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Solitonical Inequality on Submanifolds in Trans-Sasakian Manifolds Coupled with a Slant Factor

Mohd Danish Siddiqi * and Rawan Bossly

Department of Mathematics, College of Science, Jazan University, P.O. Box 114, Jazan 45142, Saudi Arabia; rbossly@jazanu.edu.sa
* Correspondence: msiddiqi@jazanu.edu.sa

Abstract: In this article, we study the Ricci soliton on slant submanifolds of trans-Sasakian manifolds with a quarter symmetric non-metric connection. Moreover, we derive a lower-bound-type inequality for the slant submanifolds of trans-Sasakian manifolds with a quarter symmetric non-metric connection in terms of gradient Ricci solitons. We also characterize anti-invariant, invariant, quasi-umbilical submanifolds of trans-Sasakian manifolds with a quarter symmetric non-metric connection for which the same inequality case holds. Finally, we deduce the above inequalities in terms of a scalar concircular field on submanifolds of trans-Sasakian manifolds with a quarter symmetric non-metric connection.

Keywords: trans-Sasakian manifolds; Ricci soliton; gradient Ricci soliton; scalar curvature; quarter symmetric non-metric connection

MSC: 53C15; 53C20; 53C25; 53C44; 53B05

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1. Introduction

Hamilton [1] originally introduced the concepts of Ricci flow in 1988. It proves that Ricci's soliton is the solution's limit for the Ricci flow. Additionally, over the past 20 years, many mathematicians have become interested in geometric flow theory, particularly the Ricci flow.

The family of metrics $g(t)$ on a Riemannian manifold $\mathcal{N}$ evolve into the Ricci flow [1] if

$$\frac{1}{2}\frac{\partial}{\partial t}g(t) = -\mathcal{S}_{Ric}(t)g(t), \quad g_0 = g(0).$$

Definition 1 ([1])**.** *On the Riemannian manifold,* $(\mathcal{N}, g)$*, a Ricci soliton (RS) is a data point* $(g, \mathcal{F}, \lambda)$ *obeying*

$$\frac{1}{2}L_{\mathcal{F}}g + \mathcal{S}_{Ric} + \lambda g = 0, \tag{1}$$

where the Ricci tensor is $\mathcal{S}_{Ric}$*, and* L_X *is the Lie-derivative across the vector field* $\mathcal{F}$*. Depending on the constant* λ*, the manifold* $(\mathcal{N}, g, \mathcal{F}, \lambda)$ *is called a Ricci shrinker, expander, or stable soliton, whether* $\lambda < 0$*,* $\lambda > 0$*, or* $\lambda = 0$*.*

The Equation (1) becomes

$$\mathrm{Hess}(\psi) + \mathcal{S}_{Ric} = \lambda g, \tag{2}$$

where Hess *denotes the Hessian operator with respect to* g*. This indicates that* $(\mathcal{N}, g, \psi, \lambda)$ *is a gradient Ricci soliton* (GRs)*.*

In 2020, Blaga and Carasmareanu [2] established an inequality for a lower bound of the geometry of metric g in terms of gradient Ricci soliton (GRs) for a smooth function ψ in ambient space $\mathcal{N}$ such as

$$||\mathcal{S}_{Ric}||_g^2 \geq ||\text{Hess}||_g^2 - \frac{1}{n}(\Delta\psi)^2, \tag{3}$$

where $\mathcal{S}_{Ric}$ is Ricci tensor, Hess is the Hessian of a smooth function ψ on $\mathcal{N}$.

On the other hand, Hayden [3] deserves recognition for popularizing the understanding of a metric connection on a Riemannian manifold. Several geometers have studied the features of Riemannian manifolds with non-metric and semi-symmetric (symmetric) connections ([4–6]). Golab [7] discussed the fundamental concept of quarter-symmetric linear connection. If a linear connection's torsion tensor $\mathcal{T}$ has the following form, it is considered quarter-symmetric.

$$\mathcal{T}(u,v) = \omega(v)\Phi u - \omega(u)\Phi v,$$

where ω is a 1—form, Φ represents a tensor of type $(1,1)$ and u,v are vector fields on a manifold $\mathcal{N}$. If $\Phi = I$, the quarter-symmetric connection simplifies to a semi-symmetric connection.

Therefore, it is possible to operate quarter-symmetric connections as an extension of semi-symmetric metric connections (ssm-connection).

If a Riemannian metric g exists in $\mathcal{N}$ such that

$$\overline{\nabla} g = 0,$$

then $\overline{\nabla}$ is a metric connection; if not, it is non-metric.

