

Perturbed Dirac Operators and Boundary Value Problems

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Abstract: In this paper, the time-independent Klein-Gordon equation in $\mathbb{R}^3$ is treated with a decomposition of the operator $\Delta - \gamma^2 I$ by the Clifford algebra $Cl(V_{3,3})$. Some properties of integral operators associated the kind of equations and some Riemann-Hilbert boundary value problems for perturbed Dirac operators are investigated.

Keywords: Dirac operators; Clifford algebra; boundary value problems

MSC: 30G35; 34B15; 11F85; 32A55

1. Introduction

The elliptic operator $\Delta - \gamma^2 I$ where I means the identical operator in $\mathbb{R}^3$ and $\gamma > 0$, corresponds to the time-independent Klein-Gordon equation in $\mathbb{R}^3$:

$$(\Delta - \gamma^2)u = 0. \quad (1)$$

There exists a fundamental solution with the form of

$$E_1(\mathbf{x}, \gamma^2) = \frac{e^{-\gamma|\mathbf{x}|}}{4\pi|\mathbf{x}|}, \quad (2)$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ is identified with $\mathbf{x} = e_1 x_1 + e_2 x_2 + e_3 x_3$. A variety of problems in scientific computing, quantum mechanics and engineering require the efficient solution of the kind of equation [1–15]. This equation is also sometimes referred to as the Yukawa equation and the modified Helmholtz equation. The kind of partial differential equations in bounded domains of $\mathbb{R}^3$ and associated to boundary value problems in Clifford analysis have been studied. We refer to [2–4,9,10,16–28]. Especially, L. Gu, J. Du and D. Cai considered some properties of pseudo-harmonic functions and study the R_m ($m > 0$) Riemann boundary value problem with values in a Clifford algebra $Cl(V_{n,n})$ in [22]. Based on obtain generalized Cauchy type integral representation formulas, authors established Schwarz lemmas for the null solutions of perturbed Dirac operators in $\mathbb{R}^3$, which can be used to process a kind of Dirichlet boundary value problem with perturbed Dirac operators in [25], and derived the nonlinear Riemann-Hilbert problem in Clifford Hölder space in [26]. In [29], based on integral representations for harmonic functions and regular functions with value in Clifford algebra, Z. Zhang proved for some properties of operators in Clifford analysis.

Motivated by [25,26], we consider nonlinear Riemann-Hilbert problems by the Clifford algebra $Cl(V_{3,3})$. The structure is set as follows: In Section 2, we recall some involved theories about Clifford algebra. In Section 3, we investigate some properties of integral operators associated to perturbed Dirac operators. In Section 4, we obtain the existence of solutions of perturbed Dirac operators with linear boundary value conditions. In Sections 5 and 6, some nonlinear Riemann-Hilbert problems for perturbed Dirac operators are studied and error estimations for the approximate solutions are given by Newton embedding method in Hölder space.



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2. Preliminaries

Denote the free $\mathbb{R}$ -algebra with indeterminants $\{e_1, e_2, e_3\}$ by $\mathcal{A} := \mathbb{R}(e_1, e_2, e_3)$ and the two-sided ideal in $\mathcal{A}$ by J . Assume the free $\mathbb{R}$ -algebra $\mathcal{A}$ is generated by the elements

$$\{e_i^2 - 1, i = 1, \dots, s; e_i^2 + 1, i = s + 1, \dots, n; e_i e_j + e_j e_i, 1 \leq i < j \leq 3\}.$$

The quotient algebra $Cl(V_{3,s}) := \mathcal{A}/J$ is named the Clifford algebra with parameters $3, s$. With no potential for ambiguity, we take the conventional approach with the consistent symbol to denote an indeterminant e_i in $\mathcal{A}$ and its equivalent class in $\mathcal{A}/J$.

Hence, e_1, e_2, e_3 considered as elements of $\mathcal{A}/J$ satisfy the following relations:

$$\begin{cases} e_i^2 = 1, & i = 1, \dots, s, \\ e_i^2 = -1, & i = s + 1, \dots, 3, \\ e_i e_j + e_j e_i = 0, & i \neq j. \end{cases}$$

Let

$$e_{l_1 \dots l_r} := e_{l_1} \cdots e_{l_r}, \quad \text{for } 1 \leq l_1 < \cdots < l_r \leq 3.$$

For more information on $Cl(V_{3,s})$, it can be referred to [2–4,30,31].

