Article

Generalized Bounded Turning Functions Connected with Gregory Coefficients

Huo Tang 1, Zeeshan Mujahid 2, Nazar Khan 2,*, Fairouz Tchier 3 and Muhammad Ghaffar Khan 4

1 School of Mathematics and Computer Sciences, Chifeng University, Chifeng 024000, China; tanghuo@cfxy.edu.cn
2 Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad 22500, Pakistan; zeeshanmujahid985@gmail.com
3 Mathematics Department, College of Science, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia; ftchier@ksu.edu.sa
4 Institute of Numerical Sciences, Kohat University of Sciences and Technology, Kohat 26000, Pakistan; ghaffarkhan020@gmail.com
* Correspondence: nazarmaths@gmail.com

Abstract: In this research article, we introduce new family $R_G$ of holomorphic functions, which is related to the generalized bounded turning and generating functions of Gregory coefficients. Leveraging the concept of functions with positive real parts, we acquire the first five coefficients for the functions belonging to this newly defined family, demonstrating their sharpness. Furthermore, we find the third Hankel determinant for functions in the class $R_G$. Moreover, the sharp bounds for logarithmic and inverse coefficients of functions belonging to the under-considered class $R_G$ are estimated.

Keywords: analytic functions; bounded turning functions; gregory coefficients; hankel determinant; logarithmic functions

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We need to review some basic concepts in order to properly understand the core ideas behind our new research. Let $H(U)$ symbolize the family of holomorphic functions in the open unit disc $U = \{ \xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1 \}$. Furthermore, $A$ define a subclass of $H(U)$, fulfilling the requirements for normalization:

$$f(0) = f'(0) - 1 = 0.$$ 

Because of this normalization, $f \in A$ is guaranteed to have the Taylor’s series expansion:

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n. \quad (1)$$

Recall also that an analytic function $f$ is considered univalent in region $U$ if it repeats no values in $U$. This implies that for each $\xi_1, \xi_2 \in U$, if $f(\xi_1) = f(\xi_2)$ then $\xi_1 = \xi_2$. Therefore, we use the symbol $S$ to represent the family of univalent functions that the series expansion given in Equation (1). Originally presented by Köebe in 1907, this class is now a fundamental component of groundbreaking research in this area. This idea attracted a lot of attention, and Bieberbach [1] produced a paper where the well-known coefficient hypothesis was put out. According to this conjecture, for every $n \geq 2$, $|a_n| \leq n$ if $f(\xi) \in S$ and has the series form (1). Numerous mathematicians have put a lot of effort into solving this problem. De-Branges [2] was the one who resolved this enduring hypothesis.
in 1985. Plenty of research on this conjecture and associated coefficient problems have been published throughout the span of these 69 years. Furthermore, $S^*$ and $R$ are the common subclasses of $S$, whose members are starlike and bounded turning, respectively. In 1977, Chichra [3] proved that the functions belonging to the class $R$ are univalent.

For the given functions $f_1, f_2 \in A$, $f_1$ is said to be subordinated to $f_2$ (mathematically written as $f_1 \prec f_2$) if an holomorphic function $w$ appears in $U$ with the limitations $w(0) = 0$ and $|w(z)| < 1$ in such a manner that $f_1(z) = f_2(w(z))$ holds.

Using subordination terminology, Ma and Minda [4] produced a unified version of the class $S^*(\phi)$ in 1992. It was described by:

$$S^*(\phi) = \{ f \in A : \frac{\xi f'(\xi)}{f(\xi)} \prec \phi(\xi) \},$$

where $\phi$ that has a positive real part and is normalized by the conditions $\phi(0) = 1, \phi(0) > 0,$ and maps $H(U)$ onto a region that is univalently starlike regarding 1 and symmetric with reference to the real axis.

