_nP}{{}_nQ+{}_nu^*+{}_na^k}\right).$$

Thus, the assumption in the form of inequality (13) being satisfied in Theorem 1 is now expressed as

$$\|\mathbf{\Phi}\|_{l^2}<\mathrm{U}.$$

Note that inequality (12) is rewritten as

$$\left\|\mathbf{U}^k\right\|_{l^2}\leq\zeta_2e^{-\zeta_1t^k}\|\mathbf{\Phi}\|_{l^2}+\zeta_3\max_{0\leq s<k}\left(|\Delta^s|\right),\quad k\in\{1,2,\cdots\},$$

Proof of Theorem 1. Further, in the process of proving Theorem 1, we consider only the case of the matrix inequality

$$\mathbf{U}^*>0.$$

Since the initial data $\mathbf{U}^0 \geq 0$, according to the discrete system (15) and the CFL condition in Equation (10), we have $\mathbf{U}^k \geq 0, \;\; k \in \{0, 1, \cdots\}$.

Consider the following candidate for a discrete Lyapunov function for any $\vec{\phi} \in \mathrm{R}^{n\times J}$:

$$\mathbf{L}(\vec{\phi}) = h\sum_{j=1}^{J}(\bar{\phi}_j, \bar{\phi}_j)^2 e^{-\bar{\alpha}x_j} + h\sum_{j=0}^{J-1}(\bar{\bar{\phi}}_j, \bar{\bar{\phi}}_j)^2 e^{\bar{\bar{\alpha}}x_j} + \left(h\sum_{j=1}^{J}\bar{B}\bar{\phi}_j, h\sum_{j=1}^{J}\bar{\phi}_j\right) + \left(h\sum_{j=0}^{J-1}\bar{\bar{B}}\bar{\bar{\phi}}_j, h\sum_{j=0}^{J-1}\bar{\bar{\phi}}_j\right).$$

Here,

$$\vec{\phi} = \begin{pmatrix} \bar{\bar{\phi}}_0 \\ \phi_1 \\ \cdots \\ \phi_{J-1} \\ \bar{\phi}_J \end{pmatrix}, \; \phi_j = \begin{pmatrix} \bar{\phi}_j \\ \bar{\bar{\phi}}_j \end{pmatrix}, \; \bar{\bar{\phi}}_j = \begin{pmatrix} {}_{m+1}\phi_j \\ {}_{m+2}\phi_j \\ \cdots \\ {}_{n}\phi_j \end{pmatrix}, \; j = \overline{0, J-1}; \; \bar{\phi}_j = \begin{pmatrix} {}_{1}\phi_j \\ {}_{2}\phi_j \\ \cdots \\ {}_{m}\phi_j \end{pmatrix} \; j = \overline{1, J}.$$

where $\bar{\alpha} > 0$ and $\bar{\bar{\alpha}} > 0$. In particular, as the values of $\bar{B}$ and $\bar{\bar{B}}$, we take the following parameters:

$$\bar{B} = \bar{\Theta}(\bar{E} - \bar{R})^{-1}(\bar{R} - e^{-\bar{\alpha}}\bar{E}) < 0, \;\; \bar{\bar{B}} = \bar{\bar{\Theta}}\left(\bar{\bar{E}} - \bar{\bar{R}}\right)^{-1}\left(e^{\bar{\bar{\alpha}}}\bar{\bar{R}}\left[1 + \bar{\bar{\alpha}}^2\right] - \bar{\bar{E}}\right) < 0\,. \tag{16}$$

and since $\bar{\Theta} \leq 1, \;\; \bar{\bar{\Theta}} \leq 1$, there are quite small $\bar{\alpha}^*$ and $\bar{\bar{\alpha}}^*$ such that for $0 < \bar{\alpha} < \bar{\alpha}^*$ and $0 < \bar{\bar{\alpha}} < \bar{\bar{\alpha}}^*$, the following inequalities are fulfilled: $-\frac{\bar{\alpha}}{e^{\bar{\alpha}}-1} < \bar{B} < 0$ and $-\frac{\bar{\bar{\alpha}}}{1-e^{-\bar{\bar{\alpha}}}} < \bar{\bar{B}} < 0$.

According to (10), the values $\mathbf{L}$ on the solution $\mathbf{U}^k$ at the moment t^k for $k \geq 0$ are defined by the expression

$$\mathrm{L}^k = \bar{\mathrm{L}}^k + \bar{\bar{\mathrm{L}}}^k, \;\; k \in \{0, 1, \cdots\},$$

where

$$\bar{\mathrm{L}}^k = \left\|\bar{\mathbf{U}}^k\right\|_{\bar{\alpha}}^2 + \left(\bar{B}\bar{A}^k, \bar{A}^k\right), \bar{\bar{\mathrm{L}}}^k = \left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2 + \left(\bar{\bar{B}}\bar{\bar{A}}^k, \bar{\bar{A}}^k\right),$$
$$\left\|\bar{\mathbf{U}}^k\right\|_{\bar{\alpha}}^2 = h\sum_{j=1}^{J}\left(\bar{U}_j^k, \bar{U}_j^k\right)e^{-\bar{\alpha}x_j}, \left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2 = h\sum_{j=0}^{J-1}\left(\bar{\bar{U}}_j^k, \bar{\bar{U}}_j^k\right)e^{\bar{\bar{\alpha}}x_j}, k \in \{0, 1, \cdots\}.$$

For fixed $\bar{R}$ and $\bar{\bar{R}}$ satisfying matrix inequalities $0 \leq \bar{R} < \bar{E}$ and $0 \leq \bar{\bar{R}} < \bar{\bar{E}}$, we assume that there exist $\bar{\alpha}^{**}$ and $\bar{\bar{\alpha}}^{**}$, such that for $0 < \bar{\alpha} < \bar{\alpha}^{**}$ and $0 < \bar{\bar{\alpha}} < \bar{\bar{\alpha}}^{**}$, the following inequalities are satisfied:

$$\exp(-\bar{\alpha}) > \bar{R} > \bar{R}^2, \;\; \bar{\alpha} < \bar{E} - \bar{R}, \quad \frac{\exp(-\bar{\bar{\alpha}})}{(1 + \bar{\bar{\alpha}}^2)} > \bar{\bar{R}} > \bar{\bar{R}}^2, \;\; \bar{\bar{\alpha}} < \bar{\bar{E}} - \bar{\bar{R}}$$

and

$$0 < h < 1.$$

As a first step, we prove that L^k is equivalent to $\left\|\mathbf{U}^k\right\|_{\alpha}^2 \triangleq \left\|\bar{\mathbf{U}}^k\right\|_{\bar{\alpha}}^2 + \left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2$.

According to the Taylor decomposition, the following chain of inequalities is valid:

$$\frac{h(e^{\bar{\alpha}} - 1)}{1 - e^{-\bar{\alpha}h}} \leq (1 + \bar{\alpha})^2 \leq (1 + 3\bar{\alpha}), \quad \frac{h(e^{-\bar{\bar{\alpha}}} - 1)}{e^{-\bar{\bar{\alpha}}h} - 1} \leq (1 + \bar{\bar{\alpha}})^2 \leq (1 + 3\bar{\bar{\alpha}}).$$

Therefore, for all $k \geq 1$,

$$\begin{aligned}
&(1) \;\; \left(\bar{A}^k, \bar{A}^k\right) = \left(h\sum_{j=1}^{J}\bar{U}_j^k, h\sum_{j=1}^{J}\bar{U}_j^k\right) \leq h^2\sum_{j=1}^{J}e^{-\bar{\alpha}x_j}\left(\bar{U}_j^k, \bar{U}_j^k\right)\sum_{j=1}^{J}e^{\bar{\alpha}x_j} = \\
&\qquad \frac{h\left(e^{\bar{\alpha}}-1\right)}{1-e^{-\bar{\alpha}h}}\left\|\bar{\mathbf{U}}^k\right\|_{\bar{\alpha}}^2 \leq (1 + \bar{\alpha})^2\left\|\bar{\mathbf{U}}^k\right\|_{\bar{\alpha}}^2 \leq (1 + 3\bar{\alpha})\left\|\bar{\mathbf{U}}^k\right\|_{\bar{\alpha}}^2, \\
&(2) \;\; \left(\bar{\bar{A}}^k, \bar{\bar{A}}^k\right) = \left(h\sum_{j=1}^{J}\bar{\bar{U}}_j^k, h\sum_{j=1}^{J}\bar{\bar{U}}_j^k\right) \leq h^2\sum_{j=0}^{J-1}e^{\bar{\bar{\alpha}}x_j}\left(\bar{\bar{U}}_j^k, \bar{\bar{U}}_j^k\right)\sum_{j=0}^{J-1}e^{-\bar{\bar{\alpha}}x_j} = \\
&\qquad \frac{h\left(e^{-\bar{\bar{\alpha}}}-1\right)}{e^{-\bar{\bar{\alpha}}h}-1}\left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2 \leq (1 + \bar{\bar{\alpha}})^2\left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2 \leq (1 + 3\bar{\bar{\alpha}})\left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2.
\end{aligned} \tag{17}$$

