Skew Cyclic and Skew Constacyclic Codes over a Mixed Alphabet

Karthick Gowdhaman 1, Cruz Mohan 2,*, Chinnapillai Durairajan 3, Selda Çalkavur 4 and Patrick Solé 5

Abstract: In this note, we study skew cyclic and skew constacyclic codes over the mixed alphabet $\mathcal{R} = \mathbb{F}_q R_1 R_2$, where $q = p^m$, $p$ is an odd prime with $m$ odd and $R_1 = \mathbb{F}_q + u\mathbb{F}_q$ with $u^2 = u$, and $R_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q$ with $u^2 = u, v^2 = v, uv = vu = 0$. Such codes consist of the juxtaposition of three codes of the same size over $\mathbb{F}_q, R_1$, and $R_2$, respectively. We investigate the generator polynomial for skew cyclic codes over $\mathcal{R}$. Furthermore, we discuss the structural properties of the skew cyclic and skew constacyclic codes over $\mathcal{R}$. We also study their $q$-ary images under suitable Gray maps.

Keywords: linear codes; skew-cyclic codes; Gray map; skew constacyclic codes

MSC: 94B05; 94B15; 94B35; 94B60

1. Introduction

The most widely used family of linear codes consists of cyclic codes. Inspired by codes for the Lee metric [1], Berlekamp adapted them to constacyclic codes. Since then, as the following paragraphs demonstrate, they have happened in a number of circumstances.

Skew cyclic codes were first introduced as ideals in the skew polynomial ring $\mathbb{F}[x; \theta]$ in 2007 by Boucher et al. [2], where $\theta$ represents an automorphism of the finite field $\mathbb{F}$. The tables of the most well-known codes were enhanced by the numerous numerical samples that this technique created. The fact that the factorization of the polynomial $x^q - 1$ is not unique gives skew polynomial rings an advantage over commutative polynomial rings. For a given length, these numerous factorizations produce a large number of additional codes. Boucher et al. further extended this technique to skew constacyclic codes in [3]. Siap et al. [4] investigated skew cyclic codes of any length in 2011 and produced maps using both classical and quasi-cyclic codes.

In 2012, Jitman et al. [5] studied skew constacyclic codes over finite chain rings and described the algebraic structure of Euclidean and Hermitian dual codes. Abualrub et al. [6] studied $\theta$-cyclic codes over the semilocal ring $\mathbb{F}_2 + u\mathbb{F}_2, v^2 = v$ with respect to Euclidean and Hermitian inner products.

These codes over semilocal rings were further studied in many contexts. The rings $\mathbb{F}_3 + v\mathbb{F}_3$ in [7], $\mathbb{F}_7 + v\mathbb{F}_7, v^2 = v$ in [8], and $\mathbb{F}_9 + u\mathbb{F}_9 + v\mathbb{F}_9, u^2 = u, v^2 = v, uv = vu = 0$ in [9] were utilized as alphabets for skew cyclic codes, for example. Dertli and Cengellenmis [10] and Yao et al. [11] also examined these codes over $\mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q, u^2 = u, v^2 = v, uv = vu$ in [12]. Skewed $(-1 + 2v)$-constacyclic codes were developed in 2017 by Gao et al.

After deriving the structure of skew constacyclic codes over the semilocal ring $\mathbb{F}_q + u\mathbb{F}_q, v^2 = v$.  

---

1 Department of Mathematics, Presidency University, Bengaluru 560064, Karnataka, India; karthygowtham@gmail.com
2 Department of Mathematics, Bishop Heber College, Bharathidasan University, Tiruchirapalli 620017, Tamil Nadu, India
3 Department of Mathematics, Bharathidasan University, Tiruchirapalli 620024, Tamil Nadu, India; cdura66@rediffmail.com
4 Department of Mathematics, Faculty of Arts and Science, Kocaeli University, Kocaeli 41001, Turkey; selda.calkavur@kocaeli.edu.tr
5 I2M, Aix Marseille University, CNRS, 13009 Marseille, France; patrick.sole@telecom-paris.fr

* Correspondence: cruzmohan@gmail.com
In [13] and [14], respectively, Islam and Prakash established the algebraic structure of skew constacyclic codes over \( \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q + u\mathbb{F}_q \), \( u^2 = u, v^2 = v, uv = vu \).

Using two non-trivial automorphisms, Bhardwaj and Raka [15] investigated the skew constacyclic codes over the ring \( \mathbb{F}_q[u, v] \{ f(u), g(v), uv - vu \} \) in 2019. Alternatively, \( \mathbb{Z}_2 \mathbb{Z}_4 \)-linear codes, or codes over the mixed alphabet \( \mathbb{Z}_2 \mathbb{Z}_4 \), where a subset of coordinates is binary and the complement is quaternary, were introduced by Borges et al. [16]. They have calculated their generator matrices and described their dual codes. Fernandez-Cordoba et al. [17] obtained the rank and kernel of \( \mathbb{Z}_2 \mathbb{Z}_4 \)-linear codes in a follow-up experiment. Steganography is one field in which these codes have found industrial use [18].

In [19], additive codes over the mixed alphabet \( \mathbb{Z}_2 \mathbb{Z}_2^q \) were examined. Next, Refs. [20–23] examined the mixed alphabet \( \mathbb{Z}_p \mathbb{Z}_p^r \) and, more broadly, \( \mathbb{Z}_p \mathbb{Z}_p^r \). Conversely, Abualrub et al. [24] defined \( \mathbb{Z}_2 \mathbb{Z}_4 \) in 2014, in line with the advancement of cyclic codes on mixed alphabets. The code for -additive cyclics is \( \mathbb{Z}_4[x] \)-submodule of \( \mathbb{Z}_2[x]/(x^4 - 1) \times \mathbb{Z}_4[x]/(x^2 - 1) \), from which the smallest spanning set and unique set of generators for these codes, where \( s \) is an odd integer, were obtained.

Furthermore, generator polynomials and duals for \( \mathbb{Z}_2 \mathbb{Z}_4 \)-additive cyclic codes were discovered by Borges et al. [25]. In [26], Aydogdu et al. [27] introduced the novel mixed alphabets \( \mathbb{Z}_2 \mathbb{Z}_2[u] \)-additive codes, where \( u^2 = 0 \). They also studied constacyclic codes over mixed alphabets by defining them as \( \mathbb{Z}_2[u][x] \times \mathbb{Z}_2[u][x]/(x^k - (1 + u)) \), -submodules of \( \mathbb{Z}_2[x]/(x^4 - 1) \).

As the Gray images of \( \mathbb{Z}_2 \mathbb{Z}_2[u] \)-cyclic codes, they were able to derive several optimum binary linear codes. In the meanwhile, \( \mathbb{Z}_2 \mathbb{Z}_2[u] \)-additive cyclic and constacyclic codes with the unit 1 + \( u \), respectively, were explored algebraically by [28]. Consequently, the predicted generalization in the continuation of these research should be \( \mathbb{Z}_2^r \mathbb{Z}_2[u] \)-additive cyclic codes, \( u^2 = 0 \), and constacyclic codes.

