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A Progressive Outlook on Possibility Multi-Fuzzy Soft Ordered Semigroups: Theory and Analysis

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Abstract: The concept of possibility fuzzy soft sets is a step in a new direction towards a soft set approach that can be used to solve decision-making issues. In this piece of research, an innovative and comprehensive conceptual framework for possibility multi-fuzzy soft ordered semigroups by making use of the notions that are associated with possibility multi-fuzzy soft sets as well as ordered semigroups is introduced. Possibility multi-fuzzy soft ordered semigroups mark a newly developed theoretical avenue, and the central aim of this paper is to investigate it. The focus lies on investigating this newly developed theoretical direction, with practical examples drawn from decision-making and diagnosis practices to enhance understanding and appeal to researchers' interests. We strictly build the notions of possibility multi-fuzzy soft left (right) ideals, as well as l-idealistic and r-idealistic possibility multi-fuzzy soft ordered semigroups. Furthermore, various algebraic operations, such as union, intersection, as well as AND and OR operations are derived, while also providing a comprehensive discussion of their properties. To clarify these innovative ideas, the theoretical constructs are further reinforced with a set of demonstrative examples in order to guarantee deep and improved comprehension of the proposed framework.



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1. Introduction

The concept of possibility multi-fuzzy soft ordered semigroups is emerging as a promising avenue for addressing uncertainty in decision-making processes. The quest to address the challenges posed by uncertainty in problem solving, traditional mathematical methodologies have often proven insufficient as these methods do not always work in ambiguous and unpredictable situations. Realizing this fact has led scientists to create new approaches for dealing with uncertainty. In this context, we introduce the novel concept of possibility multi-fuzzy soft ordered semigroups. These structures combine elements of possibility theory, fuzzy sets and soft computing. By incorporating graded membership and possibility measures, they provide a robust framework for handling uncertainty in ordered semigroups.

One of the pioneers was Lotfi Zadeh, who in 1965 introduced fuzzy set theory, which opened up many fields to new ways of thinking about them. Fuzzy logic reflects the way that things are uncertain by allowing for degrees of membership rather than just yes/no inclusion or exclusion [1]. Moldstov [2,3] built on Zadeh's ideas when he put forward the concept of soft sets in 1999; these include tools for parameterization designed to cope with decision-making queries under conditions of limited information or lots of alternatives. Soft

set theory has been shown by subsequent applications to be useful practically speaking; it can handle imprecise data and deal with incompleteness (P.K. Maji et al.) [4–6]. Alkalzaleh et al. [7,8] extended the framework by introducing ideas like soft multisets and interval-valued fuzzy soft sets. At the same time, fuzzy mathematics developed further and led to fuzzy soft set theory as a better way of dealing with uncertainty.

Currently, the fuzzy soft set approach has been bolstered by relating different new techniques such as interval-valued fuzzy soft sets, possibility intuitionistic fuzzy soft sets, intuitionistic fuzzy soft expert sets, soft expert sets, possibility fuzzy soft expert sets, neutrosophic sets, complex neutrosophic soft expert sets, neutrosophic vague soft expert sets, multi-fuzzy soft sets and many more [9–29].

Multi-fuzzy sets are an extension of ordinary fuzzy set theory that provided a new approach for certain problems that were previously unable to be resolved, such as color pixels [30]. Later, Yang et al. [31] combined the multi-fuzzy set theory with a soft set approach and provided its more precise applications in decision-making problems.

The concept of possibility fuzzy soft sets was broached by Alkalzaleh [32]. He also described some of the applications of this notion in decision making. Possibility fuzzy soft set theory states that for every element of set of parameters, there exists a degree of membership as well as a degree of possible membership value for all the elements of the universe U . Thus, the existence of two membership values will help the experts to choose well in decision-making problems. Recently, a new generalization to this study known as possibility multi-fuzzy soft sets is determined [33].

Rosenfeld in 1971 [12] was the first to study algebraic structures in terms of Zadeh's approach. He named these structures as fuzzy groups. One of the most adapted algebraic structures, i.e., the ordered semigroup, has a close connection with theoretical computer science including different code-correcting languages, sequential machines and arithmetical study, etc. The initial concept of ordered semigroups was introduced by Kehayopulu [34]. Jun et al. [35] instigated a new generalization of ordered semigroups by relating them with soft set theory. Kehayopulu [36] further generalized the concept of the ordered semigroup by relating its ideals with Green's relation. Later, the concept of fuzzy soft ordered semigroups, fuzzy soft left (resp. right) ideals and many more were introduced, and readers refer to [37–39]. Recently Habib et al. give the concept of possibility fuzzy soft ordered semigroups and described their applications [40,41].

The main purpose of this paper is to introduce a new theory by compiling possibility multi-fuzzy soft sets and ordered semigroups. For every element of a set of parameters in a possibility multi-fuzzy soft set, there exists multiple degrees of membership as well as multiple degrees of possible membership for all the elements of the universal set. This quantitative analysis will lead to obtaining an appropriate value that would further help experts in decision making. This paper is structured into multiple sections, beginning with a preliminary segment that establishes fundamental definitions and concepts necessary for grasping the new theory. Subsequent sections delve into the core principles and practical applications of the proposed theory. These discussions include an exploration of possibility multi-fuzzy soft ordered semigroups, emphasizing their significant roles in medical diagnosis and decision-making contexts. Moreover, the paper examines the process of homogenization of possibility multi-fuzzy soft sets and possibility multi-fuzzy soft l -ideals (resp. r -ideals) of ordered semigroups, and the notion of l -idealistic (resp. r -idealistic) possibility multi-fuzzy soft ordered semigroups is proposed, and some of the related properties using these notions are determined. Each section contributes to a comprehensive understanding of the proposed theory, elucidating its implications and potential applications. Finally, this paper concludes with a recapitulation of key insights and concluding remarks, providing a cohesive summary of the theory's significance and suggesting future research directions.

2. Preliminaries

This section relates some important notions and definitions, in light of which some important results are generated. An ordered semigroup is defined as a combination of a partially ordered set $(S, \leq)$ and a semigroup $(S, \cdot)$ with $x \leq y \ (\forall x, y \in S)$, implying $ax \leq ay, xa \leq ya, \forall a \in S$ [23]. Throughout the paper, an ordered semigroup $(S, \cdot, \leq)$ is denoted by S . A non-empty subset X of an ordered semigroup S is called a left (resp. right) ideal of S [20] and is denoted by $X \triangleleft_l S$ (resp. $X \triangleleft_r S$) if it satisfies the following:

- $SX \subseteq X$ (resp. $XS \subseteq X$);
- $(\forall x \in X, y \in S) (y \leq x \Rightarrow y \in X)$.

Let S and T be two ordered semigroups. Then, a mapping $H : S \rightarrow T$ is called homomorphism if it satisfies the following:

- $H(ab) = H(a)H(b) \ \forall a, b \in S$;
- If $a \leq b \Rightarrow H(a) \leq H(b) \ \forall a, b \in S$.

A set of all the elements in S which are mapped to the identity element of T under a homomorphism H is known as the kernel of H , represented as $\ker(H)$. More precisely, $\ker(H) = \{s \in S | H(s) = e_T\}$, where e_T is the identity element of T .

Let U be a universal set, Q be the set of parameters and let $A \subseteq Q$. Then, a pair (f, A) is called a soft set of U if f is a mapping from A to $P(U)$, i.e., $f : A \rightarrow P(U)$, where $P(U)$ is the power set of U [5].

Let A be a non-empty subset of a universal set U . Then, the characteristic function for A is defined as

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Definition 1 ([6]). A pair of set (U, Q) is known as a soft universe, where U is a universal set and Q is a set of parameters. If $A \subseteq Q, f : A \rightarrow F(U)$, where $F(U)$ is the power set of all fuzzy subset of U , then (f, A) is called a fuzzy soft set.

Definition 2 ([30]). For any universal set U , a multi-fuzzy set $(\bar{\psi}, A)$ with index n is defined by a mapping $\bar{\psi} : A \rightarrow M^n FS(U)$, where n is a positive integer and $\bar{\psi}$ is of ordered sequences, $\bar{\psi} = \{u/v_1(u), v_2(u), \dots, v_n(u)\} \forall u \in U$, where $v_i(u)$ is the multi-membership values of the multi-fuzzy soft set.

Definition 3 ([11]). If $U = \{u_1, u_2, \dots, u_n\}$ is the universal set and $Q = \{e_1, e_2, \dots, e_n\}$ is the set of parameters, then a mapping $\bar{f}_{\bar{v}} : Q \rightarrow F(U) \times I(U)$ ($\bar{v}$ is the fuzzy subset of Q) is defined by $\bar{f}_{\bar{v}}(e_i) = (\bar{f}(e_i)(u), \bar{v}(e_i)(u)) \forall i = 1, 2, \dots, m$ and is known as a possibility fuzzy soft set denoted by $(\bar{f}_{\bar{v}}, Q)$. A possibility fuzzy soft set is used to introduce a degree of membership value coupled with the possibility of a degree of membership of element denoted by $\bar{f}(e_i)$ and $\bar{v}(e_i)$, respectively.

Note that if (f, Q) is a soft set over an ordered semigroup S and $f(e)$ is a subsemigroup of S such that $\forall e \in Q \ f(e) \neq \emptyset$, then (f, Q) is called a soft ordered semigroup over S [38].

Definition 4 ([35]). If $(\bar{f}, Q|_1)$ and $(\bar{g}, Q|_2)$ are two soft sets over U , then their union is represented as $(\bar{f}, Q|_1) \cup (\bar{g}, Q|_2) = (\bar{h}, Q)$, which satisfies the following conditions:

- $Q = Q|_1 \cup Q|_2$;
- $\bar{h}(e) = \begin{cases} \bar{f}(e) & e \in Q|_1 - Q|_2 \\ \bar{g}(e) & \text{if } e \in Q|_2 - Q|_1 \\ \bar{f}(e) \cup \bar{g}(e) & e \in Q|_1 \cap Q|_2 \end{cases} \forall e \in Q.$

Definition 5 ([35]). If $(\bar{f}, Q|_1)$ and $(\bar{g}, Q|_2)$ are two soft sets over U , then their intersection is represented as $(\bar{f}, Q|_1) \cap (\bar{g}, Q|_2) = (\bar{h}, Q)$, which satisfies the following conditions:

- $Q = Q|_1 \cap Q|_2$;
- $\bar{h}(e) = \bar{f}(e) \text{ or } \bar{h}(e) = \bar{g}(e) \ \forall e \in Q.$

Definition 6 ([11]). If (U, Q) is a pair of sets known as a soft universe, where $U = \{u_1, u_2, \dots, u_n\}$ is the universal set and $Q = \{e_1, e_2, \dots, e_m\}$ is the set of parameters, then define a mapping $\bar{\psi}_{\bar{f}} : Q \rightarrow M^n FS(U) \times M^n FS(U)$, where $\bar{\psi} : Q \rightarrow M^n FS(U)$ and $\bar{f} : Q \rightarrow M^n FS(U)$ ($\bar{\psi}$ and $\bar{f}$ are the fuzzy subsets of Q , and $M^n FS(U)$ represents the set of all multi-fuzzy sets with dimension n). $\bar{\psi}_{\bar{f}}(e_i) = (\bar{\psi}(e_i)(u), \bar{f}(e_i)(u)) \forall u \in U$ is known as a possibility multi-fuzzy soft set denoted by $(\bar{\psi}_{\bar{f}}, Q)$ with dimension n . Note that $\bar{\psi}_{\bar{f}}(e_i)$ represents the degree of multi-membership value and $\bar{f}(e_i)$ represents the possibility of a degree of multi-membership for all the elements of U .