Due to restrictions on $\bar{B}$ and $\bar{\bar{B}}$, we obtain estimates, respectively, on $\bar{\mathrm{L}}^k$, and $\bar{\bar{\mathrm{L}}}^k$ for all $k \geq 0$:

$$(1)\ \left\|\bar{\mathbf{U}}^k\right\|_{\bar{\alpha}}^2 \geq \bar{\mathrm{L}}^k \geq h\sum_{j=1}^{J}\left(\left\{\bar{E}+\bar{\Theta}[\bar{E}-\bar{R}]^{-1}[\bar{R}-e^{-\alpha}\bar{E}]\right\}\bar{U}_j^k,\bar{U}_j^k\right)e^{-\bar{\alpha}x_j} \geq$$
$$h\sum_{j=1}^{J}\left(\left\{\bar{E}-(1+3\alpha)\bar{\Theta}\right\}\bar{U}_j^k,\bar{U}_j^k\right)e^{-\bar{\alpha}x_j} \geq \frac{1}{2}h\sum_{j=1}^{J}\left([\bar{E}-\bar{\Theta}]\bar{U}_j^k,\bar{U}_j^k\right)e^{-\bar{\alpha}x_j},$$
$$(2)\ \left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2 \geq \bar{\bar{\mathrm{L}}}^k \geq \frac{h\left(e^{-\bar{\bar{\alpha}}}-1\right)}{\left(e^{-\bar{\bar{\alpha}}h}-1\right)}h\sum_{j=0}^{J-1}\left(\left\{\begin{array}{c}\bar{\bar{E}}+\bar{\bar{\Theta}}\left[\bar{\bar{E}}-\bar{\bar{R}}\right]^{-1}\\ \left[e^{\bar{\bar{\alpha}}}\bar{\bar{R}}(1+\bar{\bar{\alpha}}^2)-\bar{\bar{E}}\right]\end{array}\right\}\bar{\bar{U}}_j^k,\bar{\bar{U}}_j^k\right)e^{\bar{\bar{\alpha}}x_j} \geq$$
$$h\sum_{j=0}^{J-1}\left(\left\{\bar{\bar{E}}-(1+3\bar{\bar{\alpha}})\bar{\bar{\Theta}}\right\}\bar{\bar{U}}_j^k,\bar{\bar{U}}_j^k\right)e^{\bar{\bar{\alpha}}x_j} \geq \frac{1}{2}h\sum_{j=0}^{J-1}\left(\left[\bar{\bar{E}}-\bar{\bar{\Theta}}\right]\bar{\bar{U}}_j^k,\bar{\bar{U}}_j^k\right)e^{\bar{\bar{\alpha}}x_j}.$$

In summary, respectively, for the left, middle, and right parts of the last inequalities above, we obtain

$$\left\|\mathbf{U}^k\right\|_{\alpha}^2 \geq \mathrm{L}^k \geq \frac{1}{2}h\left\{\sum_{j=1}^{J}\left([\bar{E}-\bar{\Theta}]\bar{U}_j^k,\bar{U}_j^k\right)e^{-\bar{\alpha}x_j}+\sum_{j=0}^{J-1}\left(\left[\bar{\bar{E}}-\bar{\bar{\Theta}}\right]\bar{\bar{U}}_j^k,\bar{\bar{U}}_j^k\right)e^{\bar{\bar{\alpha}}x_j}\right\}.$$

where the last inequality is true provided that

$$\begin{aligned}&0<\bar{\alpha}\leq\min\left\{1,\bar{\alpha}^*,\bar{\alpha}^{**},\min_{1\leq i\leq m}\frac{1-\lambda_i(\bar{\Theta})}{6\lambda_i(\bar{\Theta})}\right\},\\ &0<\bar{\bar{\alpha}}\leq\min\left\{1,\bar{\bar{\alpha}}^*,\bar{\bar{\alpha}}^{**},\min_{m+1\leq i\leq n}\frac{1-\lambda_i\left(\bar{\bar{\Theta}}\right)}{6\lambda_i\left(\bar{\bar{\Theta}}\right)}\right\}.\end{aligned} \tag{18}$$

Here, $\lambda_i(\bar{\Theta})$, where $i=\overline{1,m}$, and $\lambda_i\left(\bar{\bar{\Theta}}\right)$, where $i=\overline{m+1,n}$, are eigenvalues of the matrices, respectively, $\bar{\Theta}$ and $\bar{\bar{\Theta}}$.

In addition, the discrete weight norm is equivalent to l^2-norm for all $k \geq 0$:

$$e^{-\bar{\alpha}}\left\|\bar{\mathbf{U}}^k\right\|_{l^2}^2 \leq \left\|\bar{\mathbf{U}}^k\right\|_{\bar{\alpha}}^2 \leq \left\|\bar{\mathbf{U}}^k\right\|_{l^2}^2, \qquad \left\|\bar{\bar{\mathbf{U}}}^k\right\|_{l^2}^2 \leq \left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2 \leq e^{\bar{\bar{\alpha}}}\left\|\bar{\bar{\mathbf{U}}}^k\right\|_{l^2}^2$$

Therefore, summing up these two inequalities and reinforcing them, we have

$$e^{-\bar{\alpha}}\left\|\mathbf{U}^k\right\|_{l^2}^2 \leq e^{-\bar{\alpha}}\left\|\bar{\mathbf{U}}^k\right\|_{l^2}^2+\left\|\bar{\bar{\mathbf{U}}}^k\right\|_{l^2}^2 \leq \left\|\mathbf{U}^k\right\|_{\alpha}^2 \leq \left\|\bar{\mathbf{U}}^k\right\|_{l^2}^2+e^{\bar{\bar{\alpha}}}\left\|\bar{\bar{\mathbf{U}}}^k\right\|_{l^2}^2 \leq e^{\bar{\bar{\alpha}}}\left\|\mathbf{U}^k\right\|_{l^2}^2,$$

As a second step, we evaluate the finite-difference approximation of the time derivative L^k in time. To this end, we will use the inequalities

$$\frac{\bar{\mathrm{L}}^{k+1}-\bar{\mathrm{L}}^k}{\tau}=\frac{h}{\tau}\sum_{i=1}^{I}\left[\left(\bar{U}_j^{k+1},\bar{U}_j^{k+1}\right)-\left(\bar{U}_j^k,\bar{U}_j^k\right)\right]e^{-\alpha x_i}+$$
$$\frac{1}{\tau}\left[\left(\bar{B}h\sum_{j=1}^{J}\bar{U}_j^{k+1},h\sum_{j=1}^{J}\bar{U}_j^{k+1}\right)-\left(\bar{B}h\sum_{j=1}^{J}\bar{U}_j^k,h\sum_{j=1}^{J}\bar{U}_j^k\right)\right]=\frac{1}{\tau}\bar{L}_1^k+\frac{1}{\tau}\bar{L}_2^k,$$

$$\frac{\bar{\bar{\mathrm{L}}}^{k+1}-\bar{\bar{\mathrm{L}}}^k}{\tau}=\frac{h}{\tau}\sum_{j=0}^{J-1}\left[\left(\bar{\bar{U}}_j^{k+1},\bar{\bar{U}}_j^{k+1}\right)-\left(\bar{\bar{U}}_j^k,\bar{\bar{U}}_j^k\right)\right]e^{\bar{\bar{\alpha}}x_j}+$$
$$\frac{h^2}{\tau}\left[\left(\sum_{j=0}^{J-1}\bar{\bar{B}}\bar{\bar{U}}_j^{k+1},\sum_{j=0}^{J-1}\bar{\bar{U}}_j^{k+1}\right)-\left(\sum_{j=0}^{J-1}\bar{\bar{B}}\bar{\bar{U}}_j^k,\sum_{j=0}^{J-1}\bar{\bar{U}}_j^k\right)\right]=\frac{1}{\tau}\bar{\bar{L}}_1^k+\frac{1}{\tau}\bar{\bar{L}}_2^k.$$

where

$$\bar{L}_1^k=h\sum_{i=1}^{I}\left[\left(\bar{U}_j^{k+1},\bar{U}_j^{k+1}\right)-\left(\bar{U}_j^k,\bar{U}_j^k\right)\right]e^{-\bar{\alpha}x_i},$$

$$\bar{L}_2^k=\left[\left(\bar{B}h\sum_{j=1}^{J}\bar{U}_j^{k+1},h\sum_{j=1}^{J}\bar{U}_j^{k+1}\right)-\left(\bar{B}h\sum_{j=1}^{J}\bar{U}_j^k,h\sum_{j=1}^{J}\bar{U}_j^k\right)\right],$$

$$\bar{\bar{L}}_1^k=h\sum_{j=0}^{J-1}\left[\left(\bar{\bar{U}}_j^{k+1},\bar{\bar{U}}_j^{k+1}\right)-\left(\bar{\bar{U}}_j^k,\bar{\bar{U}}_j^k\right)\right]e^{\bar{\bar{\alpha}}x_j},$$

$$\bar{\bar{L}}_2^k = h^2\left[\left(\sum_{j=0}^{J-1}\bar{\bar{B}}\bar{\bar{U}}_j^{k+1}, \sum_{j=0}^{J-1}\bar{\bar{U}}_j^{k+1}\right) - \left(\sum_{j=0}^{J-1}\bar{\bar{B}}\bar{\bar{U}}_j^{k}, \sum_{j=0}^{J-1}\bar{\bar{U}}_j^{k}\right)\right].$$

Hence, we have

$$\frac{\mathrm{L}^{k+1}-\mathrm{L}^k}{\tau} = \frac{1}{\tau}L_1^k + \frac{1}{\tau}L_2^k. \tag{19}$$

where

$$L_1^k = \bar{L}_1^k + \bar{\bar{L}}_1^k, \quad L_2^k = \bar{L}_2^k + \bar{\bar{L}}_2^k.$$

According to Jensen's inequality, the inequality is valid for convex maps $y \to y^2$:

$$[q_1y_1 + q_2y_2]^2 \le q_1(y_1)^2 + q_2(y_2)^2,$$

where $q_1, q_2 > 0$ 8 $q_1 + q_2 = 1$.