In this article, we examine a mixed alphabet \( \mathcal{R} = \mathbb{F}_q \mathcal{R}_1 \mathcal{R}_2 \), where \( \mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q \) with \( u^2 = u \) and \( \mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q \) with \( u^2 = u, v^2 = v, uv = vu = 0 \). Moreover, we examine the cyclic codes \( \theta \) and \( (\theta, a) \) over \( \mathcal{R} \). The algebraic structure of these codes is fully determined. We examine their \( q \)-ary representations under Gray maps and provide a few brief numerical instances.

The contents are arranged as follows. The next section gathers some background information. Gray maps are examined in Section 3. Skewed cyclic codes are covered in Section 4, and skew constacyclic codes are covered in Section 5. The essay is concluded at Section 6.

2. Preliminaries

Let \( p \) be an odd prime, and let \( q = p^m \) with \( m \) being odd. Denote by \( \mathbb{F}_q \) the finite field of size \( q \). The set \( \mathbb{F}_q^n \) of all ordered \( n \)-tuples over \( \mathbb{F}_q \) is equipped with the structure of an \( \mathbb{F}_q \) vector space by the usual addition and scalar multiplication of vectors.

A code of length \( n \) over \( \mathbb{F}_q \) is just any non-empty subset \( C \) of \( \mathbb{F}_q^n \). It is said to be linear if \( C \) is an \( \mathbb{F}_q \) subspace of \( \mathbb{F}_q^n \). From now on, we write \( \mathcal{R}_1 = \mathbb{F}_q + u\mathbb{F}_q \), with \( u^2 = u \) and \( \mathcal{R}_2 = \mathbb{F}_q + u\mathbb{F}_q + v\mathbb{F}_q \), with \( u^2 = u, v^2 = v, uv = vu = 0 \).

Note that \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are finite non-chain rings. Let \( a + ub + uc \) be an element of \( \mathcal{R}_2 \). Then, we define two maps \( \eta \) and \( \delta \) as follows:

\[
\eta : \mathcal{R}_2 \to \mathbb{F}_q, \quad \delta : \mathcal{R}_2 \to \mathcal{R}_1,
\]

\[
\eta(a + ub + uc) = a, \quad \delta(a + ub + uc) = a + ub,
\]

It is clear that \( \eta \) and \( \delta \) are ring homomorphisms. We consider the ring \( \mathcal{R} \):

\[
\mathcal{R} = \mathbb{F}_q \mathcal{R}_1 \mathcal{R}_2 = \{(x, y, z) \mid x \in \mathbb{F}_q, y \in \mathcal{R}_1 \text{ and } z \in \mathcal{R}_2\}
\]
We define a $\mathcal{R}_2$-multiplication in this ring as follows:

$$\ast : \mathcal{R}_2 \times \mathcal{R} \to \mathcal{R}$$

$$r \ast (x, y, z) = (\eta(r)x, \delta(r)y, rz)$$

This is a well defined multiplication and it can be extended componentwise to $\mathcal{R}_0 = F_q^n_1 \times R_{\mathcal{I}_2}^{n_2} \times \mathcal{R}_0^{n_3}$ by:

$$\ast : \mathcal{R}_2 \times \mathcal{R}_0 \to \mathcal{R}_0$$

$$r \ast (x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}, z_1, \cdots, z_{n_3}) = (\eta(r)x_1, \cdots, \eta(r)x_{n_1}, \delta(r)y_1, \cdots, \delta(r)y_{n_2}, rz_1, \cdots, rz_{n_3})$$

where $(x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}, z_1, \cdots, z_{n_3}) \in \mathcal{R}_\gamma$. Equipped with this multiplication, $\mathcal{R}_0$ becomes an $\mathcal{R}_2$ module. A non-empty subset $C$ of $\mathcal{R}_0$ is said to be a $\mathcal{R}$-linear code of length $(n_1, n_2, n_3)$ if $C$ is an $\mathcal{R}_2$-submodule of $\mathcal{R}_0$. Now we define the inner product by the formula:

$$\langle c, c' \rangle = \sum_{i=1}^{n_1} x_i x'_i + \sum_{j=1}^{n_2} y_j y'_j + \sum_{k=1}^{n_3} z_k z'_k,$$

where $c = (x_1, \cdots, x_{n_1}, y_1, \cdots, y_{n_2}, z_1, \cdots, z_{n_3})$, $c' = (x'_1, \cdots, x'_{n_1}, y'_1, \cdots, y'_{n_2}, z'_1, \cdots, z'_{n_3})$ are in $\mathcal{R}_\gamma$. Let $C$ be an $\mathcal{R}$-linear code of length $(n_1, n_2, n_3)$. Then, the dual code of $C$ is defined as:

$$C^\perp = \{ c' \in \mathcal{R}_\gamma | \langle c, c' \rangle = 0 \ \forall \ c \in C \}$$

3. Decomposition and Properties of Gray Maps

Recall that, $\mathcal{R}_1 = F_q + uF_q$, with $u^2 = u$. Consider the idempotent orthogonal elements $e_1 = u$ and $e_2 = 1 - u$. Then, we have the decomposition:

$$\mathcal{R}_1 = e_1 \mathcal{R}_1 \oplus e_2 \mathcal{R}_1 \equiv e_1 F_q \oplus e_2 F_q,$$

where $e_1 e_2 = 0, e_1^2 = e_1, e_1 + e_2 = 1$. Hence, $\mathcal{R}_1 = \{ ae_1 + be_2 | a, b \in F_q \}$. We now define the Gray map:

$$\varphi_1 : \mathcal{R}_1 \to F_q^2$$

$$\varphi_1(ae_1 + be_2) = (a, b)$$

It can be extended to the length $n$ by:

$$\varphi_1 : \mathcal{R}_1^n \to F_q^{2^n}$$

$$\varphi_1((a_1, \cdots, a_n)e_1 + (b_1, \cdots, b_n)e_2) = (a_1, \cdots, a_n, b_1, \cdots, b_n)$$

Note that it is a linear map. We define the Gray weight of a codeword in $\mathcal{R}_1$ as:

$$wt_G(ae_1 + be_2) = wt_H(a, b)$$

where $wt_H$ denotes the Hamming weight. If $x, y$ lie in $\mathcal{R}_1^n$, then their mutual distance is given by:

$$d_G(x, y) = \sum_{i=1}^{n} wt_G(x_i - y_i) = \sum_{i=1}^{2^n} wt_H(\varphi_1(x) - \varphi_1(y)) = d_H(\varphi_1(x), \varphi_1(y)).$$

Hence, $\varphi_1$ is a weight preserving map. A non-empty subset $C$ of $\mathcal{R}_1^n$ is said to be a linear code of length $n$ if $C$ is $\mathcal{R}_2$-submodule of $\mathcal{R}_1^n$.