In this article, we only consider $s = 3$, which yields that $Cl(V_{3,3})$ is a real linear non-commutative algebra. Then define the involution as

$$\begin{cases} \bar{e}_A = (-1)^{\frac{n(A)(n(A)+3)}{2}} e_A, & \text{if } A \in \mathcal{P}N, \\ \bar{\lambda} = \sum_{A \in \mathcal{P}N} \lambda_A \bar{e}_A, & \text{if } \lambda = \sum_{A \in \mathcal{P}N} \lambda_A e_A, \end{cases}$$

where

$$\{e_A, A = \{l_1, \dots, l_r\} \in \mathcal{P}N, 1 \leq l_1 < \cdots < l_r \leq n\},$$

$n(A)$ is the cardinal number of the set A , N means the set $\{1, 2, 3\}$ and $\mathcal{P}N$ denotes the family of all order-preserving subsets of N . Define the norm of λ as

$$|\lambda| = \left(\sum_{A \in \mathcal{P}N} |\lambda_A|^2 \right)^{\frac{1}{2}}.$$

If $\Re(\lambda)$ is the scalar portion of $\lambda \in Cl(V_{3,3})$, then we note that

$$\Re(\lambda \bar{\lambda}) = \Re(\bar{\lambda} \lambda) = \sum_{A \in \mathcal{P}N} |\lambda_A|^2 = |\lambda|^2.$$

Denote D as the Dirac operator in $\mathbb{R}^3$:

$$D = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}.$$

Using Clifford algebra $Cl(V_{3,3})$ and the Dirac operator, a formal factorization of $\Delta - \gamma^2$ is given by

$$\Delta - \gamma^2 I = (D - \gamma I)(D + \gamma I).$$

Then the time-independent Klein-Gordon Equation (1) can be written as

$$(\Delta - \gamma^2 I)u = (D - \gamma I)(D + \gamma I)u = 0, \quad (3)$$

here notice that $(D - \gamma I)(D + \gamma I) = (D + \gamma I)(D - \gamma I)$. Combining (2) with (3), we obtain that

$$K_\gamma(\mathbf{x}) = \frac{1}{4\pi} \left(\frac{\mathbf{x}}{|\mathbf{x}|^3} + \frac{\gamma \mathbf{x}}{|\mathbf{x}|^2} + \frac{\gamma}{|\mathbf{x}|} \right) e^{-\gamma|\mathbf{x}|}, \quad (4)$$

and

$$K_{*\gamma}(\mathbf{x}) = \frac{1}{4\pi} \left(\frac{\mathbf{x}}{|\mathbf{x}|^3} + \frac{\gamma \mathbf{x}}{|\mathbf{x}|^2} - \frac{\gamma}{|\mathbf{x}|} \right) e^{-\gamma|\mathbf{x}|}. \quad (5)$$

It is clear that $K_\gamma(\mathbf{x})$ and $K_{*\gamma}(\mathbf{x})$ are fundamental solutions of $D + \gamma I$ and $D - \gamma I$, respectively. Furthermore, elementary properties of the perturbed Dirac operators and function theory can also be found in [4,21,23,25].

3. Integral Operators Associated to Perturbed Dirac Operators

Throughout the following sections, we denote Ω as a bounded, open set of $\mathbb{R}^3$ with a Lyapunov boundary $\partial\Omega$, where $\Omega^+ = \Omega$, $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega}$.

Definition 1. The function $f: \partial\Omega(\subset \mathbb{R}^3) \rightarrow Cl(V_{3,3})$ is called as Hölder continuous if and only if for arbitrary $\mathbf{x}, \mathbf{y} \in \partial\Omega$, there are constants $C > 0$ and $0 < \alpha < 1$ satisfying

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq C|\mathbf{x} - \mathbf{y}|^\alpha.$$

Denote by $\mathbf{C}^{0,\alpha}(\partial\Omega)$ ($0 < \alpha < 1$) the set of Hölder continuous with values in $Cl(V_{3,3})$ on $\partial\Omega$. We define the norm in $\mathbf{C}^{0,\alpha}(\partial\Omega)$ as

$$\|f\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \triangleq \sup_{\mathbf{x} \in \partial\Omega} |f(\mathbf{x})| + \sup_{\substack{\mathbf{x}_1, \mathbf{x}_2 \in \partial\Omega \\ \mathbf{x}_1 \neq \mathbf{x}_2}} \frac{|f(\mathbf{x}_1) - f(\mathbf{x}_2)|}{|\mathbf{x}_1 - \mathbf{x}_2|^\alpha}.$$

It is easy to check that $\mathbf{C}^{0,\alpha}(\partial\Omega)$ is a Banach space with respect to the above norm. Next, we consider the following integral operators $\mathbb{F}_{\pm\gamma}$ and singular integral operators $\mathbb{S}_{\pm\gamma}$:

$$\mathbb{F}_\gamma[f](\mathbf{x}) \triangleq \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} f(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega, \quad (6)$$

$$\mathbb{F}_{-\gamma}[f](\mathbf{x}) \triangleq \int_{\partial\Omega} K_\gamma(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} f(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega, \quad (7)$$

$$\mathbb{S}_\gamma[f](\mathbf{x}) \triangleq 2 \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} f(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega, \quad (8)$$

$$\mathbb{S}_{-\gamma}[f](\mathbf{x}) \triangleq 2 \int_{\partial\Omega} K_\gamma(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} f(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega, \quad (9)$$

where $\gamma > 0$, $f \in \mathbf{C}^{0,\alpha}(\partial\Omega)$ and $d\sigma_{\mathbf{y}} = e_1 dy_2 \wedge dy_3 + e_2 dy_3 \wedge dy_1 + e_3 dy_1 \wedge dy_2$.