Using the difference scheme (15) and the CFL condition (10), which ensures the fulfillment of the inequalities $0 < \bar{\Lambda}^k < \bar{E}$ and $0 < \bar{\bar{\Lambda}}^k < \bar{\bar{E}}$, as well as the convexity of the quadratic mapping, we evaluate the quadratic shapes $\left(\bar{U}_j^{k+1}, \bar{U}_j^{k+1}\right)\Big|_{j=1}^{j=J}$ and $\left(\bar{\bar{U}}_j^{k+1}, \bar{\bar{U}}_j^{k+1}\right)\Big|_{j=0}^{j=J-1}$ for all $k \ge 0$:

$$\begin{array}{ll}(1) & \left(\bar{U}_j^{k+1}, \bar{U}_j^{k+1}\right) = \left(\left\{\left[1-\bar{\Lambda}^k\right]\bar{U}_j^k + \bar{\Lambda}^k\bar{U}_{j-1}^k\right\}, \left\{\left[1-\bar{\Lambda}^k\right]\bar{U}_j^k + \bar{\Lambda}^k\bar{U}_{j-1}^k\right\}\right) \le \\ & \left(\left[1-\bar{\Lambda}^k\right]\bar{U}_j^k, \bar{U}_j^k\right) + \left(\bar{\Lambda}^k\bar{U}_{j-1}^k, \bar{U}_{j-1}^k\right), \\ (2) & \left(\bar{\bar{U}}_j^{k+1}, \bar{\bar{U}}_j^{k+1}\right) = \left(\left\{\left[1-\bar{\bar{\Lambda}}^k\right]\bar{\bar{U}}_j^k + \bar{\bar{\Lambda}}^k\bar{\bar{U}}_{j+1}^k\right\}, \left\{\left[1-\bar{\bar{\Lambda}}^k\right]\bar{\bar{U}}_j^k + \bar{\bar{\Lambda}}^k\bar{\bar{U}}_{j+1}^k\right\}\right) \le \\ & \left(\left[1-\bar{\bar{\Lambda}}^k\right]\bar{\bar{U}}_j^k, \bar{\bar{U}}_j^k\right) + \left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_{j+1}^k, \bar{\bar{U}}_{j+1}^k\right).\end{array} \tag{20}$$

Lemma 1. *The inequality is valid for the solutions of the initial boundary value difference problem:* (15)

$$\begin{array}{l}\bar{L}_1^k \le h\left(e^{-\alpha h}-1\right)\sum_{j=1}^{J}\left(\bar{\Lambda}^k\bar{U}_j^k, \bar{U}_j^k\right)e^{-\alpha x_j} + \tau e^{-\alpha h}\left[\left(\bar{\mathrm{M}}^k\bar{U}_0^k, \bar{U}_0^k\right) - e^{-\alpha}\left(\bar{\mathrm{M}}^k\bar{U}_J^k, \bar{U}_J^k\right)\right], \\ \bar{\bar{L}}_1^k \le h\left(e^{-\bar{\bar{\alpha}}h}-1\right)\sum_{j=0}^{J-1}\left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_j^k, \bar{\bar{U}}_j^k\right)e^{\bar{\bar{\alpha}}x_j} + he^{-\bar{\bar{\alpha}}h}\left[e^{\bar{\bar{\alpha}}}\left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_J^k, \bar{\bar{U}}_J^k\right) - \left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_0^k, \bar{\bar{U}}_0^k\right)\right].\end{array} \tag{21}$$

Here, $\bar{\mathrm{M}}^k \triangleq \bar{\mathrm{M}}_{\bar{A}^k}^k$.

Proof of Lemma 1. The proof is carried out separately for $\bar{L}_1^k$ and $\bar{\bar{L}}_1^k$. Let us prove the inequality for $\bar{\bar{L}}_1^k$. Using the second inequality in (20), we estimate from above the quadratic form $\left(\bar{\bar{U}}_j^{k+1}, \bar{\bar{U}}_j^{k+1}\right)$ on the left side of inequality (21) that

$$\begin{array}{l}\bar{\bar{L}}_1^k = h\sum_{j=0}^{J-1}\left[\left(\bar{\bar{U}}_j^{k+1}, \bar{\bar{U}}_j^{k+1}\right) - \left(\bar{\bar{U}}_j^k, \bar{\bar{U}}_j^k\right)\right]e^{\bar{\bar{\alpha}}x_j} \le \\ h\sum_{j=0}^{J-1}\left\{\left(\left[1-\bar{\bar{\Lambda}}^k\right]\bar{\bar{U}}_j^k, \bar{\bar{U}}_j^k\right) + \left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_{j+1}^k, \bar{\bar{U}}_{j+1}^k\right) - \left(\bar{\bar{U}}_j^k, \bar{\bar{U}}_j^k\right)\right\}e^{\bar{\bar{\alpha}}x_j} = \\ h\sum_{j=0}^{J-1}\left\{\left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_{j+1}^k, \bar{\bar{U}}_{j+1}^k\right) - \left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_j^k, \bar{\bar{U}}_j^k\right)\right\}e^{\bar{\bar{\alpha}}x_j}.\end{array} \tag{22}$$

We transform the right-hand part of inequality (23) using the following well-known formula for differential differentiation in parts for any grid function v_i:

$$\left\{v_{j+1} - v_j\right\}e^{\bar{\bar{\alpha}}x_j} = \left(v_{j+1}e^{\bar{\bar{\alpha}}x_{j+1}} - v_je^{\bar{\bar{\alpha}}x_j}\right) + v_{j+1}e^{\bar{\bar{\alpha}}x_{j+1}}\left(e^{-\bar{\bar{\alpha}}h}-1\right) \tag{23}$$

the validity of which can be verified through direct verification:

$$\left\{v_{j+1} - v_j\right\}e^{\bar{\bar{\alpha}}x_j} = v_{j+1}e^{\bar{\bar{\alpha}}x_{j+1}} - v_je^{\bar{\bar{\alpha}}x_j} - v_{j+1}e^{\bar{\bar{\alpha}}x_{j+1}} + v_{j+1}e^{\bar{\bar{\alpha}}x_j} = \left(v_{j+1}e^{\bar{\bar{\alpha}}x_{j+1}} - v_je^{\bar{\bar{\alpha}}x_j}\right) + v_{j+1}e^{\bar{\bar{\alpha}}x_{j+1}}\left(e^{-\bar{\bar{\alpha}}h}-1\right).$$

Let us apply the formula of difference differentiation (23) in the right-hand part of inequality (22). Assuming $v_j \triangleq \left(\bar{\bar{\Lambda}}^k \bar{U}_j^k, \bar{U}_j^k\right)$, we have

$$\begin{aligned} &h\sum_{j=0}^{J-1}\left\{\left(\bar{\bar{\Lambda}}^k \bar{U}_{j+1}^k, \bar{U}_{j+1}^k\right) - \left(\bar{\bar{\Lambda}}^k \bar{U}_j^k, \bar{U}_j^k\right)\right\} e^{\bar{\bar{\alpha}} x_j} = \\ &h\sum_{j=0}^{J-1}\left[\left(\bar{\bar{\Lambda}}^k \bar{U}_{j+1}^k, \bar{U}_{j+1}^k\right) e^{\bar{\bar{\alpha}} x_{j+1}} - \left(\bar{\bar{\Lambda}}^k \bar{U}_j^k, \bar{U}_j^k\right) e^{\bar{\bar{\alpha}} x_j}\right] + h\left(e^{-\bar{\bar{\alpha}} h} - 1\right)\sum_{j=0}^{J-1}\left(\bar{\bar{\Lambda}}^k \bar{U}_{j+1}^k, \bar{U}_{j+1}^k\right) e^{\bar{\bar{\alpha}} x_{j+1}}. \end{aligned} \tag{24}$$

To calculate the first sum on the right-hand side of inequality (24), we apply the following summation formula

$$\sum_{j=0}^{J-1}\left(w_{j+1} - w_j\right) = w_J - w_0. \tag{25}$$

In identity (25), we assume $w_j = h\left(\bar{\bar{\Lambda}}^k \bar{U}_j^k, \bar{U}_j^k\right) e^{\bar{\bar{\alpha}} x_j}$, and we obtain

$$h\sum_{j=0}^{J-1}\left[\left(\bar{\bar{\Lambda}}^k \bar{U}_{j+1}^k, \bar{U}_{j+1}^k\right) e^{\bar{\bar{\alpha}} x_{j+1}} - \left(\bar{\bar{\Lambda}}^k \bar{U}_j^k, \bar{U}_j^k\right) e^{\bar{\bar{\alpha}} x_j}\right] = h\left[e^{\bar{\bar{\alpha}}}\left(\bar{\bar{\Lambda}}^k \bar{U}_J^k, \bar{U}_J^k\right) - \left(\bar{\bar{\Lambda}}^k \bar{U}_0^k, \bar{U}_0^k\right)\right].$$

Then, equality (24) will take the form

$$\begin{aligned} &h\sum_{j=0}^{J-1}\left\{\left(\bar{\bar{\Lambda}}^k \bar{U}_{j+1}^k, \bar{U}_{j+1}^k\right) - \left(\bar{\bar{\Lambda}}^k \bar{U}_j^k, \bar{U}_j^k\right)\right\} e^{\bar{\bar{\alpha}} x_j} = \\ &h\left(e^{-\bar{\bar{\alpha}} h} - 1\right)\sum_{j=0}^{J-1}\left(\bar{\bar{\Lambda}}^k \bar{U}_{j+1}^k, \bar{U}_{j+1}^k\right) e^{\bar{\bar{\alpha}} x_{j+1}} + h\left[e^{\bar{\bar{\alpha}}}\left(\bar{\bar{\Lambda}}^k \bar{U}_J^k, \bar{U}_J^k\right) - \left(\bar{\bar{\Lambda}}^k \bar{U}_0^k, \bar{U}_0^k\right)\right]. \end{aligned} \tag{26}$$

In order to form $\left\|\bar{\mathbf{U}}^k\right\|_{\bar{\bar{\alpha}}}^2 = h\sum_{j=0}^{J-1}\left(\bar{U}_j^k, \bar{U}_j^k\right) e^{\bar{\bar{\alpha}} x_j}$ on the right-hand part of inequality (26), we use the following formula to shift the index backward in the sum:

$$\sum_{j=0}^{J-1}\left(\bar{\bar{\Lambda}}^k \bar{U}_{j+1}^k, \bar{U}_{j+1}^k\right) e^{\bar{\bar{\alpha}} x_{j+1}} = \sum_{j=0}^{J-1}\left(\bar{\bar{\Lambda}}^k \bar{U}_j^k, \bar{U}_j^k\right) e^{\bar{\bar{\alpha}} x_j} + e^{\bar{\bar{\alpha}}}\left(\bar{\bar{\Lambda}}^k \bar{U}_J^k, \bar{U}_J^k\right) - \left(\bar{\bar{\Lambda}}^k \bar{U}_0^k, \bar{U}_0^k\right).$$