In [8] authors proposed an ssm-connection in almost contact manifold by the relation

$$\mathcal{T}(u,v) = \omega(v)u - \omega(u)v.$$

On the contrary, trans-Sasakian manifolds (TSM) naturally resulted from Gonzales and Chinae's category of almost contact metric structures [9]. TSM is a new type of virtual contact Riemannian manifold that was introduced in 1985 by Oubina [10]. $\alpha-$Sasakian, Sasakian, $\beta-$Kenmotsu, and Kenmotsu, cosymplectic, structures are examples of trans-Sasakian structures. A class W_4 of Hermitian manifolds that are closely connected to locally conformal Kaehler manifolds appears in the Gray–Hervella classification of nearly Hermitian manifolds [11]. When the product manifold $\mathcal{N} \times \mathbb{R}$ belongs to the class W_4, then an almost contact metric structure is referred to as a TSM structure [12]. The class of TSM structures of type (α, β) coincides with the $C_6 \oplus C_5$ class.

Integrating intrinsic and extrinsic invariants, which are useful tools for researching submanifolds categorization, is one of the most important problems in submanifold theory.

With the aid of severe inequality, Chen [13] initiated a new framework in the study of the relationship between intrinsic and extrinsic invariants in the early 1990s, and he also presented a novel tool called δ-invariants (for more information, see [14–17]). Numerous researchers ([18–27], etc.) carried out relevant research from various viewpoints in different spaces.

2. Preliminaries

An $n(= 2m+1)$-dimensional manifold $\mathcal{N}$ equipped with an almost contact metric structure $(\Phi, \varsigma, \omega, g)$ can be defined as follows: Φ is a $(1,1)$ tensor field, ς is a vector field, ω is a 1—form, and g is a compatible Riemannian metric such that

$$\Phi^2 u = -u + \omega(u)\varsigma, \quad \omega(\varsigma) = 1, \quad \Phi(\varsigma) = 0, \quad \omega o \Phi = 0, \tag{4}$$

$$g(\Phi u, \Phi v) = g(u,v) - \omega(u)\omega(v),$$

$$g(u, \Phi v) = -g(\Phi u, v), \qquad g(u, \varsigma) = \omega(u),$$

for all $u, v \in T\mathcal{N}$.

Definition 2 ([18]). *An almost-contact metric structure* $(\Phi, \varsigma, \omega, g)$ *on* $\mathcal{N}$ *is said to be a TSM structure if*

$$(\nabla_u \Phi)(v) = \alpha(g(u,v)\varsigma - \omega(v)u) + \beta(g(\Phi u, v)\varsigma - \omega(v)\Phi u), \tag{5}$$

where α *and* β *are smooth functions on* $\mathcal{N}$, *and we state that the TSM is of the* (α, β) *type.*

Type $(0,0)$ TSM structures are cosymplectic, type $(\alpha, 0)$ TSM structures are α-Sasakian, and type $(0, \beta)$ TSM structures are β-Kenmotsu.

In [28], TSM is examined, and the following results are obtained:

$$\nabla_u \varsigma = -\alpha \Phi u + \beta(u - \omega(u)\varsigma), \tag{6}$$

$$(\nabla_u \omega)(v) = -\alpha g(\Phi u, v) + \beta g(\Phi u, \Phi v),$$

$$\begin{aligned} \mathcal{R}_{im}(u,v)\varsigma &= (\alpha^2 - \beta^2)[\omega(v)u - \omega(u)v] - (u\alpha)\Phi v - (u\beta)\Phi^2 v \\ &\quad +2\alpha\beta[\omega(v)\Phi u - \omega(u)\Phi v] + (v\alpha)\Phi u + (v\beta)\Phi^2 u, \end{aligned}$$

$$\mathcal{R}_{im}(\varsigma, u)\varsigma = (\alpha^2 - \beta^2 - \varsigma\beta)[\omega(u)\varsigma - u],$$

$$\mathcal{S}_{Ric}(u, \varsigma) = [(n-1)(\alpha^2 - \beta^2) - \varsigma\beta]\omega(u) - (\Phi u)\alpha - (n-2)(u\beta), \tag{7}$$

if $\Phi(grad\alpha) = (n-2)grad\beta$, then in view of (7) the following holds:

$$\mathcal{S}_{Ric}(u, \varsigma) = (n-1)(\alpha^2 - \beta^2)\omega(u),$$

$$\mathcal{S}_{Ric}(\varsigma, \varsigma) = (n-1)(\alpha^2 - \beta^2),$$

$$\mathcal{R}_{scal} = n(n-1)(\alpha^2 - \beta^2).$$

3. Characteristics of Curvatures

In this part, we will discover the link between $\widetilde{\mathcal{R}}_{im}$ and $\mathcal{R}_{im}$, where $\widetilde{\mathcal{R}}_{im}$ and $\mathcal{R}_{im}$ are the curvature tensors with respect to the $qsnm$ connection $\widetilde{\nabla}$ and the Levi–Civita connection ∇ on a TSM $\mathcal{N}$, respectively.

