

Article

Steady Solutions to Equations of Viscous Compressible Multifluids

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Abstract: For the differential equations of the barotropic dynamics of compressible viscous multifluids in a bounded three-dimensional domain with an immobile rigid boundary, a study of the solvability of the boundary value problem is made. Weak generalized solutions to the boundary value problem are shown to exist with weak constraints on the types of viscosity matrices and constitutive equations for pressure and momentum exchange.

Keywords: viscous compressible multifluid; viscosity matrix; barotropic flow; boundary value problem; weak generalized solution; existence theorem

MSC: 35D30



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1. Introduction

Since the breakthrough results of P.-L. Lions [1–4] and E. Feireisl and co-authors [5–9], a high-priority problem in mathematical hydrodynamics in recent decades has been constructing the theory of the global solvability of boundary and initial-boundary value problems for equations of the dynamics of compressible fluids and gases. In general, this theory has been significantly developed but only in the case of single-component media. However, an important branch of this theory concerning multicomponent media (multifluids) has remained much less developed (for more details on modeling the multifluids discussed in this article, cf. [10–12]). One of the first multidimensional solvability results is presented in the work of J. Frehse, S. Goj and J. Malek [13]. The work mentioned shows the solvability of the Cauchy problem for a system without convective terms in the case of the general relation between the pressures and densities of the constituents. The same authors in [14] obtained a uniqueness result for weak solutions to the Cauchy problem under the additional assumption that the mass forces are missing as well as terms that describe the exchange of momentum between different components. In the work of J. Frehse and W. Weigant [15], the existence and uniqueness are proved for a classical solution to the boundary value problem for a quasi-stationary system without convective terms and with special boundary conditions. The results on the existence of solutions with the consideration of convective terms were obtained by A. E. Mamontov and D. A. Prokudin in [16–19].

The results for similar models of multifluids are obtained in [20–27]. In more detail, these results are as follows. Reference [20] contains a global solvability result for the problem of motion of two incompressible capillary fluids in a container. In [21], the Cauchy problem for a system of equations governing the flow of an isothermal reactive mixture

of compressible gases is considered. The main contribution is to prove the sequential stability of weak solutions when the state equation essentially depends on the species concentration and the viscosity coefficients vanish under vacuum. Moreover, under the additional assumption of the “cold” pressure component in the regions of small density, the existence of weak solutions is proved for arbitrary large initial data. In [22], the global-in-time existence of weak solutions is proved for zero Mach number systems arising in fluid mechanics with periodic boundary conditions. In [23], a result is obtained for the global existence of so-called κ -entropy solutions to the Navier–Stokes equations for viscous compressible and barotropic fluids with degenerate viscosities. Reference [24] presents a study of systems of equations governing a steady flow of a polyatomic, heat-conducting reactive gas mixture. It is shown that the corresponding system of PDEs admits a weak solution and renormalized solution to the continuity equation provided the adiabatic exponent for the mixture is greater than $\frac{5}{3}$. In [25], a coupling between the compressible Navier–Stokes–Fourier system and the full Maxwell–Stefan equations is studied. This model describes the motion of a chemically reacting heat-conducting gaseous mixture. The viscosity coefficients are density-dependent functions vanishing in a vacuum, and the internal pressure depends on species concentrations. By several levels of approximation, the global-in-time existence is proved for weak solutions on the three-dimensional torus. Reference [26] discusses the Euler–Cahn–Hilliard system proposed by Lowengrub and Truskinovsky describing the motion of a binary mixture of compressible fluids. It is shown that the associated initial-value problem possesses infinitely many global-in-time weak solutions for any initial finite energy data. In [27], the solvability of the inhomogeneous boundary value problem for nonlinear equations of radiation heat transfer with Fresnel conjugation conditions on refractive index gap surfaces is analyzed. The nonlocal single-valued solvability of the boundary value problem is proved.

The results listed above, however, do not cover cases comprising the general type of rheological relations, which are the most common forms of viscosity matrices and constitutive equations for pressures and for exchange terms. This case will be considered in this paper.

The outline of the paper is as follows. Section 2 contains the differential equations of the dynamics of a compressible multicomponent medium with non-diagonal viscosity matrices, the mathematical formulation of the problem to be solved and the main result. In Section 3, the necessary auxiliary results are formulated and proved, namely, the formulation of the approximate problem, its solvability and further necessary properties of the resulting approximate solution. In Section 4, we derive uniform estimates of solutions to the approximate problem, on the basis of which we perform the passage to the limit with respect to the approximation parameter in Sections 5 and 6. The main result is formulated in Section 2 in the form of Theorem 1.

In the equations of multicomponent media dynamics, additional indices responsible for the numbering of the medium components appear. To avoid cluttering the text, it is worth employing the invariant notation. It excludes explicit mention of vector and tensor components, which we will adhere to throughout this paper and whose rules we will specify here in order to avoid discrepancies. Namely, if $\mathbf{u}$ and $\mathbf{v}$ are vectors (“columns”) of dimension n (with the components u_i, v_i), and $\mathbb{U}$ and $\mathbb{V}$ are tensors of the second rank (“matrices”) acting in $\mathbb{R}^n$ (with the entries U_{ij}, V_{ij}), then

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^n u_i v_i, \quad \mathbb{U} : \mathbb{V} = \sum_{i,j=1}^n U_{ij} V_{ij}, \quad \operatorname{div} \mathbf{u} = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i};$$

$\mathbb{U}\mathbf{u}$ and $\operatorname{div} \mathbb{U}$ are vectors (“columns”) with the components

$$(\mathbb{U}\mathbf{u})_i = \sum_{j=1}^n U_{ij} u_j, \quad (\operatorname{div} \mathbb{U})_j = \sum_{i=1}^n \frac{\partial U_{ij}}{\partial x_i};$$

and, finally, $\mathbb{U}^T$ and $\mathbf{u} \otimes \mathbf{v}$ are tensors with the entries $(\mathbb{U}^T)_{ij} = U_{ji}$ and $(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i v_j$, respectively. When applying these notations in the paper, in most cases, $n = 3$ (dimension of the flow), but sometimes, $n = N$ (number of medium components), $N \geq 2$.

This paper will use the generally accepted (see, e.g., [28,29]) notations of function spaces: namely, $L_p(\Omega)$ ($W_p^l(\Omega)$) is the space of functions integrable in the power of $p \geq 1$ (together with the generalized derivatives up to the order $l \geq 0$ inclusive); $C^l(\overline{\Omega})$ ($C_0^l(\Omega)$) is the space of functions having continuous partial derivatives up to the order $l \geq 0$ inclusive in $\overline{\Omega}$ (with compact supports in Ω); $C^\infty(\Omega)$ ($C^\infty(\overline{\Omega})$) is the space of functions which belong to all $C^l(\Omega)$ ($C^l(\overline{\Omega})$), $l = 0, 1, \dots$ ($C^0(\overline{\Omega}) = C(\overline{\Omega})$); $C_0^\infty(\Omega)$ is the space of functions from $C^\infty(\Omega)$ whose supports are compact subsets of Ω ; $W_p^l(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W_p^l(\Omega)$. We will not distinguish between the notations of the spaces of scalar-valued and vector-valued functions.

2. Statement of Boundary Value Problem and Formulation of Main Result

Let us formulate the mathematical problem considered in this paper. Let a multi-component medium (having N constituents, $N = \text{const} \geq 2$) occupy a bounded domain Ω of the Euclidean space $\mathbb{R}^3$ of points $\mathbf{x} = (x_1, x_2, x_3)$ with the boundary $\partial\Omega$ of class C^2 . The sought values are the following physical quantities described by $N + 1$ scalar and vector-valued functions ($3N + 1$ scalar functions, taking into account vector dimensions) defined in Ω : namely, the scalar total density field of the multicomponent medium $\rho(\mathbf{x}) \geq 0$ and the velocity vector fields $\mathbf{u}_i(\mathbf{x}) = (u_{i1}(\mathbf{x}), u_{i2}(\mathbf{x}), u_{i3}(\mathbf{x}))$ for each component with the number $i = 1, \dots, N$. To find these quantities, the continuity and momentum equations are to be solved

$$\text{div}(\rho \mathbf{v}) = 0, \quad (1)$$

$$\text{div}(\rho_i \mathbf{v} \otimes \mathbf{u}_i) + \alpha_i \nabla p = \text{div} \mathbb{S}_i + \mathbf{J}_i + \rho_i \mathbf{f}_i, \quad i = 1, \dots, N, \quad (2)$$

in which the following notations are used: $\mathbf{v} = \sum_{j=1}^N \alpha_j \mathbf{u}_j$ is the average velocity of the multicomponent medium, where $\alpha_j \in \mathbb{R}$, $\alpha_j \in (0, 1)$, $\alpha_1 + \dots + \alpha_N = 1$; $\rho_i = \alpha_i \rho$ are the densities of the constituents; p is the total pressure of the medium which is defined by the total density ρ , i.e., the function $p(\cdot)$ is supposed to be prescribed, provided that

$$p \in C^1[0, +\infty), \quad \frac{1}{c_1} s^{\gamma-1} - c_2 \leq p'(s) \leq c_1 s^{\gamma-1} + c_2 \quad \text{and} \quad p'(s) \geq 0 \quad \forall s \geq 0, \quad p(0) = 0, \quad (3)$$

with certain constants $\gamma > 3$, $c_1 \geq 1$ and $c_2 > 0$ (the simplest example of a situation when the formulated requirements for the pressure are fulfilled is the polytropic law $p(s) = cs^\gamma$ with a constant $c > 0$), and ∇p is the vector ("column") with the components $(\nabla p)_i = \frac{\partial p}{\partial x_i}$; $\mathbb{S}_i$ are the viscous stress tensors of the constituents which are de-

fined by the equalities $\mathbb{S}_i = \sum_{j=1}^N (2\mu_{ij} \mathbb{D}(\mathbf{u}_j) + \lambda_{ij} (\text{div} \mathbf{u}_j) \mathbb{I})$, where $\mathbb{I}$ is the identity tensor,

$\mathbb{D}(\mathbf{u}_j) = \frac{1}{2} ((\nabla \otimes \mathbf{u}_j)^T + (\nabla \otimes \mathbf{u}_j))$ is the rate of deformations tensor of the vector field $\mathbf{u}_j$, and the numerical viscosity coefficients λ_{ij} and μ_{ij} form the matrices $\mathbf{\Lambda} = \{\lambda_{ij}\}$ and $\mathbf{M} = \{\mu_{ij}\}$ such that

$$\mathbf{M} > 0, \quad \mathbf{\Lambda} + 2\mathbf{M} > 0; \quad (4)$$

the vector fields $J_i = \sum_{j=1}^N a_{ij}(\mathbf{u}_j - \mathbf{u}_i)(|\mathbf{u}_i - \mathbf{u}_j|^2 + 1)^\beta$ describe the intensity of the momentum exchange between the components, where the constants $a_{ij} > 0$, $a_{ij} = a_{ji}$, $\beta \in (0, 1]$; and finally, the known vector field of the external forces

$$\mathbf{f}_i \in C(\overline{\Omega}). \quad (5)$$

In (4), the expression $\mathbf{M} > 0$ means the positive definiteness of the matrix $\mathbf{M}$, i.e., for all $\boldsymbol{\xi} \in \mathbb{R}^N \setminus \{0\}$, the inequality $(\mathbf{M}\boldsymbol{\xi}, \boldsymbol{\xi}) > 0$ holds, and similarly for $\boldsymbol{\Lambda} + 2\mathbf{M}$.

Equations (1) and (2) need to be supplemented by boundary conditions for the velocity fields, e.g.,

$$\mathbf{u}_i|_{\partial\Omega} = \mathbf{0}, \quad (6)$$

i.e., the boundary $\partial\Omega$ of the domain Ω is assumed to be a non-permeable solid wall, and an additional density condition, which is standardly accepted as follows

$$\int_{\Omega} \rho \, dx = m, \quad (7)$$

where the positive constant m expresses the total mass of the multicomponent medium and is assumed to be known.

As a result, the subject of the research is formulated as the boundary value problems (1), (2), (6) and (7).

Remark 1. It follows from the conditions (3), in particular, that for all $s \geq 0$

$$\frac{1}{c_1\gamma}s^\gamma - c_2s \leq p(s) \leq \frac{c_1}{\gamma}s^\gamma + c_2s, \quad (8)$$

from which, in turn, it follows that

$$B_1s^\gamma - B_2 \leq s \int_1^s \frac{p(\eta)}{\eta^2} \, d\eta \leq B_3s^\gamma + B_4, \quad (9)$$

where the positive constants B_1 , B_2 , B_3 and B_4 depend only on c_1 , c_2 and γ . Further on, we will use $B_k(\cdot)$, $k \in \mathbb{N}$ to denote the quantities taking finite positive values and depending on the objects specified in brackets or listed in the comments.