Then, equality (26) is transformed as follows:

$$\begin{aligned} &h\sum_{j=0}^{J-1}\left\{\left(\bar{\bar{\Lambda}}^k \bar{U}_{j+1}^k, \bar{U}_{j+1}^k\right) - \left(\bar{\bar{\Lambda}}^k \bar{U}_j^k, \bar{U}_j^k\right)\right\} e^{\bar{\bar{\alpha}} x_j} = h\left(e^{-\bar{\bar{\alpha}} h} - 1\right)\sum_{j=0}^{J-1}\left(\bar{\bar{\Lambda}}^k \bar{U}_j^k, \bar{U}_j^k\right) e^{\bar{\bar{\alpha}} x_j} + \\ &h\left(e^{-\bar{\bar{\alpha}} h} - 1\right)\left[e^{\bar{\bar{\alpha}}}\left(\bar{\bar{\Lambda}}^k \bar{U}_J^k, \bar{U}_J^k\right) - \left(\bar{\bar{\Lambda}}^k \bar{U}_0^k, \bar{U}_0^k\right)\right] + h\left[e^{\bar{\bar{\alpha}}}\left(\bar{\bar{\Lambda}}^k \bar{U}_J^k, \bar{U}_J^k\right) - \left(\bar{\bar{\Lambda}}^k \bar{U}_0^k, \bar{U}_0^k\right)\right]. \end{aligned}$$

Or, simplifying the right part, we have inequality (21). The proof of inequality for $\bar{L}_1^k$ in (21) is carried out in a similar way.

Lemma 1 is proved. □

In proceeding similarly to the proof of Lemma 1, the validity of the following lemma is established.

Lemma 2. *The inequality is valid for solutions to the initial boundary value difference problem* (15)*:*

$$\begin{aligned} \bar{L}_2^k &= \left[\left(\bar{B}\left[\bar{A}^k + h\bar{\Lambda}^k\left(\bar{U}_0^k - \bar{U}_J^k\right)\right], \left[\bar{A}^k + h\bar{\Lambda}^k\left(\bar{U}_0^k - \bar{U}_J^k\right)\right]\right) - \left(\bar{B}\bar{A}^k, \bar{A}^k\right)\right], \\ \bar{\bar{L}}_2^k &= \left(\bar{\bar{B}}\left[\bar{\bar{A}}^k + h\bar{\bar{\Lambda}}^k\left(\bar{\bar{U}}_J^k - \bar{\bar{U}}_0^k\right)\right], \left[\bar{\bar{A}}^k + h\bar{\bar{\Lambda}}^k\left(\bar{\bar{U}}_J^k - \bar{\bar{U}}_0^k\right)\right]\right) - \left(\bar{\bar{B}}\bar{\bar{A}}^k, \bar{\bar{A}}^k\right). \end{aligned} \tag{27}$$

Lemma 3. *Let the CFL condition* (10) *be fulfilled. Then, the inequalities are valid for the solutions of the initial boundary value difference problem* (15) *for the right differential derivative of* $\bar{\mathrm{L}}^k$ *and* $\bar{\bar{\mathrm{L}}}^k$ *on time:*

$$\begin{aligned} \frac{\bar{\mathrm{L}}^{k+1} - \bar{\mathrm{L}}^k}{\tau} &\le \left(e^{-\bar{\alpha} h} - 1\right)\sum_{j=1}^{J}\left(\bar{\mathrm{M}}^k \bar{U}_j^k, \bar{U}_j^k\right) e^{-\bar{\alpha} x_j} + \bar{B}_1^k, \\ \frac{\bar{\bar{\mathrm{L}}}^{k+1} - \bar{\bar{\mathrm{L}}}^k}{\tau} &\le \left(e^{-\bar{\bar{\alpha}} h} - 1\right)\sum_{j=0}^{J-1}\left(\bar{\bar{\mathrm{M}}}^k \bar{\bar{U}}_j^k, \bar{\bar{U}}_j^k\right) e^{\bar{\bar{\alpha}} x_j} + \bar{\bar{B}}_1^k, \end{aligned} \tag{28}$$

where

$$\bar{B}_1^k = e^{-\bar{\alpha}h}\left[\left(\bar{\mathrm{M}}^k\bar{U}_0^k,\bar{U}_0^k\right) - e^{-\bar{\alpha}}\left(\bar{\mathrm{M}}^k\bar{U}_J^k,\bar{U}_J^k\right)\right] + \frac{1}{\tau}\left\{\left(\bar{B}\left[\bar{A}^k + h\bar{\Lambda}^k\left(\bar{U}_0^k - \bar{U}_J^k\right)\right],\left[\bar{A}^k + h\bar{\Lambda}^k\left(\bar{U}_0^k - \bar{U}_J^k\right)\right]\right) - \left(\bar{B}\bar{A}^k,\bar{A}^k\right)\right\},$$
$$\bar{\bar{B}}_1^k = e^{-\bar{\alpha}h}\left[e^{\bar{\alpha}}\left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{U}}_J^k,\bar{\bar{U}}_J^k\right) - \left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{U}}_0^k,\bar{\bar{U}}_0^k\right)\right] + \frac{1}{\tau}\left\{\left(\bar{\bar{B}}\left[\bar{\bar{A}}^k + h\bar{\bar{\Lambda}}^k\left(\bar{\bar{U}}_J^k - \bar{\bar{U}}_0^k\right)\right],\left[\bar{\bar{A}}^k + h\bar{\bar{\Lambda}}^k\left(\bar{\bar{U}}_J^k - \bar{\bar{U}}_0^k\right)\right]\right) - \left(\bar{\bar{B}}\bar{\bar{A}}^k,\bar{\bar{A}}^k\right)\right\}.$$

Proof. The proof is carried out separately for $\bar{\mathrm{L}}^k$and $\bar{\bar{\mathrm{L}}}^k$. First, we prove the inequality for $\bar{\bar{\mathrm{L}}}^k$.

The difference derivative of $\bar{\bar{\mathrm{L}}}^k$ is calculated using a formula similar to (19):

$$\frac{\bar{\bar{\mathrm{L}}}^{k+1} - \bar{\bar{\mathrm{L}}}^k}{\tau} = \frac{1}{\tau}\bar{\bar{L}}_1^k + \frac{1}{\tau}\bar{\bar{L}}_2^k.$$

We estimate the value $\bar{\bar{L}}_1^k$ from above using inequality (21) of Lemma 1 and transform the value $\bar{\bar{L}}_2^k$ using equality (27) of Lemma 2. Then, we have

$$\begin{aligned}
&\tfrac{\bar{\bar{\mathrm{L}}}^{k+1} - \bar{\bar{\mathrm{L}}}^k}{\tau} = \tfrac{1}{\tau}\bar{\bar{L}}_1^k + \tfrac{1}{\tau}\bar{\bar{L}}_2^k \le \\
&\tfrac{1}{\tau}\left\{h\left(e^{-\bar{\alpha}h} - 1\right)\textstyle\sum_{j=0}^{J-1}\left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_j^k,\bar{\bar{U}}_j^k\right)e^{\bar{\alpha}x_j} + he^{-\bar{\alpha}h}\left[e^{\bar{\alpha}}\left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_J^k,\bar{\bar{U}}_J^k\right) - \left(\bar{\bar{\Lambda}}^k\bar{\bar{U}}_0^k,\bar{\bar{U}}_0^k\right)\right]\right\} + \\
&\tfrac{1}{\tau}\left\{\left(\bar{\bar{B}}\left[\bar{\bar{A}}^k + h\bar{\bar{\Lambda}}^k\left(\bar{\bar{U}}_J^k - \bar{\bar{U}}_0^k\right)\right],\left[\bar{\bar{A}}^k + h\bar{\bar{\Lambda}}^k\left(\bar{\bar{U}}_J^k - \bar{\bar{U}}_0^k\right)\right]\right) - \left(\bar{\bar{B}}\bar{\bar{A}}^k,\bar{\bar{A}}^k\right)\right\} = \\
&\left(e^{-\bar{\alpha}h} - 1\right)\textstyle\sum_{j=0}^{J-1}\left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{U}}_j^k,\bar{\bar{U}}_j^k\right)e^{\bar{\alpha}x_j} + \bar{\bar{B}}_1^k,
\end{aligned}$$

Similarly, the inequality for $\bar{\mathrm{L}}^k$ is proved in (28).

Lemma 3 is proved. □

Let us separate the boundary conditions as follows:

$$\begin{aligned}
&for\ \ x = 0\ \ \bar{U}_0^{k+1} = \hat{\bar{U}}_0^{k+1} + \bar{R}\bar{\Delta}^{k+1},\ \ \hat{\bar{U}}_0^{k+1} = \bar{R}\bar{U}_J^{k+1} + (\bar{E} - \bar{R})\bar{\Theta}\bar{A}^{k+1}, \\
&for\ \ x = 1\ \ \bar{\bar{U}}_J^{k+1} = \hat{\bar{\bar{U}}}_J^{k+1} + \bar{\bar{R}}\bar{\bar{\Delta}}^{k+1},\ \ \hat{\bar{\bar{U}}}_J^{k+1} = \bar{\bar{R}}\bar{\bar{U}}_0^{k+1} + \left(\bar{\bar{E}} - \bar{\bar{R}}\right)\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}.
\end{aligned} \tag{29}$$