Furthermore, we shall determine the relationship between $\widetilde{\mathcal{S}}_{Ric}$ and $\mathcal{S}_{Ric}$, where $\widetilde{\mathcal{S}}_{Ric}$ is the Ricci tensor with respect to the $\widetilde{\nabla}$ on $\mathcal{N}$ and $\mathcal{S}_{Ric}$ is the Ricci tensor with respect to the ∇ on $\mathcal{N}$.

In [29], Tripathi introduced the subsequent relationship:

$$\widetilde{\nabla}_u v = \nabla_u v - \omega(u)\Phi v - \omega(u)v - \omega(v)u + g(u,v)\varsigma,$$

where $\widetilde{\nabla}$ is $qsnm$ connection. Adopting Definition 2 and (5), (6), we gain

$$(\widetilde{\nabla}_u \Phi)(v) = \alpha(g(u,v)\varsigma - \omega(v)u) + (\beta - 1)(g(\Phi u, v)\varsigma - \omega(v)\Phi u), \tag{8}$$

$$\widetilde{\nabla}_u \varsigma = -\alpha \Phi u + (\beta - 1)u - \beta\omega(u)\varsigma. \tag{9}$$

With this $qsnm$-connection, we can now extract the Ricci tensor $\widetilde{\mathcal{S}}_{Ric}$ and the curvature tensor. Here is the curvature tensor $\widetilde{\mathcal{R}}_{im}$:

Theorem 1 ([30])**.** *Let $\mathcal{N}$ be a TSM with the qsnm connection $\widetilde{\nabla}$. Then, the equality is shown for any vector fields u, v, and w on $\mathcal{N}$.*

$$\begin{aligned}\widetilde{\mathcal{R}}_{im}(u,v)w = \;& \mathcal{R}_{im}(u,v)w + \alpha[(g(v,w)\omega(u) - g(u,w)\omega(v)]\varsigma + (\omega(v)\omega(w) - g(\Phi v,w))u \\ & +(-\omega(u)\omega(w) + g(\Phi u,w))v - 2g(u,\Phi v)w - g(v,w)\Phi u + g(u,w)\Phi v \\ & -2g(u,\Phi v)\Phi w] + (\beta-1)[(g(\Phi v,w)\omega(u) - g(v,w)\omega(u) - g(\Phi u,w)\omega(v) \\ & +g(u,w)\omega(v))\varsigma - (\omega(v)\omega(w) - g(v,w))u + (\omega(u)\omega(w) - g(u,w))v \\ & +\omega(v)\omega(w)\Phi u - \omega(u)\omega(w)\Phi v] + \beta[g(v,w)u - g(u,w)v]\end{aligned}$$

where $\mathcal{R}_{im}$ and $\widetilde{\mathcal{R}}_{im}$ are the curvature tensors with respect to the ∇ and $\widetilde{\nabla}$, respectively.

Theorem 2 ([30])**.** *Let $\mathcal{N}$ be a TSM with the qsnm connection $\widetilde{\nabla}$. Then, the ensuing equality is given:*

$$\begin{aligned}\widetilde{\mathcal{R}}_{im}(u,v,w,z) = \;& \mathcal{R}_{im}(u,v,w,z) + \alpha[(g(v,w)\omega(u) - g(u,w)\omega(v))g(\varsigma,z) \\ & +(\omega(v)\omega(w) - g(\Phi v,w))g(u,z) + (-\omega(u)\omega(w) \\ & +g(\Phi u,w))g(v,z) - 2g(u,\Phi v)g(w,z) \\ & -g(v,w)g(\Phi u,z) + g(u,w)g(\Phi v,z) - 2g(u,\Phi v)g(\Phi w,z)] \\ & +(\beta-1)[(g(\Phi v,w)\omega(u) - g(v,w)\omega(u) - g(\Phi u,w)\omega(v) \\ & +g(u,w)\omega(v))g(\varsigma,z) - (\omega(v)\omega(w) - g(v,w))g(u,z) \\ & +(\omega(u)\omega(w) - g(u,w))g(v,z) \\ & +\omega(v)\omega(w)g(\Phi u,z) - \omega(u)\omega(w)g(\Phi v,z)] \\ & +\beta[g(v,w)g(u,z) - g(u,w)g(v,z)].\end{aligned} \tag{10}$$

Based on Theorem 2, we obtain the subsequent outcomes.