Several well-known families are included as special instances in the following generic class that Ma and Minda established:

$$S^*(\phi) = \{ f \in A : \frac{\xi f'(\xi)}{f(\xi)} \prec \phi(\xi) \}.$$

These type functions are referred to as Ma-Minda starlike functions in literature. As a special example of $S^*(\phi)$ a variety of subfamilies of the generalized analytic functions have been examined in recent years (see, [5,6]). For example, Kumar and co-authors [7] took:

$$\phi(\xi) = e^{e^\xi - 1} \quad (\xi \in \mathbb{C}),$$

as the generated function of Bell numbers. Mendiratta et al. [8] and Goel and Kumar [9] recently obtained the results of growth and distortion, inclusion relations, coefficient estimates, the structural formula, subordination theorems, and various radii constants for the exponential function and sigmoid function as well, that is:

$$\phi(\xi) = e^\xi \quad (\xi \in \mathbb{C}),$$

and

$$\phi(\xi) = \frac{2}{1 + e^{-\xi}} \quad (\xi \in \mathbb{C}),$$
correspondingly.

The conventional telephone numbers (or involution numbers) are quantified by the recurrence relation:

$$\zeta(n) = \zeta(n - 1) + (n - 1)\zeta(n - 2) \quad \text{for} \quad n \geq 2$$

with initial conditions:

$$\zeta(0) = \zeta(1) = 1$$

Wlochand Wolowiec-Musial [10] familiarized generalized telephone numbers $\zeta(l, n)$ defined for integers $n \geq 0$ and $l \geq 1$ by the ensuing recursion:

$$\zeta(l, n) = lT(l, n - 1) + (n - 1)\zeta(l, n - 2)$$

with initial conditions:

$$\zeta(l, 0) = 1, \zeta(l, 1) = l.$$
\[ \varphi(\xi) = e^{\varphi_1 + \frac{1}{2} \varphi_2^2} = \sum_{\lambda \geq 1} \zeta_l(n) \frac{(\lambda)^n}{n!} (\xi \in \mathbb{C}) \] 

As we can observe, if \( l = 1 \), then we obtain \( \zeta_l(n) = \zeta(n) \) classical telephone numbers. Clearly, \( \zeta_l(n) \) is for some values of \( n \) as:

1. \( \zeta_l(0) = \zeta_l(1) = 1; \)
2. \( \zeta_l(2) = 1 + l; \)
3. \( \zeta_l(3) = 1 + 3l, \)

which gave accessible generalization of telephone numbers.

By using the similar concept used in [4], authors [12] defined the following classes of functions:

\[ R_{\text{sin}} = \{ f \in A : f'(\xi) + \xi f''(\xi) < 1 + \sin \xi \}, \]

\[ R_{\text{card}} = \{ f \in A : f'(\xi) + \xi f''(\xi) < 1 + \frac{4}{3} \xi + \frac{2}{3} \xi^2 \}. \]

Encouraged by the previously described studies, we examine the function \( \varphi \), whose coefficients are the Gregory coefficients, and thus, in reference to 1, \( \varphi(U) \) is starlike. Gregory coefficients, which play a similar function to the Bernoulli numbers and are found in many situations, particularly those pertaining to numerical analysis and number theory, are decreasing rational numbers \( \frac{1}{2}, \frac{1}{12}, \frac{1}{24}, \frac{19}{720}, \ldots \). They were often found after making their initial appearance in the writings of Scottish mathematician James Gregory in 1671.

Rediscovered by a number of eminent mathematicians, including Laplace, Mascheroni, Fontana, Bessel, Clausen, Hermite, Pearson, and Fisher, Gregory’s coefficients are really among the most often rediscovered in mathematics. The literary explanation for the most recent rediscovery, which occurred in our century, is given. These go by a variety of names (e.g., reciprocal logarithmic numbers, second-kind Bernoulli numbers, Cauchy numbers, etc.). In this paper, we considered the generating function of the Gregory coefficients \( G_n \) (see [13,14]) as follows:

\[ \frac{\xi}{\ln(1 + \xi)} = \Psi(\xi) = \sum_{n=0}^{\infty} G_n \xi^n \quad (|\xi| < 1). \]

Clearly, \( G_n \) for some values of \( n \) as \( G_0 = 1, G_1 = \frac{1}{2}, G_2 = -\frac{1}{12}, G_3 = \frac{1}{24}, G_4 = -\frac{19}{720}, G_5 = \frac{3}{540}, \) and \( G_6 = -\frac{863}{60480} \).