Remark 2. In view of the equality (see (6))

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} \mathbb{S}_i : (\nabla \otimes \mathbf{u}_i) \, dx &= \sum_{i,j=1}^N \int_{\Omega} \mu_{ij} (\operatorname{curl} \mathbf{u}_i) \cdot (\operatorname{curl} \mathbf{u}_j) \, dx \\ &\quad + \sum_{i,j=1}^N \int_{\Omega} (\lambda_{ij} + 2\mu_{ij}) (\operatorname{div} \mathbf{u}_i) (\operatorname{div} \mathbf{u}_j) \, dx, \end{aligned} \quad (10)$$

in which $(\operatorname{curl} \mathbf{u}_i)_1 = \left(\frac{\partial u_{i3}}{\partial x_2} - \frac{\partial u_{i2}}{\partial x_3} \right)$, $(\operatorname{curl} \mathbf{u}_i)_2 = \left(\frac{\partial u_{i1}}{\partial x_3} - \frac{\partial u_{i3}}{\partial x_1} \right)$, $(\operatorname{curl} \mathbf{u}_i)_3 = \left(\frac{\partial u_{i2}}{\partial x_1} - \frac{\partial u_{i1}}{\partial x_2} \right)$, the conditions (4) provide an important relation

$$B_5(\boldsymbol{\Lambda}, \mathbf{M}) \sum_{i=1}^N \int_{\Omega} |\nabla \otimes \mathbf{u}_i|^2 \, dx \leq \sum_{i=1}^N \int_{\Omega} \mathbb{S}_i : (\nabla \otimes \mathbf{u}_i) \, dx. \quad (11)$$

The aim of this article is to construct a weak solution to problems (1), (2), (6) and (7), understood in the standard way, which is in line with the theory of one-component viscous compressible media. To be exact, let us formulate the following.

Definition 1. A weak solution to the problems (1), (2), (6) and (7) is a collection of functions

$$\rho \in L_{2\gamma}(\Omega), \quad \rho \geq 0, \quad \mathbf{u}_i \in \overset{\circ}{W}_2^1(\Omega),$$

which satisfy (7) and the following conditions:

(1) The density ρ satisfies the continuity Equation (1) in the sense that the integral identity

$$\int_{\Omega} \nabla \xi \cdot \rho \mathbf{v} \, dx = 0$$

is valid for all $\xi \in C^\infty(\overline{\Omega})$,

(2) The velocities $\mathbf{u}_i$ satisfy the momentum Equation (2) in the sense that the integral identities

$$\int_{\Omega} \left((\rho_i \mathbf{v} \otimes \mathbf{u}_i) : (\nabla \otimes \boldsymbol{\psi}_i) + \alpha_i p \operatorname{div} \boldsymbol{\psi}_i - \mathbb{S}_i : (\nabla \otimes \boldsymbol{\psi}_i) + \mathbf{J}_i \cdot \boldsymbol{\psi}_i + \rho_i \mathbf{f}_i \cdot \boldsymbol{\psi}_i \right) dx = 0$$

are valid for all $\boldsymbol{\psi}_i \in C_0^\infty(\Omega)$ (the boundary conditions (6) are immediately satisfied in the sense of the functional class $\overset{\circ}{W}_2^1(\Omega)$).

Remark 3. As it is well-known from the transport and Navier–Stokes theories (see, e.g., [30–32]), all weak solutions to the continuity equation in the sense of item (1) of Definition 1 immediately prove to be so-called renormalized solutions, i.e., they satisfy the renormalized Equation (1), formally obtained from (1) by multiplying by $R'(\rho)$ for all functions R of a certain class (namely, with sufficient smoothness and growth properties at zero and the infinity).

The main result of this paper is summarized below.

Theorem 1. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, $\partial\Omega \in C^2$, suppose that the pressure p satisfies the restrictions (3), the viscosity coefficients forming the matrices $\mathbf{\Lambda}$ and $\mathbf{M}$ satisfy (4), all other numerical coefficients $\alpha_i \in (0, 1)$, $\alpha_1 + \dots + \alpha_N = 1$, $a_{ij} > 0$, $a_{ij} = a_{ji}$, $\beta \in (0, 1]$, $m > 0$. Then, for all input data $\mathbf{f}_i$ of the class (5), the boundary value problems (1), (2), (6), and (7) possess at least one weak solution in the sense of Definition 1.

Proof of Theorem 1. This proof is the subject of the remainder of this paper. The proof follows two main steps. The first step is demonstrating the existence of a strong solution to the problem while simultaneously obtaining estimates uniform with respect to the regularization parameter, and this is the focus of the third and fourth sections. The second step consists of passing to the limit with respect to the regularization parameter on the basis of the estimates to be obtained at the first step, which takes place in the fifth and sixth sections. $\square$

3. Formulation of Approximate Boundary Value Problem and Proof of its Strong Solvability

In accordance with the above plan for proving Theorem 1, we will seek a weak solution to problems (1)–(7) as the limit of the approximate solutions, namely, the strong solutions to the following boundary value problem (we will omit the index ε for quantities depending on ε for now)

$$\operatorname{div}(\rho \mathbf{v}) = \varepsilon \Delta \rho - \varepsilon \rho + \varepsilon \frac{m}{|\Omega|}, \quad (12)$$

$$\frac{1}{2} \operatorname{div}(\rho_i \mathbf{v} \otimes \mathbf{u}_i) + \alpha_i \nabla p + \frac{1}{2} \rho_i (\mathbf{v} \cdot \nabla) \mathbf{u}_i + \frac{\varepsilon}{2} \rho_i \mathbf{u}_i + \frac{\varepsilon \alpha_i m}{2|\Omega|} \mathbf{u}_i = \operatorname{div} \mathbb{S}_i + J_i + \rho_i f_i, \quad (13)$$

$$\nabla \rho \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{u}_i|_{\partial\Omega} = \mathbf{0}, \quad (14)$$

where the parameter $\varepsilon \in (0, 1]$ is to be tending to zero later; $|\Omega|$ is the Lebesgue measure of Ω ; and $\mathbf{n}$ is the unit normal vector for $\partial\Omega$. In (12), the symbol Δ means, as usual, the Laplace operator $\Delta = \sum_{k=1}^3 \frac{\partial^2}{\partial x_k^2}$.

The problems (12)–(14) include an elliptic regularization of the boundary value problems (1), (2), (6) and (7) with additional terms and conditions, aimed at preserving the useful properties of the original problem, such as the integral orthogonality of the convective terms and velocities.

A strong solution to the approximate problems (12)–(14) is understood as follows.

Definition 2. A strong solution to the boundary value problem (12)–(14) is a non-negative function $\rho \in W_q^2(\Omega)$ with certain $q > 3$, commonly with the vector fields $\mathbf{u}_i \in W_q^2(\Omega)$, such that Equations (12) and (13) are satisfied a.e. in Ω , and the boundary conditions (14) are valid a.e. at $\partial\Omega$.

The existence of a strong solution to the boundary value problems (12)–(14) will be proved by using Leray–Schauder’s fixed point principle (see, e.g., [33], Theorem 11.3, p. 280), which we apply to the operator $\mathbf{T}$ formed hereafter.

3.1. Formation of Operator $\mathbf{T}$

First, we introduce several intermediate operators, the superposition of which will be the operator $\mathbf{T}$.

To accomplish this, consider the boundary value problem for the second-order elliptic equation

$$\operatorname{div}(\rho \mathbf{w}) = \varepsilon \Delta \rho - \varepsilon \rho + \varepsilon g, \quad \nabla \rho \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad (15)$$

with an unknown ρ and the given vector field $\mathbf{w}$ and function g .

As it is known (see e.g., [32], Proposition 4.29, p. 213 and [34], Lemma 3.1, p. 795), if $\mathbf{w} \in W_\infty^1(\Omega)$, $\mathbf{w}|_{\partial\Omega} = 0$, $g \in L_\infty(\Omega)$, then the following hold:

- (1) There exists a unique strong solution ρ to the problem (15) of the class $W_q^2(\Omega)$;
- (2) The estimate

$$\|\rho\|_{W_q^2(\Omega)} \leq B_6(\Omega, q, \varepsilon) \left(\|\mathbf{w}\|_{W_\infty^1(\Omega)} + 1 \right) \|g\|_{L_q(\Omega)} \quad (16)$$

is valid;

- (3) The equality $\int_\Omega g \, dx = \int_\Omega \rho \, dx$ holds;

- (4) If $g \geq 0$ a.e. in Ω , then $\rho \geq 0$ a. e. in Ω .

Thus, the first intermediate operator is defined as $\mathcal{P} : W_q^2(\Omega) \rightarrow W_q^2(\Omega)$, where $W_q^2(\Omega) = \{ \mathbf{w} \in W_q^2(\Omega) : \mathbf{w}|_{\partial\Omega} = 0 \}$, which acts as $\mathcal{P} : \mathbf{w} \mapsto \rho$, and such that $\forall \mathbf{w} \in W_q^2(\Omega)$, the function $\rho \in W_q^2(\Omega)$ is uniquely defined as the solution to the boundary value problem

$$\operatorname{div}(\rho \mathbf{w}) = \varepsilon \Delta \rho - \varepsilon \rho + \varepsilon \frac{m}{|\Omega|}, \quad \nabla \rho \cdot \mathbf{n}|_{\partial\Omega} = 0. \quad (17)$$

It is clear that $\rho = \mathcal{P}(w) \geq 0$ a.e. in Ω and $\int_{\Omega} \rho dx = m$ (note that the space $W_q^2(\Omega)$ is embedded into the space $C^1(\overline{\Omega})$). Moreover, for all $w_1, w_2 \in W_q^2(\Omega)$, the inequalities

$$\begin{aligned} \|\rho_1 - \rho_2\|_{W_q^2(\Omega)} &\leq B_7(B_6, \Omega, q, m, \varepsilon) (\|w_1\|_{C^1(\overline{\Omega})} + 1) (\|w_2\|_{C^1(\overline{\Omega})} + 1) \|w_1 - w_2\|_{C^1(\overline{\Omega})} \\ &\leq B_8(B_7, q, \Omega) (\|w_1\|_{C^1(\overline{\Omega})} + 1) (\|w_2\|_{C^1(\overline{\Omega})} + 1) \|w_1 - w_2\|_{W_q^2(\Omega)} \end{aligned} \quad (18)$$

hold, where $\rho_i = \mathcal{P}(w_i)$. Indeed, for $\rho_i = \mathcal{P}(w_i)$, the following equalities are valid:

$$\begin{aligned} -\varepsilon \Delta(\rho_1 - \rho_2) + \operatorname{div}((\rho_1 - \rho_2)w_1) + \varepsilon(\rho_1 - \rho_2) \\ = \varepsilon \left(-\frac{\operatorname{div}(\rho_2(w_1 - w_2))}{\varepsilon} \right), \quad \nabla(\rho_1 - \rho_2) \cdot \mathbf{n}|_{\partial\Omega} = 0. \end{aligned}$$

Using that, we directly obtain (18) from the embedding of $W_q^2(\Omega)$ into $C^1(\overline{\Omega})$ and from the estimate (16) with $g = -\frac{\operatorname{div}(\rho_2(w_1 - w_2))}{\varepsilon}$.

The second intermediate operator is defined as $\mathcal{E} : L_q(\Omega) \rightarrow W_q^2(\Omega)$, which acts as $\mathcal{E} : \mathbf{g} \mapsto \mathbf{u}$, where a given vector $\mathbf{g} = (g_1, \dots, g_N)$ with the vector-valued components $g_i \in L_q(\Omega)$ is mapped into a unique vector $\mathbf{u} = (u_1, \dots, u_N)$ with the vector-valued components $u_i \in W_q^2(\Omega)$, which is the solution to the boundary value problem for a strongly elliptic system of second-order equations

$$-\sum_{j=1}^N ((\lambda_{ij} + \mu_{ij}) \nabla \operatorname{div} u_j + \mu_{ij} \Delta u_j) = g_i, \quad u_i|_{\partial\Omega} = 0 \quad (19)$$

(note that $\operatorname{div} \mathbb{S}_i = \sum_{j=1}^N ((\lambda_{ij} + \mu_{ij}) \nabla \operatorname{div} u_j + \mu_{ij} \Delta u_j)$). The classic results for elliptic systems (see, e.g., [35–37]) imply that for all $\mathbf{g}_1, \mathbf{g}_2 \in L_q(\Omega)$, the inequality

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{W_q^2(\Omega)} \leq B_9(q, \Lambda, \mathbf{M}, \Omega) \|\mathbf{g}_1 - \mathbf{g}_2\|_{L_q(\Omega)} \quad (20)$$

is valid, where $\mathbf{u}_i = \mathcal{E}(\mathbf{g}_i)$.