For $i \in \{1, 2\}$, $A_i \subseteq \mathcal{R}_1$:

$$A_1 \oplus A_2 = \{ a_1 + a_2 | a_i \in A_i \} \text{ and } A_1 \otimes A_2 = \{ (a_1, a_2) | a_i \in A_i \}.$$
Let $C_r$ be a linear code of length $n$ over $R_1$. Then, we define:

$$C_{r1} = \{ y_1 \in F_q^n | c_1 y_1 + c_2 y_2 \in C_r, \text{ for some } y_2 \in F_q^n \}$$

$$C_{r2} = \{ y_2 \in F_q^n | c_1 y_1 + c_2 y_2 \in C_r, \text{ for some } y_1 \in F_q^n \}$$

Therefore, any linear code $C_r$ over $R_1$ can be represented as $C_r = e_1 C_{r1} \oplus e_2 C_{r2}$ and $\varphi_1(C_r) = C_{r1} \oplus C_{r2}$. Hence, $C_{r1}$ and $C_{r2}$ are $F_q$-linear codes. Also note that $\varphi_1(C_r) = \varphi_1(C_r)'.

Recall that $R_2 = F_q + uF_q + vF_q$, with $u^2 = u, v^2 = v, uv = vu = 0$. Let $o_1 = (1 - u - v), o_2 = u, o_3 = v$ be idempotent orthogonal elements in $R_2$, then:

$$R_2 = o_1 R_2 \oplus o_2 R_2 \oplus o_3 R_2 \cong o_1 F_q \oplus o_2 F_q \oplus o_2 F_q,$$

where $o_i o_j = 0$ ($i \neq j$), $o_i^2 = o_i$, $o_1 + o_2 + o_3 = 1$. Hence, any element in $R_2$ can be written as $ao_1 + bo_2 + c o_3$. We now define a weight preserving linear Gray map $\varphi_2$:

$$\varphi_2 : R_2 \rightarrow F_q^3$$

$$\varphi_2(a o_1 + b o_2 + c o_3) = (a, b, c)$$

It can be extended to length $n$ by the formula:

$$\varphi_2((a_1, \ldots, a_n) o_1 + (b_1, \ldots, b_n) o_2 + (c_1, \ldots, c_n) o_3) = (a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n)$$

We define the Gray weight of a codeword in $R_2$ as:

$$wt_G(a o_1 + b o_2 + c o_3) = wt_H(a, b, c)$$

where $wt_H$ denotes the Hamming weight. If $x, y$ are in $R_2^n$, then their Gray distance is given by:

$$d_G(x, y) = \sum_{i=1}^{n} wt_G(x_i - y_i) = \sum_{i=1}^{3n} wt_H(\varphi_2(x) - \varphi_2(y)) = d_H(\varphi_2(x), \varphi_2(y))$$

For $i \in \{1, 2, 3\}, A_i \subseteq R_2$:

$$A_1 \oplus A_2 \oplus A_3 = \{ a_1 + a_2 + a_3 | a_i \in A_i \}$$ and $A_1 \otimes A_2 \otimes A_3 = \{ (a_1, a_2, a_3) | a_i \in A_i \}$.

Let $C_o$ be a linear code of length $n$ over $R_2$. We define the three codes:

$$C_{o1} = \{ z_1 \in F_q^n | o_1 z_1 + o_2 z_2 + o_3 z_3 \in C_o, \text{ for some } z_2, z_3 \in F_q^n \},$$

$$C_{o2} = \{ z_2 \in F_q^n | o_1 z_1 + o_2 z_2 + o_3 z_3 \in C_o, \text{ for some } z_1, z_3 \in F_q^n \},$$

$$C_{o3} = \{ z_3 \in F_q^n | o_1 z_1 + o_2 z_2 + o_3 z_3 \in C_o, \text{ for some } z_1, z_2 \in F_q^n \}.$$

Then, any linear code $C_o$ over $R_2$ can be represented as $C_o = o_1 C_{o1} \oplus o_2 C_{o2} \oplus o_3 C_{o3}$ and $\varphi_2(C_o) = C_{o1} \oplus C_{o2} \oplus C_{o3}$, where $C_{o1}, C_{o2},$ and $C_{o3}$ are $F_q$-linear codes. Also note that $\varphi_2(C_o) = \varphi_2(C_o)'$.

Henceforth, we define the Gray map $\varphi$ on $R$ using the maps defined previously:

$$\varphi : R \rightarrow F_q^6$$

$$\varphi(x, y, z) = (x, \varphi_1(y), \varphi_2(z))$$

now we can extend this map to $R_1$:

$$\varphi(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n) = (x_1, \ldots, x_n, \varphi_1(y_1), \ldots, \varphi_1(y_n), \varphi_2(z_1), \ldots, \varphi_2(z_n))$$
Theorem 2. The Gray map \( \phi \) defined above is linear and weight preserving.

**Proof.** Let \( x = (x_1, x_2, x_3), x' = (x'_1, x'_2, x'_3) \) be in \( \mathcal{R}_2 \) where \( x_1, x'_1 \in \mathbb{F}_q^3, x_2, x'_2 \in \mathbb{F}_q^{n_2}, x_3, x'_3 \in \mathbb{F}_q^{n_3} \). We have:

\[
\begin{align*}
\phi(x + x') &= \phi(x_1 + x'_1, x_2 + x'_2, x_3 + x'_3) \\
&= (x_1 + x'_1, \phi_1(x_2 + x'_2), \phi_2(x_3 + x'_3)) \\
&= (x_1, \phi_1(x_2), \phi_2(x_3)) + (x'_1, \phi_1(x'_2), \phi_2(x'_3)) \quad (\phi_1 \text{ and } \phi_2 \text{ are linear}) \\
&= \phi(x) + \phi(x')
\end{align*}
\]

Using the linear map \( \phi \),

\[
d_G(x, x') = \text{wt}_G(x - x') = \text{wt}_H(\phi(x) - \phi(x')) = d_H(\phi(x), \phi(x'))
\]

Hence, \( \phi \) is a weight preserving linear map. \( \square \)

The following theorem gives the parameters of the Gray image of a linear code.

**Theorem 2.** If \( C \subseteq \mathcal{R}_2 \) is an \( (n_1 + n_2 + n_3, d_C) \) linear code then \( \phi(C) \) is an \( (n_1 + 2(n_2) + 3(n_3), d_H) \) linear code over \( \mathbb{F}_q \), where \( d_G = d_H \).