Substitute decomposition (29) into expressions $\bar{B}_1^k$ and $\bar{\bar{B}}_1^k$ to obtain

$$\begin{aligned}
&\bar{B}_1^k = e^{-\alpha h}\left\{\left(\bar{\mathrm{M}}^k\left[\hat{\bar{U}}_0^k + \bar{R}\bar{\Delta}^k\right],\left[\hat{\bar{U}}_0^k + \bar{R}\bar{\Delta}^k\right]\right) - e^{-\alpha}\left(\bar{\mathrm{M}}^k\bar{U}_J^k,\bar{U}_J^k\right)\right\} + \\
&\left\{h\left(\bar{B}\bar{\mathrm{M}}^k\left\langle\left[\hat{\bar{U}}_0^k + \bar{R}\bar{\Delta}^k\right] - \bar{U}_J^k\right\rangle,\bar{\Lambda}^k\left\langle\left[\hat{\bar{U}}_0^k + \bar{R}\bar{\Delta}^k\right] - \bar{U}_J^k\right\rangle\right) + 2\left(\bar{B}\bar{A}^k,\bar{\mathrm{M}}^k\left\langle\left[\hat{\bar{U}}_0^k + \bar{R}\bar{\Delta}^k\right] - \bar{U}_J^k\right\rangle\right)\right\}, \\
&\bar{\bar{B}}_1^k = e^{-\bar{\alpha}h}\left\{e^{\bar{\alpha}}\left(\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k + \bar{\bar{R}}\bar{\bar{\Delta}}^k\right],\left[\hat{\bar{\bar{U}}}_J^k + \bar{\bar{R}}\bar{\bar{\Delta}}^k\right]\right) - \left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{U}}_0{}^k,\bar{\bar{U}}_0{}^k\right)\right\} + \\
&\left\{\tau\left(\bar{\bar{B}}\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k + \bar{\bar{R}}\bar{\bar{\Delta}}^k - \bar{\bar{U}}_0{}^k\right],\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k + \bar{\bar{R}}\bar{\bar{\Delta}}^k - \bar{\bar{U}}_0{}^k\right]\right) + 2\left(\bar{\bar{B}}\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k + \bar{\bar{R}}\bar{\bar{\Delta}}^k - \bar{\bar{U}}_0{}^k\right]\right)\right\}.
\end{aligned} \tag{30}$$

Lemma 4. *For expressions $\bar{B}_1^k$ and $\bar{\bar{B}}_1^k$, the following inequalities are valid:*

$$\bar{B}_1^k \le \bar{B}_2^k + \bar{B}_3^k + \bar{B}_4^k, \bar{\bar{B}}_1^k \le \bar{\bar{B}}_2^k + \bar{\bar{B}}_3^k + \bar{\bar{B}}_4^k \le \bar{\bar{\bar{B}}}_2^k + \bar{\bar{\bar{B}}}_3^k + \bar{\bar{B}}_3^k + \bar{\bar{B}}_4^k. \tag{31}$$

Here,

$$\bar{B}_2^k = e^{-\alpha h}\left[\left(\bar{\mathrm{M}}^k\hat{\bar{U}}_0^k,\hat{\bar{U}}_0^k\right) - e^{-\alpha}\left(\bar{\mathrm{M}}^k\bar{U}_J^k,\bar{U}_J^k\right)\right] + 2\left(\bar{B}\bar{A}^k,\bar{\mathrm{M}}^k\left[\hat{\bar{U}}_0^k - \bar{U}_J^k\right]\right) + h\left(|\bar{B}|\bar{\mathrm{M}}^k\left[\hat{\bar{U}}_0^k - \bar{U}_J^k\right],\bar{\Lambda}^k\left[\hat{\bar{U}}_0^k - \bar{U}_J^k\right]\right);$$

$$\bar{B}_3^k = \alpha^2 e^{-\alpha h}\left(\bar{\mathrm{M}}^k\hat{\bar{U}}_0^k,\hat{\bar{U}}_0^k\right) + h\alpha^2\left(|\bar{B}|\bar{\mathrm{M}}^k\left[\hat{\bar{U}}_0^k - \bar{U}_J^k\right],\bar{\Lambda}^k\left[\hat{\bar{U}}_0^k - \bar{U}_J^k\right]\right) + \alpha^2\left(\bar{A}^k,|\bar{B}|\bar{\mathrm{M}}^k\bar{A}^k\right);$$

$$\bar{B}_4^k = e^{-\alpha h}\left(1 + \frac{1}{\alpha^2}\right)\left(\bar{\mathrm{M}}^k\bar{R}\bar{\Delta}^k,\bar{R}\bar{\Delta}^k\right) + h\left(1 + \frac{1}{\alpha^2}\right)\left(|\bar{B}|\bar{\mathrm{M}}^k\bar{R}\bar{\Delta}^k,\bar{\Lambda}^k\bar{R}\bar{\Delta}^k\right) + \frac{1}{\alpha^2}\left(\bar{R}\bar{\Delta}^k,|\bar{B}|\bar{\mathrm{M}}^k\bar{R}\bar{\Delta}^k\right);$$

$$\bar{\bar{B}}_2^k = e^{-\bar{\alpha}h}\left[e^{\bar{\alpha}}\left(\bar{\bar{\mathrm{M}}}^k\hat{\bar{\bar{U}}}_J^k,\hat{\bar{\bar{U}}}_J^k\right) - \left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{U}}_0{}^k,\bar{\bar{U}}_0{}^k\right)\right] + 2\left(\bar{\bar{B}}\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0{}^k\right]\right) + \tau\left(\bar{\bar{B}}\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0{}^k\right],\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right]\right);$$

$$\bar{\bar{B}}_3^k = e^{\bar{\bar{\alpha}}(1-h)}\alpha^2\left(\bar{\bar{\mathrm{M}}}^k\hat{\bar{\bar{U}}}_J^k,\hat{\bar{\bar{U}}}_J^k\right) + \tau\alpha^2\left(\left|\bar{\bar{B}}\right|\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0{}^k\right],\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0{}^k\right]\right) + \alpha^2\left(\left|\bar{\bar{B}}\right|\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{A}}^k\right);$$

$$\bar{\bar{B}}_4^k = e^{\bar{\bar{\alpha}}(1-h)}\left(1+\frac{1}{\alpha^2}\right)\left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k,\bar{\bar{R}}\bar{\bar{\Delta}}^k\right) + \tau\left(\left[\bar{\bar{B}} + \frac{1}{\alpha^2}\left|\bar{\bar{B}}\right|\right]\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k\right) + \frac{1}{\alpha^2}\left(\left|\bar{\bar{B}}\right|\bar{\bar{R}}\bar{\bar{\Delta}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k\right);$$

$$\bar{\bar{\bar{B}}}_3^k = -2\left(\bar{\bar{\Theta}}e^{\bar{\bar{\alpha}}}\bar{\bar{R}}\bar{\bar{\alpha}}^2\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\left[\bar{\bar{U}}_0^{k+1} - \bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right]\right).$$

Proof of Lemma 4. First, we prove the inequality for $\bar{\bar{B}}_1^k$.

It is not difficult to prove the inequality:

$$\left(\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k + \bar{\bar{R}}\bar{\bar{\Delta}}^k\right],\left[\hat{\bar{\bar{U}}}_J^k + \bar{\bar{R}}\bar{\bar{\Delta}}^k\right]\right) \le \left(1+\bar{\bar{\alpha}}^2\right)\left(\bar{\bar{\mathrm{M}}}^k\hat{\bar{\bar{U}}}_J^k,\hat{\bar{\bar{U}}}_J^k\right) + \left(1+\frac{1}{\bar{\bar{\alpha}}^2}\right)\left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k,\bar{\bar{R}}\bar{\bar{\Delta}}^k\right).$$

Similarly, let us upper bound the expressions $2\left(\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k\right)$ and $\left(\bar{\bar{\mathrm{M}}}^k\left\{\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right] + \bar{\bar{R}}\bar{\bar{\Delta}}^k\right\},\right.$ $\left.\bar{\bar{\mathrm{M}}}^k\left\{\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right] + \bar{\bar{R}}\bar{\bar{\Delta}}^k\right\}\right)$, respectively, as follows:

1. $2\left(\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k\right) \le \alpha^2\left(\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{A}}^k\right) + \frac{1}{\alpha^2}\left(\bar{\bar{R}}\bar{\bar{\Delta}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k\right)$,
2. $\begin{array}{c}\left(\bar{\bar{\mathrm{M}}}^k\left\{\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right] + \bar{\bar{R}}\bar{\bar{\Delta}}^k\right\},\bar{\bar{\mathrm{M}}}^k\left\{\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right] + \bar{\bar{R}}\bar{\bar{\Delta}}^k\right\}\right) \le \\ (1+\alpha^2)\left(\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right],\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right]\right) + \left(1+\frac{1}{\alpha^2}\right)\left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k\right).\end{array}$;

Then, from (30), we obtain

$$\begin{array}{l}\bar{\bar{B}}_1^k \le e^{-\bar{\bar{\alpha}}h}\left\{\ e^{\bar{\bar{\alpha}}}\left\langle(1+\bar{\bar{\alpha}}^2)\left(\bar{\bar{\mathrm{M}}}^k\hat{\bar{\bar{U}}}_J^k,\hat{\bar{\bar{U}}}_J^k\right) + \left(1+\frac{1}{\bar{\bar{\alpha}}^2}\right)\left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k,\bar{\bar{R}}\bar{\bar{\Delta}}^k\right)\right\rangle - \left(\bar{\bar{\mathrm{M}}}^k\bar{\bar{U}}_0^k,\bar{\bar{U}}_0^k\right)\ \right\} + \\ \tau\left\{\left(\left[\bar{\bar{B}} + \bar{\bar{\alpha}}^2\left|\bar{\bar{B}}\right|\right]\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right],\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right]\right) + \left(\left[\bar{\bar{B}} + \frac{1}{\bar{\bar{\alpha}}^2}\left|\bar{\bar{B}}\right|\right]\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k\right)\right\} + \\ \left\{\bar{\bar{\alpha}}^2\left(\left|\bar{\bar{B}}\right|\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{A}}^k\right) + \frac{1}{\bar{\bar{\alpha}}^2}\left(\left|\bar{\bar{B}}\right|\bar{\bar{R}}\bar{\bar{\Delta}}^k,\bar{\bar{\mathrm{M}}}^k\bar{\bar{R}}\bar{\bar{\Delta}}^k\right)\right\} + 2\left(\bar{\bar{B}}\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right]\right).\end{array} \tag{32}$$