Finally, according to the structure of Equation (13), the third set of intermediate operators $\mathcal{D}_i : W_q^2(\Omega) \rightarrow C(\overline{\Omega})$, $i = 1, \dots, N$ is defined as follows:

$$\begin{aligned} \mathcal{D}_i(\mathbf{w}) &= -\frac{1}{2} \operatorname{div} \left(\alpha_i \rho \left(\sum_{j=1}^N \alpha_j w_j \right) \otimes w_i \right) - \alpha_i \nabla p \\ &\quad - \frac{1}{2} \alpha_i \rho \left(\left(\sum_{j=1}^N \alpha_j w_j \right) \cdot \nabla \right) w_i - \frac{\varepsilon}{2} \alpha_i \rho w_i - \frac{\varepsilon \alpha_i m}{2|\Omega|} w_i \\ &\quad + \sum_{j=1}^N a_{ij} (w_j - w_i) (|w_i - w_j|^2 + 1)^\beta + \alpha_i \rho f_i, \end{aligned} \quad (21)$$

where a given vector $\mathbf{w} = (w_1, \dots, w_N)$ with the vector-valued components $w_i \in W_q^2(\Omega)$ is mapped into the value $\rho = \mathcal{P} \left(\sum_{j=1}^N \alpha_j w_j \right) \in W_q^2(\Omega)$. Due to the embedding of $W_q^2(\Omega)$ into $C^1(\overline{\Omega})$, the conditions $\mathbf{w} \in W_q^2(\Omega)$, $\rho \in W_q^2(\Omega)$ imply that $\mathcal{D}_i \in C(\overline{\Omega})$. Moreover, for all $\mathbf{w}_1 = (w_1^1, \dots, w_N^1)$, $\mathbf{w}_2 = (w_1^2, \dots, w_N^2) \in W_q^2(\Omega)$ and the corresponding $\rho_1 = \mathcal{P} \left(\sum_{j=1}^N \alpha_j w_j^1 \right)$, $\rho_2 = \mathcal{P} \left(\sum_{j=1}^N \alpha_j w_j^2 \right) \in W_q^2(\Omega)$, the estimates

$$\begin{aligned} \|\mathcal{D}_i(\mathbf{w}_1) - \mathcal{D}_i(\mathbf{w}_2)\|_{L_q(\Omega)} &\leq B_{10}(\Omega, q) \|\mathcal{D}_i(\mathbf{w}_1) - \mathcal{D}_i(\mathbf{w}_2)\|_{C(\overline{\Omega})} \leq B_{11} \|\rho_1 - \rho_2\|_{C^1(\overline{\Omega})} \\ &+ B_{12} \sum_{i=1}^N \|\mathbf{w}_i^1 - \mathbf{w}_i^2\|_{C^1(\overline{\Omega})} + B_{10} \|\rho_2\|_{C^1(\overline{\Omega})} \|p'(\rho_1) - p'(\rho_2)\|_{C(\overline{\Omega})}, \quad i = 1, \dots, N \end{aligned} \quad (22)$$

hold, where $B_{11} = B_{11}(B_{10}, \|\mathbf{w}_1\|_{C^1(\overline{\Omega})}, \|\rho_1\|_{C^1(\overline{\Omega})}, \{\|f_i\|_{C(\overline{\Omega})}\}, c_1, c_2, N, \{a_{ij}\}, \gamma)$, $B_{12} = B_{12}(B_{10}, \|\mathbf{w}_1\|_{C^1(\overline{\Omega})}, \|\mathbf{w}_2\|_{C^1(\overline{\Omega})}, \|\rho_2\|_{C^1(\overline{\Omega})}, \{\|f_i\|_{C(\overline{\Omega})}\}, m, N, \{a_{ij}\}, \beta, \Omega)$, and the dependence of B_{11} and B_{12} on $\|\rho_1\|_{C^1(\overline{\Omega})}, \|\rho_2\|_{C^1(\overline{\Omega})}, \|\mathbf{w}_1\|_{C^1(\overline{\Omega})}$ and $\|\mathbf{w}_2\|_{C^1(\overline{\Omega})}$ is locally bounded, i.e., the values $\sup B_{11}$ and $\sup B_{12}$ are bounded over every set of the type $\|\rho_1\|_{C^1(\overline{\Omega})} + \|\rho_2\|_{C^1(\overline{\Omega})} + \|\mathbf{w}_1\|_{C^1(\overline{\Omega})} + \|\mathbf{w}_2\|_{C^1(\overline{\Omega})} \leq \text{const}$.

As a result, we define the operator $\mathbf{T} : \mathcal{W}_q^2(\Omega) \rightarrow \mathcal{W}_q^2(\Omega)$, which acts as $\mathbf{T} = \mathcal{E} \circ (\mathcal{D}_1, \dots, \mathcal{D}_N)$, i.e., for all $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_N) \in \mathcal{W}_q^2(\Omega)$, we set

$$\mathbf{T}(\mathbf{w}) = \mathcal{E}(\mathcal{D}_1(\mathbf{w}), \dots, \mathcal{D}_N(\mathbf{w})). \quad (23)$$

Note that a fixed point $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_N)$ of the operator $\mathbf{T}$ (provided it exists), together with the corresponding function $\rho = \mathcal{P}\left(\sum_{j=1}^N \alpha_j \mathbf{u}_j\right)$, provides a solution to the auxiliary problem (12)–(14), since the construction of the image $\mathbf{T}(\mathbf{w})$, $\mathbf{w} \in \mathcal{W}_q^2(\Omega)$ is solving first the problem

$$\text{div}(\rho \mathbf{v}) = \varepsilon \Delta \rho - \varepsilon \rho + \varepsilon \frac{m}{|\Omega|}, \quad \mathbf{v} = \sum_{j=1}^N \alpha_j \mathbf{w}_j, \quad \nabla \rho \cdot \mathbf{n}|_{\partial \Omega} = 0,$$

producing $\rho = \mathcal{P}(\mathbf{v})$ and then the boundary value problem

$$\begin{aligned} - \sum_{j=1}^N ((\lambda_{ij} + \mu_{ij}) \nabla \text{div} \mathbf{u}_j + \mu_{ij} \Delta \mathbf{u}_j) &= -\frac{1}{2} \text{div}(\rho_i \mathbf{v} \otimes \mathbf{w}_i) - \alpha_i \nabla p \\ &- \frac{1}{2} \rho_i (\mathbf{v} \cdot \nabla) \mathbf{w}_i - \frac{\varepsilon}{2} \rho_i \mathbf{w}_i - \frac{\varepsilon \alpha_i m}{2|\Omega|} \mathbf{w}_i \\ &+ \sum_{j=1}^N a_{ij} (\mathbf{w}_j - \mathbf{w}_i) (|\mathbf{w}_i - \mathbf{w}_j|^2 + 1)^\beta + \rho_i \mathbf{f}_i, \quad \mathbf{u}_i|_{\partial \Omega} = \mathbf{0}, \end{aligned}$$

where $\rho_i = \alpha_i \rho$.

We are to prove that the operator $\mathbf{T}$ meets the conditions of Leray–Schauder’s fixed point principle.

First, we prove that the operator $\mathbf{T}$ is continuous. Let $\mathbf{w}^m = (\mathbf{w}_1^m, \dots, \mathbf{w}_N^m) \in \mathcal{W}_q^2(\Omega)$, $m \in \mathbb{N}$, $\mathbf{w}^m \rightarrow \mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_N)$ as $m \rightarrow \infty$ strongly in $\mathcal{W}_q^2(\Omega)$. Then, due to the continuity of $\mathcal{P}$ from $\mathcal{W}_q^2(\Omega)$ into $W_q^2(\Omega)$ (see (18)), we have that

$$\rho^m = \mathcal{P}\left(\sum_{j=1}^N \alpha_j \mathbf{w}_j^m\right) \rightarrow \rho = \mathcal{P}\left(\sum_{j=1}^N \alpha_j \mathbf{w}_j\right),$$

as $m \rightarrow \infty$, in the norm of the space $W_q^2(\Omega)$. The continuous embedding of $W_q^2(\Omega)$ into $C^1(\overline{\Omega})$ and the properties (22) of the operators $\mathcal{D}_i$ imply that

$$\mathcal{D}_i(\mathbf{w}^m) \rightarrow \mathcal{D}_i(\mathbf{w}),$$

as $m \rightarrow \infty$, in the norms of the spaces $L_q(\Omega)$ and $C(\overline{\Omega})$. Finally, the continuity of $\mathcal{E}$ from $L_q(\Omega)$ into $\mathcal{W}_q^2(\Omega)$ (see (20)) leads to the convergence

$$\mathbf{T}(\mathbf{w}^m) \rightarrow \mathbf{T}(\mathbf{w})$$

as $m \rightarrow \infty$ strongly in $\mathcal{W}_q^2(\Omega)$, which means the continuity of the operator $\mathbf{T}$.

In order to prove that the operator $\mathbf{T}$ is compact (completely continuous), let us take in $\mathcal{W}_q^2(\Omega)$ any bounded sequence $\mathbf{w}^m$, $m \in \mathbb{N}$. Since the embedding of $\mathcal{W}_q^2(\Omega)$ into $C^1(\overline{\Omega})$ is compact, the sequence $\mathbf{w}^m$, $m \in \mathbb{N}$ admits the selection of a subsequence (preserving the initial notation) such that

$$\mathbf{w}^m \longrightarrow \mathbf{w} \text{ strongly in } C^1(\overline{\Omega})$$

as $m \rightarrow \infty$. Thus, the estimate (18) implies that

$$\rho^m = \mathcal{P} \left(\sum_{j=1}^N \alpha_j \mathbf{w}_j^m \right) \longrightarrow \rho = \mathcal{P} \left(\sum_{j=1}^N \alpha_j \mathbf{w}_j \right)$$

as $m \rightarrow \infty$ in the norm of the space $\mathcal{W}_q^2(\Omega)$. Iterating the previous argument, we find that

$$\mathbf{T}(\mathbf{w}^m) \longrightarrow \mathbf{T}(\mathbf{w})$$

as $m \rightarrow \infty$ strongly in $\mathcal{W}_q^2(\Omega)$, which concludes the proof of the compactness of the operator $\mathbf{T}$.

To complete the verification of the conditions of Leray–Schauder’s fixed point principle and, thus, to prove the existence of a strong solution to the boundary value problem (12)–(14), we need just to show that the set of all $\mathcal{W}_q^2(\Omega)$ -solutions to the operator equation

$$\lambda \mathbf{T}(\mathbf{u}) = \mathbf{u}, \quad \lambda \in (0, 1] \quad (24)$$

is bounded in $\mathcal{W}_q^2(\Omega)$. In other words, we obtain an a priori estimate of the solutions to Equation (24) in the space $\mathcal{W}_q^2(\Omega)$ uniformly with respect to the parameter λ .

3.2. Problem with Parameter λ and First Estimate

To obtain an a priori estimate of a solution to Equation (24) in the space $\mathcal{W}_q^2(\Omega)$ uniformly in λ means to estimate a supposed solution $(\rho, \mathbf{u}_1, \dots, \mathbf{u}_N)$ to the boundary value problem (we omit the index λ for the values depending on λ)

$$\operatorname{div}(\rho \mathbf{v}) = \varepsilon \Delta \rho - \varepsilon \rho + \varepsilon \frac{m}{|\Omega|}, \quad (25)$$

$$\begin{aligned} \frac{\lambda}{2} \operatorname{div}(\rho_i \mathbf{v} \otimes \mathbf{u}_i) + \lambda \alpha_i \nabla p + \frac{\lambda}{2} \rho_i (\mathbf{v} \cdot \nabla) \mathbf{u}_i + \frac{\lambda \varepsilon}{2} \rho_i \mathbf{u}_i + \frac{\lambda \varepsilon \alpha_i m}{2|\Omega|} \mathbf{u}_i \\ = \operatorname{div} \mathbb{S}_i + \lambda \mathbf{J}_i + \lambda \rho_i \mathbf{f}_i, \end{aligned} \quad (26)$$

$$\nabla \rho \cdot \mathbf{n}|_{\partial \Omega} = 0, \quad \mathbf{u}_i|_{\partial \Omega} = \mathbf{0}, \quad (27)$$

in the space $\mathcal{W}_q^2(\Omega)$, uniformly with respect to λ , where $\mathbf{v} = \sum_{j=1}^N \alpha_j \mathbf{u}_j$, $\rho_i = \alpha_i \rho$,

$$\mathbb{S}_i = \sum_{j=1}^N (2\mu_{ij} \mathbb{D}(\mathbf{u}_j) + \lambda_{ij} (\operatorname{div} \mathbf{u}_j) \mathbb{I}), \quad \mathbf{J}_i = \sum_{j=1}^N a_{ij} (\mathbf{u}_j - \mathbf{u}_i) (|\mathbf{u}_i - \mathbf{u}_j|^2 + 1)^\beta.$$