**Proof.** The proof can extended from the proof of Theorem 1. \( \square \)

The following Theorem characterizes \( \phi(C) \):

**Theorem 3.** If \( C \subseteq \mathcal{R}_2 \) is linear, then

\[
\phi(C) = C_1 \otimes_{i=1}^{r_2} C_i \otimes_{j=1}^{r_3} C_{o_j}, \quad |C| = |C_1| |\prod_{i=1}^{r_2} |C_i| |\prod_{j=1}^{r_3} |C_{o_j}|.
\]

**Proof.** Let

\[
\phi(x, y, z) = (x, \phi_1(y), \phi_3(z)) = (a_1, \ldots, a_6) \in \phi(C) \subseteq \mathbb{F}_q^6.
\]
Note that \( \varphi \) is bijective and \( C = C_1 \otimes C_e \otimes C_o \) is linear. Thus, \( a_1 = x \in C_1 \). Also note that \( \varphi_1(y) = (a_2, a_3), \varphi_2(z) = (a_4, a_5, a_6) \). Since \( \varphi_1 \)'s are bijective, \( a_2 e_1 + a_3 e_2 \in e_1 C_1 + e_2 C_e = C_e \). Hence, \((a_2, a_3) \in C_1 \otimes C_e \) and similarly \((a_4, a_5, a_6) \in C_1 \otimes C_e \otimes C_o \). The converse holds in a similar way. The second part of the statement follows from the fact that \( \varphi \) is bijective. □

The following Theorem furnishes the decomposition of the dual of the linear code \( C \).

**Theorem 4.** If \( C = C_1 \otimes C_e \otimes C_o \) is a linear code over \( \mathcal{R} \) then \( C^\perp = C_1^\perp \otimes C_e^\perp \otimes C_o^\perp \), where \( C_1^\perp, C_e^\perp \) and \( C_o^\perp \) are duals for the respective linear codes.

**Proof.** Let \( C^\perp = \{ c' \in \mathcal{R}_e \mid (c, c') = 0 \text{ for all } c \in C \} = \{ (x', y', z') \in \mathcal{R}_e \mid x' \in \mathcal{R}_e^{n_1}, y' \in \mathcal{R}_e^{n_2}, z' \in \mathcal{R}_e^{n_3} \}. \) Let \( c = (x, y, z) \in C = C_1 \otimes C_e \otimes C_o \). Then:

\[
\langle c, c' \rangle = xx' + yy' + zz' = 0.
\]

Thus, \( x' \in C_1^\perp, y' \in C_e^\perp, z' \in C_o^\perp \) and so \( C^\perp \subseteq C_1^\perp \otimes C_e^\perp \otimes C_o^\perp \). Since \( |C^\perp| = |C_1^\perp| \cdot |C_e^\perp| \cdot |C_o^\perp| \), the statement holds. □

The next result shows that the Gray maps is compatible with duality.

**Theorem 5.** If \( C \subseteq \mathcal{R}_e \) is linear, then \( \varphi(C^\perp) = \varphi(C)^\perp \).

**Proof.** Let \( (x, y, z) \in C \) and \( (x', y', z') \in C^\perp \), where \( x \in C_1, y \in C_e, z \in C_o \) and \( x' \in C_1^\perp, y' \in C_e^\perp, z' \in C_o^\perp \), then \( \langle (x|y|z), (x'|y'|z') \rangle = 0 \). Using Theorem 4, \( C_1^\perp, C_e^\perp \) and \( C_o^\perp \) are duals for \( C_1, C_e, \) and \( C_o \). Now, we have \( \varphi(x, y, z) = (x, \varphi_1(y), \varphi_2(z)), \varphi(x', y', z') = (x', \varphi_1(y'), \varphi_2(z')) \), then the inner product is given by:

\[
\langle \varphi(x, y, z), \varphi(x', y', z') \rangle = \langle (x, \varphi_1(y), \varphi_2(z)), (x', \varphi_1(y'), \varphi_2(z')) \rangle = \langle (x, 0, 0), (x', 0, 0) \rangle + \langle (0, \varphi_1(y), 0), (0, \varphi_1(y'), 0) \rangle + \langle (0, 0, \varphi_2(z)), (0, 0, \varphi_2(z')) \rangle = 0.
\]

Thus, \( \varphi(C^\perp) \subseteq \varphi(C)^\perp \). Since the cardinality is the same on both sides, the statement holds. □

The following result provides the self duality nature of the linear code and its Gray image.

**Corollary 1.** If \( C \) is a linear \( \mathcal{R}_e \)-code, then \( C \) is self-dual if \( \varphi(C) \) is self-dual. Moreover, \( \varphi(C) \) is a self-orthogonal code over \( \mathbb{F}_q \) if \( C \) is self-orthogonal.

**Proof.** Let \( C \) be a self-dual linear code of length \( n \) over \( \mathcal{R} \). Thus, \( C = C^\perp \). Then, \( \varphi(C) = \varphi(C^\perp) \), and hence, by Theorem 5, we have \( \varphi(C) = (\varphi(C))^\perp \). Thus, \( \varphi(C) \) is a self-dual linear code of length \( n_1 + 2n_2 + 3n_3 \) over \( \mathbb{F}_q \). Conversely, let \( \varphi(C) \) be a self-dual linear code of length \( n_1 + 2n_2 + 3n_3 \) over \( \mathbb{F}_q \). Then, \( \varphi(C) = (\varphi(C))^\perp \), and hence, by Theorem 5, we have \( \varphi(C) = \varphi(C^\perp) \). Since \( \varphi \) is bijection, \( C = C^\perp \). Therefore, \( C \) is a self-dual linear code over \( \mathcal{R}_e \). Similarly, the self orthogonal case holds. □

### 4. Skew Cyclic \( \mathcal{R} \)-Codes

Let \( \theta \) be a non-trivial Frobenius automorphism defined by:

\[
\theta : \mathbb{F}_q \to \mathbb{F}_q, \theta(a) = a^{q^k},
\]
where \( t \) divides \( m \). It can be extended to \( R_1 \) and \( R_2 \) by:
\[
\theta_i(a + ub) = \theta_i(a) + u\theta_i(b), \theta_i(a + ub + vc) = \theta_i(a) + u\theta_i(b) + v\theta_i(c).
\]

Since \( t | m \), the order of automorphism \( \theta_i \) is \( \frac{m}{t} \). We define a polynomial ring \( R_i[x, \theta_i] \) (\( 1 \leq i \leq 2 \)) as follows:
\[
R_i[x, \theta_i] = \{ a_1 + \cdots + a_nx^n : a_j \in R, 1 \leq j \leq n \}
\]
Clearly, \( R_i[x, \theta_i] \) is a ring with respect to usual addition and the multiplication defined by:
\[
ax^mbx^n = a\theta_i^m(b)x^{m+n}
\]
Note that it is a non-commutative ring unless \( \theta_i \) is an identity map. A non-empty set \( \mathcal{C} \) is said to be a linear code of length \( n_i \) over \( R_i \) if it is a \( R_i \) submodule of \( R^n_i \). Using the above polynomial rings above, we extend the polynomial ring to \( R \) by:
\[
R[x, \theta_i] = \{ (a(x), b(x), c(x)) : a(x) \in \mathbb{F}_q[x], b(x) \in R_1[x], c(x) \in R_2[x] \}.
\]
It can be seen that \( R[x, \theta_i] \) is a \( R_2[x; \theta_i] \) submodule with respect to usual addition and multiplication defined by:
\[
\times : R_2[x] \times R[x, \theta_i] \rightarrow R[x, \theta_i]
\]
\[
(ax^i)(b_1x^j, b_2x^k, b_3x^l) = (\eta(a)x^ib_1x^j, \delta(a)x^ib_2x^k, ax^ib_3x^l) = (\eta(a)\theta_i^j(b_1)x^j, \delta(a)\theta_i^k(b_2)x^k, a\theta_i^l(b_3)x^{l+k})
\]
However, under associative and distributive laws, the multiplication can be extended to \( R_\gamma[x; \theta_i] = R_2[x; \theta_i] \times R[x, \theta_i] \) as follows:
\[
r(x) \times (f_1(x) + \langle x^{n_1} - 1 \rangle, f_2(x) + \langle x^{n_2} - 1 \rangle, f_3(x) + \langle x^{n_3} - 1 \rangle) = (\eta(r(x))f_1(x) + \langle x^{n_1} - 1 \rangle, \delta(r(x))f_2(x) + \langle x^{n_2} - 1 \rangle, \gamma(r(x))f_3(x) + \langle x^{n_3} - 1 \rangle).
\]