As it provides various features of functions, determining the upper bound for coefficients has been one of the main areas of study in geometric function theory. Specifically, growth and distortion theorems for functions in the class are provided by the bound for the second coefficient. Another one is the coefficient problem related with Hankel determinants. The Hankel determinant [15] \( H_q(n) \) \( (n = 1, 2, \ldots, q = 1, 2, \ldots) \) of the function \( f \) are defined by:

\[ H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-1} \end{vmatrix} \quad (a_1 = 1). \]

This determinant was discussed by several authors with \( q = 2 \). For example, we know that the functional \( H_2(1) := [a_3 - a_2^2] \) is known as the Fekete–Szegő functional and they consider the further generalized functional of \( [a_3 - \mu a_2^2] \), where \( \mu \) is real or complex number. For the class \( S^* \), Keogh and Merkes [16] solved the Fekete-Szegő problem in 1969. The formula for the second Hankel determinant, \( H_2(2) \), is \( H_2(2) = |a_2a_4 - a_3^2| \) and this bound found for the class \( S^* \) in [17]. Lee et al. [18] established the sharp bound to \( |H_2(2)| \) by generalizing their classes using subordination. Moreover, the quantity given by \( H_3(1) := a_3(a_2a_4 - a_2^2a_3) - a_4(a_4 - a_2^2a_3) + a_5(a_3 - a_2^2) \) is called the third Hankel determi-
nant. Zaprawa [19] proved that $|H_3(1)| < 1$ for $f \in S^*$. The Zaprawa result was enhanced in 2019 by Kwon, Lecko, and Sim [20], as $|H_3(1)| < \frac{n}{8}$. According to this finding, the class $S^*$ has the best upper bound on $|H_3(1)|$. Fekete–Szegö problem and Hankel determinants have been discussed recently in numerous articles by Deniz and their coauthors (see, [11,16]) and Srivastava and their coauthors (see, for example, [21–24]).

2. Logarithmic Function

The logarithmic coefficients $\beta_n$ of $f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n$ that belong to $S$ are defined by the following formula:

$$G_f(\xi) := \log \left( \frac{f(\xi)}{\xi} \right) = 2 \sum_{n=2}^{\infty} \beta_n \xi^n \quad \text{for } \xi \in U.$$ (4)

These coefficients make a substantial contribution to the idea of univalent functions in various estimations. De Branges [2] established in 1985 that:

$$\sum_{k=1}^{n} K(n - K + 1)|\beta|^2 \leq \sum_{k=1}^{n} \frac{n - K + 1}{K} \quad n \geq 1$$ (5)

and equality will be attained if $f$ takes the form $\xi / (1 - e^{i\theta})^2$ for some $\theta \in R$. The Bieberbach–Robertson–Milin conjectures concerning the Taylor coefficients of $f \in S$ are provided by this inequality in the form that is most comprehensive. For further information on the evidence supporting De Branges’ discovery, see [25–27]. In 2005, Kayumov [28] proved Brennan’s conjecture for conformal mappings by taking into consideration the logarithmic coefficients. We include a few works that have significantly advanced our understanding of logarithmic coefficients for various holomorphic univalent function subclasses.

As stated in the definition, it is easy to figure out that the logarithmic coefficients for $f \in S$ are calculated by:

$$\beta_1 = \frac{a_2}{2},$$ (6)

$$\beta_2 = \frac{1}{2} \left( a_3 - \frac{1}{2} a_2^2 \right),$$ (7)

$$\beta_3 = \frac{1}{2} \left( a_4 - a_2 a_3 - \frac{1}{2} a_3^2 \right),$$ (8)

$$\beta_4 = \frac{1}{2} \left( a_5 - a_2 a_4 + a_2 a_2 a_3 - \frac{1}{2} a_3^2 - \frac{1}{4} a_2^2 \right).$$ (9)