Theorem 3 ([30])**.** *Let $\mathcal{N}$ be a TSM with the connection $\widetilde{\nabla}$; then, the Ricci tensor $\widetilde{\mathcal{S}}_{Ric}$ and the scalar curvature $\widetilde{\mathcal{R}}_{scal}$ with respect to the $\widetilde{\nabla}$ are obtained as follows:*

$$\begin{aligned}\widetilde{\mathcal{S}}_{Ric}(u,v) = \;& \mathcal{S}_{Ric}(u,v) + \alpha[-g(u,v) + n\omega(u)\omega(v) - ng(\Phi u,v)] \\ & +(\beta-1)[g(\Phi u,v) + (n-2)g(\Phi u,\Phi v)] \\ & +\beta(n-1)g(u,v).\end{aligned} \tag{11}$$

$$\widetilde{\mathcal{R}}_{scal} = \mathcal{R}_{scal} + 2\beta(n-1)^2 - (n-1)(n-2), \tag{12}$$

where the Ricci tensor and the scalar curvature concerning the Levi–Civita connection ∇ on $\mathcal{N}$ are denoted by $\mathcal{S}_{Ric}$ and $\mathcal{R}_{scal}$, respectively.

4. Slant Submanifolds of a Trans-Sasakian Manifold with *qsnm* Connection

Let Ξ be a submanifold of a *TSM* $(\mathcal{N},\Phi,\omega,\varsigma,g)$ of (α,β) type. For each non-zero vector U tangent to Ξ at any point $p \in \Xi$, $u \in T_p\Xi$, if the slant angle between $T\mathcal{N}$ and ΦU is independent of the choice of p, then Ξ is said to be *slant submanifold*. Note that if the slant angle is $\Theta = 0$ or $\Theta = \frac{\pi}{2}$, then submanifold Ξ becomes Φ-*invariant* and Φ-*anti-invariant*, respectively. Proper slant, sometimes known as Θ-slant proper, submanifolds are slant submanifolds that are neither anti-invariant nor invariant.

The following characteristics of slant submanifolds in *TSM* hold according to the findings of ([31,32]).

Theorem 4 ([31])**.** *Let Ξ be a submanifold of an almost contact metric manifold $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ such that $\varsigma \in T\mathcal{N}$. Then*

1. *Ξ is slant if and only if there exists a constant $\gamma \in [0, 1]$ such that*

$$\Phi^2 = -\gamma(I - \omega \otimes \varsigma).$$

In addition, if the Θ is the slant angle of Ξ, then $\gamma = \mathrm{Cos}^2\Theta$.

2. *$g(\Phi u, \Phi v) = \mathrm{Cos}^2\Theta[g(u, v) - \omega(u)\omega(v)]$, for any $u, v \in \Gamma(T\Xi)$.*

Theorem 5. *Let Ξ be an m-dimensional Θ-slant submanifold of a TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ with a qsnm connection. Then, the scalar curvature with respect to the qsnm connection is given as*

$$\tilde{\mathcal{R}}_{scal} = 2m(\alpha^2 - \beta^2) + \beta(m-1)[(m-2)\mathrm{Cos}^2\Theta + m] - \mathrm{Cos}^2\Theta(m-2)(m-1). \tag{13}$$

Proof. In view of (11) and Theorem 4, after contraction over u and v, we turn to (13). ☐

Let Ξ be a submanifold of a *TSM* $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ with a *qsnm* connection. Then,

$$\Phi u = Tu + Fu, \quad \forall \quad u \in \Gamma(T\Xi), \tag{14}$$

where Tu and Fu are the tangential and the normal components of ΦX, respectively. It is noted that Ξ is known as invariant if F is identically zero (that is, $\Phi \in \Gamma(T\Xi)$) and anti-invariant if T is identically zero (that is, $\Phi u \in \Gamma(T^{\perp}\Xi)), \forall u \in \Gamma(T\Xi)$.

Similarly,

$$\Phi v = tv + fv, \quad \forall \quad v \in \Gamma(T^{\perp}\Xi),$$

where Φv has tangential and normal components, denoted by tv and fv, respectively. Furthermore, $\{b_1, \ldots, b_m\}$ represents the submanifold Ξ in $\mathcal{N}$, a local orthonormal frame of the tangent bundle $T\Xi$ of Ξ, and $\{b_{m+1}, \ldots, b_{2m+1}\}$ indicates the local orthonormal frame of the normal bundle $T^{\perp}\Xi$ of Ξ in $\mathcal{N}$, a local orthonormal frame of the normal bundle $T^{\perp}\Xi$ of Ξ in $\mathcal{N}$.