Lemma 1 ([22]). Let Ω be an open nonempty bounded subset of $\mathbb{R}^3$ with a Lyapunov boundary $\partial\Omega$, $f \in \mathbf{C}^{0,\alpha}(\partial\Omega)$ $0 < \alpha < 1$. Then

$$\lim_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \in \partial\Omega \\ \mathbf{x} \in \Omega}} \mathbb{F}_\gamma[f](\mathbf{x}) = \frac{f(\mathbf{x}_0)}{2} + \frac{1}{2} \mathbb{S}_\gamma[f](\mathbf{x}_0), \quad (10)$$

$$\lim_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \in \partial\Omega \\ \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}}} \mathbb{F}_\gamma[f](\mathbf{x}) = -\frac{f(\mathbf{x}_0)}{2} + \frac{1}{2} \mathbb{S}_\gamma[f](\mathbf{x}_0), \quad (11)$$

$$\lim_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \in \partial\Omega \\ \mathbf{x} \in \Omega}} \mathbb{F}_{-\gamma}[f](\mathbf{x}) = \frac{f(\mathbf{x}_0)}{2} + \frac{1}{2} \mathbb{S}_{-\gamma}[f](\mathbf{x}_0), \quad (12)$$

$$\lim_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \in \partial\Omega \\ \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}}} \mathbb{F}_{-\gamma}[f](\mathbf{x}) = -\frac{f(\mathbf{x}_0)}{2} + \frac{1}{2} \mathbb{S}_{-\gamma}[f](\mathbf{x}_0). \quad (13)$$

Lemma 2 ([23]). Let Ω be an open nonempty bounded subset of $\mathbb{R}^3$ with a Lyapunov boundary $\partial\Omega$. Then the singular integral operators $\mathbb{S}_{\pm\gamma} : \mathbf{C}^{0,\alpha}(\partial\Omega) \rightarrow \mathbf{C}^{0,\alpha}(\partial\Omega)$ defined by (8) and (9) is bounded, i.e.,

$$\|\mathbb{S}_{\pm\gamma}[f]\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \leq C\|f\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)},$$

where C is a positive constant depending only on $|\partial\Omega|$, α and γ .

Remark 1. Notice that $\mathbb{S}_{+\gamma} = \mathbb{S}_{\gamma}$.

For the sake of simplicity, we denote

$$f^{\pm}(\mathbf{x}) = \lim_{\substack{\mathbf{y} \rightarrow \mathbf{x} \in \partial\Omega \\ \mathbf{y} \in \Omega^{\pm}}} f(\mathbf{y}).$$

Theorem 1. For a given function $u \in \mathbf{C}^{0,\alpha}(\partial\Omega)$, there exists a function $f \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\overline{\Omega})$ which satisfies the following conditions

$$\begin{cases} (D + \gamma I)f = 0, & \text{in } \Omega, \\ f|_{\partial\Omega} = u, \end{cases}$$

if and only if u is a solution of the homogeneous integral equation

$$u - \mathbb{S}_{\gamma}[u] = 0, \text{ on } \partial\Omega.$$

The solutions is given by

$$f(\mathbf{x}) = \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} u(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega. \quad (14)$$

Proof. Due to $(D + \gamma I)f = 0$ in Ω and $f|_{\partial\Omega} = u$. Then with the aid of integral representation formula in Theorem 1 in [25] we obtain that

$$\begin{aligned} f(\mathbf{x}) &= \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} f(\mathbf{y}) \\ &= \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} u(\mathbf{y}), \quad \mathbf{x} \in \Omega. \end{aligned}$$

Using (10) in Lemma 1, it follows that

$$\begin{aligned} u(\mathbf{x}_0) &= f^+(\mathbf{x}_0) \\ &= \frac{u(\mathbf{x}_0)}{2} + \frac{1}{2} \mathbb{S}_{\gamma}[u](\mathbf{x}_0), \quad \mathbf{x}_0 \in \partial\Omega. \end{aligned}$$

Thus $u - \mathbb{S}_{\gamma}[u] = 0$.

Conversely, if u is a solution of $u - \mathbb{S}_\gamma[u] = 0$, then by Lemma 1 in [32] and (10) in Lemma 1 the function f defined (14) has

$$(D + \gamma I)f(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega,$$

and boundary values

$$\begin{aligned} f^+(\mathbf{x}_0) &= \frac{u(\mathbf{x}_0)}{2} + \frac{1}{2}\mathbb{S}_\gamma[u](\mathbf{x}_0) \\ &= u(\mathbf{x}_0), \quad \mathbf{x}_0 \in \partial\Omega. \end{aligned}$$

The proof is finished. $\square$

Similarly, we have the following theorem.