It is well-known that the function of the form (1) has an inverse $f^{-1}$, which is holomorphic in $|w| < 1/4$, as we know the Koebe one quarter theorem [36] verify that the image of $D$ under every univalent function $f \in A$ having a disk of radius $1/4$. If $f \in S$, then:

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \ldots, |w| < 1/4.$$ (10)

It was demonstrated by Lowner [37] that if $f \in S$ and its inverse is provided by (10), following that, the sharp estimate holds:

$$|A_n| \leq \frac{(2n)!}{n!(n + 1)!},$$ (11)

It has been demonstrated that the inverse of the Koebe function $k(\xi) = \xi / (1 - \xi)^2$ provides the best bounds for all $|A_n|$ $(n = 2, 3, \ldots)$ in (11) over all members of $S$. Determining the behavior of the inverse coefficients of $f$ given in (10) when the corresponding
function $f$ is restricted to certain suitable geometric subclasses of $S$ has attracted a lot of attention. Many authors have provided different proofs of the inequality (11), but Yang [38] provided a straightforward proof.

Since $f(f^{-1}(w)) = w$, using (10) it is obvious to observe that:

$$A_2 = -a_2,$$

$$A_3 = 2a_2^2 - a_3.$$  \hspace{1cm} (12)

Motivated by the abovementioned work, we now define new family of functions. Furthermore, for functions $H_3(1)$ that belong to the newly defined family of functions, we are searching for sharp upper bounds for the coefficients $a_j$ ($j = 2, 3, 5, 6$) and functionals $H_2(2)$ and $H_3(1)$.

**Definition 1.** An analytic function $f$ given in (1) belongs to the functions class $R_{C_0}$, if the following subordination condition satisfied:

$$f'(\xi) + \xi f''(\xi) \prec \Psi(\xi),$$  \hspace{1cm} (13)

where $\Psi(\xi)$ is given in (2).

### 3. A Set of Lemmas

For the primary findings, the following lemmas are required.

**Lemma 1** ([39]). Let $f(\xi) = 1 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \cdots \in P$ ($p_1 \geq 0$), then:

$$2p_2 = p_1^2 + x(4-p_1^2),$$  \hspace{1cm} (14)

$$4p_3 = p_1^3 + 2(4-p_1^2)p_1x - p_1(4-p_1^2)x^2 + 2(4-p_1^2)1 - |x|2y$$  \hspace{1cm} (15)

for some $x, y \in C$ with $|x| \leq 1$ and $|y| \leq 1$.

**Lemma 2.** Let $f(\xi) = 1 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \cdots \in P$ ($p_1 \geq 0$), then:

$$|p_n| \leq 2 \quad (n \geq 1),$$  \hspace{1cm} (16)

and if $Q \in [0,1]$ and $Q(2Q - 1) \leq R \leq Q$, then:

$$|p_3 - 2Qp_1p_2 + Rp_3^4| \leq 2.$$  \hspace{1cm} (17)

Also:

$$|p_{n+k} - \mu p_np_k| \leq 2\max\{1, |2\mu - 1|\}$$

$$= 2\left\{ \begin{array}{ll}
1 & \text{for } 0 \leq \mu \leq 1 \\
|2\mu - 1| & \text{otherwise}
\end{array} \right\}.$$  \hspace{1cm} (18)

The inequalities (16), (17) and (18) are taken from [39–41], respectively.

**Lemma 3** ([42]). Assume that $\tau, \psi, \rho$ and $\xi$ obey the inequalities $0 < \tau < 1, 0 < \xi < 1$ and:

$$8\xi(1-\xi)\left[(\tau\psi - 2\rho)^2 + (\tau(\xi + \tau) - \psi)^2\right] + \tau(1-\xi)(\psi - 2\xi)^2 \leq 4\xi^2(1-\tau)^2(1-\xi).$$  \hspace{1cm} (19)

If $p(\xi) = f(\xi) = \xi + a_1\xi + a_2\xi^2 + a_3\xi^3 + \cdots \in P$ ($\xi_1 \geq 0$), then:

$$\left|\rho p_4^4 + \xi p_2^4 + 2\tau p_1 p_3 - \frac{3}{2} \psi p_1^2 p_2 - p_4\right| \leq 2.$$ \hspace{1cm} (20)
Let $f$ be an analytic function of the form (1) belongs to $\mathcal{Y}$ where:

$w$ in $H$.