First, we compose the scalar product of (26) and $\frac{\mathbf{u}_i}{\lambda}$, integrate over Ω and sum over $i = 1, \dots, N$, and we arrive at

$$\begin{aligned} \frac{1}{\lambda} \sum_{i=1}^N \int_{\Omega} \mathbb{S}_i : (\nabla \otimes \mathbf{u}_i) dx + \frac{\varepsilon m}{2|\Omega|} \sum_{i=1}^N \alpha_i \int_{\Omega} |\mathbf{u}_i|^2 dx + \frac{\varepsilon}{2} \sum_{i=1}^N \int_{\Omega} \rho_i |\mathbf{u}_i|^2 dx \\ + \frac{1}{2} \sum_{i,j=1}^N a_{ij} \int_{\Omega} |\mathbf{u}_i - \mathbf{u}_j|^2 (|\mathbf{u}_i - \mathbf{u}_j|^2 + 1)^\beta dx \\ = \int_{\Omega} p \operatorname{div} \mathbf{v} dx + \sum_{i=1}^N \int_{\Omega} \mathbf{u}_i \cdot \rho_i \mathbf{f}_i dx. \end{aligned} \quad (28)$$

Then, we integrate Equation (25) over Ω and obtain the equality

$$\int_{\Omega} \rho \, dx = m. \quad (29)$$

Multiplying Equation (25) by the function $R'(\rho)$ (where $R : \mathbb{R} \rightarrow \mathbb{R}$ is an arbitrary double continuously differentiable function), we obtain the equality

$$(\rho R'(\rho) - R(\rho)) \operatorname{div} v + \operatorname{div}(R(\rho)v) = -\varepsilon R''(\rho) |\nabla \rho|^2 + \varepsilon \operatorname{div}(R'(\rho) \nabla \rho) - \varepsilon \rho R'(\rho) + \varepsilon \frac{m}{|\Omega|} R'(\rho). \quad (30)$$

Setting $R(\rho) = (\rho + \delta) \int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds$ in (30) with arbitrary $\delta \in (0, 1]$, we derive

$$\begin{aligned} \int_{\Omega} p(\rho + \delta) \operatorname{div} v \, dx &= \delta \int_{\Omega} \frac{p(\rho + \delta)}{\rho + \delta} \operatorname{div} v \, dx + \delta \int_{\Omega} \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) \operatorname{div} v \, dx \\ &+ \frac{\varepsilon m}{|\Omega|} \int_{\Omega} \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) dx + \frac{\varepsilon m}{|\Omega|} \int_{\Omega} \frac{p(\rho + \delta)}{\rho + \delta} \, dx - \varepsilon \int_{\Omega} \rho \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) dx \\ &- \varepsilon \int_{\Omega} \frac{\rho p(\rho + \delta)}{\rho + \delta} \, dx - \varepsilon \int_{\Omega} \frac{p'(\rho + \delta)}{\rho + \delta} |\nabla \rho|^2 \, dx. \end{aligned} \quad (31)$$

Let us take a closer look at the summands in the right-hand side of (31). The first term satisfies the estimate (see the conditions (3) and Remark 1)

$$\delta \int_{\Omega} \frac{p(\rho + \delta)}{\rho + \delta} \operatorname{div} v \, dx \leq \delta B_{13}(\gamma, B_3, B_4) \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i| (\rho^{\gamma-1} + 1) \, dx. \quad (32)$$

The second term in the right-hand side of (31), in view of the conditions (3) and Remark 1, can be estimated as

$$\begin{aligned} \delta \int_{\Omega} \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) \operatorname{div} v \, dx &= \delta \int_{\{\Omega : \rho \geq 1-\delta, \operatorname{div} v \geq 0\}} \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) \operatorname{div} v \, dx \\ &+ \delta \int_{\{\Omega : \rho \geq 1-\delta, \operatorname{div} v < 0\}} \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) \operatorname{div} v \, dx + \delta \int_{\{\Omega : 0 \leq \rho < 1-\delta, \operatorname{div} v \geq 0\}} \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) \operatorname{div} v \, dx \\ &+ \delta \int_{\{\Omega : 0 < \rho < 1-\delta, \operatorname{div} v < 0\}} \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) \operatorname{div} v \, dx + \delta \int_{\{\Omega : \rho = 0, \operatorname{div} v < 0\}} \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) \operatorname{div} v \, dx \\ &\leq \delta B_{14}(c_1, \gamma, B_3, B_4) \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i| (\rho^{\gamma-1} + 1) \, dx + \delta (\ln \delta) B_{15}(c_2) \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i| \, dx \\ &- \delta B_2 \int_{\{\Omega : 0 < \rho < 1-\delta, \operatorname{div} v < 0\}} \frac{\operatorname{div} v}{\rho + \delta} \, dx. \end{aligned} \quad (33)$$

The third summand in the right-hand side of (31), using (3), Remark 1 and Young's inequality, satisfies

$$\frac{\varepsilon m}{|\Omega|} \int_{\Omega} \left(\int_1^{\rho+\delta} \frac{p(s)}{s^2} \, ds \right) dx \leq B_{16}(\Omega, c_1, m, \gamma, B_1, B_3, B_4) + \frac{\varepsilon}{4} \left(\frac{1}{c_1 \gamma} + B_1 \right) \int_{\Omega} \rho^{\gamma} \, dx. \quad (34)$$

Using Remark 1 and Young's inequality again, we estimate the fourth summand in the right-hand side of (31) as

$$\frac{\varepsilon m}{|\Omega|} \int_{\Omega} \frac{p(\rho + \delta)}{\rho + \delta} dx \leq B_{17}(\Omega, c_1, c_2, m, \gamma, B_1) + \frac{\varepsilon}{4} \left(\frac{1}{c_1 \gamma} + B_1 \right) \int_{\Omega} \rho^{\gamma} dx. \quad (35)$$

In view of Remark 1, the fifth summand in the right-hand side of (31) satisfies the estimate

$$-\varepsilon \int_{\Omega} \rho \left(\int_1^{\rho + \delta} \frac{p(s)}{s^2} ds \right) dx \leq B_{18}(\Omega, B_2) - \varepsilon B_1 \int_{\Omega} \rho^{\gamma} dx. \quad (36)$$

Using Remark 1 and formula (29), the penultimate term in the right-hand side of (31) can be estimated as

$$-\varepsilon \int_{\Omega} \frac{\rho p(\rho + \delta)}{\rho + \delta} dx \leq B_{19}(m, c_2) - \frac{\varepsilon}{c_1 \gamma} \int_{\Omega} \rho^{\gamma} dx. \quad (37)$$

The last term is (31) non-positive (see (3)).

As a result, from (31), in view of (32)–(37) and the sign of the last term is (31), it follows that

$$\begin{aligned} \int_{\Omega} p(\rho + \delta) \operatorname{div} v dx &\leq -\frac{\varepsilon}{2} \left(\frac{1}{c_1 \gamma} + B_1 \right) \int_{\Omega} \rho^{\gamma} dx + \delta B_{13} \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i| (\rho^{\gamma-1} + 1) dx \\ &+ \delta B_{14} \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i| (\rho^{\gamma-1} + 1) dx + \delta (\ln \delta) B_{15} \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i| dx \\ &- \delta B_2 \int_{\{\Omega: 0 < \rho < 1 - \delta, \operatorname{div} v < 0\}} \frac{\operatorname{div} v}{\rho + \delta} dx + B_{20}(B_{16}, B_{17}, B_{18}, B_{19}). \end{aligned} \quad (38)$$

Let us pass to the limit in (38) as $\delta \rightarrow +0$, using Lebesgue's dominated convergence theorem, so that we obtain the estimate

$$\int_{\Omega} p \operatorname{div} v dx \leq B_{20} - \frac{\varepsilon}{2} \left(\frac{1}{c_1 \gamma} + B_1 \right) \int_{\Omega} \rho^{\gamma} dx. \quad (39)$$

Further, the relation (28), in view of (11), (39) and the fact that $\frac{1}{\lambda} \geq 1$, imply the inequality

$$\begin{aligned} B_5 \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i|^2 dx + \frac{\varepsilon m}{2|\Omega|} \sum_{i=1}^N \alpha_i \int_{\Omega} |u_i|^2 dx + \frac{\varepsilon}{2} \sum_{i=1}^N \int_{\Omega} \rho_i |u_i|^2 dx \\ + \frac{\varepsilon}{2} \left(\frac{1}{c_1 \gamma} + B_1 \right) \int_{\Omega} \rho^{\gamma} dx + \frac{1}{2} \sum_{i,j=1}^N a_{ij} \int_{\Omega} |u_i - u_j|^2 (|u_i - u_j|^2 + 1)^{\beta} dx \\ \leq \sum_{i=1}^N \int_{\Omega} u_i \cdot \rho_i f_i dx + B_{20}, \end{aligned} \quad (40)$$

from which, in view of the estimates

$$B_{21}(\Omega, B_5) \sum_{i=1}^N \|u_i\|_{W_2^1(\Omega)}^2 \leq B_5 \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i|^2 dx,$$

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} \mathbf{u}_i \cdot \rho_i \mathbf{f}_i d\mathbf{x} &\leq \frac{B_{21}}{2} \sum_{i=1}^N \|\mathbf{u}_i\|_{W_2^1(\Omega)}^2 \\ &+ \frac{\varepsilon}{4} \left(\frac{1}{c_1 \gamma} + B_1 \right) \|\rho\|_{L_\gamma(\Omega)}^\gamma + B_{22} \left(\Omega, c_1, N, \gamma, \varepsilon, B_1, B_{21}, \left\{ \|\mathbf{f}_i\|_{C(\overline{\Omega})} \right\} \right), \end{aligned}$$

we obtain the estimate

$$\sum_{i=1}^N \|\mathbf{u}_i\|_{W_2^1(\Omega)} + \|\rho\|_{L_\gamma(\Omega)} \leq B_{23}(c_1, N, \gamma, \varepsilon, B_1, B_{20}, B_{21}, B_{22}) \quad (41)$$

uniform in λ . In particular, this implies (in view of the embedding of $W_2^1(\Omega)$ into $L_6(\Omega)$) that

$$\|\mathbf{u}_i\|_{L_6(\Omega)} \leq B_{24}(\Omega, B_{23}). \quad (42)$$

3.3. Further Estimates

Let us rewrite Equation (25) as

$$-\varepsilon \Delta \rho = \operatorname{div}(\mathbf{w}) - \operatorname{div}(\rho \mathbf{v}), \quad (43)$$

where the vector field $\mathbf{w}$ is the solution to the problem

$$\operatorname{div} \mathbf{w} = -\varepsilon \rho + \frac{\varepsilon m}{|\Omega|}, \quad \mathbf{w}|_{\partial \Omega} = \mathbf{0}. \quad (44)$$

From the estimate (41), the properties of solutions to the problem (44) (see, e.g., [32], p. 169) and the embedding of $W_\gamma^1(\Omega)$ into $C(\overline{\Omega})$, it follows that

$$\|\mathbf{w}\|_{C(\overline{\Omega})} \leq B_{25}(\Omega, \gamma) \|\mathbf{w}\|_{W_\gamma^1(\Omega)} \leq B_{26}(\Omega, \gamma, B_{25}) \left\| -\varepsilon \rho + \frac{\varepsilon m}{|\Omega|} \right\|_{L_\gamma(\Omega)} \leq B_{27}, \quad (45)$$

where $B_{27} = B_{27}(\Omega, m, \gamma, B_{23}, B_{26})$. In view of (41), (42) and (45), from (25) (i.e., (43)) and (27), due to the estimates for the solutions of elliptic boundary value problems (see, e.g., [32], p. 211), it follows that

$$\|\nabla \rho\|_{L_{\frac{6\gamma}{\gamma+6}}(\Omega)} \leq B_{28}(\Omega, \varepsilon, \gamma) \left(\sum_{i=1}^N \|\rho \mathbf{u}_i\|_{L_{\frac{6\gamma}{\gamma+6}}(\Omega)} + \|\mathbf{w}\|_{C(\overline{\Omega})} \right) \leq B_{29}, \quad (46)$$

where $B_{29} = B_{29}(N, B_{23}, B_{24}, B_{27}, B_{28})$, and hence

$$\|\rho\|_{W_{\frac{6\gamma}{\gamma+6}}^1(\Omega)} \leq B_{30}(\Omega, \gamma, B_{23}, B_{29}). \quad (47)$$

Since the space $W_{\frac{6\gamma}{\gamma+6}}^1(\Omega)$ is embedded into the space $C(\overline{\Omega})$ provided that $\gamma \in (6, +\infty)$, then in this case

$$\|\rho\|_{C(\overline{\Omega})} \leq B_{31}(\Omega, \gamma, B_{30}). \quad (48)$$