**Definition 1** ([2]). We say that an \( R \)-submodule \( \mathcal{C} \) of \( R^n \) is a \( \theta_i \)-cyclic code if for any \( c = (c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}, c_1 \in \mathcal{C} \) is \( (\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in \mathcal{C} \). The operator \( \sigma_1 \) is then said to be a \( \theta_i \)-cyclic shift operator on \( R^n \).

**Definition 2.** A non-trivial \( R_2 \)-submodule \( \mathcal{C} \) of \( R_\gamma \) is called a \( \theta_i \)-cyclic code if for any \( c = (c_0, c_1, \ldots, c_{n-1}, c_{n-1}, c_0, c_1, \ldots, c_{n-1}, 1, 0, \ldots, 0, c_{n-1}) \in \mathcal{C}, c_1 \in \mathcal{C} \) is \( \theta(c_{n-1}, 1), \theta(c_0, 1), \ldots, \theta(c_{n-2}, 1), \theta(c_{n-1}, 1), \theta(c_0, 1), \ldots, \theta(c_{n-2}, 1) \in C \). The operator \( \sigma \) is called a \( \theta_i \)-cyclic shift operator on \( R^n \).

The following result yields the relationship between the \( \theta_i \)-cyclic codes over \( R \) and \( \mathbb{F}_q \).

**Theorem 6.** Let \( \mathcal{C} = C_1 \otimes C_e \otimes C_o \subseteq \mathcal{R}_\gamma \) be linear. Then \( \mathcal{C} \) is a \( \theta_i \)-cyclic code if and only if \( C_1, C_e \) and \( C_o \) are \( \theta_i \)-cyclic codes of length \( n_1, n_2 \) and \( n_3 \) over \( \mathbb{F}_q, R_1 \) and \( R_2 \) respectively.

**Proof.** Let \( \mathcal{C} = C_1 \otimes C_e \otimes C_o \) be a \( \theta_i \)-cyclic code over \( R \). Let \( z = (z_1, z_2, z_o) \in \mathcal{C} \), that is:
\[
z = (z_{0,1}, z_{1,1}, \ldots, z_{0,n_1}, z_{1,n_1}, z_{0,n_2}, z_{1,n_2}, \ldots, z_{0,n_3}, z_{1,n_3}) \in \mathcal{C}.
\]
Then, \( \sigma(z) = (\theta(z_{n_1-1,1}), \theta(z_{0,1}), \ldots, \theta(z_{n_1-2,1}), \theta(z_{n_1-1,0}), \theta(z_{n_2-1,1}), \theta(z_{n_2-1,0}), \ldots, \theta(z_{n_3-2,1}), \theta(z_{n_3-1,0}), \theta(z_{0,0}), \ldots, \theta(z_{n_3-2,0})) = (\sigma(z_1), \sigma(z_e), \sigma(z_o)) \in \mathcal{C} \). From this, we can conclude that:
\[
\sigma(z_1) \in C_1, \sigma(z_e) \in C_e \text{ and } \sigma(z_o) \in C_o
\]
Hence, $C_1, C_e$, and $C_0$ are $\theta_1$-cyclic code of length $n_i$. The converse holds in a similar way. 

We recall the following Theorem from [9].

**Theorem 7** ([9]). Let $C_0 = \omega_1 C_{21} \oplus \omega_2 C_{22} \oplus \omega_3 C_{23}$ be a linear code over $R_2$ of length $n_3$, then $C_0$ is $\theta_1$-cyclic code iff $C_{o,i}(1 \leq i \leq 3)$ is a $\theta_1$-cyclic code of length $n_3$ over $F_q$.

The analogue of this result in our setting is as follows.

**Theorem 8** ([8]). Let $C_e = e_1 C_{e_1} \oplus e_2 C_{e_2}$ be a linear code over $R_1$ of length $n_2$ then $C_e$ is $\theta_1$-cyclic code iff $C_{e,i}(1 \leq i \leq 2)$ is a $\theta_1$-cyclic code of length $n_2$ over $F_q$.

**Theorem 9.** If $C = C_1 \otimes C_e \otimes C_0$ is a linear code of length $\gamma = n_1 + n_2 + n_3$, then $C$ is $\theta_1$-cyclic iff $C_1, C_e, C_0(1 \leq i \leq 2, l \leq j \leq 3)$ are $\theta_1$-cyclic code of length $n_1, n_2, n_3$ over $F_q$ respectively.

**Proof.** We obtain the proof on combining proofs of Theorems 6–8. 

These notions are well-behaved with respect to duality as the next result shows.

**Theorem 10.** If $C$ is a $\theta_1$-cyclic code of length $n$, then its dual $C^\perp$ is also a $\theta_1$-cyclic code.

**Proof.** From Theorem 9, $C_1, C_e, C_0 (1 \leq i \leq 2, 1 \leq j \leq 3)$ are $\theta_1$-cyclic codes over $F_q$. Then, $C_1^\perp, C_e^\perp, C_0^\perp (1 \leq i \leq 2, 1 \leq j \leq 3)$ are $\theta_1$-cyclic codes over $F_q$ from [29] and once again by using Theorem 9, $C^\perp$ becomes a $\theta_1$-cyclic code. 

Recall the following result from [4].

**Lemma 1** ([4]). Let $C$ be a $\theta_1$-cyclic code of length $n$ over $F_q$. Then, there exists a polynomial $f(x) \in F_q[x; \theta_1]$ such that $C = \{f(x)\}$ and $x^n - 1 = g(x)f(x)$ in $F_q[x; \theta_1]$.

By assuming $o(\theta_1) | n$, the counterpart follows.

**Theorem 11.** Let $C = C_1 \otimes C_e \otimes C_0$ be a $\theta_1$-cyclic code of length $n$ over $R$ and assume that the order of $\theta_1$ divides $n$. Then, $C = \langle B_1, B_2, B_3 \rangle$, where $B_1 = \langle (f_1(x), 0, 0) \rangle, B_3 = \langle (0, f_3(x), 0) \rangle$, and $B_2 = \langle (b_1(x), b_2(x), f_0(x)) \rangle$, such that $C_1 = \langle f_1(x) \rangle, C_e = \langle f_e(x) \rangle, C_0 = \langle f_0(x) \rangle, b_1(x) \in C_1$ and $b_2(x) \in C_e$.