Consider in (30)–(33).

Moreover, if $\alpha \delta < 0$, then:

$$
Y(\alpha, \beta, \delta) = \begin{cases} 
1 + |\beta| + \frac{\beta^2}{4(1-|\beta|)} & (-4 \alpha \delta (\delta^{-2} - 1) \leq \beta^2; |\beta| \geq 2(1 - |\delta|)) \\
1 + |\beta| + \frac{\beta^2}{4(1-|\beta|)} & (\beta^2 < \min\{4(1 - |\delta|); -4 \alpha \delta (\delta^{-2} - 1)\}) \\
\mathcal{R}(\alpha, \beta, \delta) & \text{otherwise}
\end{cases}
$$

where:

$$
Y(\alpha, \beta, \delta) = \begin{cases} 
|\alpha| + |\beta| + |\delta| & (|\delta|(|\beta| + 4|\alpha|) \leq |\alpha \beta|) \\
-|\alpha| + |\beta| + |\delta| & (|\alpha \beta| \leq |\delta|(|\beta| - 4|\alpha|)) \\
(|\alpha| + |\delta|)\sqrt{1 - \frac{\beta^2}{4 \alpha \delta}} & \text{otherwise}
\end{cases}
$$

4. Main Results

Our first result is related to find bounds for the function $f$ to be in the functions of class $\mathcal{R}_G$.

**Theorem 1.** Let $f$ be an analytic function of the form (1) belongs to $\mathcal{R}_G$, then:

$$
|a_n| \leq \frac{1}{2n^2} \quad (n = 2, 3, 4, 5),
$$

$$
|a_6| \leq \frac{593}{7680}.
$$

Every estimate listed above, with the exception of that on $a_6$, are sharp for the functions given in (30)–(33).

**Proof.** Consider $f \in \mathcal{R}_G$. Then, the Schwarz function $w$ with $w(0) = 0$ and $|w(\xi)| < 1$ in $U$ such that:

$$
f'(\xi) + \sim f''(\xi) = \Psi(w(\xi)) = \frac{w(\xi)}{\ln(1 + w(\xi))}.
$$

Define the function $q$ by:

$$
q(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + p_1 \xi + p_2 \xi^2 + p_3 \xi^3 + \ldots,
$$

then $q \in \mathcal{P}$. This implies that:

$$
w(\xi) = \frac{q(\xi) - 1}{q(\xi) + 1} = \frac{p_1 \xi + p_2 \xi^2 + p_3 \xi^3 + \ldots}{2 + p_1 \xi + p_2 \xi^2 + p_3 \xi^3 + \ldots}
$$

in $H(U)$. Clearly $q$ is holomorphic in $H(U)$ with $q(0) = 1$ and has positive real part in $U$. By using (22) along with $\frac{w(\xi)}{\ln(1 + w(\xi))}$, it is clear that:

$$
\Psi(w(\xi)) = 1 + \frac{1}{2} p_1 \xi + \frac{1}{24} (12 p_2 - 7 p_1^2) \xi^2 + \frac{1}{120} (17 p_3^2 - 56 p_1 p_2 + 48 p_3^3) \xi^3 + \frac{1}{720} (-649 p_2^3 + 3060 p_1^2 p_2 - 3360 p_1 p_3 - 1680 p_2^2 + 2880 p_3^4) \xi^4 + \frac{1}{34560} (1739 p_3^3 - 10384 p_1^3 p_2 + 12240 p_2^2 p_3 + 12240 p_1^2 p_3 - 13440 p_2 p_3 + 13440 p_1 p_4 + 11520 p_3^2) \xi^5 + \ldots,
$$