Theorem 2. For a given function $u \in \mathbf{C}^{0,\alpha}(\partial\Omega)$, there exists a function $f \in \mathbf{C}^1(\Omega^-) \cap \mathbf{C}(\overline{\Omega^-})$ which satisfies the following conditions

$$\begin{cases} (D + \gamma I)f = 0, & \text{in } \Omega^-, \\ f|_{\partial\Omega} = u, \end{cases}$$

if and only if u is a solution of the homogeneous integral equation

$$u + \mathbb{S}_\gamma[u] = 0, \text{ on } \partial\Omega.$$

The solutions is given by

$$f(\mathbf{x}) = - \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} u(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega. \quad (15)$$

Proof. In view of $(D + \gamma I)f = 0$ in Ω^- and $f|_{\partial\Omega} = u$. Using integral representation formula in Theorem 1 in [25], it yields that

$$\begin{aligned} f(\mathbf{x}) &= - \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} f(\mathbf{y}) \\ &= - \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} u(\mathbf{y}), \quad \mathbf{x} \in \Omega^-. \end{aligned}$$

By (10) in Lemma 1, we obtain that

$$\begin{aligned} u(\mathbf{x}_0) &= f^-(\mathbf{x}_0) \\ &= \frac{u(\mathbf{x}_0)}{2} - \frac{1}{2}\mathbb{S}_\gamma[u](\mathbf{x}_0), \quad \mathbf{x}_0 \in \partial\Omega. \end{aligned}$$

Therefore $u - \mathbb{S}_\gamma[u] = 0$.

Conversely, if u is a solution of $u + \mathbb{S}_\gamma[u] = 0$, then by Lemma 1 in [32] and (11) in Lemma 1 the function f defined (15) has

$$(D + \gamma I)f(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega^-,$$

and boundary values

$$\begin{aligned} f^-(\mathbf{x}_0) &= \frac{u(\mathbf{x}_0)}{2} - \frac{1}{2}\mathbb{S}_\gamma[u](\mathbf{x}_0) \\ &= u(\mathbf{x}_0), \quad \mathbf{x}_0 \in \partial\Omega. \end{aligned}$$

The proof is completed. $\square$

Theorem 3. The singular integral operator $\mathbb{S}_\gamma$ satisfies $\mathbb{S}_\gamma^2 = I$.

Proof. For $f \in \mathbf{C}^{0,\alpha}(\partial\Omega)$ we define

$$u(\mathbf{x}) = \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} f(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega.$$

Using Lemma 1, Theorems 1 and 2, we get

$$\begin{cases} u^+ = \frac{f}{2} + \frac{1}{2}\mathbb{S}_\gamma[f], \\ u^- = -\frac{f}{2} + \frac{1}{2}\mathbb{S}_\gamma[f], \end{cases}$$

and

$$\begin{cases} u^+ - \mathbb{S}_\gamma[u^+] = 0, \\ u^- + \mathbb{S}_\gamma[u^-] = 0. \end{cases}$$

Therefore we derive

$$\mathbb{S}_\gamma^2[f] = \mathbb{S}_\gamma[u^+ + u^-] = u^+ - u^- = f.$$

The proof is done. $\square$

By the same method we obtain the following results.

Theorem 4. For a given function $u \in \mathbf{C}^{0,\alpha}(\partial\Omega)$, there exists a function $f \in \mathbf{C}^1(\Omega) \cap \mathbf{C}(\overline{\Omega})$ which satisfies the following conditions

$$\begin{cases} (D - \gamma I)f = 0, & \text{in } \Omega, \\ f|_{\partial\Omega} = u, \end{cases}$$

if and only if u is a solution of the homogeneous integral equation

$$u - \mathbb{S}_{-\gamma}[u] = 0, \text{ on } \partial\Omega.$$

The solutions is given by

$$f(\mathbf{x}) = \int_{\partial\Omega} K_\gamma(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} u(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega.$$

Theorem 5. For a given function $u \in \mathbf{C}^{0,\alpha}(\partial\Omega)$, there exists a function $f \in \mathbf{C}^1(\Omega^-) \cap \mathbf{C}(\overline{\Omega^-})$ which satisfies the following conditions

$$\begin{cases} (D - \gamma I)f = 0, & \text{in } \Omega^-, \\ f|_{\partial\Omega} = u, \end{cases}$$

if and only if u is a solution of the homogeneous integral equation

$$u + \mathbb{S}_{-\gamma}[u] = 0, \text{ on } \partial\Omega.$$

The solutions is given by

$$f(\mathbf{x}) = \int_{\partial\Omega} K_\gamma(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} u(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega.$$

Theorem 6. The singular integral operator $\mathbb{S}_\gamma$ satisfies $\mathbb{S}_{-\gamma}^2 = I$.

Remark 2. Theorems 1–6 imply that the singular integral operator $\mathbb{S}_{\pm\gamma}$ are not compact, and also mean that $I \pm \mathbb{S}_{\pm\gamma}$ cannot be regularized.