In generalized form, a possibility multi-fuzzy soft set is represented as

$$\bar{\psi}_{\bar{f}}(e_i) = \left\{ (u/v_{\bar{\psi}(e_i)}(u), v_{\bar{f}(e_i)}(u)) : u \in U \right\}$$

where

$$v_{\bar{\psi}(e_i)}(u) = (v_{\bar{\psi}(e_i)}^1(u), v_{\bar{\psi}(e_i)}^2(u), \dots, v_{\bar{\psi}(e_i)}^n(u))$$

and

$$v_{\bar{f}(e_i)}(u) = (v_{\bar{f}(e_i)}^1(u), v_{\bar{f}(e_i)}^2(u), \dots, v_{\bar{f}(e_i)}^n(u)) \quad i = 1, 2, \dots, m$$

The possibility multi-fuzzy soft set is called a possibility fuzzy soft set with index $n = 1$.

Definition 7 ([11]). If $\bar{\psi}_{\bar{f}}$ and $\bar{\xi}_{\bar{g}}$ are two possibility multi-fuzzy soft sets over U , then their union is denoted by $\bar{\psi}_{\bar{f}} \cup \bar{\xi}_{\bar{g}} = \bar{\eta}_{\bar{h}}$, where $\bar{\eta}_{\bar{h}} : Q \rightarrow M^n FS(U) \times M^n FS(U)$ is defined as $\bar{\eta}_{\bar{h}}(e) = (\bar{\eta}(e)(u), \bar{h}(e)(u))$. The union must satisfy the following conditions:

- $\bar{\psi}(e) \cup \bar{\xi}(e) = \bar{\eta}(e)$;
- $\bar{h}(e) = \bar{f}(e) \cup \bar{g}(e) \forall e \in Q$.

Definition 8 ([11]). If $\bar{\psi}_{\bar{f}}$ and $\bar{\xi}_{\bar{g}}$ are two possibility multi-fuzzy soft sets over U , then their intersection is denoted as $\bar{\psi}_{\bar{f}} \cap \bar{\xi}_{\bar{g}} = \bar{\eta}_{\bar{h}}$, where $\bar{\eta}_{\bar{h}} : Q \rightarrow M^n FS(U) \times M^n FS(U)$ is defined as $\bar{\eta}_{\bar{h}}(e) = (\bar{\eta}(e)(u), \bar{h}(e)(u))$. The intersection must satisfy the following:

- $\bar{\psi}(e) \cap \bar{\xi}(e) = \bar{\eta}(e)$;
- $\bar{h}(e) = \bar{f}(e) \cap \bar{g}(e) \forall e \in Q$.

Definition 9 ([11]). For two possibility multi-fuzzy soft sets $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\xi}_{\bar{g}}, B)$ over U with index j , the similarity measure is defined as

$$\bar{s}(\bar{\psi}_{\bar{f}}, \bar{\xi}_{\bar{g}}) = \frac{\sum_{j=1}^n (\varphi_j(\bar{\psi}, \bar{\xi})) \cdot (\phi_j(\bar{f}, \bar{g}))}{j}$$

where

$$\varphi_j(\bar{\psi}, \bar{\xi}) = \frac{\sum_{i=1}^n \max_{x \in U} \left\{ \min \left(v_{\bar{\psi}(e_i)}^j(x), v_{\bar{\xi}(e_i)}^j(x) \right) \right\}}{\sum_{i=1}^n \max_{x \in U} \left\{ \max \left(v_{\bar{\psi}(e_i)}^j(x), v_{\bar{\xi}(e_i)}^j(x) \right) \right\}},$$

$$\phi_j(\bar{f}, \bar{g}) = \frac{\sum_{i=1}^n \max_{x \in U} \left\{ \min \left(v_{\bar{f}(e_i)}^j(x), v_{\bar{g}(e_i)}^j(x) \right) \right\}}{\sum_{i=1}^n \max_{x \in U} \left\{ \max \left(v_{\bar{f}(e_i)}^j(x), v_{\bar{g}(e_i)}^j(x) \right) \right\}}.$$

For all $v_{\bar{\psi}(e_i)}^j(x) \neq 0, v_{\bar{\xi}(e_i)}^j(x) \neq 0, v_{\bar{f}(e_i)}^j(x) \neq 0$ and $v_{\bar{g}(e_i)}^j(x) \neq 0$.

The possibility multi-fuzzy soft set is the most appropriate method used to solve decision-making problems more efficiently.

3. Possibility Multi-Fuzzy Soft Ordered Semigroups

This section provides an overview of possibility multi-fuzzy soft ordered semigroups. Additionally, it introduces new generalized concepts such as homomorphism in possibility multi-fuzzy soft ordered semigroups and possibility multi-fuzzy soft l-ideals (or r-ideals) of ordered semigroups. These concepts will be elucidated further through algebraic results employing fundamental operations like union, intersection, AND and OR operations.

Definition 10. Let $(S, \cdot, \leq)$ be an ordered semigroup and R be a set of parameters of S , then a mapping $\bar{\psi}_{\bar{f}} : R \rightarrow M^n FS(S) \times M^n FS(S)$ is defined, where $\bar{\psi} : R \rightarrow M^n FS(S)$ and $\bar{f} : R \rightarrow M^n FS(S)$. Then, $(\bar{\psi}_{\bar{f}}, R)$ is called a possibility multi-fuzzy soft ordered semigroup of S if it satisfies $\forall e \in R (\bar{\psi}(e) \neq \emptyset, \bar{f}(e) \neq \emptyset)$ or $\bar{\psi}_{\bar{f}}(e_i) \neq \emptyset$ (if $\bar{\psi}_{\bar{f}}(e)$ are fuzzy subsemigroups of S). Possibility multi-fuzzy soft ordered semigroup is represented as PMFSS unless otherwise stated.

Definition 11 . Let $\bar{\psi}$ and $\bar{f}$ be two multi-fuzzy subsets of S and $(\bar{\psi}_{\bar{f}}, R)$ be a possibility multi-fuzzy soft ordered semigroup over S . Then, $\forall t \in [0, 1]$; we define possibility multi-fuzzy soft level set of $\bar{\psi}_{\bar{f}}$ as $U(\bar{\psi}_{\bar{f}}; t) = \{s \in S \mid \bar{\psi}(s) \geq t, \bar{f}(s) \geq t\}$.

Example 1. Let $S = \{a_1, a_2, a_3, a_4\}$ be an ordered semigroup under the following multiplication table and ordered relation:

$$\leq := \{(a_1, a_1), (a_2, a_2), (a_3, a_3), (a_4, a_4), (a_1, a_2)\}$$

Multiplication Table

	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_1
a_3	a_1	a_1	a_2	a_1
a_4	a_1	a_1	a_2	a_2

For a set of parameters $R = \{e_1, e_2, e_3\}$, a possibility multi-fuzzy soft set $(\bar{\psi}_{\bar{f}}, R)$ is defined by mapping

$$\bar{\psi} : R \rightarrow M^n FS(S), \bar{f} : R \rightarrow M^n FS(S)$$

Here, we obtain

$$\bar{\psi}(e) = \begin{bmatrix} (0.8, 0.7, 0.5) & (0.7, 0.6, 0.3) & (0.6, 0.4, 0.2) & (0.6, 0.5, 0.1) \\ (0.7, 0.7, 0.9) & (0.6, 0.6, 0.8) & (0.3, 0.5, 0.7) & (0.4, 0.5, 0.2) \\ (0.9, 0.8, 0.8) & (0.6, 0.7, 0.7) & (0.4, 0.6, 0.2) & (0.4, 0.3, 0.1) \end{bmatrix},$$

$$\bar{f}(e) = \begin{bmatrix} (0.7, 0.6, 0.5) & (0.6, 0.5, 0.4) & (0.4, 0.3, 0.3) & (0.5, 0.3, 0.2) \\ (0.8, 0.7, 0.8) & (0.7, 0.3, 0.6) & (0.5, 0.1, 0.3) & (0.6, 0.2, 0.5) \\ (0.8, 0.8, 0.7) & (0.6, 0.7, 0.5) & (0.4, 0.6, 0.2) & (0.2, 0.3, 0.1) \end{bmatrix}.$$

Thus,

$$\bar{\psi}_{\bar{f}}(e) = \begin{bmatrix} (0.8, 0.7, 0.5), (0.7, 0.6, 0.5) & (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2), (0.4, 0.3, 0.3) & (0.6, 0.5, 0.1), (0.5, 0.3, 0.2) \\ (0.7, 0.7, 0.9), (0.8, 0.7, 0.8) & (0.6, 0.6, 0.8), (0.7, 0.3, 0.6) & (0.3, 0.5, 0.7), (0.5, 0.1, 0.3) & (0.4, 0.5, 0.2), (0.6, 0.2, 0.5) \\ (0.9, 0.8, 0.8), (0.8, 0.8, 0.7) & (0.6, 0.7, 0.7), (0.6, 0.7, 0.5) & (0.4, 0.6, 0.2), (0.4, 0.6, 0.2) & (0.4, 0.3, 0.1), (0.2, 0.3, 0.1) \end{bmatrix}$$

Here, $\bar{\psi}_{\bar{f}}(e) \neq \emptyset$ and $\bar{\psi}_{\bar{f}}(e)(a), \forall a \in S$ are fuzzy subsemigroups of S . Hence, by Definition 10, $(\bar{\psi}_{\bar{f}}, R)$ is a PMFSS over S .

Theorem 1. Let $(\bar{\psi}_{\bar{f}}, A)$ be a possibility multi-fuzzy soft ordered semigroup of S . Then, for any subset B of A , $(\bar{\psi}_{\bar{f}}, B)$ is also a possibility multi-fuzzy soft ordered semigroup of S .

Proof. The proof follows directly from Definition 10. $\square$

Example 2. If S is an ordered semigroup and $(\bar{\psi}_{\bar{f}}, A)$ is a possibility multi-fuzzy soft ordered semigroup over S as defined in Example 1, then

$$\bar{\psi}_{\bar{f}}(e) = \begin{bmatrix} (0.8, 0.7, 0.5), (0.7, 0.6, 0.5) & (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2), (0.4, 0.3, 0.3) & (0.6, 0.5, 0.1), (0.5, 0.3, 0.2) \\ (0.7, 0.7, 0.9), (0.8, 0.7, 0.8) & (0.6, 0.6, 0.8), (0.7, 0.3, 0.6) & (0.3, 0.5, 0.7), (0.5, 0.1, 0.3) & (0.4, 0.5, 0.2), (0.6, 0.2, 0.5) \\ (0.9, 0.8, 0.8), (0.8, 0.8, 0.7) & (0.6, 0.7, 0.7), (0.6, 0.7, 0.5) & (0.4, 0.6, 0.2), (0.4, 0.6, 0.2) & (0.4, 0.3, 0.1), (0.2, 0.3, 0.1) \end{bmatrix}$$

Let $B \subseteq A$; then, we define another mapping $\bar{\xi}_{\bar{g}} : B \rightarrow M^nFS(S) \times M^nFS(S)$ defined as $\bar{\xi}_{\bar{g}}(e) = (\bar{\xi}(e)(s), \bar{g}(e)(s))$ for all $s \in S$.