Therefore, from here, we obtain an estimate:

$$\bar{\bar{B}}_1^k \le \bar{\bar{B}}_2^k + \bar{\bar{B}}_3^k + \bar{\bar{B}}_4^k.$$

From the boundary conditions (29) and taking into account Jensen's inequality for convex maps, we obtain the following inequality and equality:

$$\begin{array}{l}\left(\hat{\bar{\bar{U}}}_J^{k+1},\hat{\bar{\bar{U}}}_J^{k+1}\right) \le \left(\bar{\bar{R}}\bar{\bar{U}}_0^{k+1},\bar{\bar{U}}_0^{k+1}\right) + \left(\left[\bar{\bar{E}} - \bar{\bar{R}}\right]\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1},\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right), \\ \hat{\bar{\bar{U}}}_J^{k+1} - \bar{\bar{U}}_0^{k+1} = \left(\bar{\bar{R}} - \bar{\bar{E}}\right)\bar{\bar{U}}_0^{k+1} + \left(\bar{\bar{E}} - \bar{\bar{R}}\right)\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}.\end{array}$$

Taking into account these relations of the boundary conditions for the expression $\bar{\bar{B}}_2^k$, we obtain the following inequality:

$$\begin{array}{l}\bar{\bar{B}}_2^k \le e^{-\bar{\bar{\alpha}}h}\left\{\left(\bar{\bar{\mathrm{M}}}^k\left[e^{\bar{\bar{\alpha}}}\bar{\bar{R}} - \bar{\bar{E}}\right]\bar{\bar{U}}_0^{k+1},\bar{\bar{U}}_0^{k+1}\right) + e^{\bar{\bar{\alpha}}}\left(\bar{\bar{\mathrm{M}}}^k\left[\bar{\bar{E}} - \bar{\bar{R}}\right]\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1},\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right)\right\} + \\ -2\left(\bar{\bar{B}}\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\left[\bar{\bar{E}} - \bar{\bar{R}}\right]\left[\bar{\bar{U}}_0^{k+1} - \bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right]\right) + \tau\left(\bar{\bar{B}}\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right],\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right]\right);\end{array}$$

Note that $\left(\bar{\bar{B}}\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right],\bar{\bar{\mathrm{M}}}^k\left[\hat{\bar{\bar{U}}}_J^k - \bar{\bar{U}}_0^k\right]\right) \le 0$. Therefore, for the expression $\bar{\bar{B}}_2^k$, we obtain the following inequality:

$$\bar{\bar{B}}_2^k \le e^{-\bar{\bar{\alpha}}h}\left\{\left(\bar{\bar{\mathrm{M}}}^k\left[e^{\bar{\bar{\alpha}}}\bar{\bar{R}} - \bar{\bar{E}}\right]\bar{\bar{U}}_0^{k+1},\bar{\bar{U}}_0^{k+1}\right) + e^{\bar{\bar{\alpha}}}\left(\bar{\bar{\mathrm{M}}}^k\left[\bar{\bar{E}} - \bar{\bar{R}}\right]\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1},\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right)\right\} + -2\left(\bar{\bar{B}}\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\left[\bar{\bar{E}} - \bar{\bar{R}}\right]\left[\bar{\bar{U}}_0^{k+1} - \bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right]\right).$$

Substituting the value $\bar{\bar{B}}$ from expression (16) to express $\bar{\bar{B}}_2^k$, we obtain the following inequality:

$$\bar{\bar{B}}_2^k \le e^{-\bar{\bar{\alpha}}h}\left\{\left(\bar{\bar{\mathrm{M}}}^k\left[e^{\bar{\bar{\alpha}}}\bar{\bar{R}} - \bar{\bar{E}}\right]\bar{\bar{U}}_0^{k+1},\bar{\bar{U}}_0^{k+1}\right) + e^{\bar{\bar{\alpha}}}\left(\bar{\bar{\mathrm{M}}}^k\left[\bar{\bar{E}} - \bar{\bar{R}}\right]\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1},\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right)\right\} - 2\left(\bar{\bar{\Theta}}\left(e^{\bar{\bar{\alpha}}}\bar{\bar{R}}\left[1+\bar{\bar{\alpha}}^2\right] - \bar{\bar{E}}\right)\bar{\bar{A}}^k,\bar{\bar{\mathrm{M}}}^k\left[\bar{\bar{U}}_0^{k+1} - \bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right]\right).$$

Then, taking the right-hand side of the last inequality above to express $\bar{\bar{B}}_2^k$, we split it into two parts:

$$\bar{\bar{B}}_2^k \le e^{-\bar{\bar{\alpha}}h}\left\{\left(\bar{\mathrm{M}}^k\left[e^{\bar{\bar{\alpha}}}\bar{\bar{R}}-\bar{\bar{E}}\right]\bar{\bar{U}}_0^{k+1},\bar{\bar{U}}_0^{k+1}\right)+e^{\bar{\bar{\alpha}}}\left(\bar{\mathrm{M}}^k\left[\bar{\bar{E}}-\bar{\bar{R}}\right]\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1},\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right)\right\}+\bar{\bar{\bar{B}}}_3^k-2\left(\bar{\bar{\Theta}}\left(e^{\bar{\bar{\alpha}}}\bar{\bar{R}}-\bar{\bar{E}}\right)\bar{\bar{A}}^k,\bar{\mathrm{M}}^k\left[\bar{\bar{U}}_0^{k+1}-\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right]\right).$$

where $\bar{\bar{\bar{B}}}_3^k=-2\left(\bar{\bar{\Theta}}e^{\bar{\bar{\alpha}}}\bar{\bar{R}}\bar{\bar{\alpha}}^2\bar{\bar{A}}^k,\bar{\mathrm{M}}^k\left[\bar{\bar{U}}_0^{k+1}-\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right]\right)$.

Taking the right-hand part of the last inequality to express $\bar{\bar{B}}_2^k$, we transform it as follows:

$$\bar{\bar{B}}_2^k \le \bar{\bar{\bar{B}}}_3^k+e^{-\bar{\bar{\alpha}}h}\left(\bar{\mathrm{M}}^k\left[e^{\bar{\bar{\alpha}}}\bar{\bar{R}}-\bar{\bar{E}}\right]\left[\bar{\bar{U}}_0^k-e^{\bar{\bar{\alpha}}h}\bar{\bar{\Theta}}\bar{\bar{A}}^k\right],\left[\bar{\bar{U}}_0^k-e^{\bar{\bar{\alpha}}h}\bar{\bar{\Theta}}\bar{\bar{A}}^k\right]\right)+$$
$$e^{\bar{\bar{\alpha}}(1-h)}\left(\bar{\mathrm{M}}^k\left[\bar{\bar{E}}-\bar{\bar{R}}\right]\bar{\bar{\Theta}}\bar{\bar{A}}^k,\bar{\bar{\Theta}}\bar{\bar{A}}^k\right)-e^{\bar{\bar{\alpha}}h}\left(\bar{\mathrm{M}}^k\left[e^{\bar{\bar{\alpha}}}\bar{\bar{R}}-\bar{\bar{E}}\right]\bar{\bar{\Theta}}\bar{\bar{A}}^k,\bar{\bar{\Theta}}\bar{\bar{A}}^k\right)+2\left(\left(e^{\bar{\bar{\alpha}}}\bar{\bar{R}}-\bar{\bar{E}}\right)\bar{\bar{\Theta}}\bar{\bar{A}}^k,\bar{\mathrm{M}}^k\bar{\bar{\Theta}}\bar{\bar{A}}^k\right).$$

For the fixed $\bar{\bar{R}}$, we choose $\bar{\bar{\alpha}}$ so that $e^{\bar{\bar{\alpha}}}\bar{\bar{R}}-\bar{\bar{E}}\le 0$. Then,

$$e^{-\bar{\bar{\alpha}}h}\left(\bar{\mathrm{M}}^k\left[e^{\bar{\bar{\alpha}}}\bar{\bar{R}}-\bar{\bar{E}}\right]\left[\bar{\bar{U}}_0^k-e^{\bar{\bar{\alpha}}h}\bar{\bar{\Theta}}\bar{\bar{A}}^k\right],\left[\bar{\bar{U}}_0^k-e^{\bar{\bar{\alpha}}h}\bar{\bar{\Theta}}\bar{\bar{A}}^k\right]\right)\le 0$$

and therefore, strengthening the inequality for the expression $\bar{\bar{B}}_2^k$, we obtain

$$\bar{\bar{B}}_2^k \le\ \bar{\bar{\bar{B}}}_3^k+\bar{\bar{\bar{B}}}_2^k.$$

Here, $\bar{\bar{\bar{B}}}_2^k=\left(\left\{e^{\bar{\bar{\alpha}}(1-h)}\left[\bar{\bar{E}}-\bar{\bar{R}}\right]+\left(2-e^{\bar{\bar{\alpha}}h}\right)\left[e^{\bar{\bar{\alpha}}}\bar{\bar{R}}-\bar{\bar{E}}\right]\right\}\bar{\bar{\Theta}}\bar{\bar{A}}^k,\bar{\mathrm{M}}^k\bar{\bar{\Theta}}\bar{\bar{A}}^k\right)$.

Thus, evaluating the expression$\bar{\bar{B}}_2^k$ from (32) from above, we obtain inequality (31) for the expression $\bar{\bar{B}}_1^k$.

Similarly, inequality (31) is proved for the expression $\bar{B}_1^k$.