4. Riemann-Hilbert Boundary Value Problems for Perturbed Dirac Operators

In this section, we consider the following Riemann-Hilbert boundary value problem:

$$\begin{cases} (D + \gamma I)f = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ f^+(\mathbf{x}) = f^-(\mathbf{x})u(\mathbf{x}) + g(\mathbf{x}), & \mathbf{x} \in \partial\Omega \\ \lim_{|\mathbf{x}| \rightarrow \infty} |f(\mathbf{x})| = 0, \end{cases} \quad (16)$$

where $u(\mathbf{x})$ and $g(\mathbf{x})$ are Clifford value functions in $\mathbf{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha < 1$.

Theorem 7. Suppose $u(\mathbf{x})$ satisfies the following condition

$$\|1 - u\|_{\mathbf{C}^\alpha(\partial\Omega)} < \frac{C + 1}{2}, \quad (17)$$

where C is a positive constant mentioned in Theorem 2. Then (16) there exists a unique solution to Riemann type problem (16).

Proof. In view of Lemma 1 in [32], the solution to this Riemann-Hilbert problem (7) can be written in the form

$$f(\mathbf{x}) = \int_{\partial\Omega} K_{*\gamma}(\mathbf{y} - \mathbf{x}) d\sigma_{\mathbf{y}} \varphi(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega \quad (18)$$

where $\varphi(\mathbf{y})$ is a Hölder continuous function to be determined on $\partial\Omega$. Using Lemma 1 and boundary value condition in (16), the (18) can be reduced to an equivalent singular integral equation for φ ,

$$\varphi(\mathbf{x}) = (\varphi(\mathbf{x}) - \mathbb{S}_\gamma[\varphi](\mathbf{x})) \left(\frac{1 - u(\mathbf{x})}{2} \right) + g(\mathbf{x}). \quad (19)$$

Let $\mathcal{M}$ be an integral operator defined by the right hand side of (19). We get

$$(\mathcal{M}\varphi)(\mathbf{x}) = (\varphi(\mathbf{x}) - \mathbb{S}_\gamma[\varphi](\mathbf{x})) \left(\frac{1 - u(\mathbf{x})}{2} \right) + g(\mathbf{x}).$$

For any $\varphi_1, \varphi_2 \in \mathbf{C}^{0,\alpha}(\partial\Omega)$, by Lemma 2, it follows that

$$\|\mathcal{M}\varphi_1 - \mathcal{M}\varphi_2\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \leq \frac{1}{2}(1 + C)\|\varphi_1 - \varphi_2\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)}\|1 - u\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)}.$$

From (17), the operator $\mathcal{M}$ is a contraction operator mapping the Banach space $\mathbf{C}^{0,\alpha}(\partial\Omega)$ into itself. Then the operator $\mathcal{M}$ has a unique fixed point. Thus there exists a unique solution to (16). The proof is finished. $\square$

Remark 3. By the same technique in the above proof of Theorem 7, we also consider the following Riemann-Hilbert boundary value problem:

$$\begin{cases} (D - \gamma I)f = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ f^+(\mathbf{x}) = f^-(\mathbf{x})u(\mathbf{x}) + g(\mathbf{x}), & \mathbf{x} \in \partial\Omega \\ \lim_{|\mathbf{x}| \rightarrow \infty} |f(\mathbf{x})| = 0, \end{cases} \quad (20)$$

where $u(\mathbf{x})$ and $g(\mathbf{x})$ are Clifford value functions in $\mathbf{C}^{0,\alpha}(\partial\Omega)$, $0 < \alpha < 1$. Under the condition (17), The problem (20) has a unique solution.

5. The Clifford Hölder Spaces and a Non-Linear Riemann-Hilbert Type Problems

To research nonlinear Riemann-Hilbert type boundary value problem, we note the following lemmas.

Lemma 3. *The integral transform $\mathbb{F}_{\pm\gamma}f$, defined in (6) and (7), can be Hölder continuously extended from Ω into $\overline{\Omega}$ and from $\mathbb{R}^3 \setminus \overline{\Omega}$ into $\mathbb{R}^3 \setminus \Omega$, respectively, with limiting values in (10)–(13) and then we have*

$$\|\mathbb{F}_{\pm\gamma}u\|_{\mathbf{C}^{0,\alpha}(\overline{\Omega})} \leq \tilde{C}\|u\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)}, \text{ for } u \in \mathbf{C}^{0,\alpha}(\partial\Omega), 0 < \alpha < 1 \quad (21)$$

and

$$\|\mathbb{F}_{\pm\gamma}u\|_{\mathbf{C}^{0,\alpha}(\mathbb{R}^3 \setminus \Omega)} \leq \tilde{C}\|u\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)}, \text{ for } u \in \mathbf{C}^{0,\alpha}(\mathbb{R}^3 \setminus \Omega), 0 < \alpha < 1 \quad (22)$$

where some constant $\tilde{C}$ depends on $|\partial\Omega|$, α and γ .