$$\bar{\xi}(e) = \begin{bmatrix} (0.7, 0.6, 0.3) & (0.6, 0.4, 0.2) & (0.5, 0.3, 0.1) & (0.5, 0.3, 0.1) \\ (0.6, 0.6, 0.7) & (0.5, 0.5, 0.6) & (0.2, 0.4, 0.5) & (0.4, 0.4, 0.1) \\ (0.8, 0.7, 0.6) & (0.5, 0.6, 0.5) & (0.3, 0.5, 0.1) & (0.3, 0.2, 0.1) \end{bmatrix}$$

$$\bar{g}(e) = \begin{bmatrix} (0.6, 0.5, 0.4) & (0.5, 0.5, 0.3) & (0.4, 0.3, 0.2) & (0.4, 0.2, 0.1) \\ (0.5, 0.4, 0.6) & (0.4, 0.3, 0.5) & (0.3, 0.1, 0.2) & (0.5, 0.1, 0.3) \\ (0.7, 0.6, 0.5) & (0.5, 0.5, 0.4) & (0.2, 0.3, 0.1) & (0.2, 0.2, 0.1) \end{bmatrix}.$$

Combining above two matrices we obtain

$$\bar{\xi}_{\bar{g}}(e) = \begin{bmatrix} (0.7, 0.6, 0.3)(0.6, 0.5, 0.4) & (0.6, 0.4, 0.2)(0.5, 0.5, 0.3) & (0.5, 0.3, 0.1)(0.4, 0.3, 0.2) & (0.5, 0.3, 0.1)(0.4, 0.2, 0.1) \\ (0.6, 0.6, 0.7)(0.5, 0.4, 0.6) & (0.5, 0.5, 0.6)(0.4, 0.3, 0.5) & (0.2, 0.4, 0.5)(0.3, 0.1, 0.2) & (0.4, 0.4, 0.1)(0.5, 0.1, 0.3) \\ (0.8, 0.7, 0.6)(0.7, 0.6, 0.5) & (0.5, 0.6, 0.5)(0.5, 0.5, 0.4) & (0.3, 0.5, 0.1)(0.2, 0.3, 0.1) & (0.3, 0.2, 0.1)(0.2, 0.2, 0.1) \end{bmatrix}$$

Here, $\bar{\xi}_{\bar{g}}(e) \neq \emptyset$ and $\bar{\xi}_{\bar{g}}(e)(a), \forall a \in S$ are fuzzy subsemigroups of S . Hence, by Definition 10, $(\bar{\xi}_{\bar{g}}, B)$ is a PMFSS over S .

Theorem 2. Let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\xi}_{\bar{g}}, B)$ be two PMFSSs over S . If $A \cap B = \emptyset$, then $(\bar{\psi}_{\bar{f}}, A) \cup (\bar{\xi}_{\bar{g}}, B)$ is also possibility multi-fuzzy soft ordered semigroup over S .

Proof. Union of any two possibility multi-fuzzy soft sets is denoted by $(\bar{\psi}_{\bar{f}}, A) \cup (\bar{\xi}_{\bar{g}}, B)$. Let $(\bar{\psi}_{\bar{f}}, A) \cup (\bar{\xi}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$ where $C = A \cup B$; then, $\forall e \in C$.

$$(\bar{\sigma}_{\bar{h}}, C) = \begin{cases} (\bar{\psi}_{\bar{f}}, A) & e \in A - B, \\ (\bar{\xi}_{\bar{g}}, B) & \text{if } e \in B - A, \\ (\bar{\psi}_{\bar{f}}, A) \cup (\bar{\xi}_{\bar{g}}, B) & e \in A \cap B. \end{cases}$$

As $A \cap B = \emptyset \Rightarrow$ either $e \in A - B$ or $e \in B - A$; in other words, either $\bar{\sigma}_{\bar{h}}(e) = \bar{\psi}_{\bar{f}}(e)$ or $\bar{\sigma}_{\bar{h}}(e) = \bar{\xi}_{\bar{g}}(e)$. Then, a mapping $\bar{\sigma}_{\bar{h}} : C \rightarrow M^nFS(S) \times M^nFS(S)$ is defined as $\bar{\sigma}_{\bar{h}}(e) = (\bar{\sigma}(e)(s), \bar{h}(e)(s)) \forall s \in S$. Hence, $\bar{\sigma}_{\bar{h}}(e) \neq \emptyset$ also implies $\bar{\sigma}(e)(s)$, and $\bar{h}(e)(s)$ is a fuzzy subsemigroup over S . Then, using Definition 10, $(\bar{\sigma}_{\bar{h}}, C)$ is a PMFSS over S . Thus, the union of two PMFSSs over S is also a PMFSS over S . $\square$

Example 3. Let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\xi}_{\bar{g}}, B)$ be two possibility multi-fuzzy soft sets over S . Where S is defined under the same ordered relation as defined in Example 1,

$$\bar{\psi}_{\bar{f}}(e) = \begin{bmatrix} (0.8, 0.7, 0.5), (0.7, 0.6, 0.5) & (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2), (0.4, 0.3, 0.3) & (0.6, 0.5, 0.1), (0.5, 0.3, 0.2) \\ (0.7, 0.7, 0.9), (0.8, 0.7, 0.8) & (0.6, 0.6, 0.8), (0.7, 0.3, 0.6) & (0.3, 0.5, 0.7), (0.5, 0.1, 0.3) & (0.4, 0.5, 0.2), (0.6, 0.2, 0.5) \\ (0.9, 0.8, 0.8), (0.8, 0.8, 0.7) & (0.6, 0.7, 0.7), (0.6, 0.7, 0.5) & (0.4, 0.6, 0.2), (0.4, 0.6, 0.2) & (0.4, 0.3, 0.1), (0.2, 0.3, 0.1) \end{bmatrix}$$

$$\bar{\xi}_{\bar{g}}(e) = \begin{bmatrix} (0.7, 0.6, 0.3)(0.6, 0.5, 0.4) (0.6, 0.4, 0.2)(0.5, 0.5, 0.3) (0.5, 0.3, 0.1)(0.4, 0.3, 0.2) (0.5, 0.3, 0.1)(0.4, 0.2, 0.1) \\ (0.6, 0.6, 0.7)(0.5, 0.4, 0.6) (0.5, 0.5, 0.6)(0.4, 0.3, 0.5) (0.2, 0.4, 0.5)(0.3, 0.1, 0.2) (0.4, 0.4, 0.1)(0.5, 0.1, 0.3) \\ (0.8, 0.7, 0.6)(0.7, 0.6, 0.5) (0.5, 0.6, 0.5)(0.5, 0.5, 0.4) (0.3, 0.5, 0.1)(0.2, 0.3, 0.1) (0.3, 0.2, 0.1)(0.2, 0.2, 0.1) \end{bmatrix}.$$

Then, $(\bar{\psi}_{\bar{f}}, A) \cup (\bar{\xi}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$; using Theorem 2, we obtain

$$\bar{\sigma}_{\bar{h}}(e) = \begin{bmatrix} (0.8, 0.7, 0.5), (0.7, 0.6, 0.5) (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) (0.6, 0.4, 0.2), (0.4, 0.3, 0.3) (0.6, 0.5, 0.1), (0.5, 0.3, 0.2) \\ (0.7, 0.7, 0.9), (0.8, 0.7, 0.8) (0.6, 0.6, 0.8), (0.7, 0.3, 0.6) (0.3, 0.5, 0.7), (0.5, 0.1, 0.3) (0.4, 0.5, 0.2), (0.6, 0.2, 0.5) \\ (0.9, 0.8, 0.8), (0.8, 0.8, 0.7) (0.6, 0.7, 0.7), (0.6, 0.7, 0.5) (0.4, 0.6, 0.2), (0.4, 0.6, 0.2) (0.4, 0.3, 0.1), (0.2, 0.3, 0.1) \end{bmatrix}.$$

Here, it is easily notified that $\bar{\sigma}_{\bar{h}}(e) \neq \emptyset$ also implies $\bar{\sigma}(e)(s)$ and $\bar{h}(e)(s)$ are fuzzy subsemigroups over S . Then, by Definition 10, $(\bar{\sigma}_{\bar{h}}, C)$ is a PMFSS over S . Thus, the union of two PMFSSs over S is also a PMFSS over S .

Theorem 3. Let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\xi}_{\bar{g}}, B)$ be two PMFSS over S . If $A \cap B \neq \emptyset$, then their intersection $(\bar{\psi}_{\bar{f}}, A) \cap (\bar{\xi}_{\bar{g}}, B)$ is also possibility multi-fuzzy soft ordered semigroup over S .

Proof. Intersection of any two possibility multi-fuzzy soft set is denoted as $(\bar{\psi}_{\bar{f}}, A) \cap (\bar{\xi}_{\bar{g}}, B)$. Let $(\bar{\psi}_{\bar{f}}, A) \cap (\bar{\xi}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$ where $C = A \cap B$ then $\forall e \in C$ implies $e \in A$ and $e \in B$ or in other words $\bar{\psi}_{\bar{f}}(e) \cap \bar{\xi}_{\bar{g}}(e) = \bar{\sigma}_{\bar{h}}(e)$ implies $(\bar{\sigma}(e), \bar{h}(e)) = (\bar{\psi}(e), \bar{f}(e))$ and $(\bar{\sigma}(e), \bar{h}(e)) = (\bar{\xi}(e), \bar{g}(e))$. Thus, there exists a mapping $\bar{\sigma}_{\bar{h}}: C \rightarrow M^n FS(S) \times M^n FS(S)$ defined as $\bar{\sigma}_{\bar{h}}(e) = \bar{\sigma}(e)(s), \bar{h}(e)(s) \forall s \in S$. Where, $\bar{\sigma}_{\bar{h}}(e) \neq \emptyset$ also implies $\bar{\sigma}(e)(s)$ and $\bar{h}(e)(s)$ are fuzzy subsemigroups over S then by Definition 10 $(\bar{\sigma}_{\bar{h}}, C)$ is a PMFSS over S . Thus intersection of two PMFSS over S is also a PMFSS over S . $\square$

Example 4. Let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\xi}_{\bar{g}}, B)$ be two possibility multi-fuzzy soft sets over S . Where S is defined under the same ordered relation as defined in Example 1,

$$\bar{\psi}_{\bar{f}}(e) = \begin{bmatrix} (0.8, 0.7, 0.5), (0.7, 0.6, 0.5) (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) (0.6, 0.4, 0.2), (0.4, 0.3, 0.3) (0.6, 0.5, 0.1), (0.5, 0.3, 0.2) \\ (0.7, 0.7, 0.9), (0.8, 0.7, 0.8) (0.6, 0.6, 0.8), (0.7, 0.3, 0.6) (0.3, 0.5, 0.7), (0.5, 0.1, 0.3) (0.4, 0.5, 0.2), (0.6, 0.2, 0.5) \\ (0.9, 0.8, 0.8), (0.8, 0.8, 0.7) (0.6, 0.7, 0.7), (0.6, 0.7, 0.5) (0.4, 0.6, 0.2), (0.4, 0.6, 0.2) (0.4, 0.3, 0.1), (0.2, 0.3, 0.1) \end{bmatrix},$$

$$\bar{\xi}_{\bar{g}}(e) = \begin{bmatrix} (0.7, 0.6, 0.3)(0.6, 0.5, 0.4) (0.6, 0.4, 0.2)(0.5, 0.5, 0.3) (0.5, 0.3, 0.1)(0.4, 0.3, 0.2) (0.5, 0.3, 0.1)(0.4, 0.2, 0.1) \\ (0.6, 0.6, 0.7)(0.5, 0.4, 0.6) (0.5, 0.5, 0.6)(0.4, 0.3, 0.5) (0.2, 0.4, 0.5)(0.3, 0.1, 0.2) (0.4, 0.4, 0.1)(0.5, 0.1, 0.3) \\ (0.8, 0.7, 0.6)(0.7, 0.6, 0.5) (0.5, 0.6, 0.5)(0.5, 0.5, 0.4) (0.3, 0.5, 0.1)(0.2, 0.3, 0.1) (0.3, 0.2, 0.1)(0.2, 0.2, 0.1) \end{bmatrix}$$

Then, their intersection is denoted as $(\bar{\psi}_{\bar{f}}, A) \cap (\bar{\xi}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$.