**Proof.** Let $C = C_1 \otimes C_e \otimes C_0$ be a $\theta_1$-cyclic code of length $\gamma = n_1 + n_2 + n_3$ over $R$. Then, by Theorem 6, $C_1, C_e, C_0$ are $\theta_1$-cyclic codes of length $n_i$ over $F_q, R_1$ and $R_2$. Define a homomorphism from $C$ to $R$ as follows:

$$\psi : C \to R$$

$$(c_1(x), c_e(x), c_0(x)) = (0, 0, c_0(x))$$

Define:

$$\ker(\psi) = \{ (c_1(x), c_e(x), 0) : c_1(x) \in C_1, c_e(x) \in C_e \}$$

$$I = \{ (c_1(x), c_e(x)) \in F_q[x; \theta_1] \times R_1[x; \theta_1] : (c_1(x), c_e(x), 0) \in \ker(\psi) \}.$$ 

Clearly, $I = I_1 \times I_2$ forms a submodule of $F_q[x; \theta_1] \times R_1[x; \theta_1]$. Therefore, there exist a polynomial $f_1(x)$ and $f_e(x)$ in $F_q[x; \theta_1]$ and $R_1[x; \theta_1]$, respectively, generating $I_1$ and $I_2$ with $f_1(x)|x^{n_1} - 1$ and $f_e(x)|x^{n_2} - 1$. Thus, $I = \langle (f_1(x), 0), (0, f_e(x)) \rangle$, then for any $(c_1(x), c_e(x), 0) \in \ker(\psi)$, $(c_1(x), c_e(x)) = \nu(x) \ast (f_1(x), 0), (0, f_e(x))$ for some $\nu(x) \in$
Then, there exists a polynomial:

\[ \psi(x) = (f_1(x), f_2(x), f_3(x)) = (0, 0, f_0(x)). \]

From this, any \( \theta \)-cyclic code of length \( n \) can be represented by \( C = \langle (f_1(x), 0, 0), (f_2(x), 0), (b_1(x), b_2(x), f_3(x)) \rangle \), where \( f_1(x) | (x^{n_1} - 1), f_2(x) | (x^{n_2} - 1) \) and \( f_3(x) | (x^{n_3} - 1) \).

Furthermore, we have \( C \) is \( \theta \)-cyclic, then \( C_k \), where \( k \in \{1, e_1, e_2, o_1, o_2, o_3\} \) is skew \( \theta \)-cyclic code over \( \mathbb{F}_q \) with respective lengths. From Theorem 3, \( |C| = |C_1| \prod_{i=2}^{m} |C_{e_i}| \prod_{i=1}^{3} |C_{o_i}| \), since each \( C_k \) is \( \theta \)-cyclic it is generated by a polynomial \( f_k(x) \), and thus, \( |C| = q^{\gamma - \sum_{i=1}^{1} e_i} \), where \( \gamma = n_1 + 2(n_2) + 3(n_3) \). The following theorem provides the generator polynomials for \( \theta \)-cyclic codes over \( \mathbb{F}_q \).

**Theorem 12.** Let \( C = C_1 \otimes C_e \otimes C_o \) be a skew cyclic code over \( \mathcal{R} \) of length \( \gamma = n_1 + n_2 + n_3 \). Then, there exists a polynomial:

\[ f_1(x) \in \mathbb{F}_q[x; \theta_1] \text{ such that } C_1 = \langle f_1(x) \rangle \text{ and } x^{n_1} - 1 = g_1(x)f_1(x). \]

\[ f_2(x) \in \mathcal{R}_1[x; \theta_1] \text{ such that } C_e = \langle f_2(x) \rangle \text{ and } x^{n_2} - 1 = g_e(x)f_2(x) \text{ where } f_2(x) = \sum_{i=1}^{e_1} e_if_1(x). \]

\[ f_3(x) \in \mathcal{R}_2[x; \theta_1] \text{ such that } C_o = \langle f_3(x) \rangle \text{ and } x^{n_3} - 1 = g_o(x)f_3(x) \text{ where } f_3(x) = \sum_{i=1}^{o_1} o_if_1(x). \]

**Proof.** Let \( C \) be a \( \theta \)-cyclic code of length \( \gamma = n_1 + n_2 + n_3 \). From Theorem 6, we have that \( C_1, C_e, \text{ and } C_o \) are \( \theta \)-cyclic codes. Using Lemma 1, (i) follows.

Then, the proof of (ii) is as follows. Let \( C_e = e_1C_1 \oplus e_2C_2 \) be a \( \theta \)-cyclic code of length \( n_2 \) over \( \mathcal{R}_1 \). Theorem 7 says that, \( C_1 \oplus C_2 \) are \( \theta \)-cyclic codes of length \( n_2 \) over \( \mathbb{F}_q \). Lemma 1 says that we have \( C_1 = \langle f_1(x) \rangle \) and \( x^{n_2} - 1 = g_1(x)f_1(x) \) in \( \mathbb{F}_q[x; \theta_1] \) for \( i \in \{1, 2\} \). Then, \( e_1f_i(x) \in C \) for \( i \in \{1, 2\} \). Also, for any \( f_2(x) \in C \), we have \( f_3(x) = \sum_{i=1}^{e_1} e_ih_i(x)f_i(x), \) where \( h_i(x) \in \mathbb{F}_q[x; \theta_1] \) for \( i \in \{1, 2\} \). Thus, \( f_2(x) \in \langle e_1f_1(x), e_2f_2(x) \rangle \). Therefore, \( C = \langle e_1f_1(x), e_2f_2(x) \rangle \). As \( x^{n_2} - 1 = g_1(x)f_1(x) \) in \( \mathbb{F}_q[x; \theta_1] \) for \( i \in \{1, 2\} \). Let \( f_3(x) = e_1f_1(x) + e_2f_2(x) \in \mathcal{R}_1[x; \theta_1] \). Then, \( f_3(x) \in C \). On the other hand \( e_if_1(x) = e_if_1(x) \in \langle f_1(x) \rangle \) for \( i = 1, 2 \). Consequently, \( C = \langle f_1(x) \rangle \).

Furthermore, \( \sum_{i=1}^{e_1} e_ih_i(x)f_i(x) = \sum_{i=1}^{e_1} e_ih_i(x)f_i(x) = \sum_{i=1}^{o_1} o_1e_if_1(x) = x^{n_2} - 1 = x^{n_2} - 1 \). Then, \( x^{n_2} - 1 = g_1(x)f_1(x) \) in \( \mathcal{R}_1[x; \theta_1] \), where \( g_1(x) = \sum_{i=1}^{e_1} e_ih_i(x) \). Thus, (ii) follows. (iii) is similar to the proof of (ii).