In order to obtain a similar estimate in the case $\gamma \in (3, 6]$, we repeat the above argument. Since the space $W_{\frac{6\gamma}{\gamma+6}}^1(\Omega)$ is embedded into the space $L_{\frac{6\gamma}{6-\gamma}}(\Omega)$ provided that $\gamma \in (3, 6)$, or into the space $L_{12}(\Omega)$ provided that $\gamma = 6$, then in the case $\gamma \in (3, 6)$ we have

$$\|\rho\|_{L_{\frac{6\gamma}{6-\gamma}}(\Omega)} \leq B_{32}(\Omega, \gamma, B_{30}), \quad (49)$$

and, if $\gamma = 6$,

$$\|\rho\|_{L_{12}(\Omega)} \leq B_{33}(\Omega, B_{30}). \quad (50)$$

Using the estimates (49), (50), the properties of the solutions to the problem (44) and the embedding of the spaces $W_{\frac{6\gamma}{6-\gamma}}^1(\Omega)$ and $W_{12}^1(\Omega)$ into $C(\overline{\Omega})$, we arrive at the inequalities

$$\begin{aligned} \|w\|_{C(\overline{\Omega})} &\leq B_{34}(\Omega, \gamma) \|w\|_{W_{\frac{6\gamma}{6-\gamma}}^1(\Omega)} \leq B_{35}(\Omega, \gamma, B_{34}) \left\| -\varepsilon\rho + \frac{\varepsilon m}{|\Omega|} \right\|_{L_{\frac{6\gamma}{6-\gamma}}(\Omega)} \\ &\leq B_{36}(\Omega, m, \gamma, B_{32}, B_{35}), \quad \gamma \in (3, 6), \end{aligned} \quad (51)$$

$$\begin{aligned} \|w\|_{C(\overline{\Omega})} &\leq B_{37}(\Omega) \|w\|_{W_{12}^1(\Omega)} \leq B_{38}(\Omega, B_{37}) \left\| -\varepsilon\rho + \frac{\varepsilon m}{|\Omega|} \right\|_{L_{12}(\Omega)} \\ &\leq B_{39}(\Omega, m, B_{33}, B_{38}), \quad \gamma = 6. \end{aligned} \quad (52)$$

Taking into account (42), (49), (50), (51) and (52), from (25) (i.e., (43)) and (27), in view of the estimates for the solutions of elliptic boundary value problems, we obtain that

$$\begin{aligned} \|\nabla\rho\|_{L_\gamma(\Omega)} &\leq B_{40}(\Omega, \varepsilon, \gamma) \left(\sum_{i=1}^N \|\rho u_i\|_{L_\gamma(\Omega)} + \|w\|_{C(\overline{\Omega})} \right) \\ &\leq B_{41}(N, B_{24}, B_{32}, B_{35}, B_{40}), \quad \gamma \in (3, 6), \end{aligned} \quad (53)$$

$$\begin{aligned} \|\nabla\rho\|_{L_4(\Omega)} &\leq B_{42}(\Omega, \varepsilon) \left(\sum_{i=1}^N \|\rho u_i\|_{L_4(\Omega)} + \|w\|_{C(\overline{\Omega})} \right) \\ &\leq B_{43}(N, B_{24}, B_{33}, B_{39}, B_{42}), \quad \gamma = 6. \end{aligned} \quad (54)$$

Then

$$\|\rho\|_{W_\gamma^1(\Omega)} \leq B_{44}(\gamma, B_{23}, B_{43}), \quad \gamma \in (3, 6) \quad \text{and} \quad \|\rho\|_{W_4^1(\Omega)} \leq B_{45}(\Omega, B_{33}, B_{43}), \quad \gamma = 6. \quad (55)$$

In view of the embedding of the spaces $W_\gamma^1(\Omega)$ and $W_4^1(\Omega)$ into $C(\overline{\Omega})$, we obtain the estimate we intended

$$\|\rho\|_{C(\overline{\Omega})} \leq B_{46}(\Omega, \gamma, B_{44}, B_{45}), \quad \gamma \in (3, 6]. \quad (56)$$

In particular, from (42), (48) and (56), it follows that

$$\|\rho u_i\|_{L_6(\Omega)} \leq B_{47}(B_{24}, B_{46}). \quad (57)$$

Let us introduce the notations

$$\zeta_i = \frac{1}{|\Omega|} \int_{\Omega} \left(\frac{\alpha_i}{2} \rho(v \cdot \nabla) u_i - J_i \right) dx,$$

$$V_i = \lambda \left(\rho_i f_i - \frac{\varepsilon}{2} \rho_i u_i - \frac{\varepsilon m}{2|\Omega|} \alpha_i u_i - \zeta_i \right).$$

Denote as $\mathbb{W}_i$, the solutions to the boundary value problems

$$\operatorname{div} \mathbb{W}_i = \frac{\alpha_i}{2} \rho(v \cdot \nabla) u_i - J_i - \zeta_i, \quad \mathbb{W}_i|_{\partial\Omega} = 0, \quad (58)$$

and set $\mathbb{U}_i = \lambda \left(-\frac{\alpha_i}{2} \rho v \otimes u_i - \alpha_i p \mathbb{I} - \mathbb{W}_i \right)$. In these notations, Equation (26) takes the form

$$-\operatorname{div} \mathbb{S}_i = V_i + \operatorname{div} \mathbb{U}_i. \quad (59)$$

Since the right-hand side of Equation (58) is bounded in the space $L_{\frac{3}{2}}(\Omega)$ uniformly in λ , then, due to the properties of the solutions to the problems (58) (see, e.g., [32], p. 169), for $\mathbb{W}_i, i = 1, \dots, N$, the following inequalities are valid:

$$\|\mathbb{W}_i\|_{W_{\frac{3}{2}}^1(\Omega)} \leq B_{48}(\Omega, N, \beta, B_{23}, B_{24}, B_{31}, B_{46}, \{a_{ij}\}). \quad (60)$$

As a result, the following estimates hold (here, we use the embedding of $W_{\frac{3}{2}}^1(\Omega)$ into $L_3(\Omega)$)

$$\begin{aligned} & \|V_i\|_{L_6(\Omega)} + \|U_i\|_{L_3(\Omega)} \\ & \leq B_{49}(\Omega, N, \beta, m, \gamma, c_1, c_2, B_{23}, B_{24}, B_{31}, B_{46}, B_{47}, B_{48}, \{\|f_i\|_{C(\overline{\Omega})}\}, \{a_{ij}\}). \end{aligned}$$

Hence, Equation (26) (i.e., (59)) and the boundary conditions (27), due to the estimates for the solutions of elliptic boundary value problems (see, e.g., [35–37]) provide that

$$\|u_i\|_{W_3^1(\Omega)} \leq B_{50}(\Omega, \Lambda, \mathbf{M}, N, B_{49}), \quad (61)$$

and, in view of the embedding of $W_3^1(\Omega)$ into $L_{2q}(\Omega)$, we arrive at the relations

$$\|u_i\|_{L_{2q}(\Omega)} \leq B_{51}(\Omega, q, B_{50}). \quad (62)$$

From (25) (i.e., (43)) and (27), due to the estimates for the solutions of elliptic boundary value problems, we now obtain that

$$\|\nabla \rho\|_{L_{2q}(\Omega)} \leq B_{52}(\Omega, N, \varepsilon, q, B_{27}, B_{31}, B_{46}, B_{51}), \quad (63)$$

and hence

$$\|\rho\|_{W_{2q}^1(\Omega)} \leq B_{53}(\Omega, q, B_{31}, B_{46}). \quad (64)$$

Then, using again the estimates for the solutions of elliptic boundary value problems, from (25) (i.e., (43)) and (27), we deduce

$$\|\nabla \rho\|_{W_3^1(\Omega)} \leq B_{54}(\Omega, N, q, m, \varepsilon, B_{24}, B_{27}, B_{31}, B_{46}, B_{50}, B_{51}, B_{52}). \quad (65)$$

Hence

$$\|\rho\|_{W_3^2(\Omega)} \leq B_{55}(\Omega, B_{31}, B_{46}, B_{54}). \quad (66)$$

From here, due to the embedding of $W_3^2(\Omega)$ into $W_{2q}^1(\Omega)$, it follows that

$$\|\rho\|_{W_{2q}^1(\Omega)} \leq B_{56}(\Omega, q, B_{55}). \quad (67)$$

Thus, for the functions V_i, U_i , we obtain the estimates

$$\|V_i\|_{L_{2q}(\Omega)} + \|U_i\|_{L_q(\Omega)} \leq B_{57}, \quad (68)$$

where $B_{57} = B_{57}(\Omega, c_1, c_2, \{\|f_i\|_{C(\overline{\Omega})}\}, \{a_{ij}\}, N, \beta, m, \gamma, q, B_{31}, B_{46}, B_{50}, B_{51})$, which enable us to write the relations

$$\|u_i\|_{W_q^1(\Omega)} \leq B_{58}(\Omega, \Lambda, q, \mathbf{M}, N, B_{57}), \quad (69)$$

and

$$\|u_i\|_{C(\overline{\Omega})} \leq B_{59}(\Omega, q, B_{58}). \quad (70)$$

Using these inequalities and the estimates for the solutions of elliptic boundary value problems, from (25) (i.e., (43)) and (27), we derive that

$$\|\rho\|_{W_q^2(\Omega)} \leq B_{60}(\Omega, N, m, q, \varepsilon, B_{27}, B_{31}, B_{46}, B_{56}, B_{58}, B_{59}),$$

and hence

$$\|\nabla \rho\|_{C(\overline{\Omega})} \leq B_{61}(\Omega, q, B_{60}). \quad (71)$$

Now, for the functions $\mathbb{U}_i$, the following inequalities are valid:

$$\|\operatorname{div} \mathbb{U}_i\|_{L_q(\Omega)} \leq B_{62}(\Omega, c_1, c_2, \{a_{ij}\}, N, \beta, \gamma, q, B_{31}, B_{46}, B_{58}, B_{59}, B_{61}). \quad (72)$$

Finally, from (26) (i.e., (59)), (27) and the estimates for the solutions of elliptic boundary value problems, it follows that

$$\|u_i\|_{W_q^2(\Omega)} \leq B_{63}(\Omega, \Lambda, N, \mathbf{M}, q, B_{57}, B_{62}). \quad (73)$$

3.4. Solvability of Approximate Boundary Value Problem

All the conditions of Leray–Schauder’s fixed point principle are satisfied; hence, we can assert the following.

Theorem 2. *Under the conditions of Theorem 1, for all $\varepsilon \in (0, 1]$, the approximate boundary value problems (12)–(14) possess at least one strong solution in the sense of Definition 2.*

To prove the solvability of the original boundary value problems (1), (2), (6), and (7) (see Definition 1), it remains to pass to the limit with respect to the parameter ε , which distinguishes problems (12)–(14) from the original problem (1), (2), (6), (7). This limit is made below on the basis of the estimates which are uniform with respect to this parameter.