Proof. By the weakly singular property of the kernels (4) and (5) and Theorem 3.2 in [29], we can obtain the estimates (21) and (22). $\square$

Theorem 8. *Let f be the solution of the following Riemann-Hilbert type problem:*

$$\begin{cases} (D \pm \gamma I)f = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ f^+(x) = f^-(x) + g(x), & x \in \partial\Omega, \\ \lim_{|x| \rightarrow \infty} f(x) = 0, \end{cases}$$

where $g(x)$ is Clifford value function in $\mathbf{C}^{0,\alpha}(\partial\Omega)$ $0 < \alpha < 1$. Then it yields

$$\|f\|_{\mathbf{C}^{0,\alpha}(\overline{\Omega})} \leq \tilde{C}\|g\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \quad (23)$$

and

$$\|f\|_{\mathbf{C}^{0,\alpha}(\mathbb{R}^3 \setminus \Omega)} \leq \tilde{C}\|g\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)}. \quad (24)$$

Proof. Using Lemma 3, it can be directly proved the result. $\square$

Remark 4. *For a bounded u in $\mathbf{C}^{0,\alpha}(\overline{\Omega}) \cap \mathbf{C}^{0,\alpha}(\mathbb{R}^3 \setminus \Omega)$, define the norm*

$$\|u\|_{\alpha} := \|u\|_{\mathbf{C}^{0,\alpha}(\overline{\Omega})} + \|u\|_{\mathbf{C}^{0,\alpha}(\mathbb{R}^3 \setminus \Omega)},$$

then (23) and (24) in Theorem 8 can be substituted for

$$\|f\|_{\alpha} \leq C\|g\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)}$$

where $C = 2\tilde{C}$.

Next, we consider the following nonlinear Riemann-Hilbert type problem with the form of

$$\begin{cases} (D + \gamma I)f = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ f^+(x) = f^-(x) + g(x, f^+, f^-), & x \in \partial\Omega, \\ \lim_{|x| \rightarrow \infty} f(x) = 0. \end{cases} \quad (25)$$

Firstly, we assume the following conditions is fulfilled:

(C1) For each f_1, f_2 in $C^{0,\alpha}(\partial\Omega)$ $0 < \alpha < 1$, the function

$$g(\mathbf{x}, f_1, f_2) = g(\mathbf{x}, f_1(\mathbf{x}), f_2(\mathbf{x}))$$

is a function of $\mathbf{x}$ in $C^{0,\alpha}(\partial\Omega)$.

Moreover there exists a nonnegative constant N such that $CN < 1$, where C has the same form as in Remark 4, and it yields that

$$\begin{aligned} & \|g(\cdot, f_1, f_2) - g(\cdot, \tilde{f}_1, \tilde{f}_2)\|_{C^{0,\alpha}(\partial\Omega)} \\ & \leq N[\|f_1 - \tilde{f}_1\|_{C^{0,\alpha}(\partial\Omega)} + \|f_2 - \tilde{f}_2\|_{C^{0,\alpha}(\partial\Omega)}] \end{aligned}$$

for all $f_1, \tilde{f}_1, f_2, \tilde{f}_2$ in $C^{0,\alpha}(\partial\Omega)$.

Then, we shall consider the existence of solution for the boundary value problem (25).

Theorem 9. Suppose g satisfies the above conditions (C1). Then the problem (25) has exactly one solution provided that the constant N in (C1).

Proof. Firstly, for each $t(0 \leq t \leq 1)$, we consider the problem

$$\begin{cases} (D + \gamma)[f_t] = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ f_t^+(x) = f_t^-(x) + tg(x, f_t^+, f_t^-), & x \in \partial\Omega, \\ \lim_{|\mathbf{x}| \rightarrow \infty} f_t(\mathbf{x}) = 0. \end{cases} \quad (26)$$

For $t = 1$, the problem (26) directly become to (25).

For $t = 0$, $u_0(\mathbf{x})$ is a monogenic in $\mathbb{R}^3$ vanishing at infinity so that $f_0(\mathbf{x}) \equiv 0$ is the unique solution. We now assume $f_{t_0}(\mathbf{x})$ to be a solution of (26) for a given t_0 with $0 \leq t_0 < 1$.

With the help of a combination method between an imbedding method and a Newton's method, we will show the existence of a solution of (26) for all t in $t_0 \leq t \leq t_0 + \delta$ for some $\delta > 0$ that is independent of t_0 . Then we can conclude there is a solution for $t = 1$.

Denote

$$f_t^0(\mathbf{x}) \triangleq f_{t_0}(\mathbf{x})$$

and let $f_t^{k+1}(\mathbf{x})$ to be the solution of the linear problem

$$\begin{cases} (D + \gamma)f_t^{k+1} = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ (f_t^{k+1})^+(\mathbf{x}) = (f_t^{k+1})^-(\mathbf{x}) + tg(\mathbf{x}, (f_t^k)^+, (f_t^k)^-), & \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}| \rightarrow \infty} f_t^{k+1}(\mathbf{x}) = 0. \end{cases} \quad (27)$$