(23)
since:
\[ f'(\xi) + \xi f''(\xi) = 1 + 4a_2\xi + 9a_3\xi^2 + 16a_4\xi^3 + 25a_5\xi^4 + \ldots, \tag{24} \]
it follows by (21), (22), and (24) that:
\[ a_2 = \frac{p_1}{16}, \tag{25} \]
\[ a_3 = \frac{1}{432} \left( 12p_2 - 7p_1^2 \right), \tag{26} \]
\[ a_4 = \frac{1}{3072} \left( 17p_1^3 - 56p_1p_2 + 48p_3 \right), \tag{27} \]
\[ a_5 = \frac{1}{288000} \left( -649p_1^4 + 3060p_1^2p_2 - 3360p_1p_3 - 1680p_2^2 + 2880p_4 \right), \tag{28} \]
\[ a_6 = \frac{1}{1638880} \left( 1739p_1^5 - 10384p_1^3p_2 + 12240p_1^2p_3 + 12240p_1p_2^2 - 13440p_1p_3 + 13440p_1p_4 + 11520p_5 \right). \tag{29} \]

Thus, using Lemma (2) we get:
\[ |a_2| \leq \frac{1}{8} \text{ (from 16)} \]
\[ |a_3| = \frac{1}{36} \left( p_2 - \frac{7}{12}p_1^2 \right) \leq \frac{1}{18}, \]
by rearranging (27) and (28) it gives:
\[ |a_4| = \frac{1}{64} \left| \left( p_3 - \frac{56}{48}p_1p_2 + \frac{17}{48}p_1^3 \right) \right| \leq \frac{1}{32}, \]
\[ |a_5| = \frac{1}{100} \left| \left( \frac{649}{288000}p_1^4 + \frac{7}{12}p_2^2 - \frac{17}{16}p_1^2p_2 + \frac{7}{6}p_1p_3 - p_4 \right) \right| \leq \frac{1}{50}. \]

We now find the estimates on $|a_6|$. Therefore, from the Lemma (3), $|p_n| \leq 2$ and (18), we find that:
\[ |a_6| = \frac{1}{36 \times 46080} \left| \left( 1739p_1^5 - 10384p_1^3p_2 + 12240p_1^2p_3 + 12240p_1p_2^2 - 13440p_1p_3 + 13440p_1p_4 + 11520p_5 \right) \right| \leq \frac{593}{7680}. \]

The initial five outcomes are precise for the functions: $f: U \to \mathbb{C}$ provided by:
\[ f_1(\xi) = \xi + \frac{1}{8}\xi^2 + \ldots \tag{30} \]
\[ f_2(\xi) = \xi + \frac{1}{18}\xi^3 + \ldots \tag{31} \]
\[ f_3(\xi) = \xi + \frac{1}{32}\xi^4 + \ldots \tag{32} \]
\[ f_4(\xi) = \xi + \frac{1}{50}\xi^5 + \ldots \tag{33} \]

This completes the proof. \(\square\)

**Conjecture 1.** Let $f(\xi) = \xi + a_1\xi + a_2\xi^2 + a_3\xi^3 + \ldots \in R_G^*$. It very easy to observe from the functions $f_i(\xi)$, $i = 1, 2, 3, 4$, that our initial four coefficients estimate are sharp. As seen from the structure of the extremal function $f_5(\xi)$, the bound for $|a_6|$ is expected to be extremal.
Theorem 2. Let \( f \) be given in (1) belongs to \( R_G \). Then, the following sharp estimates holds:

\[
|a_3 - \delta a_2^2| \leq \frac{1}{18} \max \left\{ 1, \frac{1}{18} |27 \delta + 12| \right\}, \quad \delta \in \mathbb{C}.
\]

(34)

\[
|a_2 a_3 - a_4| \leq \frac{1}{32}
\]

(35)

\[
|a_2 a_4 - a_3^2| \leq \frac{1}{324}.
\]

(36)

Results (34) and (36) are sharp for the function given in (31) and the result (35) is sharp for the function (32).