So here,

$$\bar{\sigma}_{\bar{h}}(e) = \begin{bmatrix} (0.7, 0.6, 0.3)(0.6, 0.5, 0.4) (0.6, 0.4, 0.2)(0.5, 0.5, 0.3) (0.5, 0.3, 0.1)(0.4, 0.3, 0.2) (0.5, 0.3, 0.1)(0.4, 0.2, 0.1) \\ (0.6, 0.6, 0.7)(0.5, 0.4, 0.6) (0.5, 0.5, 0.6)(0.4, 0.3, 0.5) (0.2, 0.4, 0.5)(0.3, 0.1, 0.2) (0.4, 0.4, 0.1)(0.5, 0.1, 0.3) \\ (0.8, 0.7, 0.6)(0.7, 0.6, 0.5) (0.5, 0.6, 0.5)(0.5, 0.5, 0.4) (0.3, 0.5, 0.1)(0.2, 0.3, 0.1) (0.3, 0.2, 0.1)(0.2, 0.2, 0.1) \end{bmatrix}$$

As $\bar{\sigma}_{\bar{h}}(e) \neq \emptyset$ also implies $\bar{\sigma}(e)$ and $\bar{h}(e)$ are fuzzy subsemigroups over S , then by Definition 10, $(\bar{\sigma}_{\bar{h}}, C)$ is a PMFSS over S .

Next, we discuss the logical operators, i.e., AND and OR, for possibility multi-fuzzy soft sets and characterized possibility multi-fuzzy soft ordered semigroups by the properties of these newly defined notions.

Theorem 4. If $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\xi}_{\bar{g}}, B)$ are two possibility multi-fuzzy soft ordered semigroups over S , then $(\bar{\psi}_{\bar{f}}, A) \wedge (\bar{\xi}_{\bar{g}}, B)$ is also a PMFSS over S .

Proof. As AND operation in possibility multi-fuzzy soft sets is defined as $(\bar{\psi}_{\bar{f}}, A) \wedge (\bar{\zeta}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$, where $C = A \times B$ and $\bar{\sigma}(a, b) = \bar{\psi}(a) \cap \bar{\zeta}(b)$, similarly $\bar{h}(a, b) = \bar{f}(a) \cap \bar{g}(b) \forall (a, b) \in A \times B$. Since $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ are PMFSSs over S , we can say that $\bar{\psi}_{\bar{f}}(a)$ and $\bar{\zeta}_{\bar{g}}(b)$ are multi-fuzzy subsemigroups of S , and the intersection of $(\bar{\psi}_{\bar{f}}(a)) \cap (\bar{\zeta}_{\bar{g}}(b)) \forall (a, b) \in A \times B$ is also a fuzzy subsemigroup of S . Hence, $\bar{\sigma}_{\bar{h}}(a, b) = (\bar{\sigma}(a, b)(s), \bar{h}(a, b)(s)) \forall s \in S$ is also a fuzzy subsemigroup of $S \forall (a, b) \in A \times B$. Thus, $(\bar{\psi}_{\bar{f}}, A) \wedge (\bar{\zeta}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$ is also a PMFSS over S . $\square$

Example 5. Let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ be two PMFSSs over S as defined in Example 2. Then, we can define its AND operation as $(\bar{\psi}_{\bar{f}}, A) \wedge (\bar{\zeta}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$, where $\bar{\sigma}_{\bar{h}}$ for all pair of parameters can be concluded, and we obtain

$$\begin{aligned} \bar{\sigma}_{\bar{h}}(e_1, e_1) &= \left\{ \left(\frac{s_1}{(0.7, 0.6, 0.3)}, (0.6, 0.5, 0.4) \right), \left(\frac{s_2}{(0.6, 0.4, 0.2)}, (0.5, 0.5, 0.3) \right), \left(\frac{s_3}{(0.5, 0.3, 0.1)}, (0.4, 0.3, 0.2) \right), \left(\frac{s_4}{(0.5, 0.3, 0.1)}, (0.4, 0.2, 0.1) \right) \right\}, \\ \bar{\sigma}_{\bar{h}}(e_1, e_2) &= \left\{ \left(\frac{s_1}{(0.6, 0.6, 0.5)}, (0.5, 0.4, 0.5) \right), \left(\frac{s_2}{(0.5, 0.5, 0.3)}, (0.4, 0.3, 0.4) \right), \left(\frac{s_3}{(0.2, 0.4, 0.2)}, (0.3, 0.1, 0.2) \right), \left(\frac{s_4}{(0.4, 0.4, 0.1)}, (0.5, 0.1, 0.2) \right) \right\}, \\ \bar{\sigma}_{\bar{h}}(e_1, e_3) &= \left\{ \left(\frac{s_1}{(0.8, 0.8, 0.5)}, (0.7, 0.6, 0.5) \right), \left(\frac{s_2}{(0.5, 0.6, 0.3)}, (0.5, 0.5, 0.4) \right), \left(\frac{s_3}{(0.3, 0.4, 0.1)}, (0.2, 0.3, 0.1) \right), \left(\frac{s_4}{(0.3, 0.2, 0.1)}, (0.2, 0.2, 0.1) \right) \right\}. \end{aligned}$$

Similarly, we can calculate the values for every pair of parameters.

In matrix form,

$$\bar{\sigma}_{\bar{h}} = \begin{bmatrix} (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2), (0.5, 0.5, 0.3) & (0.5, 0.3, 0.1), (0.4, 0.3, 0.2) & (0.5, 0.3, 0.1), (0.4, 0.2, 0.1) \\ (0.6, 0.6, 0.5), (0.5, 0.4, 0.5) & (0.5, 0.5, 0.3), (0.4, 0.3, 0.4) & (0.2, 0.4, 0.2), (0.3, 0.1, 0.2) & (0.4, 0.4, 0.1), (0.5, 0.1, 0.2) \\ (0.8, 0.7, 0.5), (0.7, 0.6, 0.5) & (0.5, 0.6, 0.3), (0.5, 0.5, 0.4) & (0.3, 0.4, 0.1), (0.2, 0.3, 0.1) & (0.3, 0.2, 0.1), (0.2, 0.2, 0.1) \\ (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2), (0.5, 0.3, 0.3) & (0.3, 0.3, 0.1), (0.4, 0.1, 0.2) & (0.4, 0.3, 0.1), (0.4, 0.2, 0.1) \\ (0.6, 0.6, 0.7), (0.5, 0.4, 0.6) & (0.5, 0.5, 0.6), (0.4, 0.3, 0.5) & (0.2, 0.4, 0.5), (0.3, 0.1, 0.2) & (0.4, 0.4, 0.1), (0.5, 0.1, 0.3) \\ (0.7, 0.7, 0.6), (0.7, 0.6, 0.5) & (0.5, 0.6, 0.5), (0.5, 0.3, 0.4) & (0.3, 0.5, 0.1), (0.2, 0.1, 0.1) & (0.3, 0.2, 0.1), (0.2, 0.2, 0.1) \\ (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2), (0.5, 0.5, 0.3) & (0.4, 0.3, 0.1), (0.4, 0.3, 0.2) & (0.4, 0.3, 0.1), (0.2, 0.2, 0.1) \\ (0.6, 0.6, 0.7), (0.5, 0.4, 0.6) & (0.5, 0.5, 0.6), (0.4, 0.3, 0.5) & (0.2, 0.4, 0.2), (0.3, 0.1, 0.2) & (0.4, 0.3, 0.1), (0.2, 0.1, 0.1) \\ (0.8, 0.7, 0.6), (0.7, 0.6, 0.5) & (0.5, 0.6, 0.5), (0.5, 0.5, 0.4) & (0.3, 0.5, 0.1), (0.2, 0.3, 0.1) & (0.3, 0.2, 0.1), (0.2, 0.2, 0.1) \end{bmatrix}$$

As $\bar{\sigma}_{\bar{h}}(e_i, e_j) \neq \phi$, $\bar{\sigma}_{\bar{h}}(e_i, e_j)$ is a fuzzy subsemigroup of S . Thus, $(\bar{\psi}_{\bar{f}}, A) \wedge (\bar{\zeta}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$ is a PMFSS over S .

Definition 12. Let $(\bar{\psi}_{\bar{f}}, A)$ be a PMFSS over S . Then, $(\bar{\psi}_{\bar{f}}, A)$ is said to be trivial if $\bar{\psi}_{\bar{f}}(e) = \{T\}$ for all $e \in A$, where T stands for a trivial ordered semigroup.

Lemma 1. Let us define homomorphism as a mapping $H : S \rightarrow T$ from an ordered semigroup S to a trivial ordered semigroup T . If $(\bar{\psi}_{\bar{f}}, R)$ is a PMFSS over S , then $(H(\bar{\psi}_{\bar{f}}), R)$ also defines a PMFSS over T .

Proof. As the definition of homomorphism states that $\forall e \in R, H(\bar{\psi}_{\bar{f}})(e) = H(\bar{\psi}_{\bar{f}}(e)) = H(\bar{\psi}(e), \bar{f}(e))$ is a subsemigroup of T . If $(\bar{\psi}_{\bar{f}}, R)$ defines a PMFSS over S , then by definition its homomorphic image is a fuzzy subsemigroup of T . Hence, $H(\bar{\psi}_{\bar{f}}(e))$ is a fuzzy subsemigroup of T . Thus, it implies that $(H(\bar{\psi}_{\bar{f}}), R)$ is a possibility multi-fuzzy soft ordered semigroup over T . $\square$

Theorem 5. Let $(\bar{\psi}_{\bar{f}}, R)$ be a PMFSS over S and $H : S \rightarrow T$ be a homomorphic image from an ordered semigroup S to a trivial ordered semigroup T . Then, if $\bar{\psi}_{\bar{f}}(e) \subseteq \ker(H), \forall e \in R$, then $(H(\bar{\psi}_{\bar{f}}), R)$ is a trivial PMFSS over T .

Proof. As $\bar{\psi}_{\bar{f}}(e) \subseteq \ker(H), \forall e \in R$, also by definition of homomorphism, $H(\bar{\psi}_{\bar{f}})(e) = H(\bar{\psi}_{\bar{f}}(e)) = H(\bar{\psi}(e), \bar{f}(e)), \forall e \in R$. As defined earlier for a trivial PMFSS $(\bar{\psi}_{\bar{f}}, R)$ over S ,

$\bar{\psi}_{\bar{f}}(e) = \{T\}$. Thus, by using the Lemma 1, it is concluded that $H(\bar{\psi}_{\bar{f}})(e) = H(\bar{\psi}_{\bar{f}}(e)) = \{T\}, \forall e \in R$. Hence, $(H(\bar{\psi}_{\bar{f}}), R)$ is a trivial PMFSS over T. $\square$

Theorem 6. Let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ be two PMFSSs over S. Then, for all $B \subseteq A$, $(\bar{\zeta}_{\bar{g}}, B)$ is a multi-fuzzy subsemigroup of $(\bar{\psi}_{\bar{f}}, A)$ or $(\bar{\zeta}_{\bar{g}}, B) \leq (\bar{\psi}_{\bar{f}}, A)$ if and only if $\bar{\zeta}_{\bar{g}}(e)$ is a fuzzy subsemigroup of $\bar{\psi}_{\bar{f}}(e)$ $\forall e \in R$.