Thus the linear problem (27) is uniquely solvable. The differences

$$u_t^k(\mathbf{x}) \triangleq f_t^{k+1}(\mathbf{x}) - f_t^k(\mathbf{x}), \quad k \in \mathbb{N},$$

satisfy

$$\begin{cases} D[u_t^k] = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ (u_t^k)^+(\mathbf{x}) = (u_t^k)^-(\mathbf{x}) + h_k(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}| \rightarrow \infty} u_t^k(\mathbf{x}) = 0, \end{cases}$$

where

$$h_0(\mathbf{x}) \triangleq (t - t_0)g(\mathbf{x}, (f_t^0)^+, (f_t^0)^-)$$

and

$$h_k(\mathbf{x}) \triangleq t \left[g(\mathbf{x}, (f_t^k)^+, (f_t^k)^-) - g(\mathbf{x}, (f_t^{k-1})^+, (f_t^{k-1})^-) \right], \quad k \in \mathbb{N} \setminus \{0\}.$$

By Theorem 8, we obtain that

$$\|u_t^k\|_\alpha \leq C \|h_k\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \quad k \in \mathbb{N} \setminus \{0\}. \quad (28)$$

By the condition (C1), it is easy to derive that

$$\begin{aligned} \|h_0\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} &\leq (t-t_0)N\|(f_t^0)^+\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} + (t-t_0)N\|(f_t^0)^-\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \\ &\quad + (t-t_0)\|g(\cdot, 0, 0)\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \\ &\leq (t-t_0)N\|f_t^0\|_\alpha + (t-t_0)\|g(\cdot, 0, 0)\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \end{aligned} \quad (29)$$

and

$$\begin{aligned} \|h_k\|_{H^\alpha(\partial\Omega)} &\leq tN\|(u_t^{k-1})^+\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} + tN\|(u_t^{k-1})^-\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \\ &\leq tN\|u_t^{k-1}\|_\alpha. \end{aligned} \quad (30)$$

Combining (28), (29) with (30), we have the following inequalities

$$\|u_t^0\|_\alpha \leq C(t-t_0)N\|f_t^0\|_\alpha + C(t-t_0)\|g(\cdot, 0, 0)\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \quad (31)$$

and

$$\|u_t^k\|_\alpha \leq CtN\|u_t^{k-1}\|_\alpha, \quad (32)$$

with $k \in \mathbb{N} \setminus \{0\}$.

Because $f_t^0(\mathbf{x})$ is a solution of (27) for $t = t_0$, with the method of Theorem 8, we note that the apriori estimate can be given as

$$\|f_t^0\|_\alpha \leq t_0C\|g(\cdot, (f_t^0)^+, (f_t^0)^-)\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)}.$$

Using the condition (C1), we obtain that

$$\|f_t^0\|_\alpha \leq t_0CN\|f_t^0\|_\alpha + t_0C\|g(\cdot, 0, 0)\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)}. \quad (33)$$

Define

$$\kappa \triangleq CN$$

and

$$\kappa_0 \triangleq C\|g(\cdot, 0, 0)\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)}.$$

It yields

$$\|f_t^0\|_\alpha \leq \frac{t_0\kappa_0}{1-t_0\kappa},$$

and (31) is transferred to

$$\|u_t^0\|_\alpha \leq (t-t_0) \left(\frac{t_0\kappa_0}{1-t_0\kappa} + \kappa_0 \right).$$

Then we have

$$\sum_{j=0}^k (\|f_t^{j+1}\|_\alpha - \|f_t^j\|_\alpha) \leq \sum_{k=0}^n \|u_t^k\|_\alpha$$

and use the inequalities (31)–(33) and $CN < 1$, when n tend to $+\infty$, which imply the convergence of $\{f_t^k\}_{k=0}^{+\infty}$ in the $\|\cdot\|_\alpha$.

Secondly, the limit function $f_t(\mathbf{x})$ satisfying (27) is considered. To do so we let n tend to $+\infty$ in (27).

Since convergence is with respect to the $\|\cdot\|_\alpha$ norm it follows that $f_t(\mathbf{x})$ belongs to $C^{0,\alpha}(\overline{\Omega}) \cap C^{0,\alpha}(\mathbb{R}^3 \setminus \Omega)$, and that the transmission condition of (26) is satisfied. Moreover, $\|f_t^k\|_\alpha$ are uniformly bounded, by Weierstrass' Theorem (See [30,31]), we conclude that $(D + \gamma I)f_t^{k+1}$ converge to $(D + \gamma I)f_k$ uniformly on compact subsets of $\mathbb{R}^3 \setminus \partial\Omega$ such that

$$(D + \gamma I)f_t = 0$$

in $\mathbb{R}^3 \setminus \partial\Omega$. It is clear that

$$\lim_{|\mathbf{x}| \rightarrow \infty} f_t(\mathbf{x}) = 0.$$

Hence it is easy to obtain that $f_t(\mathbf{x})$ satisfies all of (26). It follows that after finitely steps one ends up with a solution of (27) for $t = 1$, which is equal to the problem (26).