Lemma 4 is proved. □

Lemma 5. *Let the conditions of Theorem 1 be fulfilled. Then, there are positive constants* $\bar{\zeta}_1^k$, $\bar{\zeta}_2^k$, $\bar{\bar{\zeta}}_1^k$, *and* $\bar{\bar{\zeta}}_2^k$ *such that the solution* $\mathbf{U}^k$ *to the initial boundary value difference problem (11) satisfies the inequalities*

$$\frac{\bar{\mathrm{L}}^{k+1}-\bar{\mathrm{L}}^k}{\tau}\le -\bar{\zeta}_1^k\bar{\mathrm{L}}^k+\bar{\zeta}_2^k\left(\bar{\Delta}^k,\bar{\Delta}^k\right),\ \frac{\bar{\bar{\mathrm{L}}}^{k+1}-\bar{\bar{\mathrm{L}}}^k}{\tau}\le -\bar{\bar{\zeta}}_1^k\bar{\bar{\mathrm{L}}}^k+\bar{\bar{\zeta}}_2^k\left(\bar{\bar{\Delta}}^k,\bar{\bar{\Delta}}^k\right). \quad (33)$$

The Proof of Lemma 5. The proof is carried out separately for every $\bar{\mathrm{L}}^k$ and $\bar{\bar{\mathrm{L}}}^k$. First, we prove the inequality for $\bar{\bar{\mathrm{L}}}^k$.

Let us start by evaluating the expressions $\bar{\bar{\bar{B}}}_3^k+\bar{\bar{B}}_3^k$ and $\bar{\bar{B}}_4^k$. Taking into account that $\bar{\bar{B}}$, defined by (16) and $\bar{\mathrm{M}}$, is valid, we have the following inequality:

$$\left|\bar{\bar{B}}\right|\le\frac{\bar{\bar{\alpha}}}{1-e^{-\bar{\bar{\alpha}}}}\le 1,\ \ \bar{\mathrm{M}}\le diag\left(\frac{{}_1P}{{}_1Q},\frac{{}_2P}{{}_2Q},\cdots,\frac{{}_mP}{{}_mQ}\right),\ \ \text{and } \bar{\bar{\Lambda}}^k\le\bar{\bar{E}},$$

where $\bar{\bar{\Lambda}}^k$ and h 8 $\bar{\bar{\Theta}}$ all are limited from above by one.

By virtue of (17) and (18) for $0<\bar{\bar{\alpha}}\le 1$, we have

$$\left(\hat{\bar{\bar{U}}}_J^k,\hat{\bar{\bar{U}}}_J^k\right)\le 2\left(\bar{\bar{U}}_0^k,\bar{\bar{U}}_0^k\right)+2\left(\bar{\bar{A}}^k,\bar{\bar{A}}^k\right)\le(2+2(1+3))\left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2=10\left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2$$

and

$$\left(\left[\hat{\bar{\bar{U}}}_J^k-\bar{\bar{U}}_0^k\right],\left[\hat{\bar{\bar{U}}}_J^k-\bar{\bar{U}}_0^k\right]\right)\le 2\left(\hat{\bar{\bar{U}}}_J^k,\hat{\bar{\bar{U}}}_J^k\right)+2\left(\bar{\bar{U}}_0^k,\bar{\bar{U}}_0^k\right)\le 22\left\|\bar{\bar{\mathbf{U}}}^k\right\|_{\bar{\bar{\alpha}}}^2.$$

Since

$$\left(\bar{\bar{U}}_0^k-\bar{\bar{\Theta}}\bar{\bar{A}}^{k+1}\right)=\left(\bar{\bar{R}}-\bar{\bar{E}}\right)^{-1}\left(\hat{\bar{\bar{U}}}_J^{k+1}-\bar{\bar{U}}_0^{k+1}\right).$$

for the expression $\bar{\bar{\bar{B}}}_3^k$, we obtain the following inequality:

$$\bar{\bar{B}}_3^k = -2\left(\bar{\Theta}e^{\bar{\alpha}}\bar{R}\bar{\alpha}^2\bar{A}^k, \bar{M}^k\left[\bar{R}-\bar{E}\right]^{-1}\left[\hat{\bar{U}}_J^k-\bar{U}_0^k\right]\right) = 2\left(\bar{\Theta}e^{\bar{\alpha}}\bar{R}\bar{\alpha}^2\bar{A}^k, \bar{M}^k\left[\bar{E}-\bar{R}\right]^{-1}\left[\hat{\bar{U}}_J^k-\bar{U}_0^k\right]\right) \leq \left(\bar{\Theta}e^{\bar{\alpha}}\bar{R}\bar{\alpha}^2\bar{M}^k\left[\bar{E}-\bar{R}\right]^{-1}\left[\hat{\bar{U}}_J^k-\bar{U}_0^k\right], \left[\hat{\bar{U}}_J^k-\bar{U}_0^k\right]\right) + \left(\bar{\Theta}e^{\bar{\alpha}}\bar{R}\bar{\alpha}^2\bar{M}^k\left[\bar{E}-\bar{R}\right]^{-1}\bar{A}^k, \bar{A}^k\right) \leq \alpha^2\bar{C}h\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{U}_j^k, \bar{U}_j^k\right)e^{\bar{\alpha}x_j}.$$

Here, $\bar{\bar{C}}$ is some positive number satisfying the inequality

$$\bar{\Theta}e^{\bar{\alpha}}\bar{R}\left[\bar{E}-\bar{R}\right]^{-1}(23+3\bar{\alpha}) \leq \bar{C}\bar{E}.$$

Then, it is obvious that the following inequalities are true: $\bar{\bar{B}}_3^k + \bar{\bar{\bar{B}}}_3^k \leq \alpha^2\bar{\bar{\bar{C}}}h\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{U}_j^k, \bar{U}_j^k\right)e^{\bar{\alpha}x_j}$ and $\bar{B}_4^k \leq e^{\bar{\alpha}(1-h)}\left(1+\frac{3}{\bar{\alpha}^2}\right)\left(\bar{M}^k\bar{\Delta}^k, \bar{\Delta}^k\right)$.

By virtue of the previous estimates, as well as by virtue of Equation (29) for $\bar{\bar{\bar{B}}}_2^k$, we have

$$\bar{\bar{\bar{B}}}_2^k = \left(\left\{e^{\bar{\alpha}(1-h)}\left[\bar{E}-\bar{R}\right] + \left(2-e^{\bar{\alpha}h}\right)\left[e^{\bar{\alpha}}\bar{R}-\bar{E}\right]\right\}\bar{\Theta}\bar{A}^k, \bar{M}^k\bar{\Theta}\bar{A}^k\right) \leq (1+3\bar{\alpha})\bar{\alpha}h\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{\Theta}\bar{U}_j^k, \bar{\Theta}\bar{U}_j^k\right)e^{\bar{\alpha}x_j}.$$

Previous estimates allow us to estimate the difference derivative of $\bar{L}^k$ in inequality (28) for $k \geq 0$:

$$\frac{\bar{L}^{k+1}-\bar{L}^k}{\tau} \leq \left(e^{-\bar{\alpha}h}-1\right)\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{U}_j^k, \bar{U}_j^k\right)e^{\bar{\alpha}x_j} + \bar{B}_1^k \leq \left(e^{-\bar{\alpha}h}-1\right)\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{U}_j^k, \bar{U}_j^k\right)e^{\bar{\alpha}x_j} + \bar{\bar{\bar{B}}}_2^k + \bar{\bar{\bar{B}}}_3^k + \bar{\bar{B}}_3^k + \bar{B}_4^k \leq$$
$$\left(e^{-\bar{\alpha}h}-1\right)\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{U}_j^k, \bar{U}_j^k\right)e^{\bar{\alpha}x_j} + (1+3\bar{\alpha})\bar{\alpha}h\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{\Theta}\bar{U}_j^k, \bar{\Theta}\bar{U}_j^k\right)e^{\bar{\alpha}x_j} +$$
$$\bar{\alpha}^2\bar{\bar{\bar{C}}}h\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{U}_j^k, \bar{U}_j^k\right)e^{\bar{\alpha}x_j} + e^{\bar{\alpha}(1-h)}\left(1+\frac{3}{\bar{\alpha}^2}\right)\left(\bar{M}^k\bar{\Delta}^k, \bar{\Delta}^k\right) \leq (with precision\ \alpha^2)$$
$$-\bar{\alpha}h\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{U}_j^k, \bar{U}_j^k\right)e^{\bar{\alpha}x_j} + \bar{\alpha}h\sum_{j=0}^{J-1}\left(\bar{M}^k\bar{\Theta}\bar{U}_j^k, \bar{\Theta}\bar{U}_j^k\right)e^{\bar{\alpha}x_j} + \bar{\alpha}h\left(1+\frac{3}{\bar{\alpha}^2}\right)\left(\bar{M}^k\bar{\Delta}^k, \bar{\Delta}^k\right)$$

The last inequality is satisfied, provided that $0 < \bar{\alpha}$ is small enough, so that (18) is fulfilled and the following inequality is fair:

$$\bar{\alpha} \leq \frac{1}{7+2\bar{\bar{\bar{C}}}}\left(\bar{E}-\bar{\Theta}^2\right).$$

Finally, it remains to show that $\bar{M}^k$ is bounded from below by a strictly positive number. This is equivalent to the concept that $\bar{A}^k$ is bounded from above. Since

$$h\sum_{j=0}^{J-1}\left(\bar{U}_j^k, \bar{U}_j^k\right)e^{\bar{\alpha}x_j} \geq \bar{L}^k$$

we have

$$\frac{\bar{L}^{k+1}-\bar{L}^k}{\tau} \leq -\bar{b}_1\lambda_{\min}\left(\bar{M}^k\right)\bar{L}^k + \bar{b}_2\lambda_{\max}\left(\bar{M}^k\right)\left(\bar{\Delta}^k, \bar{\Delta}^k\right),\ \bar{b}_1 \triangleq \bar{b}_1(\bar{\alpha}) = \bar{\alpha}\frac{1-\lambda_{\max}\left(\bar{\Theta}^2\right)}{2},\ \ \bar{b}_2 \triangleq \bar{b}_2(\bar{\alpha}) = \bar{\alpha}\left(1+\frac{3}{\bar{\alpha}^2}\right).$$

Here,

$$\lambda_{\min}\left(\bar{M}^k\right) = \min_{m+1\leq i\leq n}\lambda_i\left(\bar{M}^k\right),\ \ \lambda_{\max}\left(\bar{M}^k\right) = \max_{m+1\leq i\leq n}\lambda_i\left(\bar{M}^k\right);\ \ \lambda_{\max}\left(\bar{\Theta}^2\right) = \max_{m+1\leq i\leq n}\lambda_i\left(\bar{\Theta}^2\right);$$

$\lambda_{m+1}\left(\bar{M}^k\right), \lambda_{m+1}\left(\bar{M}^k\right), \cdots, \lambda_n\left(\bar{M}^k\right)$ are eigenvalues of matrix $\bar{M}^k$;

$\lambda_{m+1}\left(\bar{\Theta}^2\right), \lambda_{m+2}\left(\bar{\Theta}^2\right), \cdots, \lambda_n\left(\bar{\Theta}^2\right)$ are eigenvalues of matrix $\bar{\Theta}^2$.