Proof. The theorem can be directly proved by using Theorem 1. $\square$

Example 6. Let us consider two PMFSSs $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ over S (defined as in Example 1) defined as

$$\bar{\psi}_{\bar{f}}(e) = \begin{bmatrix} (0.8, 0.7, 0.5), (0.7, 0.6, 0.5) & (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2), (0.4, 0.3, 0.3) & (0.6, 0.5, 0.1), (0.5, 0.3, 0.2) \\ (0.7, 0.7, 0.9), (0.8, 0.7, 0.8) & (0.6, 0.6, 0.8), (0.7, 0.3, 0.6) & (0.3, 0.5, 0.7), (0.5, 0.1, 0.3) & (0.4, 0.5, 0.2), (0.6, 0.2, 0.5) \\ (0.9, 0.8, 0.8), (0.8, 0.8, 0.7) & (0.6, 0.7, 0.7), (0.6, 0.7, 0.5) & (0.4, 0.6, 0.2), (0.4, 0.6, 0.2) & (0.4, 0.3, 0.1), (0.2, 0.3, 0.1) \end{bmatrix}$$

$$\bar{\zeta}_{\bar{g}}(e) = \begin{bmatrix} (0.7, 0.6, 0.3), (0.6, 0.5, 0.4) & (0.6, 0.4, 0.2), (0.5, 0.5, 0.3) & (0.5, 0.3, 0.1), (0.4, 0.3, 0.2) & (0.5, 0.3, 0.1), (0.4, 0.2, 0.1) \\ (0.6, 0.6, 0.7), (0.5, 0.4, 0.6) & (0.5, 0.5, 0.6), (0.4, 0.3, 0.5) & (0.2, 0.4, 0.5), (0.3, 0.1, 0.2) & (0.4, 0.4, 0.1), (0.5, 0.1, 0.3) \\ (0.8, 0.7, 0.6), (0.7, 0.6, 0.5) & (0.5, 0.6, 0.5), (0.5, 0.5, 0.4) & (0.3, 0.5, 0.1), (0.2, 0.3, 0.1) & (0.3, 0.2, 0.1), (0.2, 0.2, 0.1) \end{bmatrix}$$

It can easily be analyzed that for each parameter, $\bar{\psi}_{\bar{f}}(e)$ is multi-fuzzy subsemigroup of $\bar{\zeta}_{\bar{g}}(e)$ over S. Conversely, for any two $\bar{\psi}_{\bar{f}}(e)$ and $\bar{\zeta}_{\bar{g}}(e)$ that are multi-fuzzy subsets of an ordered semigroup S, $\bar{\zeta}_{\bar{g}}(e)$ is a fuzzy subsemigroup of $\bar{\psi}_{\bar{f}}(e)$, then $\bar{\zeta}(e) \leq \bar{\psi}(e)$, $\bar{g}(e) \leq \bar{f}(e) \forall e \in B$. Also, $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ satisfy the definition of PMFSS. Thus, for $B \subseteq A$, $(\bar{\zeta}_{\bar{g}}, B)$ is a fuzzy subsemigroup of $(\bar{\psi}_{\bar{f}}, A)$.

Theorem 7. Suppose $(\bar{\psi}_{\bar{f}}, A)$ is a PMFSS over S and let $(\bar{\zeta}_{\bar{g}}|_1, B_1)$ and $(\bar{\zeta}_{\bar{g}}|_2, B_2)$ be the two possibility multi-fuzzy soft ordered subsemigroups of $(\bar{\psi}_{\bar{f}}, A)$, then

- (1) $(\bar{\zeta}_{\bar{g}}|_1, B_1) \cap (\bar{\zeta}_{\bar{g}}|_2, B_2) \leq (\bar{\psi}_{\bar{f}}, A)$;
- (2) If $B_1 \cap B_2 = \phi$, then $(\bar{\zeta}_{\bar{g}}|_1, B_1) \cup (\bar{\zeta}_{\bar{g}}|_2, B_2) \leq (\bar{\psi}_{\bar{f}}, A)$.

Proof. (1) The intersection of any two possibility multi-fuzzy soft sets is defined as $(\bar{\zeta}_{\bar{g}}|_1, B_1) \cap (\bar{\zeta}_{\bar{g}}|_2, B_2) = (\bar{\zeta}_{\bar{g}}, B)$, where $B_1 \cap B_2 = B, \forall e \in B$ implies $e \in B_1$ and $e \in B_2$, or in other words, either $\bar{\zeta}_{\bar{g}}(e) = \bar{\zeta}_{\bar{g}}|_1(e)$ or $\bar{\zeta}_{\bar{g}}(e) = \bar{\zeta}_{\bar{g}}|_2(e)$. Since $(\bar{\zeta}_{\bar{g}}|_1, B_1) \leq (\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}|_2, B_2) \leq (\bar{\psi}_{\bar{f}}, A)$, $\bar{g}_{\bar{\mu}}(e)$ is a fuzzy subsemigroup of $\bar{f}_{\bar{\nu}}(e)$.

Thus, $(\bar{\zeta}_{\bar{g}}|_1, B_1) \cap (\bar{\zeta}_{\bar{g}}|_2, B_2) \leq (\bar{f}_{\bar{\nu}}, A)$.

(2) The union of any two possibility multi-fuzzy soft sets can be defined as $(\bar{\zeta}_{\bar{g}}|_1, B_1) \cup (\bar{\zeta}_{\bar{g}}|_2, B_2) = (\bar{\zeta}_{\bar{g}}, B)$, where $B_1 \cup B_2 = B$. As $\forall e \in B$, we obtain

$$\bar{\zeta}_{\bar{g}}(e) = \begin{cases} \bar{\zeta}_{\bar{g}}|_1(e) & \text{if } e \in B_1 - B_2, \\ \bar{\zeta}_{\bar{g}}|_2(e) & \text{if } e \in B_2 - B_1, \\ \bar{\zeta}_{\bar{g}}|_1(e) \cup \bar{\zeta}_{\bar{g}}|_2(e) & \text{if } e \in B_1 \cap B_2. \end{cases}$$

Here, if $B_1 \cap B_2 = \phi$, either $e \in B_1 - B_2$ or $e \in B_2 - B_1$; thus, either $\bar{\zeta}_{\bar{g}}(e) = \bar{\zeta}_{\bar{g}}|_1(e)$ or $\bar{\zeta}_{\bar{g}}(e) = \bar{\zeta}_{\bar{g}}|_2(e)$, as $(\bar{\zeta}_{\bar{g}}|_1, B_1) \leq (\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}|_2, B_2) \leq (\bar{\psi}_{\bar{f}}, A)$, so $\bar{\zeta}_{\bar{g}}(e)$ is a fuzzy subsemigroup of $\bar{\psi}_{\bar{f}}(e)$. Hence, $(\bar{\zeta}_{\bar{g}}|_1, B_1) \cup (\bar{\zeta}_{\bar{g}}|_2, B_2) \leq (\bar{\psi}_{\bar{f}}, A)$. $\square$

Theorem 8. Let $H: S \rightarrow T$ define a homomorphic mapping of ordered semigroups. And let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ be two PMFSSs over S. Then, if $(\bar{\psi}_{\bar{f}}, A) \leq (\bar{\zeta}_{\bar{g}}, B)$, this implies $H(\bar{\psi}_{\bar{f}}, A) \leq H(\bar{\zeta}_{\bar{g}}, B)$.

Proof. Let $(\bar{\psi}_{\bar{f}}, A) \leq (\bar{\zeta}_{\bar{g}}, B) \forall A \subseteq B$, then by Theorem 6 it is clear that $\bar{\psi}_{\bar{f}}(e)$ is a fuzzy subsemigroup of $\bar{\zeta}_{\bar{g}}(e)$. Also, according to the definition of homomorphism, $H(\bar{\psi}_{\bar{f}}(e))$ is a fuzzy subsemigroup of $H(\bar{\zeta}_{\bar{g}}(e))$. Therefore, it is concluded that $H(\bar{\psi}_{\bar{f}}, A) \leq H(\bar{\zeta}_{\bar{g}}, B)$. $\square$

Definition 13. Let $(\bar{\psi}_{\bar{f}}, A)$ be a PMFSS over S . Then, a possibility multi-fuzzy soft set $(\bar{\zeta}_{\bar{g}}, X)$ over S is called a possibility multi-fuzzy soft l -ideal (resp. r -ideal) of $(\bar{\psi}_{\bar{f}}, A)$ and is denoted $((\bar{\zeta}_{\bar{g}}, X) \triangleleft_l (\bar{\psi}_{\bar{f}}, A))$ (resp. $((\bar{\zeta}_{\bar{g}}, X) \triangleleft_r (\bar{\psi}_{\bar{f}}, A))$) if it follows

- (1) $X \subseteq A$;
- (2) $\forall e \in X, \bar{\zeta}_{\bar{g}}(e)$ is a fuzzy soft left ideal (resp. right ideal) of $\bar{\psi}_{\bar{f}}(e) \Rightarrow ((\bar{\zeta}_{\bar{g}}, X) \triangleleft_l (\bar{\psi}_{\bar{f}}, A))$ (resp. $((\bar{\zeta}_{\bar{g}}, X) \triangleleft_r (\bar{\psi}_{\bar{f}}, A))$).

If $(\bar{\zeta}_{\bar{g}}, B)$ is both an l -ideal and r -ideal of $(\bar{\psi}_{\bar{f}}, A)$, then we can call $(\bar{\zeta}_{\bar{g}}, B)$ a possibility multi-fuzzy soft ideal of $(\bar{\psi}_{\bar{f}}, A)$, and it is denoted as $(\bar{\zeta}_{\bar{g}}, X) \triangleleft (\bar{\psi}_{\bar{f}}, A)$.

Example 7. Suppose $S = \{s_1, s_2, s_3, s_4\}$ is an ordered semigroup under the following multiplication relation and order relation:

$$\leq := \{(s_1, s_1), (s_2, s_2), (s_3, s_3), (s_4, s_4), (s_1, s_2)\}.$$

Multiplication table

.	s_1	s_2	s_3	s_4
s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_1	s_1	s_1
s_3	s_1	s_1	s_1	s_2
s_4	s_1	s_1	s_2	s_3

Let $(\bar{\psi}_{\bar{f}}, A)$ be a possibility multi-fuzzy soft set of S , where $A = \{e_1, e_2, e_3\}$ and $\bar{\psi}_{\bar{f}} : A \rightarrow M^n \text{FSS}(S) \times M^n \text{FSS}(S)$. Let us assume a set of parameters X , where $X \subseteq A$; then, a mapping $\bar{\zeta}_{\bar{g}} : X \rightarrow M^n \text{FSS}(S) \times M^n \text{FSS}(S)$ can be defined, where $\bar{\zeta}_{\bar{g}}(e) = (\bar{\zeta}(e)(s_i), \bar{g}(e)(s_i)) \forall s_i \in S$.

$$\bar{\zeta}_{\bar{g}}(e) = \begin{bmatrix} (0.7, 0.6), (0.6, 0.5) & (0.6, 0.4), (0.5, 0.5) & (0.5, 0.3), (0.4, 0.3) & (0.5, 0.3), (0.4, 0.2) \\ (0.6, 0.6), (0.5, 0.4) & (0.5, 0.5), (0.4, 0.3) & (0.2, 0.4), (0.3, 0.1) & (0.1, 0.4), (0.2, 0.1) \\ (0.8, 0.7), (0.7, 0.6) & (0.5, 0.6), (0.5, 0.5) & (0.3, 0.5), (0.2, 0.3) & (0.3, 0.2), (0.2, 0.2) \end{bmatrix}.$$

As $\bar{\zeta}_{\bar{g}}(e)(a_i \cdot a_j) \geq \bar{\zeta}_{\bar{g}}(e)(a_i) \forall a_i, a_j \in S$, $\bar{\zeta}_{\bar{g}}(e)$ is a fuzzy soft right ideal of $\bar{\psi}_{\bar{f}}(e)$. Similarly, $\bar{\zeta}_{\bar{g}}(e)(a_i \cdot a_j) \geq \bar{\zeta}_{\bar{g}}(e)(a_j)$ implies $\bar{\zeta}_{\bar{g}}(e)$ is a fuzzy soft left ideal of $\bar{\psi}_{\bar{f}}(e)$. Thus, $(\bar{\zeta}_{\bar{g}}, X) \triangleleft_r (\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, X) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$. Thus, $(\bar{\zeta}_{\bar{g}}, X) \triangleleft (\bar{\psi}_{\bar{f}}, A)$.