4. Uniform Estimates of Solutions to Approximate Boundary Value Problem

Let us obtain the estimates of the solutions to the approximate boundary value problem which are uniform with respect to the parameter ε . Integrating Equation (12) over the domain Ω , we obtain the equality

$$\int_{\Omega} \rho \, dx = m. \quad (74)$$

Composing the scalar product of (13) and u_i , integrating over Ω (using the boundary conditions (14)) and summing over $i = 1, \dots, N$, we derive the equality

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega} (\nabla \otimes u_i) : \mathbb{S}_i \, dx + \frac{\varepsilon}{2} \sum_{i=1}^N \int_{\Omega} |u_i|^2 \rho_i \, dx + \frac{\varepsilon m}{2|\Omega|} \sum_{i=1}^N \alpha_i \int_{\Omega} |u_i|^2 \, dx \\ + \frac{1}{2} \sum_{i,j=1}^N a_{ij} \int_{\Omega} |u_i - u_j|^2 (|u_i - u_j|^2 + 1)^{\beta} \, dx \\ = \int_{\Omega} p \operatorname{div} v \, dx + \sum_{i=1}^N \int_{\Omega} u_i \cdot \rho_i f_i \, dx, \end{aligned} \quad (75)$$

from which, due to (11) and (39), it follows that

$$\begin{aligned} B_5 \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i|^2 \, dx + \frac{\varepsilon m}{2|\Omega|} \sum_{i=1}^N \alpha_i \int_{\Omega} |u_i|^2 \, dx + \frac{\varepsilon}{2} \sum_{i=1}^N \int_{\Omega} |u_i|^2 \rho_i \, dx \\ + \frac{\varepsilon}{2} \left(\frac{1}{c_1 \gamma} + B_1 \right) \int_{\Omega} \rho^{\gamma} \, dx + \frac{1}{2} \sum_{i,j=1}^N a_{ij} \int_{\Omega} |u_i - u_j|^2 (|u_i - u_j|^2 + 1)^{\beta} \, dx \\ \leq \sum_{i=1}^N \int_{\Omega} u_i \cdot \rho_i f_i \, dx + B_{20}. \end{aligned} \quad (76)$$

From (76), in view of the inequalities

$$B_{64}(\Omega, B_5) \sum_{i=1}^N \|u_i\|_{W_2^1(\Omega)}^2 \leq B_5 \sum_{i=1}^N \int_{\Omega} |\nabla \otimes u_i|^2 dx,$$

$$\sum_{i=1}^N \int_{\Omega} u_i \cdot \rho_i f_i dx \leq \frac{B_{64}}{2} \sum_{i=1}^N \|u_i\|_{W_2^1(\Omega)}^2 + B_{65} \left(\Omega, B_{64}, \{\|f_i\|_{C(\overline{\Omega})}\} \right) \|\rho\|_{L_{\frac{5}{3}}(\Omega)}^2,$$

we obtain the estimate

$$\sum_{i=1}^N \|u_i\|_{W_2^1(\Omega)}^2 \leq B_{66}(B_{20}, B_{64}, B_{65}) \left(\|\rho\|_{L_{\frac{5}{3}}(\Omega)}^2 + 1 \right). \quad (77)$$

Composing the scalar product of (13) and ψ , where the vector field ψ is the solution to the problem

$$\operatorname{div} \psi = -\frac{1}{|\Omega|} \int_{\Omega} \rho^\gamma dx + \rho^\gamma, \quad \psi|_{\partial\Omega} = 0, \quad (78)$$

and integrating over Ω , we arrive at the identities

$$\begin{aligned} \alpha_i \int_{\Omega} p \rho^\gamma dx &= \frac{\alpha_i}{|\Omega|} \left(\int_{\Omega} p dx \right) \left(\int_{\Omega} \rho^\gamma dx \right) \\ &+ \int_{\Omega} \left(-\frac{1}{2} \rho_i v \otimes u_i + \mathbb{S}_i \right) : (\nabla \otimes \psi) dx - \int_{\Omega} J_i \cdot \psi dx + \frac{1}{2} \int_{\Omega} (\rho_i (v \cdot \nabla) u_i) \cdot \psi dx \\ &- \int_{\Omega} \rho_i f_i \cdot \psi dx + \frac{\varepsilon}{2} \int_{\Omega} \rho_i u_i \cdot \psi dx + \frac{\varepsilon m \alpha_i}{2 |\Omega|} \int_{\Omega} u_i \cdot \psi dx. \end{aligned} \quad (79)$$

Due to the estimates

$$\begin{aligned} \|\psi\|_{L_6(\Omega)} + \|\nabla \otimes \psi\|_{L_2(\Omega)} &\leq B_{67}(\Omega) \|\rho\|_{L_{2\gamma}(\Omega)}^\gamma, \\ \|\mathbb{S}_i\|_{L_2(\Omega)} &\leq B_{68}(\Omega, \Lambda, N, \mathbf{M}, \gamma, B_{66}) \left(\|\rho\|_{L_{2\gamma}(\Omega)} + 1 \right), \\ \|\rho_i (v \cdot \nabla) u_i\|_{L_{\frac{5}{3}}(\Omega)} + \|\rho_i v \otimes u_i\|_{L_2(\Omega)} &\leq B_{69}(\Omega, N, \gamma, B_{66}) \|\rho\|_{L_{2\gamma}(\Omega)} \left(\|\rho\|_{L_{2\gamma}(\Omega)}^2 + 1 \right), \\ \|J_i\|_{L_{\frac{5}{3}}(\Omega)} &\leq B_{70}(\Omega, N, \beta, \gamma, B_{66}, \{a_{ij}\}) \left(\|\rho\|_{L_{2\gamma}(\Omega)}^{2\beta+1} + 1 \right), \\ \|\rho_i f_i\|_{L_2(\Omega)} &\leq B_{71}(\Omega, \gamma, \{\|f_i\|_{C(\overline{\Omega})}\}) \|\rho\|_{L_{2\gamma}(\Omega)}, \end{aligned}$$

which are valid for all $i = 1, \dots, N$ and uniform in ε , from (79), it follows that (see (8) and (74))

$$\|\rho\|_{L_{2\gamma}(\Omega)}^{2\gamma} \leq B_{72} \left(\|\rho\|_{L_{2\gamma}(\Omega)}^{\gamma+3} + \|\rho\|_{L_{2\gamma}(\Omega)}^{\frac{4\gamma(\gamma-1)}{2\gamma-1}} + 1 \right), \quad (80)$$

where $B_{72} = B_{72}(\Omega, c_1, c_2, \{\alpha_i\}, m, \beta, \gamma, B_{67}, B_{68}, B_{69}, B_{70}, B_{71})$.

Now, from (80), we obtain the estimate

$$\|\rho\|_{L_{2\gamma}(\Omega)} \leq B_{73}(\gamma, B_{72}), \quad (81)$$

and, due to (77), the estimates

$$\|u_i\|_{W_2^1(\Omega)} \leq B_{74}(\Omega, \gamma, B_{66}, B_{73}). \quad (82)$$

Finally, from (12), (14), (81) and (82), it follows that

$$\sqrt{\varepsilon} \|\nabla \rho\|_{L_2(\Omega)} \leq B_{75}(\Omega, N, m, \gamma, B_{73}, B_{74}). \quad (83)$$

Hereinafter, the solutions to the problems (12)–(14) will be denoted using the index ε .

5. Limit Everywhere Except Pressure

In view of the estimates (81)–(83), the family $\rho^\varepsilon, \mathbf{u}_i^\varepsilon$, of the solutions to the boundary value problems (12)–(14) admits the selection of a sequence (which will be denoted likewise), for which, as $\varepsilon \rightarrow +0$, the convergences hold

$$\mathbf{u}_i^\varepsilon \rightharpoonup \mathbf{u}_i \text{ weakly in } W_2^1(\Omega), \quad (84)$$

$$\rho^\varepsilon \rightharpoonup \rho \text{ weakly in } L_{2\gamma}(\Omega), \quad \rho \geq 0 \text{ a. e. in } \Omega, \quad (85)$$

$$\varepsilon \nabla \rho^\varepsilon \rightarrow 0 \text{ strongly in } L_2(\Omega), \quad (86)$$

$$p(\rho^\varepsilon) \rightarrow \overline{p(\rho)} \text{ weakly in } L_2(\Omega), \quad \overline{p(\rho)} \geq 0 \text{ a. e. in } \Omega, \quad (87)$$

where $\overline{p(\rho)}$ denotes the weak limit of the sequence $p(\rho^\varepsilon)$ in $L_2(\Omega)$.

Note that (84) immediately provides the strong convergences $\mathbf{u}_i^\varepsilon \xrightarrow{\varepsilon \rightarrow +0} \mathbf{u}_i$ in $L_p(\Omega)$ for all $p \in [1, 6)$.

Thus, we obtain that the limit functions $\rho, \mathbf{u}_i$, satisfy the equation

$$\int_{\Omega} \nabla \zeta \cdot \rho \mathbf{v} \, dx = 0 \quad \forall \zeta \in C^\infty(\overline{\Omega}) \quad (88)$$

which is the weak form of (1), where $\mathbf{v} = \sum_{j=1}^N \alpha_j \mathbf{u}_j$; the equations

$$\begin{aligned} \int_{\Omega} \left((\rho_i \mathbf{v} \otimes \mathbf{u}_i) : (\nabla \otimes \boldsymbol{\psi}_i) + \alpha_i \overline{p(\rho)} \operatorname{div} \boldsymbol{\psi}_i - \mathbb{S}_i : (\nabla \otimes \boldsymbol{\psi}_i) \right. \\ \left. + J_i \cdot \boldsymbol{\psi}_i + \rho_i f_i \cdot \boldsymbol{\psi}_i \right) dx = 0 \quad \forall \boldsymbol{\psi}_i \in C_0^\infty(\Omega) \end{aligned} \quad (89)$$

which are the weak form of Equation (2) and the boundary conditions for the velocity fields (6),

where $\rho_i = \alpha_i \rho$, $J_i = \sum_{j=1}^N a_{ij}(\mathbf{u}_j - \mathbf{u}_i)(|\mathbf{u}_i - \mathbf{u}_j|^2 + 1)^\beta$, $\mathbb{S}_i = \sum_{j=1}^N (2\mu_{ij} \mathbb{D}(\mathbf{u}_j) + \lambda_{ij}(\operatorname{div} \mathbf{u}_j) \mathbb{I})$, and $\overline{p(\rho)}$ replaces $p(\rho)$ for now, and also the integral condition (7) for the density.

Remark 4. With the accuracy of another limit, it is not difficult to verify that Equation (88) is valid $\forall \zeta \in W_{\frac{6\gamma}{5\gamma-3}}^1(\Omega)$, and Equation (89) is valid $\forall \boldsymbol{\psi}_i \in W_2^1(\Omega)$.

While passing to the limit as $\varepsilon \rightarrow +0$ in Equation (13), we used the following identities (the first of which is based on (12)):

$$\begin{aligned} \frac{1}{2} \operatorname{div}(\rho_i^\varepsilon \mathbf{v}^\varepsilon \otimes \mathbf{u}_i^\varepsilon) + \frac{1}{2} \rho_i^\varepsilon (\mathbf{v}^\varepsilon \cdot \nabla) \mathbf{u}_i^\varepsilon + \frac{\varepsilon}{2} \rho_i^\varepsilon \mathbf{u}_i^\varepsilon + \frac{\varepsilon \alpha_i m}{2|\Omega|} \mathbf{u}_i^\varepsilon \\ = \operatorname{div}(\rho_i^\varepsilon \mathbf{v}^\varepsilon \otimes \mathbf{u}_i^\varepsilon) - \frac{\varepsilon}{2} (\Delta \rho_i^\varepsilon) \mathbf{u}_i^\varepsilon + \varepsilon \rho_i^\varepsilon \mathbf{u}_i^\varepsilon, \end{aligned} \quad (90)$$

$$(\Delta \rho_i^\varepsilon) \mathbf{u}_i^\varepsilon = -(\nabla \rho_i^\varepsilon \cdot \nabla) \mathbf{u}_i^\varepsilon + \operatorname{div}((\nabla \rho_i^\varepsilon) \otimes \mathbf{u}_i^\varepsilon). \quad (91)$$

6. Limit in Pressure

In order to complete the limit as $\varepsilon \rightarrow +0$, it remains to prove that

$$\overline{p(\rho)} = p(\rho) \text{ a. e. in } \Omega. \quad (92)$$

6.1. Preliminary Constructions

Let us consider the so-called effective viscous fluxes for the mixture constituents

$$\alpha_i p(\rho) - \sum_{j=1}^N (\lambda_{ij} + 2\mu_{ij}) \operatorname{div} \mathbf{u}_j,$$

the corresponding values for the regularized problem

$$\alpha_i p(\rho^\varepsilon) - \sum_{j=1}^N (\lambda_{ij} + 2\mu_{ij}) \operatorname{div} \mathbf{u}_j^\varepsilon,$$

and their weak limits in $L_2(\Omega)$

$$\alpha_i \overline{p(\rho)} - \sum_{j=1}^N (\lambda_{ij} + 2\mu_{ij}) \operatorname{div} \mathbf{u}_j.$$

Let us note that the concept of the effective viscous flux comes from the theory of single-component fluids [32], where the effective viscous flux looks like $p - (\lambda + 2\mu) \operatorname{div} \mathbf{u}$, so that it is a scalar-valued function but not a vector as in the multifluid theory. And in general, we note that the immediate transfer of the results of the theory of compressible Navier–Stokes equations to multifluid models is possible if

$$\operatorname{div}(\operatorname{div} \mathbb{S}_i) = \operatorname{const}_i \Delta(\operatorname{div} \mathbf{u}_i),$$

but in the present case, one has

$$\begin{pmatrix} \operatorname{div}(\operatorname{div} \mathbb{S}_1) \\ \vdots \\ \operatorname{div}(\operatorname{div} \mathbb{S}_N) \end{pmatrix} = \begin{pmatrix} 2\mu_{11} + \lambda_{11} & \dots & 2\mu_{1N} + \lambda_{1N} \\ \vdots & \ddots & \vdots \\ 2\mu_{N1} + \lambda_{N1} & \dots & 2\mu_{NN} + \lambda_{NN} \end{pmatrix} \begin{pmatrix} \Delta(\operatorname{div} \mathbf{u}_1) \\ \vdots \\ \Delta(\operatorname{div} \mathbf{u}_N) \end{pmatrix}.$$

Let us introduce the operator Δ^{-1} , defined as $(\Delta^{-1}g)(\mathbf{x}) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{g(\mathbf{y}) d\mathbf{y}}{|\mathbf{y} - \mathbf{x}|}$, and apply

it to the functions $g \in L_p(\Omega)$, $p \in \left(\frac{3}{2}, +\infty\right)$, which are defined as zero outside of Ω . It is well known that $\Delta^{-1} : L_p(\Omega) \rightarrow W_p^2(\Omega)$, and $\Delta \circ \Delta^{-1} = I$.