Proof. From (25) and (26), we have:

\[
|a_3 - \delta a_2^2| = \frac{p_2}{36} - \frac{7}{432} p_2^2 - \frac{\delta p_2^2}{256} = \frac{1}{36} \left( p_2 - \frac{(112 + 27 \delta) p_1^2}{192} \right).
\]

If we take \( v = \frac{(112 + 27 \delta)}{192} \), we obtain that:

\[
|a_3 - \delta a_2^2| \leq \frac{1}{18} \max \left\{ 1, \frac{1}{18} |27 \delta + 12| \right\}.
\]

Similarly from (25), (26) and (27), we have:

\[
a_2 a_3 - a_4 = \frac{23}{1152} p_1 p_2 - \frac{181}{27648} p_3^3 - \frac{p_3}{64},
\]

and so from (17):

\[
|a_2 a_3 - a_4| = \frac{1}{64} \left| p_3 - \frac{23}{18} p_1 p_2 + \frac{181}{432} p_3^3 \right| \leq \frac{1}{32}.
\]

Now, from (25), (26), and (27) again, one can observe:

\[
a_2 a_4 - a_3^2 = \frac{2893 p_1^2}{11943936} + \frac{503 p_1^2}{497664} p_2^2 - \frac{p_1 p_3}{1024} + \frac{p_2^2}{1296} = \tau,
\]

using Lemma (1) and presuming that \( s = p_1 \in [0, 2] \), whereby:

\[
\tau = \frac{4 - s^2}{(3456)^2} \left[ \frac{8317 s^4}{4 - s^2} + 4812 x^2 + \left( 9216 - 612 s^2 \right) x^2 + 5832 s \left( 1 - |x|^2 \right) y \right].
\]

If \( s = 0 \), then \( \tau = \frac{1}{324} x^2 \). Thus, since \( |x| \leq 1 \), we have:

\[
|\tau| \leq \frac{1}{324}.
\]

If \( s = 2 \) then:

\[
|\tau| = 0.
\]

(37)

Assume that \( s \in (0, 2) \). Following that, we may write:

\[
|\tau| = \frac{4 - s^2}{(3456)^2} \left[ \frac{8317 s^4}{4 - s^2} + 4812 x^2 + \left( 9216 - 612 s^2 \right) x^2 + 5832 s \left( 1 - |x|^2 \right) y \right] \leq \frac{(4 - s^2) 5832 s}{(3456)^2} \left[ \frac{8317 s^3}{4 - s^2} \frac{4812}{5832} + \left( 9216 - 612 s^2 \right) x^2 + \left( 1 - |x|^2 \right) y \right] \leq \frac{(4 - s^2) 5832 s}{(3456)^2} \left[ |\alpha + \beta x + \delta x^2| + 1 - |x|^2 \right],
\]
where:
\[ \alpha = \frac{8317s^3}{(4 - s^2)5832}, \quad \beta = \frac{401s}{486}, \quad \delta = \frac{256 - 17s^2}{162s}. \]

It follows that \( \alpha \delta > 0 \). Furthermore, we easily see that:
\[
|\beta| - 2(1 - |\delta|) = \frac{503s^2 - 972s + 1536}{486s} > 0.
\]

Therefore, we have:
\[
\tau \leq s\left(\frac{4 - s^2}{2048}\right)(|\alpha| + |\beta| + |\delta|)
= \frac{s(4 - s^2)(2s^2 + 3)}{1944}
= \frac{-2s^4 - 5s^2 + 12}{3888}.
\]

Let \( t = s^2 (t \in (0, 4)) \). Subsequently, we examine the maximum of the function \( H_0 \) as described by:
\[
H_0(t) = \frac{-2t^2 - 5t + 12}{3888}.
\]

In that case, we have:
\[
|\tau| \leq H_0(t) \leq \frac{-2t^2 - 5t + 12}{3888} \leq \frac{1}{324}. \quad (38)
\]

\( \Box \)

**Theorem 3.** If \( f \) has the form (13), and belongs to \( R_G \). Then:
\[
|H_3(1)| \leq \frac{8441}{2332800}.
\]

**Proof.** Since by (3) we have:
\[
H_3(1) := a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).
\]

By using Theorems (1) and (2), we have:
\[
|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2| \leq \frac{8441}{2332800}.
\]

This completes the proof. \( \square \)

**Theorem 4.** If \( f \) has the form (13), and belongs to \( R_G \). Then
\[
|\beta_n| \leq \frac{1}{4n^2} \quad (n = 2, 3, 4, 5) \quad (39)
\]

These outcomes are extreme for functions \( f_i(\xi) \), \( i = 2, 3, 4, 5 \) given by (30)–(33).