Theorem 9. Let $(\bar{\psi}_{\bar{f}}, A)$ be a PMFSS over S . Then, for any two possibility multi-fuzzy soft sets $(\bar{\zeta}_{\bar{g}}|_1, X_1)$ and $(\bar{\zeta}_{\bar{g}}|_2, X_2)$ of S , where $X_1 \cap X_2 \neq \emptyset$, we can prove following:

- (1) If $(\bar{\zeta}_{\bar{g}}|_1, X_1) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}|_2, X_2) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$, then $(\bar{\zeta}_{\bar{g}}|_1, X_1) \cap (\bar{\zeta}_{\bar{g}}|_2, X_2) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$.
- (2) If $(\bar{\zeta}_{\bar{g}}|_1, X_1) \triangleleft_r (\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}|_2, X_2) \triangleleft_r (\bar{\psi}_{\bar{f}}, A)$, then $(\bar{\zeta}_{\bar{g}}|_1, X_1) \cap (\bar{\zeta}_{\bar{g}}|_2, X_2) \triangleleft_r (\bar{\psi}_{\bar{f}}, A)$.

Proof. (1) The intersection of any two possibility multi-fuzzy soft sets can be defined as $(\bar{\zeta}_{\bar{g}}|_1, X_1) \cap (\bar{\zeta}_{\bar{g}}|_2, X_2) = (\bar{\zeta}_{\bar{g}}, X)$, where $X_1 \cap X_2 = X$. Then, $\forall e \in X$ implies $e \in X_1$ and $e \in X_2$, so either $\bar{\zeta}_{\bar{g}}(e) = \bar{\zeta}_{\bar{g}}|_1(e)$ or $\bar{\zeta}_{\bar{g}}(e) = \bar{\zeta}_{\bar{g}}|_2(e)$, also $X \subseteq A$; hence, $(\bar{\zeta}_{\bar{g}}, X)$ is a

PMFSS over S . $(\bar{\zeta}_{\bar{g}}|_1, X_1) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$ implies $(\bar{\zeta}_{\bar{g}}, X) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$ as $(\bar{\zeta}_{\bar{g}}|_1, X_1) \cap (\bar{\zeta}_{\bar{g}}|_2, X_2) = (\bar{\zeta}_{\bar{g}}, X) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$.

(2) Similarly, we can prove the second relation. $\square$

Theorem 10. Let $(\bar{\psi}_{\bar{f}}, A)$ be a PMFSS over S . Then, for any two possibility multi-fuzzy soft sets $(\bar{\zeta}_{\bar{g}}, B)$ and $(\bar{\sigma}_{\bar{h}}, C)$ over S , where $B \cap C = \phi$, we can prove the following:

- (1) If $(\bar{\zeta}_{\bar{g}}, B) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\sigma}_{\bar{h}}, C) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$, then $(\bar{\zeta}_{\bar{g}}, B) \cup (\bar{\sigma}_{\bar{h}}, C) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$.
- (2) If $(\bar{\zeta}_{\bar{g}}, B) \triangleleft_r (\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\sigma}_{\bar{h}}, C) \triangleleft_r (\bar{\psi}_{\bar{f}}, A)$, then $(\bar{\zeta}_{\bar{g}}, B) \cup (\bar{\sigma}_{\bar{h}}, C) \triangleleft_r (\bar{\psi}_{\bar{f}}, A)$.

Proof. (1) The union of any two possibility multi-fuzzy soft sets is defined as $(\bar{\zeta}_{\bar{g}}, B) \cup (\bar{\sigma}_{\bar{h}}, C) = (\bar{\omega}_{\bar{k}}, K)$, where $B \cup C = K$, $\forall e \in K$.

$$\bar{\omega}_{\bar{k}}(e) = \begin{cases} \bar{\zeta}_{\bar{g}}(e) & \text{if } e \in B - C, \\ \bar{\sigma}_{\bar{h}}(e) & \text{if } e \in C - B, \\ \bar{\zeta}_{\bar{g}}(e) \cup \bar{\sigma}_{\bar{h}}(e) & \text{if } e \in B \cap C. \end{cases}$$

Here, $B \cap C = \phi$, so either $e \in B - C$ or $e \in C - B$. If $e \in B - C$, then $\bar{\omega}_{\bar{k}}(e) = \bar{\zeta}_{\bar{g}}(e)$, where $\bar{\zeta}_{\bar{g}}(e)$ is a left ideal of $\bar{\psi}_{\bar{f}}(e)$. So, $\bar{\omega}_{\bar{k}}(e)$ is also a left ideal of $\bar{\psi}_{\bar{f}}(e)$. Thus, $(\bar{\omega}_{\bar{k}}, K) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$. If $e \in C - B$, then $\bar{\omega}_{\bar{k}}(e) = \bar{\sigma}_{\bar{h}}(e)$, where $\bar{\sigma}_{\bar{h}}(e)$ is a left ideal of $\bar{\psi}_{\bar{f}}(e)$. So, $\bar{\omega}_{\bar{k}}(e)$ is also a left ideal of $\bar{\psi}_{\bar{f}}(e)$. Thus, $(\bar{\omega}_{\bar{k}}, K) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$.

Hence, we have $(\bar{\zeta}_{\bar{g}}, B) \cup (\bar{\sigma}_{\bar{h}}, C) \triangleleft_l (\bar{\psi}_{\bar{f}}, A)$.

(2) Similarly, we can prove $(\bar{\zeta}_{\bar{g}}, B) \cup (\bar{\sigma}_{\bar{h}}, C) \triangleleft_r (\bar{\psi}_{\bar{f}}, A)$. $\square$

4. Application of PMFSS in Decision Making and Medical Diagnosis

In this section, we showcase the application of PMFSSs in decision-making problems by examining the following examples.

Example 8. Let three players give a test for their selection in a cricket team. The parameters required for the players are e_1 , which is all-rounder, consisting of batsman, bowler and fielder; e_2 , which is age, consisting of old, medium and young; and e_3 , which is fitness, consisting of excellent, good and poor. Here, $U = \{P_1, P_2, P_3\}$ and $R = \{e_1, e_2, e_3\}$. We have defined the following multiplication and ordered relation on the basis of their average performance in the last ten matches.

Multiplication table			
.	P_1	P_2	P_3
P_1	P_1	P_2	P_1
P_2	P_2	P_2	P_3
P_3	P_3	P_3	P_3

$$\leq := \{(P_1, P_1), (P_2, P_2), (P_3, P_3), (P_1, P_3), (P_2, P_3)\}.$$

Keeping in view the order relation as shown in Figure 1, it is easily noticed that the third player is better than first two. We consider two observations $\bar{\psi}_{\bar{f}}$ and $\bar{\zeta}_{\bar{g}}$ as two committee members to decide the best player.

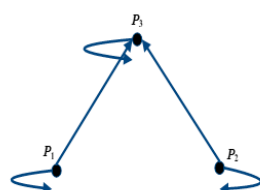


Figure 1. Hasse diagram for ordered relation of Z .

Here, possibility multi-fuzzy soft set technique is applied for both the observations. We obtain

$$\begin{aligned}\bar{\psi}_{\bar{f}}(e_1) &= \left\{ \left(\frac{P_1}{(0.2,0.5,0.5)}, (0.6, 0.9, 0.1) \right), \left(\frac{P_2}{(0.9,0.8,0.3)}, (0.8, 0.7, 0.4) \right), \left(\frac{P_3}{(0.1,0.3,0.2)}, (0.5, 0.2, 0.3) \right) \right\}, \\ \bar{\psi}_{\bar{f}}(e_2) &= \left\{ \left(\frac{P_1}{(0.3,0.4,0.6)}, (0.2, 0.3, 0.7) \right), \left(\frac{P_2}{(0.3,0.4,0.7)}, (0.2, 0.6, 0.7) \right), \left(\frac{P_3}{(0.8,0.2,0.1)}, (0.7, 0.3, 0.2) \right) \right\}\end{aligned}$$

and

$$\bar{\psi}_{\bar{f}}(e_3) = \left\{ \left(\frac{P_1}{(0.6,0.3,0.2)}, (0.7, 0.4, 0.3) \right), \left(\frac{P_2}{(0.9,0.7,0.2)}, (0.8, 0.6, 0.3) \right), \left(\frac{P_3}{(0.6,0.4,0.1)}, (0.7, 0.5, 0.1) \right) \right\},$$

In matrix form,

$$\bar{\psi}_{\bar{f}} = \begin{bmatrix} (0.2, 0.5, 0.5), (0.6, 0.9, 0.1) & (0.9, 0.8, 0.3), (0.8, 0.7, 0.4) & (0.1, 0.3, 0.2), (0.5, 0.2, 0.3) \\ (0.3, 0.4, 0.6), (0.2, 0.3, 0.7) & (0.3, 0.4, 0.7), (0.2, 0.6, 0.7) & (0.8, 0.2, 0.1), (0.7, 0.3, 0.2) \\ (0.6, 0.3, 0.2), (0.7, 0.4, 0.3) & (0.9, 0.7, 0.2), (0.8, 0.6, 0.3) & (0.6, 0.4, 0.1), (0.7, 0.5, 0.1) \end{bmatrix}.$$

Similarly, $\bar{\xi}_{\bar{g}}$ can be written as

$$\bar{\xi}_{\bar{g}} = \begin{bmatrix} (0.3, 0.5, 0.6), (0.7, 0.8, 0.2) & (0.9, 0.9, 0.2), (0.9, 0.8, 0.4) & (0.2, 0.3, 0.3), (0.6, 0.3, 0.2) \\ (0.2, 0.4, 0.6), (0.1, 0.5, 0.4) & (0.4, 0.5, 0.6), (0.2, 0.5, 0.7) & (0.7, 0.3, 0.2), (0.6, 0.4, 0.1) \\ (0.5, 0.3, 0.1), (0.6, 0.4, 0.1) & (0.9, 0.8, 0.1), (0.7, 0.5, 0.2) & (0.5, 0.6, 0.1), (0.7, 0.4, 0.3) \end{bmatrix}.$$