We compose the scalar product of Equation (13) and the function $\tau \mathbf{r}^\varepsilon$, where $\mathbf{r}^\varepsilon = \nabla \Delta^{-1} \rho^\varepsilon$, and

$$\tau \in C_0^\infty(\Omega), \quad (93)$$

and integrate over Ω . Taking into account (90) and (91), we arrive at the equalities

$$\begin{aligned} \int_{\Omega} \left(-\mathbb{S}_i^\varepsilon : (\nabla \otimes (\tau \mathbf{r}^\varepsilon)) + \tau \alpha_i p(\rho^\varepsilon) \rho^\varepsilon \right) d\mathbf{x} &= - \int_{\Omega} \tau (\rho_i^\varepsilon \mathbf{v}^\varepsilon \otimes \mathbf{u}_i^\varepsilon) : (\nabla \otimes \mathbf{r}^\varepsilon) d\mathbf{x} \\ &\quad - \alpha_i \int_{\Omega} p(\rho^\varepsilon) \nabla \tau \cdot \mathbf{r}^\varepsilon d\mathbf{x} + \frac{\varepsilon}{2} \int_{\Omega} \tau ((\nabla \rho_i^\varepsilon \cdot \nabla) \mathbf{u}_i^\varepsilon) \cdot \mathbf{r}^\varepsilon d\mathbf{x} \\ &\quad - \int_{\Omega} (\rho_i^\varepsilon \mathbf{v}^\varepsilon \otimes \mathbf{u}_i^\varepsilon) : ((\nabla \tau) \otimes \mathbf{r}^\varepsilon) d\mathbf{x} - \int_{\Omega} \tau (J_i^\varepsilon + \rho_i^\varepsilon f_i) \cdot \mathbf{r}^\varepsilon d\mathbf{x} \\ &\quad + \varepsilon \int_{\Omega} \tau \rho_i^\varepsilon \mathbf{u}_i^\varepsilon \cdot \mathbf{r}^\varepsilon d\mathbf{x} + \frac{\varepsilon}{2} \int_{\Omega} ((\nabla \rho_i^\varepsilon) \otimes \mathbf{u}_i^\varepsilon) : (\nabla \otimes (\tau \mathbf{r}^\varepsilon)) d\mathbf{x}. \quad (94) \end{aligned}$$

On the other hand, if we accept the vector fields $\psi_i = \tau r$ as test functions in (89) (see (93)), where $r = \nabla \Delta^{-1} \rho$, then the identities

$$\begin{aligned} \int_{\Omega} \left(-\mathbb{S}_i : (\nabla \otimes (\tau r)) + \tau \alpha_i \overline{p(\rho)} \rho \right) dx &= - \int_{\Omega} \tau (\rho_i v \otimes u_i) : (\nabla \otimes r) dx \\ &\quad - \alpha_i \int_{\Omega} \overline{p(\rho)} \nabla \tau \cdot r dx - \int_{\Omega} (\rho_i v \otimes u_i) : ((\nabla \tau) \otimes r) dx - \int_{\Omega} \tau r \cdot (J_i + \rho_i f_i) dx \end{aligned} \quad (95)$$

follow.

From (85) and the compactness of the embedding of $W_{2\gamma}^1(\Omega)$ into $C(\overline{\Omega})$, it follows that $r^\varepsilon \rightarrow r$ as $\varepsilon \rightarrow +0$ in $C(\overline{\Omega})$.

Subtracting equalities (95) from (94) and passing to the limit as $\varepsilon \rightarrow +0$, we obtain the relations

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} \int_{\Omega} \left(-\mathbb{S}_i^\varepsilon : (\nabla \otimes (\tau r^\varepsilon)) + \tau \alpha_i p(\rho^\varepsilon) \rho^\varepsilon \right) dx &- \int_{\Omega} \left(-\mathbb{S}_i : (\nabla \otimes (\tau r)) + \tau \alpha_i \overline{p(\rho)} \rho \right) dx \\ &= \lim_{\varepsilon \rightarrow +0} \int_{\Omega} \tau \left((\rho_i v \otimes u_i) : (\nabla \otimes r) - (\rho_i^\varepsilon v^\varepsilon \otimes u_i^\varepsilon) : (\nabla \otimes r^\varepsilon) \right) dx, \end{aligned} \quad (96)$$

the right-hand side of which contains commutators, which allows us to conclude below that the right-hand side of (96) equals zero.

6.2. Analysis of Commutative Relations

Let us prove that the right-hand side of (96) equals zero. With this aim, we introduce the operator Commutator acting as

$$\text{Commutator}(w, g) = g(\nabla \operatorname{div} \Delta^{-1} w) - (\nabla \otimes \nabla \Delta^{-1} g) w,$$

which is known to possess the following properties (see [3,32,38,39]): if $w^k \rightarrow w$ weakly in $L_s(\Omega)$, $g^k \rightarrow g$ weakly in $L_t(\Omega)$, where $\frac{1}{s} + \frac{1}{t} < 1$, then $\text{Commutator}(w^k, g^k) \rightarrow \text{Commutator}(w, g)$ weakly in $L_r(\Omega)$, where $\frac{1}{r} = \frac{1}{s} + \frac{1}{t}$.

Let us rewrite the the right-hand side of (96) (taking into account the equality (88)):

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} \int_{\Omega} \tau \left((\rho_i v \otimes u_i) : (\nabla \otimes r) - (\rho_i^\varepsilon v^\varepsilon \otimes u_i^\varepsilon) : (\nabla \otimes r^\varepsilon) \right) dx \\ = \lim_{\varepsilon \rightarrow +0} \int_{\Omega} v^\varepsilon \cdot \text{Commutator}(\tau \rho_i^\varepsilon u_i^\varepsilon, \rho^\varepsilon) dx - \int_{\Omega} v \cdot \text{Commutator}(\tau \rho_i u_i, \rho) dx \\ - \lim_{\varepsilon \rightarrow +0} \int_{\Omega} \nabla \Delta^{-1} \operatorname{div}(\rho^\varepsilon v^\varepsilon) \cdot \tau \rho_i^\varepsilon u_i^\varepsilon dx. \end{aligned} \quad (97)$$

From (84) and (85), it follows that $\rho_i^\varepsilon u_i^\varepsilon \rightarrow \rho_i u_i$ weakly in $L_s(\Omega)$ for all $s < \frac{6\gamma}{\gamma+3}$, and hence

for all $s \in \left(\frac{2\gamma}{2\gamma-1}, \frac{6\gamma}{\gamma+3} \right)$, we have

$$\text{Commutator}(\tau \rho_i^\varepsilon u_i^\varepsilon, \rho^\varepsilon) \rightarrow \text{Commutator}(\tau \rho_i u_i, \rho) \quad \text{weakly in } L_r(\Omega),$$

where $r = \frac{2s\gamma}{2\gamma+s}$. Since the embedding of $L_r(\Omega)$ into $W_2^{-1}(\Omega)$ is compact (provided the additional condition $s > \frac{6\gamma}{5\gamma-3}$, which is deliberately consistent with the above conditions), then

$$\text{Commutator}(\tau\rho_i^\varepsilon u_i^\varepsilon, \rho^\varepsilon) \rightarrow \text{Commutator}(\tau\rho_i u_i, \rho) \quad \text{strongly in } W_2^{-1}(\Omega).$$

The relations obtained, together with (84), imply the equalities

$$\lim_{\varepsilon \rightarrow +0} \int_{\Omega} v^\varepsilon \cdot \text{Commutator}(\tau\rho_i^\varepsilon u_i^\varepsilon, \rho^\varepsilon) dx = \int_{\Omega} v \cdot \text{Commutator}(\tau\rho_i u_i, \rho) dx. \quad (98)$$

Equation (12) gives the identity

$$\nabla \Delta^{-1} \text{div}(\rho^\varepsilon v^\varepsilon) = \varepsilon \nabla \Delta^{-1} \Delta \rho^\varepsilon + \varepsilon \nabla \Delta^{-1} \left(\frac{m}{|\Omega|} - \rho^\varepsilon \right),$$

which makes it clear that the last summand in (97) equals zero. Here, we used the representation $\varepsilon \nabla \Delta^{-1} \Delta \rho^\varepsilon = \nabla \otimes \nabla \Delta^{-1}(\varepsilon \nabla \rho^\varepsilon)$, the relation (86) and the boundedness of the Riesz operator $\nabla \otimes \nabla \Delta^{-1}$ in $L_2(\Omega)$ (see, e.g., [31], p. 348).

Thus, from (96) and (97), it follows that

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} \int_{\Omega} \left(-\mathbb{S}_i^\varepsilon : (\nabla \otimes (\tau r^\varepsilon)) + \tau \alpha_i p(\rho^\varepsilon) \rho^\varepsilon \right) dx \\ = \int_{\Omega} \left(-\mathbb{S}_i : (\nabla \otimes (\tau r)) + \tau \alpha_i \overline{p(\rho)} \rho \right) dx. \end{aligned} \quad (99)$$

6.3. Relations for Effective Viscous Fluxes

For all $i = 1, \dots, N$, we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} \int_{\Omega} (\nabla \otimes (\tau r^\varepsilon)) : \mathbb{S}_i^\varepsilon dx - \int_{\Omega} (\nabla \otimes (\tau r)) : \mathbb{S}_i dx \\ = \lim_{\varepsilon \rightarrow +0} \sum_{j=1}^N (\lambda_{ij} + 2\mu_{ij}) \int_{\Omega} (\text{div} u_j^\varepsilon) \tau \rho^\varepsilon dx - \sum_{j=1}^N (\lambda_{ij} + 2\mu_{ij}) \int_{\Omega} (\text{div} u_j) \tau \rho dx \\ + \lim_{\varepsilon \rightarrow +0} \sum_{j=1}^N (\lambda_{ij} + 2\mu_{ij}) \int_{\Omega} \left(2r^\varepsilon \cdot \nabla \tau + (\Delta \tau) \Delta^{-1} \rho^\varepsilon \right) (\text{div} u_j^\varepsilon) dx \\ - \sum_{j=1}^N (\lambda_{ij} + 2\mu_{ij}) \int_{\Omega} \left(2r \cdot \nabla \tau + (\Delta \tau) \Delta^{-1} \rho \right) (\text{div} u_j) dx \\ + \int_{\Omega} \left(\nabla \otimes ((\nabla \tau) \Delta^{-1} \rho) \right) : \mathbb{S}_i dx - \lim_{\varepsilon \rightarrow +0} \int_{\Omega} \left(\nabla \otimes (\nabla \tau) \Delta^{-1} \rho^\varepsilon \right) : \mathbb{S}_i^\varepsilon dx, \end{aligned} \quad (100)$$

where the last four integrals eliminate each other, and hence the equalities (99) convert to the following relations for the effective viscous fluxes of the mixture components:

$$\begin{aligned} \lim_{\varepsilon \rightarrow +0} \int_{\Omega} \left(\alpha_i p(\rho^\varepsilon) - \sum_{j=1}^N (\lambda_{ij} + 2\mu_{ij}) \text{div} u_j^\varepsilon \right) \tau \rho^\varepsilon dx \\ = \int_{\Omega} \left(\alpha_i \overline{p(\rho)} - \sum_{j=1}^N (\lambda_{ij} + 2\mu_{ij}) \text{div} u_j \right) \tau \rho dx. \end{aligned} \quad (101)$$

Using that, we can derive from (101) the following consequence:

$$\lim_{\varepsilon \rightarrow +0} \int_{\Omega} (\sigma_0 p(\rho^\varepsilon) - \operatorname{div} v^\varepsilon) \tau \rho^\varepsilon dx = \int_{\Omega} (\sigma_0 \overline{p(\rho)} - \operatorname{div} v) \tau \rho dx, \quad (102)$$

where $\sigma_0 = (\mathbf{\Lambda} + 2\mathbf{M})^{-1} \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} > 0$ (see (4)), and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N)^T$. In view of the fact that τ is arbitrary (see (93)), the equality (102) expresses the crucial relation

$$\sigma_0 \overline{\rho p(\rho)} - \overline{\rho \operatorname{div} v} = \sigma_0 \overline{\rho p(\rho)} - \rho \operatorname{div} v \quad \text{a. e. in } \Omega. \quad (103)$$

6.4. Finalizing Analysis

According to Remark 3, the renormalized Equation (1) is valid. In particular, for the functions $R \in C[0, \infty) \cap C^1(0, \infty)$ such that

$$|R'(s)| \leq B_{76} s^{-p} \quad \forall s \in (0, 1], \quad p < 1,$$

$$|R'(s)| \leq B_{77} s^q \quad \forall s \in (1, +\infty), \quad q \in (-1, \gamma - 1],$$

the following equations are valid in $\mathcal{D}'(\mathbb{R}^3)$:

$$\operatorname{div}(R(\rho)v) + (\operatorname{div} v)(\rho R'(\rho) - R(\rho)) = 0,$$

which yields

$$\int_{\Omega} (\operatorname{div} v) \rho dx = 0 \quad (104)$$

provided that $R(s) = s \ln s$.