**5. Skew Constacyclic Code over \( \mathcal{R} \)**

In this section, we study skew \( \theta \)-constacyclic codes over \( \mathcal{R} \). We choose a unit element \( a \in \mathcal{R} \) such that \( a \) satisfies the condition \( a^2 = 1, (a = 1, -1, \cdots) \).

**Definition 3.** Let \( a \in \mathbb{F}_p^\times \setminus \{0\} \). A linear code \( C \subseteq \mathcal{R}_1[x, \theta] \) is called skew \( \alpha = a \) + \( u \alpha_2 + v \alpha_3 \)-constacyclic code if it is invariant under the cyclic shift operator \( \lambda_\alpha \), which is whenever:

\[ C = \langle x_0, x_1, \cdots, x_{n_1-1}, y_0, y_1, \cdots, y_{n_2-1}, z_0, z_1, \cdots, z_{n_3-1} \rangle \subseteq C \]

\[ \lambda_\alpha(C) = \langle \alpha_1 \theta_i(x_{n_1-1}), \theta_i(x_0), \cdots, \theta_i(x_{n_2-1}), \theta_i(y_1), \cdots, \theta_i(y_{n_2-2}), \alpha_1 + u \alpha_2 + v \alpha_3 \rangle \subseteq C \]

The following two results translate symmetry conditions into algebraic constraints. We give the first result without proof.

**Theorem 13.** Let \( R_{n, \lambda} = R[x; \theta_1] / (x^n - \lambda) \). A linear code \( C \) of length \( n \) over \( R \) is \( (\theta_1, \lambda) \)-cyclic code if and only if \( C \) is a left \( R[x; \theta_1] \)-submodule of \( R_{n, \lambda} \).
The second result is less immediate.

**Theorem 14.** A code $C$ is skew $\alpha$-cyclic code over $R_2 = \frac{R_q[x,\theta]}{x^2-\alpha}$ if $C$ is a left $R_2[x,\theta]$ module over $R_2$.

**Proof.** Let $C$ be a skew $\alpha$-cyclic code. Then, by definition $x \ast (f(x)|g(x)|h(x)) \in C$:

$$x \ast (f(x)|g(x)|h(x)) = (\theta_i(f_0)x + \theta_i(f_1)x^2 + \cdots + \alpha_1\theta_i(f_{\alpha-1}),\theta_i(g_0)x + \theta_i(g_1)x^2 + \cdots + (\alpha_1 + \alpha_2)\theta_i(g_{\alpha-2}),$$

$$\theta_i(h_0)x + \theta_i(h_1)x^2 + \cdots + (\alpha_1 + \alpha_2 + \alpha_3)\theta_i(h_{\alpha-1})) \in C$$

Moreover, by using linearity of $C$:

$$r(x) \ast (g_1(x)|g_2(x)|g_3(x)) \in C$$

for some $r(x) \in R_2[x,\theta]$. Hence, $C$ is an left $R_2[x,\theta]$ submodule over $R_2$. Conversely, assume that $C$ is an left $R_2[x,\theta]$ submodule over $R_2$, then we have $x \ast (f(x)|g(x)|h(x)) \in C$ implies $C$ is skew $\alpha$-cyclic code. \hfill $\square$

**Theorem 15.** The code $C_o \subseteq \mathbb{R}_2^3$ is skew $\alpha = \alpha_1 + \alpha_2 + \alpha_3$-cyclic of length $n$ iff $C_{o_1}, C_{o_2}$, and $C_{o_3}$ are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$-cyclic codes over $\mathbb{F}_q$ of length $n$.

**Proof.** Let $C_o$ be a skew $\alpha$-cyclic code. Let $a = x_0 + y_0z + z_0z_3 \in C_o$, where $x = (x_0, x_1, \cdots, x_{n-1}) \in C_{o_1}, y = (y_0, y_1, \cdots, y_{n-1}) \in C_{o_2}$ and $x = (z_0, z_1, \cdots, z_{n-1}) \in C_{o_3}$.

Then, we have by definition, $\lambda_a(x) \in C_o$:

$$\lambda_a(x_0, x_1, \cdots, x_{n-1}) + o_2(y_0, y_1, \cdots, y_{n-1}) + o_3(z_0, z_1, \cdots, z_{n-1})$$

Hence, $C_{o_1}, C_{o_2}$, and $C_{o_3}$ are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$-cyclic codes over $\mathbb{F}_q$ of length $n$.

Conversely, assume that $C_{o_1}, C_{o_2}$, and $C_{o_3}$ are skew $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3$-cyclic codes over $\mathbb{F}_q$ of length $n$. Let $m_0, m_1, \cdots, m_{n-1}$ be an element in $C_o$, where $m_i = o_1x_0 + o_2y_0 + o_3z_0$ such that $x = (x_0, x_1, \cdots, x_{n-1}) \in C_{o_1}, y = (y_0, y_1, \cdots, y_{n-1}) \in C_{o_2}$ and $z = (z_0, z_1, \cdots, z_{n-1}) \in C_{o_3}$. Then we have $\lambda_{m_i}(x) \in C_{o_1}$, $\lambda_{m_i}(y) \in C_{o_2}$, and $\lambda_{m_i}(z) \in C_{o_3}$. So we get:

$$o_1\lambda_{m_1}(x) + o_2\lambda_{m_1+z_0}(y) + o_3\lambda_{m_1+z_0}(z) = o_1\lambda_{m_1}(x_0, x_1, \cdots, x_{n-1}) + o_2\lambda_{m_1+z_0}(y_0, y_1, \cdots, y_{n-1}) + o_3\lambda_{m_1+z_0}(z_0, z_1, \cdots, z_{n-1}) \in C$$

Hence, $C$ is skew $\alpha$-cyclic code over $\mathbb{R}_2^2$. \hfill $\square$

**Theorem 16.** $C_e$ be a skew $\alpha = \alpha_1 + \alpha_2$-cyclic code over $R_1$ iff $C_e$ is a skew $\alpha_1 + \alpha_2$ and $\alpha_1$-cyclic codes over $\mathbb{F}_q$.

**Proof.** The proof is similar to Theorem 15 taking mod $\nu$ to the above condition. \hfill $\square$

**Theorem 17.** $C$ be a skew $\alpha = \alpha_1 + \alpha_2 + \alpha_3$-cyclic code over $R$ of length $\gamma = n_1 + n_2 + n_3$ iff $C_1, C_2$, and $C_0$ are $\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3$-cyclic codes over $\mathbb{F}_q, R_1$, and $R_2$, respectively.
Proof. \( C_1, C_r \) and \( C_o \) be \( \alpha_1, \alpha_1 + u\alpha_2, \) and \( \alpha_1 + u\alpha_2 + \nu \alpha_3 \)-cyclic. Consider \( x = (x_0, x_1, \ldots, x_{n-1}), \) 
\( y = (y_0, y_1, \ldots, y_{n-1}) \) and \( z = (z_0, z_1, \ldots, z_{n-1}) \). Consider \( \alpha_1 + u\alpha_2 = \beta \). Then, we have:

\[(x,y,z) \in C \implies (\lambda_a(x), \lambda_b(y), \lambda_c(y)) \in C\]

Hence, \( C \) is skew \( \alpha \)-cyclic. The converse part holds similarly. \( \square \)

**Theorem 18.** \( C \) be a skew \( \alpha \)-cyclic code of length \( \gamma = n_1 + n_2 + n_3 \) iff \( C_1 \) is skew \( \alpha_1 \)-cyclic code of length \( n_1, C_r, \) and \( C_o \) are \( \alpha_1 + \alpha_2, \alpha_3 \)-cyclic codes of length \( n_2 \) and \( C_1, C_2, \) and \( C_3 \) are skew \( \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \nu \alpha_3 \)-cyclic codes over \( \mathbb{F}_q \) of length \( n_3 \).