In order to calculate a mutual decision by both the members of committee, AND operation of possibility multi-fuzzy soft sets is applied. Defined as $\bar{\sigma}_{\bar{h}} = \bar{\psi}_{\bar{f}} \wedge \bar{\xi}_{\bar{g}}$,

$$\begin{aligned}\bar{\sigma}_{\bar{h}}(e_1, e_1) &= \left\{ \left(\frac{P_1}{(0.2,0.5,0.5)}, (0.6, 0.8, 0.1) \right), \left(\frac{P_2}{(0.9,0.8,0.2)}, (0.8, 0.7, 0.4) \right), \left(\frac{P_3}{(0.1,0.3,0.2)}, (0.5, 0.2, 0.2) \right) \right\}, \\ \bar{\sigma}_{\bar{h}}(e_1, e_2) &= \left\{ \left(\frac{P_1}{(0.2,0.4,0.5)}, (0.1, 0.5, 0.1) \right), \left(\frac{P_2}{(0.4,0.5,0.3)}, (0.2, 0.5, 0.4) \right), \left(\frac{P_3}{(0.1,0.3,0.2)}, (0.5, 0.2, 0.1) \right) \right\}, \\ \bar{\sigma}_{\bar{h}}(e_1, e_3) &= \left\{ \left(\frac{P_1}{(0.2,0.3,0.1)}, (0.6, 0.4, 0.1) \right), \left(\frac{P_2}{(0.9,0.8,0.1)}, (0.7, 0.5, 0.2) \right), \left(\frac{P_3}{(0.1,0.3,0.1)}, (0.5, 0.2, 0.3) \right) \right\}.\end{aligned}$$

And the values for all the other pairs of parameters can be evaluated in a similar manner. The values for all the pairs of parameters in matrix form are written as

$$\bar{\sigma}_{\bar{h}} = \begin{bmatrix} (0.2, 0.5, 0.5), (0.6, 0.8, 0.1) & (0.9, 0.8, 0.2), (0.8, 0.7, 0.4) & (0.1, 0.3, 0.2), (0.5, 0.2, 0.2) \\ (0.2, 0.4, 0.5), (0.1, 0.5, 0.1) & (0.4, 0.5, 0.3), (0.2, 0.5, 0.4) & (0.1, 0.3, 0.2), (0.5, 0.2, 0.1) \\ (0.2, 0.3, 0.1), (0.6, 0.4, 0.1) & (0.9, 0.8, 0.1), (0.7, 0.5, 0.2) & (0.1, 0.3, 0.1), (0.5, 0.2, 0.3) \\ (0.3, 0.4, 0.6), (0.2, 0.3, 0.2) & (0.3, 0.4, 0.2), (0.2, 0.6, 0.4) & (0.2, 0.2, 0.1), (0.6, 0.3, 0.2) \\ (0.2, 0.4, 0.6), (0.1, 0.3, 0.4) & (0.3, 0.4, 0.6), (0.2, 0.5, 0.7) & (0.7, 0.2, 0.1), (0.6, 0.3, 0.1) \\ (0.3, 0.3, 0.1), (0.2, 0.3, 0.1) & (0.3, 0.4, 0.1), (0.2, 0.5, 0.2) & (0.5, 0.2, 0.1), (0.7, 0.3, 0.2) \\ (0.3, 0.3, 0.2), (0.7, 0.4, 0.2) & (0.9, 0.7, 0.2), (0.8, 0.6, 0.3) & (0.2, 0.3, 0.1), (0.6, 0.3, 0.1) \\ (0.2, 0.3, 0.2), (0.1, 0.4, 0.3) & (0.4, 0.5, 0.2), (0.2, 0.5, 0.3) & (0.6, 0.3, 0.1), (0.6, 0.4, 0.1) \\ (0.5, 0.3, 0.1), (0.6, 0.4, 0.1) & (0.9, 0.7, 0.1), (0.7, 0.5, 0.2) & (0.5, 0.4, 0.1), (0.7, 0.4, 0.1) \end{bmatrix}$$

As we need to find the best player suitable for the team, we would compute the grades and possibility grades by the following formulas:

$$\begin{aligned}r_{ij}(u_k) &= \sum_{u \in U} ((C_k^1 - v_{\bar{\sigma}(e_i, e_j)}^1(u)) + ((C_k^2 - v_{\bar{\sigma}(e_i, e_j)}^2(u)) + ((C_k^3 - v_{\bar{\sigma}(e_i, e_j)}^3(u))) \\ \lambda_{ij}(u_k) &= \sum_{u \in U} ((C_k^1 - v_{\bar{\sigma}(e_i, e_j)}^1(u)) + ((C_k^2 - v_{\bar{\sigma}(e_i, e_j)}^2(u)) + ((C_k^3 - v_{\bar{\sigma}(e_i, e_j)}^3(u)))\end{aligned}$$

By using these formulas, the grades and possibility grades for all possible values can be calculated.

$$\begin{aligned}
 r_{11}(P_1) &= \sum_{P_1 \in U} ((C_1^1 - v_{\bar{\sigma}(e_1, e_1)}^1(P_1)) + ((C_1^2 - v_{\bar{\sigma}(e_1, e_1)}^2(P_1)) + ((C_1^3 - v_{\bar{\sigma}(e_1, e_1)}^3(P_1)) \\
 &= (0.2 - 0.9) + (0.2 - 0.1) + (0.5 - 0.8) + (0.5 - 0.3) + (0.5 - 0.8) + (0.5 - 0.2) \\
 &= -0.7 + 0.1 - 0.3 + 0.2 + 0.3 + 0.3 \\
 &= -0.1 \\
 \lambda_{11}(P_1) &= \sum_{P_1 \in U} ((C_1^1 - v_{\bar{h}(e_1, e_1)}^1(P_1)) + ((C_1^2 - v_{\bar{h}(e_1, e_1)}^2(P_1)) + ((C_1^3 - v_{\bar{h}(e_1, e_1)}^3(P_1)) \\
 &= (0.6 - 0.8) + (0.6 - 0.5) + (0.8 - 0.7) + (0.8 - 0.2) + (0.1 - 0.4) + (0.1 - 0.2) \\
 &= 0.2
 \end{aligned}$$

Similarly, we can calculate the values for all the other possible points, which are given in Table 1. Now, we mark the highest numerical grade in each row and possibility grade related to that and then find the total score for each player by taking the sum of the product of these numerical grades with their respective possibility grades. Here, P_2 is the player with the highest score; thus, they will select P_2 for the team.

Table 1. Numerical and possibility grade table.

$\bar{\sigma}_{\bar{h}}$	P_1	P_2	P_3
(e_1, e_1)	−0.1, 0.2	0.2, 1.4	−0.1, −0.1
(e_1, e_2)	0.4, −0.5	0.7, 0.7	−1.1, −0.2
(e_1, e_3)	0.2, 0.3	0.5, 0	−0.7, −0.4
(e_2, e_1)	1.2, −0.9	0.0, 0.6	−1.2, 0.5
(e_2, e_2)	0.1, −0.6	0.4, 1.0	−0.5, −0.2
(e_2, e_3)	0, −0.5	0, −0.5	0, 1
(e_3, e_1)	0.3, 0.6	0.6, 0.3	−1.6, −0.9
(e_3, e_2)	−0.3, −0.2	0, −0.2	0.3, 0.4
(e_3, e_3)	1.5, 0.5	−0.9, −0.4	−0.6, −0.3

Example 9. Suppose that we have a patient who is suffering with certain problem in his health, and he thinks that he might be suffering with hyperthyroidism or hypothyroidism. Let us consider all the symptoms he has as set of parameters as represented in Table 2, and the universal set for this case is yes or no. Here, $R = \{e_1 + e_2 + e_3 + e_4 + e_5\}$ and $U = \{y, n\}$.

Table 2. Model table for a hyperthyroid patient.

h	y	$\bar{f}_y$	n	$\bar{f}_n$
e_1	(0,0,1)	(1,1,1)	(1,1,0)	(1,1,1)
e_2	(1,0,0)	(1,1,1)	(0,1,1)	(1,1,1)
e_3	(1,1,1)	(1,1,1)	(0,0,0)	(1,1,1)
e_4	(1,1,1)	(1,1,1)	(0,0,0)	(1,1,1)
e_5	(1,1,1)	(1,1,1)	(0,0,0)	(1,1,1)

For this case, we first have to construct a PMFSS model for a hyperthyroid and a hypothyroid patient by consulting a physician; then, we would construct a PMFSS model for the patient under observation.

Table 2 represents a possibility multi-fuzzy soft set model table for a hyperthyroid patient, and Table 3 represents a possibility multi-fuzzy soft set model table for a hypothyroid

patient. Now, after constructing a model table for the patient under observation, we find the similarity measure between Tables 2 and 3 by using the Definition 9, and we obtain

$$\begin{aligned}\varphi_1(\bar{\psi}, \bar{\xi}) &= \frac{\sum_{i=1}^5 \max_{u \in U} \left\{ \min(v_{\bar{\psi}(e_i)}^1(u), v_{\bar{\xi}(e_i)}^1(u)) \right\}}{\sum_{i=1}^5 \max_{u \in U} \left\{ \max(v_{\bar{\psi}(e_i)}^1(u), v_{\bar{\xi}(e_i)}^1(u)) \right\}} \\ &= 4.3/5 = 0.86\end{aligned}$$

$$\begin{aligned}\theta_1(\bar{f}, \bar{h}) &= \frac{\sum_{i=1}^5 \max_{u \in U} \left\{ \min(v_{\bar{f}(e_i)}^1(u), v_{\bar{h}(e_i)}^1(u)) \right\}}{\sum_{i=1}^5 \max_{u \in U} \left\{ \max(v_{\bar{f}(e_i)}^1(u), v_{\bar{h}(e_i)}^1(u)) \right\}} \\ &= 3.7/5 = 0.7\end{aligned}$$

Table 3. Model table for a hypothyroid patient.

$\bar{\omega}_{\bar{g}}$	y	$\bar{g}_y$	n	$\bar{g}_n$
e_1	(1,0,0)	(1,1,1)	(0,0,1)	(1,1,1)
e_2	(0,0,1)	(1,1,1)	(1,0,0)	(1,1,1)
e_3	(1,1,1)	(1,1,1)	(0,0,0)	(1,1,1)
e_4	(1,1,1)	(1,1,1)	(0,0,0)	(1,1,1)
e_5	(1,1,1)	(1,1,1)	(0,0,0)	(1,1,1)

Similarly, $\varphi_2(\bar{\psi}, \bar{\xi}) = 0.62$, $\theta_2(\bar{f}, \bar{h}) = 0.66$, $\varphi_3(\bar{\psi}, \bar{\xi}) = 0.7$ and $\theta_3(\bar{f}, \bar{h}) = 0.72$. Thus, we can obtain $S(\bar{\psi}_{\bar{f}}, \bar{\xi}_{\bar{h}}) = (0.6364, 0.4092, 0.504)$. Hence, $s(\bar{\psi}_{\bar{f}}, \bar{\xi}_{\bar{h}}) = 0.514 > 0.5$. The result shows that the two possibility multi-fuzzy soft set models are significantly similar; thus, the patient is suffering with hyperthyroidism.

For the second case, we consider Figure 2 of the possibility multi-fuzzy soft set model for a hypothyroid patient with Table 4 of the possibility multi-fuzzy soft set model of the patient under observation. We find the similarity measure for the two models following the same method, and we conclude that $S(\bar{\omega}_{\bar{g}}, \bar{\xi}_{\bar{h}}) = (0.37, 0.4092, 0.288)$. Hence, $s(\bar{\omega}_{\bar{g}}, \bar{\xi}_{\bar{h}}) = 0.3557 < 0.5$. The result shows that the two possibility multi-fuzzy soft set models are not significantly similar; thus, the patient is not suffering from hypothyroidism.