**Proof.** Let \( f \in H_G^* \). Then, putting the value of (25)–(28), in (7)–(10), we get:
\[
\beta_1 = \frac{p_1}{32}, \quad (40)
\]
\[
\beta_2 = \frac{1}{72}\left(p_2 - \frac{251}{384}p_1^2\right), \quad (41)
\]
\[ \beta_3 = \frac{1}{128} \left( p_3 - \frac{23}{18} p_1 p_2 + \frac{1}{192} p_1^3 \right), \quad (42) \]

\[ \beta_4 = \frac{1}{200} \left( \frac{649}{2880} p_1^4 - \frac{19}{32} p_1^2 p_2 + \frac{13}{8} p_1 p_3 + \frac{21}{64} p_2^2 - p_4 \right). \quad (43) \]

For \( \beta_1 \), using (16) in (40), we obtain:

\[ |\beta_1| \leq \frac{1}{16}. \]

For \( \beta_2 \), putting (18) in (41), we get:

\[ |\beta_2| \leq \frac{1}{36}. \]

For \( \beta_3 \), we can rewrite (42) as:

\[ \beta_3 = \frac{1}{128} \left( p_3 - \frac{23}{18} p_1 p_2 + \frac{1}{192} p_1^3 \right). \]

For \( \beta_4 \), we can rewrite (43) as:

\[ \beta_4 = \frac{1}{200} \left( \frac{649}{2880} p_1^4 + \frac{7}{6} p_1 p_3 + \frac{7}{12} p_2^2 - \frac{17}{16} p_1^2 p_2 - p_4 \right). \quad (44) \]

On comparing (44) with (20), we get \( \rho = \frac{649}{2880}, \xi = \frac{7}{12}, \tau = \frac{7}{12} \) and \( \psi = \frac{17}{24} \). It follows that:

\[ 8\xi(1 - \xi) \left[ (\tau\psi - 2\rho)^2 + (\tau(\xi + \tau) - \psi)^2 \right] + \tau(1 - \tau)(\psi - 2\xi\tau)^2 = 0.00111, \]

and

\[ 4\xi\tau^2(1 - \tau)^2(1 - \xi) = 0.0120. \]

Using (19) we deduce that:

\[ |\beta_4| \leq \frac{1}{100}. \]

\[ \square \]

**Theorem 5.** An analytic function \( f \) of the form (1) belongs to \( \mathcal{R}_G \) and has the series representation \( f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots \) then:

\[ |A_2| \leq \frac{1}{8}, \quad (45) \]

\[ |A_3| \leq \frac{1}{18}. \quad (46) \]

These bounds are sharp for the functions given in (30) and (31).

**Proof.** If \( f \in \mathcal{R}_G \) and \( f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots \). When we write Equations (25)–(27) in (12), we get:

\[ A_2 = -\frac{p_1}{16}, \]

\[ A_3 = \frac{83p_1^2}{3456} = \frac{p_2}{36}. \]
Because of $|p_1| \leq 2$, the outcome for $|A_2| \leq \frac{1}{8}$ is trivial. We now bound $|A_3|$ by using (18) in equality (46). Thus, we have:

$$|A_3| \leq \frac{1}{36} \left( p_2 - \frac{83p_2^2}{96} \right) \leq \frac{1}{18}.$$ 

5. Conclusions

In this study, we defined the class $\mathcal{R}_G$ of bounded turning functions connected with Grogory coefficients and logarithmic functions by using the technique of subordination. For the functions $f \in \mathcal{R}_G$, sharp results, such as initial coefficient bounds, the Fekete–Szegö functional, and second- and third-order Hankel determinants.

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