Thus, we proved the validity of the second inequality (33) for the expression $\bar{L}^k$ for

$$\bar{\zeta}_1^k = \bar{b}_1\lambda_{\min}\left(\bar{M}^k\right),\ \ \bar{\zeta}_2^k = \bar{b}_2\lambda_{\max}\left(\bar{M}^k\right).$$

Similarly, the first inequality (33) is proved for the expression $\bar{\mathrm{L}}^k$.
Lemma 5 is proved. □

Lemma 6. *Let the conditions of Theorem 1 be fulfilled. Then, there are such positive constants $\bar{\eta}, \bar{\nu}, \bar{\bar{\eta}}$, and $\bar{\bar{\nu}}$ that the solution $\mathbf{U}^k$ the initial boundary value difference problem (11) satisfies the inequalities*

$$\frac{\bar{\mathrm{L}}^{k+1}-\bar{\mathrm{L}}^k}{\tau} \leq -\bar{\eta}\bar{\mathrm{L}}^k + \bar{\nu}\left(\bar{\Delta}^k, \bar{\Delta}^k\right), \ \frac{\bar{\bar{\mathrm{L}}}^{k+1}-\bar{\bar{\mathrm{L}}}^k}{\tau} \leq -\bar{\bar{\eta}}\bar{\bar{\mathrm{L}}}^k + \bar{\bar{\nu}}\left(\bar{\bar{\Delta}}^k, \bar{\bar{\Delta}}^k\right). \quad (34)$$

Proof. Lemma 6 can be easily proved on the basis of Lemma 5 and [1–3]. □

Summing up both parts of inequality (34), we obtain

$$\frac{\mathrm{L}^{k+1} - \mathrm{L}^k}{\tau} \leq -\eta\mathrm{L}^k + \nu\left(\Delta^k, \Delta^k\right), \quad (35)$$

where $\eta = \bar{\bar{\eta}} + \bar{\eta}$ and $\nu = \bar{\bar{\nu}} + \bar{\nu}$.

Inequality (35) indicates the existence of a discrete Lyapunov function L^k, which provides an exponential decrease L^k.

This completes the proof of Theorem 1. □

4. Numerical Experiment

As an example for calculating and confirming theoretical results, consider the hyperbolic system (1), for $m = n$ with initial data $\Phi(x) = 5 + 2\cos(2\pi x)$. For $x = 0$, boundary condition (4) takes the following form:

$$-\mathrm{M}(A(t))U(t,1) = V(t).$$

Here, $V(t)$ is a controller that is defined with equality

$$V(t) = R \times \mathrm{M}(A(t))[U(t,0) + \Delta(t)];$$

$R \in [0,1)$ is a feedback parameter; $U^* = 0 \in \mathbb{R}$ is a predetermined state of equilibrium; $\mathrm{M}(s) = \frac{1}{1+s}$, $s \in [0,+\infty)$ is the characteristic speed; and $\Delta(t) \in \mathbb{R}$ is limited measurement perturbation.

Note that the value $\mathrm{M}(A(t))$ for $U(t,x) = U^*$ is calculated as follows:

$$\mathrm{M}(A(t))_{U(t,x)=U^*} = \mathrm{M}\left(\int_0^1 U^* dx\right) = \mathrm{M}(U^*) = \frac{1}{1+U^*} = 1.$$

Let us build a difference scheme by dividing the segment $[0,1]$ with a step $h = 10^{-3}$ into J parts such that the equality $h \cdot J = 1$ holds. We denote the grid nodes as $x_j = \left(j - \frac{1}{2}\right)h$, $j = \overline{1,J}$, and the boundary of the domain consists of two nodal points $x_0 = 0$ and $x_J = 1$. Discretize the function $A(t)$ as follows:

$$A^n = h\sum_{j=1}^{J} u_j^n, \quad n = 1,2,3,\cdots,$$

where $u_j^n = u\left(t^n, x_j\right)$. Discrete value M^n is defined as follows:

$$\mathrm{M}^n = \mathrm{M}(A^n) = \frac{1}{1+A^n},$$

Here, $t^n = n \cdot \tau, \;\; n = 1, 2, 3, \cdots$ denotes layers by time, where the time step τ satisfies the Courant–Friedrichs–Lewy condition:

$$r^n = \frac{\mathrm{M}^n \tau}{h}, \;\; 0 < r^n \le 1, \;\; \forall n \in \{0, 1, \cdots\}.$$

Since the inequalities $\mathrm{M}^n \le 1, \;\; \forall n \in \{0, 1, \cdots\}$, are valid, for the value of time step τ, we choose $\tau = 0.6 \cdot h$ so that the inequality

$$\frac{\mathrm{M}^n \tau}{h} \le a < 1, \;\; n \ge 0$$

is fulfilled. The initial data will be written in the following form:

$$U^0 = \left(U_0^0, U_1^0, \cdots, U_J^0 \right)^T, \;\; U_j^0 = 5 + 2\cos(2\pi x_j), \, j \in \{0, \cdots, J\}.$$

Then, as an upwind difference scheme, we use

$$\begin{cases} U_j^{n+1} = (1 - r^n) U_j^n + r^n U_{j+1}^n, \;\; j \in \{0, \cdots, J-1\}, \;\; n = \{0, \cdots, K\}, \\ U_J^{n+1} = 0.2 U_0^{n+1} + 0.8 \frac{U^* \mathrm{M}(U^*)}{\mathrm{M}^{n+1}} + 0.2 \times 5.5 \times 10^{-3} \;\; \cos(t_{n+1}), \;\; n = \{0, \cdots, K\}. \end{cases} \tag{36}$$

For the numerical solution of the initial boundary value difference problem (36), we use the mathematical system Mathcad. Below is the graph of the numerical solution of the initial boundary value difference problem (36) in l^2-norm.

In Figure 1, the convergence of the solution to the system of Equation (36) to the equilibrium state $U^* = 0$ is shown for various values of n. We observe that with an increase in n, the rate of decay of the Lyapunov function decreases. Additionally, we observe that below the accuracy of the grid steps τ and h, further decay is not observed. Thus, it can be concluded that the numerical results obtained in the numerical experiment fully confirm the validity of the conclusion of Theorem 1, asserting the exponential stability of the solution to the problem (1)–(4).

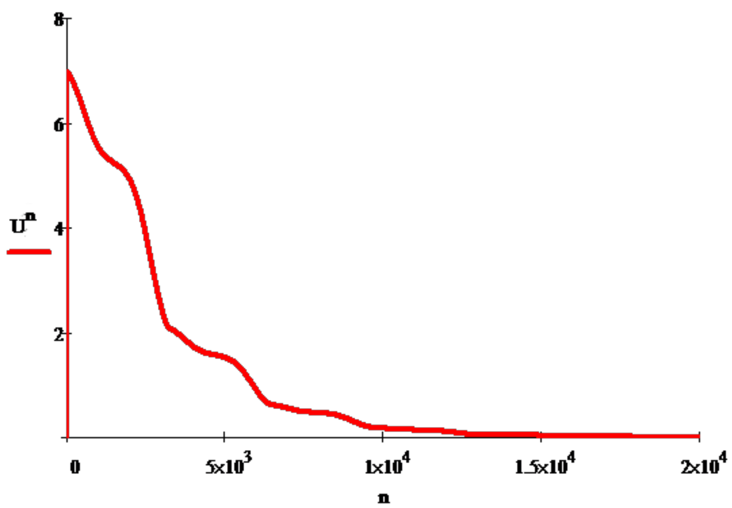

Figure 1. Numerical solution graph of the initial boundary value difference problem (36) in l^2-norm.

5. Conclusions

In this paper, a new initial boundary value problem for one-dimensional hyperbolic systems with nonlocal characteristic velocities is posed and qualitatively investigated, and an effective way was developed to prove the exponential stability of a constant equilibrium state of initial boundary value difference problems for it.

The analysis of the equilibrium state's exponential stability of the considered nonlinear hyperbolic system was based on the Lyapunov function method. A Lyapunov function was constructed. For the considered nonlinear hyperbolic system with nonlocal characteristic velocities, the corresponding constructed Lyapunov function makes it possible to solve a whole range of problems of important applied importance.

A constructive method for constructing a Lyapunov function suitable for a wide class of considered nonlinear hyperbolic systems with nonlocal characteristic velocities was developed.

In addition, the main scientific results of this work are the proposed method for constructing a discrete Lyapunov function and its justification, as well as its use to study the exponential stability of the numerical solution of the initial boundary value difference problem proposed for the initial boundary value problem of a nonlinear hyperbolic system with nonlocal characteristic velocities.

Moreover, sufficient conditions for the exponential stability of the equilibrium position of a nonlinear hyperbolic system with nonlocal characteristic velocities for both the differential problem and the discrete problem were obtained.

In this regard, the method of numerical construction of Lyapunov functions for a wide class of nonlinear systems developed in this article has theoretical and practical value.

It must be noted that in [1–3], the exponential stability of the zero solution of a mixed problem was established for only for a one-dimensional scalar equation with a positive nonlocal characteristic velocity.

And we solved this problem both for the differential problem for any hyperbolic system in Riemann invariants with positive and negative nonlocal characteristic velocities and for the initial boundary value difference problem.

A numerical experiment was provided. The results obtained in the numerical example fully confirm the theoretical results obtained in this work.

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