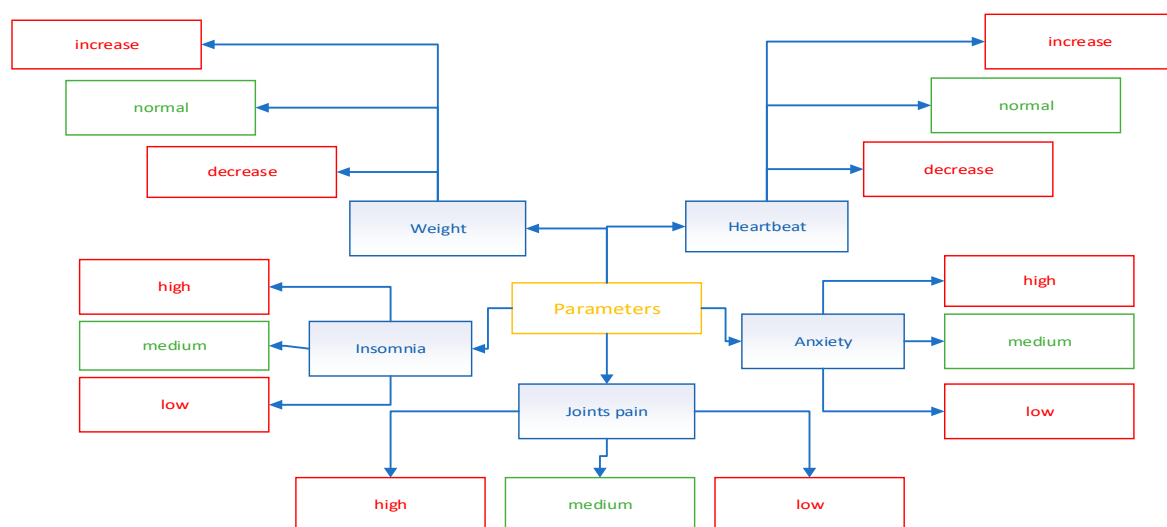


Figure 2. Set of parameters.

Table 4. Model table for the patient under observation.

$\bar{\xi}_h$	y	$\bar{h}_y$	n	$\bar{h}_n$
e_1	(0.0,0.2,0.9)	(0.1,0.2,1.0)	(1.0,0.5,0.2)	(0.9,0.1,0.1)
e_2	(0.9,0.1,0.1)	(0.8,0.2,0.1)	(0.1,0.6,0.9)	(0.2,0.7,0.6)
e_3	(0.7,0.6,0.4)	(0.6,0.7,0.5)	(0.1,0.3,0.2)	(0.2,0.5,0.3)
e_4	(0.9,0.7,0.8)	(0.8,0.8,0.9)	(0.2,0.4,0.1)	(0.1,0.8,0.2)
e_5	(0.8,0.7,0.5)	(0.6,0.7,0.6)	(0.1,0.8,0.5)	(0.1,0.9,0.1)

5. Idealistic Possibility Multi-fuzzy Soft Ordered Semigroups

This section includes a new notion of idealistic possibility multi-fuzzy soft ordered semigroups. Further, some basic results are obtained using different operations including union, intersection, AND and OR operation with an idealistic PMFSS technique.

Definition 14. Let $(\bar{\psi}_{\bar{f}}, A)$ be a possibility multi-fuzzy soft set over S . Then, $(\bar{\psi}_{\bar{f}}, A)$ is called an *l-idealistic* (resp. *r-idealistic*) possibility multi-fuzzy soft ordered semigroup over S if $\bar{\psi}_{\bar{f}}(e)$ is the left (resp. right) ideal of S , $\forall e \in A$.

The example below will give a better understanding of this new concept.

Example 10. Let S be an ordered semigroup with the multiplication and ordered relation as follows:

$$\leq := \{(s_1, s_1), (s_2, s_2), (s_3, s_3), (s_4, s_4), (s_1, s_2)\}.$$

Multiplication table

$\cdot$	s_1	s_2	s_3	s_4
s_1	s_1	s_1	s_1	s_1
s_2	s_1	s_1	s_1	s_1
s_3	s_1	s_1	s_1	s_2
s_4	s_1	s_1	s_2	s_3

$(\bar{\psi}_{\bar{f}}, A)$ is a possibility multi-fuzzy soft set over S , where

$$\bar{\psi}_{\bar{f}}(e) = \begin{bmatrix} (0.8, 0.7), (0.7, 0.6) & (0.7, 0.6), (0.6, 0.5) & (0.6, 0.5), (0.4, 0.3) & (0.6, 0.4), (0.3, 0.1) \\ (0.7, 0.7), (0.8, 0.7) & (0.6, 0.5), (0.7, 0.4) & (0.3, 0.5), (0.5, 0.1) & (0.1, 0.4), (0.2, 0.1) \\ (0.9, 0.7), (0.8, 0.6) & (0.7, 0.6), (0.6, 0.5) & (0.4, 0.5), (0.3, 0.4) & (0.4, 0.3), (0.2, 0.2) \end{bmatrix}.$$

As here, $\bar{\psi}_{\bar{f}}(e)(s_i \cdot s_j) \geq \bar{\psi}_{\bar{f}}(e)(s_i) \forall e \in A$, so $\bar{\psi}_{\bar{f}}(e)$ is the fuzzy right ideal of S by definition. Hence, $\bar{\psi}_{\bar{f}}(e)$ is an *r-idealistic* possibility multi-fuzzy soft ordered semigroup over S . Similarly, we can check it for an *l-idealistic* possibility multi-fuzzy soft ordered semigroup over S .

Theorem 11. Let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\xi}_{\bar{g}}, B)$ be *l-idealistic* (resp. *r-idealistic*) possibility multi-fuzzy soft ordered semigroups over S . If $A \cap B \neq \emptyset$, then their intersection $((\bar{\psi}_{\bar{f}}, A) \cap (\bar{\xi}_{\bar{g}}, B))$ is also an *l-idealistic* (resp. *r-idealistic*) possibility multi-fuzzy soft ordered semigroup over S .

Proof. As the intersection of any two possibility multi-fuzzy soft sets is defined as $(\bar{\psi}_{\bar{f}}, A) \cap (\bar{\xi}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$, where $C = A \cap B$, then $\forall e \in C$ either $(\bar{\sigma}(e), \bar{h}(e)) = (\bar{\psi}(e), \bar{f}(e))$ or $(\bar{\sigma}(e), \bar{h}(e)) = (\bar{\xi}(e), \bar{g}(e))$. $(\bar{\sigma}_{\bar{h}}, C)$ is a PMFSS over S as defined in Theorem 3. Since $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\xi}_{\bar{g}}, B)$ are both *l-idealistic* (resp. *r-idealistic*) PMFSSs over S , it follows that either $\bar{\sigma}_{\bar{h}}(e) = \bar{\psi}_{\bar{f}}(e)$ or $\bar{\sigma}_{\bar{h}}(e) = \bar{\xi}_{\bar{g}}(e)$; hence, $\bar{\sigma}_{\bar{h}}(e)$ is also an *l-idealistic* (resp. *r-idealistic*)

PMFSS over S . So, we can say that the intersection of two l -idealistic (resp. r -idealistic) PMFSSs over S is also an l -idealistic (resp. r -idealistic) PMFSS over S . $\square$

Theorem 12. Let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ be l -idealistic (resp. r -idealistic) possibility multi-fuzzy soft ordered semigroups over S . If $A \cap B = \phi$, then their union $(\bar{\psi}_{\bar{f}}, A) \cup (\bar{\zeta}_{\bar{g}}, B)$ is also an l -idealistic (resp. r -idealistic) PMFSS over S .

Proof. As the union of any two possibility multi-fuzzy soft sets is defined as $(\bar{\psi}_{\bar{f}}, A) \cup (\bar{\zeta}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$, where $C = A \cup B$, then $\forall e \in C$.

$$(\bar{\sigma}_{\bar{h}}, C) = \begin{cases} (\bar{\psi}_{\bar{f}}, A) & \text{if } e \in A - B \\ (\bar{\zeta}_{\bar{g}}, B) & \text{if } e \in B - A, \\ (\bar{\psi}_{\bar{f}}, A) \cup (\bar{\zeta}_{\bar{g}}, B) & \text{if } e \in A \cap B \end{cases}$$

As $A \cap B = \phi \Rightarrow$ either $e \in A - B$ or $e \in B - A$. If $e \in A - B$, then $\bar{\sigma}_{\bar{h}}(e) = \bar{\psi}_{\bar{f}}(e)$, where $\bar{\psi}_{\bar{f}}(e)$ is an l -idealistic (resp. r -idealistic) PMFSS over S , so $\bar{\sigma}_{\bar{h}}(e)$ is also an l -idealistic (resp. r -idealistic) PMFSS over S . Similarly, if $e \in B - A$, then $\bar{\sigma}_{\bar{h}}(e) = \bar{\zeta}_{\bar{g}}(e)$, where $\bar{\zeta}_{\bar{g}}(e)$ is an l -idealistic (resp. r -idealistic) PMFSS over S , so $\bar{\sigma}_{\bar{h}}(e)$ is also an l -idealistic (resp. r -idealistic) PMFSS over S . Hence, $(\bar{\psi}_{\bar{f}}, A) \cup (\bar{\zeta}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$ is an l -idealistic (resp. r -idealistic) PMFSS over S . $\square$

Theorem 13. Let $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ be l -idealistic (resp. r -idealistic) possibility multi-fuzzy soft ordered semigroups over S . Then, $(\bar{\psi}_{\bar{f}}, A) \wedge (\bar{\zeta}_{\bar{g}}, B)$ is an l -idealistic (resp. r -idealistic) possibility multi-fuzzy soft ordered semigroup over S .

Proof. As the AND operation is defined as $(\bar{\psi}_{\bar{f}}, A) \wedge (\bar{\zeta}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$, where $C = A \times B$, then by definition, $\bar{\sigma}_{\bar{h}}(a, b) = \bar{\psi}_{\bar{f}}(a) \cap \bar{\zeta}_{\bar{g}}(b) \forall (a, b) \in A \times B$. Since $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ are l -idealistic (resp. r -idealistic) PMFSSs over S , we can say that $(\bar{\psi}_{\bar{f}}, A)$ and $(\bar{\zeta}_{\bar{g}}, B)$ are possibility multi-fuzzy soft l -ideals (resp. r -ideals) over S ; therefore, the intersection of $(\bar{\psi}_{\bar{f}}(a)) \cap (\bar{\zeta}_{\bar{g}}(b)) \forall (a, b) \in A \times B$ is a possibility multi-fuzzy soft l -ideal (resp. r -ideal) over S . Thus, the intersection of $(\bar{\psi}_{\bar{f}}(a)) \cap (\bar{\zeta}_{\bar{g}}(b)) \forall (a, b) \in A \times B$ is also an l -idealistic (resp. r -idealistic) PMFSS over S . Therefore, $(\bar{\psi}_{\bar{f}}, A) \wedge (\bar{\zeta}_{\bar{g}}, B) = (\bar{\sigma}_{\bar{h}}, C)$ is an l -idealistic (resp. r -idealistic) PMFSS over S . $\square$

6. Conclusions

This paper presents a comprehensive exploration of the amalgamation of possibility multi-fuzzy soft sets with ordered semigroups. By synthesizing concepts from fuzzy mathematics, soft set theory and algebraic structures, this theory provides a powerful tool for addressing uncertainty in problem solving from different fields of applied sciences including medical diagnosis and decision making. This research will further lead to possibility multi-fuzzy soft interior ideals, possibility multi-fuzzy soft bi-ideals, possibility multi-fuzzy soft generalized bi-ideals, possibility multi-fuzzy soft quasi-ideals and several other algebraic structures, which will provide a platform for other researchers to further enhance this new theory, providing valuable insights and applications for future endeavors.

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