On the other hand, multiplying (12) by $\frac{\rho^\varepsilon}{\rho^\varepsilon + \delta} + \ln(\rho^\varepsilon + \delta)$, $\delta \in (0, 1]$, integrating over Ω , then using elementary estimates, and finally passing to the limit (first as $\delta \rightarrow +0$, and then as $\varepsilon \rightarrow +0$), we obtain the inequality

$$\int_{\Omega} \overline{(\operatorname{div} v) \rho} dx \leq 0. \quad (105)$$

Combining (104) and (105), we arrive at the inequality

$$\int_{\Omega} (\overline{(\operatorname{div} v) \rho} - (\operatorname{div} v) \rho) dx \leq 0. \quad (106)$$

In view of the monotonicity of the function $p(\cdot)$, the pointwise inequality

$$(p(\rho^\varepsilon) - p(\rho))(\rho^\varepsilon - \rho) \geq 0$$

takes place. Due to this and the Formulas (85) and (87), we derive

$$\begin{aligned} \liminf_{\varepsilon \rightarrow +0} \int_V (\rho^\varepsilon p(\rho^\varepsilon) - \rho p(\rho^\varepsilon)) dx \\ = \lim_{\varepsilon \rightarrow 0} \int_V (\rho^\varepsilon - \rho) p(\rho) dx + \liminf_{\varepsilon \rightarrow 0} \int_V (p(\rho^\varepsilon) - p(\rho))(\rho^\varepsilon - \rho) dx \geq 0, \end{aligned} \quad (107)$$

where V is an arbitrary ball in Ω , and hence $\overline{p(\rho)\rho} \geq \rho \overline{p(\rho)}$ a. e. in Ω .

Now, (103) and (107) provide that

$$\overline{(\operatorname{div} v)}\rho - (\operatorname{div} v)\rho \geq 0 \quad \text{a. e. in } \Omega.$$

Employing (106), we obtain

$$\overline{(\operatorname{div} v)}\rho - (\operatorname{div} v)\rho = 0 \quad \text{a. e. in } \Omega.$$

Now, from (103), we deduce

$$\overline{p(\rho)}\rho = \rho\overline{p(\rho)} \quad \text{a. e. in } \Omega.$$

Finally, treating temporarily the function p as an odd one defined also on $(-\infty, 0]$ (preserving the initial notation), i.e., accepting $p(s) := \operatorname{sign}(s)p(|s|)$, in order to apply Lemma 3.39 (p. 188) from [32], we arrive at the required relation (92).

As a result, it is shown that the functions ρ, u_i are a weak solution to the problems (1), (2), (6) and (7) (see Definition 1). Theorem 1 is proved.

7. Conclusions

The solvability of the boundary value problem for the differential equations of the dynamics of barotropic compressible viscous multifluids in a bounded three-dimensional domain with an immobile solid boundary is analyzed. An approximate boundary value problem with a small approximation parameter is formulated, and the existence of a strong generalized solution to this problem is proved using the Leray–Schauder fixed point principle. Next, the estimates independent of the approximation parameter are obtained and, on their basis, a weak limit transition is made as the approximation parameter tends to zero, which allowed finally to prove the existence of a weak generalized solution of the boundary value problem under the weak restrictions on the viscosity matrices and the constitutive equations for pressure and momentum exchange. In the process of the limit transition, alongside the standard methods of the theory of nonlinear partial differential equations, the method of the effective viscous flux (modified for the case of multifluids) is used. This method proved effective in the analysis of compressible Navier–Stokes (single-component) equations. However, it has not been designed for the equations of the dynamics of multifluids, since the effective viscous flux in this case, due to the composite form of viscous stress tensors, is not a scalar but a vector and contains absolutely new terms.

The obtained results can be used in further mathematical studies of the boundary value problems for three-dimensional differential equations of the dynamics of compressible viscous multifluids with non-diagonal viscosity matrices (taking into account the dependence of coefficients in the equations on the unknown values, non-zero boundary conditions, non-stationary boundary value problems, etc.). In addition, the considered models and methods can serve as a basis for the construction of numerical algorithms for solving the equations of the multifluid dynamics taking into account the intercomponent viscous friction, which can play an important role in the development of specific technological processes related to multifluids.

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References

1. Lions, P.L. Existence globale de solutions pour les équations de Navier–Stokes compressible isentropiques. *C. R. Acad. Sci. Paris* **1993**, *316*, 1335–1340.
2. Lions, P.L. Compacité des solutions des équations de Navier–Stokes compressible isentropiques. *C. R. Acad. Sci. Paris* **1993**, *317*, 115–120.
3. Lions, P.L. *Mathematical Topics in Fluid Mechanics. Compressible Models*; Oxford University Press: New York, NY, USA, 1998; Volume 2.
4. Lions, P.L. Bornes sur la densité pour les équations de Navier–Stokes compressibles isentropiques avec conditions aux limites de Dirichlet. *C. R. Acad. Sci. Paris* **1999**, *328*, 659–662. [\[CrossRef\]](#)
5. Feireisl, E.; Matusu–Necasova, S.; Petzeltova, H.; Straskraba, I. On the motion of a viscous compressible fluid driven by a time-periodic external force. *Arch. Rat. Mech. Anal.* **1999**, *149*, 69–96. [\[CrossRef\]](#)
6. Feireisl, E. On compactness of solutions to the compressible isentropic Navier–Stokes equations when the density is not square integrable. *Comment. Math. Univ. Carol.* **2001**, *42*, 83–98.
7. Feireisl, E. Some recent results on the existence of global-in-time weak solutions to the Navier–Stokes equations of a general barotropic fluid. *Math. Bohemica* **2001**, *2*, 203–209. [\[CrossRef\]](#)
8. Feireisl, E.; Novotny, A.; Petzeltova, H. On the existence of globally defined weak solutions to the Navier–Stokes equations. *J. Math. Fluid Mech.* **2001**, *3*, 358–392. [\[CrossRef\]](#)
9. Feireisl, E. *Dynamics of Viscous Compressible Fluids*; Oxford University Press: Oxford, UK, 2004.
10. Nigmatulin, R.I. *Dynamics of Multiphase Media*; Hemisphere: New York, NY, USA, 1990; Volume 1–2.
11. Rajagopal, K.L.; Tao, L. *Mechanics of Mixtures*; World Scientific Publishing: Hackensack, NJ, USA, 1995.
12. Mamontov, A.E.; Prokudin, D.A. Viscous compressible homogeneous multi-fluids with multiple velocities: Barotropic existence theory. *Sib. Electron. Math. Rep.* **2017**, *14*, 388–397.
13. Frehse, J.; Goj, S.; Malek, J. On a Stokes-like system for mixtures of fluids. *SIAM J. Math. Anal.* **2005**, *36*, 1259–1281. [\[CrossRef\]](#)
14. Frehse, J.; Goj, S.; Malek, J. A uniqueness result for a model for mixtures in the absence of external forces and interaction momentum. *Appl. Math.* **2005**, *50*, 527–541. [\[CrossRef\]](#)
15. Frehse, J.; Weigant, W. On quasi-stationary models of mixtures of compressible fluids. *Appl. Math.* **2008**, *53*, 319–345. [\[CrossRef\]](#)
16. Mamontov, A.E.; Prokudin, D.A. Solubility of a stationary boundary-value problem for the equations of motion of a one-temperature mixture of viscous compressible heat-conducting fluids. *Izv. Math.* **2014**, *78*, 554–579. [\[CrossRef\]](#)
17. Mamontov, A.E.; Prokudin, D.A. Solubility of steady boundary value problem for the equations of polytropic motion of multicomponent viscous compressible fluids. *Sib. Electron. Math. Rep.* **2016**, *13*, 664–693.
18. Mamontov, A.E.; Prokudin, D.A. Global unique solvability of the initial-boundary value problem for the equations of one-dimensional polytropic flows of viscous compressible multifluids. *J. Math. Fluid Mech.* **2019**, *21*, 9. [\[CrossRef\]](#)
19. Mamontov, A.E.; Prokudin, D.A. Solubility of unsteady equations of the three-dimensional motion of two-component viscous compressible heat-conducting fluids. *Izv. Math.* **2021**, *85*, 755–812. [\[CrossRef\]](#)
20. Denisova, I.V.; Solonnikov, V.A. Global solvability of a problem governing the motion of two incompressible capillary fluids in a container. *J. Math. Sci.* **2012**, *185*, 20–52. [\[CrossRef\]](#)
21. Zatorska, E. On the flow of chemically reacting gaseous mixture. *J. Differ. Equ.* **2012**, *253*, 3471–3500. [\[CrossRef\]](#)
22. Bresch, D.; Giovangigli, V.; Zatorska, E. Two-velocity hydrodynamics in fluid mechanics: Part I. Well posedness for zero Mach number systems. *J. Math. Pures Appl.* **2015**, *104*, 762–800. [\[CrossRef\]](#)
23. Bresch, D.; Desjardins, B.; Zatorska, E. Two-velocity hydrodynamics in fluid mechanics: Part II. Existence of global k -entropy solutions to the compressible Navier–Stokes systems with degenerate viscosities. *J. Math. Pures Appl.* **2015**, *104*, 801–836. [\[CrossRef\]](#)
24. Giovangigli, V.; Pokorný, M.; Zatorska, E. On the steady flow of reactive gaseous mixture. *Analysis* **2015**, *35*, 319–341. [\[CrossRef\]](#)
25. Mucha, P.B.; Pokorný, M.; Zatorska, E. Heat-conducting, compressible mixtures with multicomponent diffusion: Construction of a weak solution. *SIAM J. Math. Anal.* **2015**, *47*, 3747–3797. [\[CrossRef\]](#)
26. Feireisl, E. On weak solutions to a diffuse interface model of a binary mixture of compressible fluids. *Discret. Contin. Dyn. Syst. Ser. S* **2016**, *9*, 173–183. [\[CrossRef\]](#)
27. Chebotarev, A.Y. Inhomogeneous boundary-value problem of radiation heat transfer for a multicomponent medium. *Dal'nevostochnyi Matematicheskii Zhurnal* **2021**, *20*, 113–121. [\[CrossRef\]](#)
28. Maz'ja, V.G. *Sobolev Spaces*; Springer: Berlin, Germany, 1985.
29. Nikol'skii, S.M. *Approximation of Functions of Several Variables and Imbedding Theorems*; Springer: Berlin, Germany, 1975.
30. DiPerna, R.J.; Lions, P.L. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **1989**, *98*, 511–547. [\[CrossRef\]](#)
31. Feireisl, E.; Novotny, A. *Singular Limits in Thermodynamics of Viscous Fluids*; Birkhauser: Basel, Switzerland, 2009.
32. Novotny, A.; Straskraba, I. *Introduction to the Mathematical Theory of Compressible Flow*; Oxford University Press: Oxford, UK, 2004.
33. Gilbarg, D.; Trudinger, N.S. *Elliptic Partial Differential Equations of Second Order*; Springer: Berlin, Germany, 2001.

34. Mucha, P.; Pokorný, M. Weak solutions to equations of steady compressible heat conducting fluids. *Math. Model. Methods Appl. Sci.* **2010**, *20*, 785–813. [[CrossRef](#)]
35. Ladyzhenskaya, O.A. *The Boundary Value Problems of Mathematical Physics*; Springer: New York, NY, USA, 1985.
36. Solonnikov, V.A. On general boundary value problems for systems which are elliptic in the sense of A. Douglis and L. Nirenberg. II. *Proc. Steklov Inst. Math.* **1968**, *92*, 269–339.
37. Agmon, S.; Douglis, A.; Nirenberg, L. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. *Comm. Pure Appl. Math.* **1964**, *17*, 35–92. [[CrossRef](#)]
38. Coifman, R.; Meyer, Y. On commutators of singular integrals. *Trans. Am. Math. Soc.* **1975**, *212*, 315–331. [[CrossRef](#)]
39. Tartar, L. Compensated compactness and applications to partial differential equations. *Nonlinear Anal. Mech.* **1979**, *39*, 136–212.

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