**Proof.** Using Theorems 15–17 the result follows. \( \square \)

**Theorem 19.** \( C \) be a skew \( \alpha \)-cyclic code over \( R \) of length \( \gamma = n_1 + n_2 + n_3 \) iff \( C_1, C^e_r, \) and \( C^o_r \) are \( (\alpha_1)^{-1}, (\alpha_1 + u\alpha_2)^{-1}, \) and \( (\alpha_1 + u\alpha_2 + \nu \alpha_3)^{-1} \)-cyclic.

**Proof.** Let \( C \) be a skew \( \alpha \)-cyclic code, Lemma 3.1 [5] says that \( C^e_r \) is skew \( (\alpha_1 + u\alpha_2 + \nu \alpha_3)^{-1} \)-cyclic. From Theorem 17, we have \( C_1, C^e_r, \) and \( C^o_r \) are skew \( (\alpha_1)^{-1}, (\alpha_1 + u\alpha_2)^{-1} \) and \( (\alpha_1 + u\alpha_2 + \nu \alpha_3)^{-1} \)-cyclic. \( \square \)

**Corollary 2.** Let \( C = C_1 \otimes C_r \otimes C_o \) be a skew \( \alpha = \alpha_1 + u\alpha_2 + \nu \alpha_3 \)-cyclic code over \( R \) of length \( \gamma = n_1 + n_2 + n_3 \). Then, there exist polynomials:

(i) \( f_1(x) \in \mathbb{F}_q[x; \theta_1] \) such that \( C_1 = \langle f_1(x) \rangle \) and \( x^{n_1} - \alpha_1 = g_1(x) f_1(x) \).

(ii) \( f_2(x) \in \mathbb{R}_1[x; \theta_1] \) such that \( C_r = \langle f_2(x) \rangle \) and \( x^{n_2} - (\alpha_1 + u\alpha_2) = g_2(x) f_2(x) \).

(iii) \( f_3(x) \in \mathbb{R}_2[x; \theta_1] \) such that \( C_o = \langle f_3(x) \rangle \) and \( x^{n_3} - (\alpha_1 + u\alpha_2 + \nu \alpha_3) = g_3(x) f_3(x) \).

**Proof.** The proof is similar to the proof of Theorem 12. \( \square \)

**Theorem 20.** Let \( C = C_1 \otimes C_r \otimes C_o \) be a \( \theta_1 \)-constacyclic code of length \( \gamma \) over \( R \). Then \( C = \langle B_1, B_2, B_3 \rangle \), where \( B_1 = \langle (f_1(x), 0, 0) \rangle, B_1 = \langle (0, f_2(x), 0) \rangle, \) and \( B_1 = \langle (b_1(x), b_2(x), f_3(x)) \rangle \).

**Proof.** The proof is similar to the proof of Theorem 11. \( \square \)

**Example 1.** Let \( q = 9 \) and \( \mathbb{F}_q = \mathbb{F}_3[x] \) with \( z^2 + 1 = 0 \). Consider the ring \( R_\gamma = \mathbb{F}_q[x; \theta_1] \otimes \mathbb{R}_1[x; \theta_1] \otimes \mathbb{R}_2[x; \theta_1] \), where \( \theta_1 \) is the Frobenius automorphism defined by \( \theta_1(a) = a^3 \) for any \( a \in \mathbb{F}_q \).

Write:

\[x^4 - 1 = (x + 1)(x + 2)(x + z)(x + 2z) \in \mathbb{F}_q[x, \theta_1] \]
\[x^5 - 1 = (x + 2)(x^4 + x^3 + x^2 + x + 1) \in \mathbb{F}_q[x, \theta_1] \]
\[f_1(x) = \langle (x + 1) \rangle, f_2(x) = \langle (x + 2) \rangle, f_3(x) = \langle (x + z) \rangle \] By Theorem 12, we have that \( f_i \) divides \( x_3 \) for \( i = 1, 2, \) and \( 3 \), yielding a code with parameter \( [29, 18, 2] \) over \( \mathbb{F}_9 \).

**Example 2.** Let \( q = 25 \) and \( \mathbb{F}_{25} = \mathbb{F}_5[x] \) with \( z^2 + z + 1 = 0 \). Consider the ring \( R_\gamma = \mathbb{F}_{25}[x; \theta_2] \otimes \mathbb{R}_1[x; \theta_2] \otimes \mathbb{R}_2[x; \theta_2] \), where \( \theta_2 \) is the Frobenius automorphism defined by \( \theta_2(a) = a^5 \) for any \( a \in \mathbb{F}_{25} \).

Write:

\[x^4 - 1 = (x + 2)(x + 3)(x + z)(x + z + 1) \in \mathbb{F}_{25}[x, \theta_2] \]
\[x^5 - 1 = (x^2 + x + 1)(x^3 + x^2 + x + 1) \in \mathbb{F}_{25}[x, \theta_2] \]
\[f_1(x) = \langle (x + 2) \rangle, f_2(x) = \langle (x^2 - 1) \rangle, f_3(x) = \langle (x + z + 1) \rangle \] By Theorem 12, we have that \( f_i \) divides \( x_3 \) for \( i = 1, 2, \) and \( 3 \), yielding a code with parameter \( [28, 20, 2] \) over \( \mathbb{F}_{25} \).

6. Conclusions and Open Problems

In this note, we have studied the algebraic and metric structure of skew cyclic and skew constacyclic codes over a special mixed alphabet. Thus, our codes have a structure of
module over the largest of the three alphabets $R_2$. Codes over the product ring $F_q \times R_1 \times R_2$ would be modules over that larger ring. The two algebraic structures are different and should not be confused.

The present work leads itself to two paths of generalization: consider different mixed alphabets or replace the concepts of cyclicity by that of quasi-cyclicity. The former path seems easier than the latter, in view of the many examples of rings that have been used as alphabets of cyclic codes in recent years. On the other hand, the structure of quasi-cyclic codes is always more subtle than that of cyclic codes.

**Author Contributions:** Conceptualization, K.G., C.D., S.C., and P.S.; Methodology, C.M. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** No new data were created or analyzed in this study. Data sharing is not applicable to this article.

**Conflicts of Interest:** The authors declare no conflicts of interest.


Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.