Finally, aim to complete the proof of Theorem 9, we still need to derive the uniqueness of solution. Let f_1 and f_2 be two solutions of (25). It is obvious that $f = f_1 - f_2$ is a solution of the linear

$$\begin{cases} (D + \gamma I)f = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ f^+(\mathbf{x}) = f^-(\mathbf{x}) + \tilde{g}(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}| \rightarrow \infty} f(\mathbf{x}) = 0. \end{cases}$$

where

$$\tilde{g}(\mathbf{x}) \triangleq g(\mathbf{x}, f_1^+(\mathbf{x}), f_1^-(\mathbf{x})) - g(\mathbf{x}, f_2^+(\mathbf{x}), f_2^-(\mathbf{x})).$$

Using Theorem 8 and the condition (C1) again, it is obtained that

$$\|f\|_\alpha \leq CN\|f\|_\alpha.$$

For $CN < 1$, we conclude that $f_1 = f_2$. The proof is ended. $\square$

6. Error Estimation

In this section, we shall compute the difference of the solution of (25) and its approximation $f_t^k(\mathbf{x})$.

Let

$$v_k(\mathbf{x}) \triangleq f_t(\mathbf{x}) - f_t^k(\mathbf{x}), \quad v(\mathbf{x}) \triangleq f(\mathbf{x}) - f_t(\mathbf{x})$$

with $f(\mathbf{x}) = f_1(\mathbf{x})$.

By (26) and (27), it yields

$$\begin{cases} (D + \gamma I)v_{k+1} = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ v_{k+1}^+(\mathbf{x}) = v_{k+1}^-(\mathbf{x}) + \tilde{g}_k(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \\ \lim_{|\mathbf{x}| \rightarrow \infty} v_{k+1}(\mathbf{x}) = 0, \end{cases}$$

where

$$\tilde{g}_k(\mathbf{x}) \triangleq t[g(\mathbf{x}, u_t^+, u_t^-) - g(\mathbf{x}, (u_t^k)^+, (u_t^k)^-)].$$

According to Theorem 8, we have

$$\begin{aligned}\|v_{n+1}\|_\alpha &\leq C\|\tilde{g}_k\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \\ &\leq tCN\|f_t^+ - (f_t^k)^+\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} + tCN\|f_t^- - (f_t^n)^-\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \\ &\leq tCN\|f_t - f_t^k\|_\alpha \\ &\leq tCN[\|v_{k+1}\|_\alpha + \|f_t^{k+1} - f_t^k\|_\alpha] \\ &= t\kappa[\|v_{k+1}\|_\alpha + \|u_t^k\|_\alpha].\end{aligned}$$

Further we obtain

$$\|v_{k+1}\|_\alpha \leq \frac{t\kappa}{1-t\kappa}\|u_t^k\|_\alpha.$$

By (31) and (32), we derive that

$$\|v_{k+1}\|_\alpha \leq c(t\kappa)^{k+1} \quad (34)$$

with $c = \kappa_0 \frac{2-\kappa}{(1-\kappa)^2}$.

On the other hand, the function v is a solution of

$$\begin{cases} (D + \gamma I)v = 0, & \text{in } \mathbb{R}^3 \setminus \partial\Omega, \\ v^+ = v^- + \hat{g}, & \text{on } \partial\Omega, \\ \lim_{|\mathbf{x}| \rightarrow \infty} v(\mathbf{x}) = 0, \end{cases}$$

where

$$\hat{g} \triangleq g(\mathbf{x}, f^+, f^-) - g(\mathbf{x}, f_t^+, f_t^-) + (1-t)g(\mathbf{x}, f_t^+, f_t^-).$$

Using Theorem 8 and the condition (C1) again, it can be obtained that

$$\begin{aligned}\|v\|_\alpha &\leq \kappa\|v\|_\alpha + (1-t)\kappa\|f_t\|_\alpha + (1-t)C\|g(\cdot, 0, 0)\|_{\mathbf{C}^{0,\alpha}(\partial\Omega)} \\ &= \kappa\|v\|_\alpha + (1-t)\kappa\|f_t\|_\alpha + (1-t)\kappa_0,\end{aligned}$$

furthermore

$$\begin{aligned}\|v\|_\alpha &\leq \frac{1-t}{1-\kappa}(\kappa_0 + \kappa\|f_t\|_\alpha) \\ &\leq (1-t)\frac{2-\kappa}{(1-\kappa)^2}\kappa_0 \\ &= (1-t)c.\end{aligned} \quad (35)$$

Combining (34) with (35), it is obvious to obtain the following result:

Theorem 10. The error between the solution $u(x)$ of (25) and its approximation $u_t^k(x)$ can be estimated by

$$\|f - f_t^k\|_\alpha \leq c[(t\kappa)^{k+1} + (1-t)],$$

where $c = \kappa_0 \frac{2-\kappa}{(1-\kappa)^2}$.

7. Conclusions

In the article, we have considered that Riemann-Hilbert problems on $\mathbb{R}^3$ in the sense of Hölder space. With the method of Clifford analytic approach and Newton embedding method, we prove the existence and uniqueness of solutions of the nonlinear Riemann-

Hilbert problems, which means the linear and nonlinear Riemann-Hilbert approach in higher dimensional space is feasible from Theorems 7 and 9.

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