In order to derive the normalized scalar curvature σ, we first express the second fundamental form of submanifold Ξ using Λ.

$$\sigma = \frac{2\tilde{\mathcal{R}}_{scal}}{m(m-1)}.$$

Find the mean curvature using the equation

$$\Pi = \sum_{\mu=1}^{m} \frac{1}{m} \Lambda(b_\mu, b_\mu).$$

Let $\Lambda^a_{\mu\nu} = g(\Lambda(b_\mu, b_\nu), b_a)$, for $\mu, \nu = \{1, \ldots, m\}$ and $a = \{m+1, \ldots, 2m+1\}$. Next, the mean curvature vector's squared norm is provided by

$$||\Pi||^2 = \frac{1}{m^2} \sum_{a=m+1}^{2m+1} \left(\sum_{\mu=1}^{m} \Lambda^a_{\mu\nu} \right)^2 \tag{15}$$

and Λ, the second fundamental form, has the equation

$$||\Lambda||^2 = \sum_{a=m+1}^{2m+1} \sum_{\mu,\nu=1}^{m} (\Lambda^a_{\mu\nu})^2.$$

Moreover, the divergence of any vector field u on $\Gamma(T\Xi)$ is denoted by $\mathcal{D}iv(u)$ and defined by

$$\mathcal{D}iv(u)=\sum_{\mu=1}^{m} g(\nabla_{b_\mu}u,b_\mu), \tag{16}$$

where $\{b_1,\ldots,b_m\}$ a local orthonormal tangent frame of the tangent bundle $T\Xi$ of Ξ. The scalar curvature expressed by

$$\widetilde{\mathcal{R}}_{scal}=\sum_{1\leq\mu<\nu\leq q}\widetilde{\mathcal{R}}_{im}(b_\mu,b_\nu,b_\nu,b_\mu).$$

Then, considering the Gauss equation and (13), we obtain

$$\begin{aligned}2\widetilde{\mathcal{R}}_{scal} \;=\;& 2m(\alpha^2-\beta^2)+\beta(m-1)[(m-2)\mathrm{Cos}^2\Theta+m]\\ &-(m-2)(m-1)\mathrm{Cos}^2\Theta+m^2||\Pi||^2+m||\Lambda||^2,\end{aligned}$$

5. Ricci Soliton on Slant Submanifold

To obtain a relationship between the intrinsic and extrinsic invariants, we give the scalar curvature of submanifold Ξ of Ricci soliton $(\mathcal{N},g,\mathcal{F},\lambda)$ in this section. Then, to describe such a submanifold Ξ, we establish an inequality for the Ricci soliton $(\mathcal{N},g,\mathcal{F},\lambda)$ and gradient Ricci soliton.

Let $(\mathcal{N},g,\mathcal{F},\Lambda)$ be a $(2m+1)$-dimensional TSM manifold and $\varphi:\Xi\rightarrow\mathcal{N}$ be an isometric immersion from an m-dimensional Riemannian manifold (Ξ,g) into $(\mathcal{N},g)$ manifold. Then, the Ricci tensor $\mathcal{S}_{Ric}$ can be written as

$$\tilde{\mathcal{S}}_{Ric}(u,v)=\tilde{\mathcal{S}}_{Ric}|_{T_p\Xi}(u,v)+\tilde{\mathcal{S}}_{Ric}|_{T_p^{\perp}\Xi}(u,v) \tag{17}$$

for any $u,v\in T_p\Xi$.

Now, the previous Lemma marks the start of this section:

Lemma 1. *Let $(\mathcal{N},g,\mathcal{F},\lambda)$ be an RS and let Ξ be an m-dimensional Θ-slant submanifold of a $(2m+1)$-dimensional TSM $(\mathcal{N},\Phi,\omega,\varsigma,g)$ equipped with a qsnm, connection $\tilde{\nabla}$. Then, we have*

$$\begin{aligned}\widetilde{\mathcal{D}iv(\mathcal{F})} \;=\;& (m-2)(m-1)\mathrm{Cos}^2\Theta+m^2||\Pi||^2-||\Lambda||^2\\ &-2m(\alpha^2-\beta^2)-\beta(m-1)[(m-2)\mathrm{Cos}^2\Theta+m]-m\lambda.\end{aligned}$$

Proof. Since $(\mathcal{N},g,\mathcal{F},\Lambda)$ is a RS. Then in view of (17), one get

$$\frac{1}{2}\sum_{\mu=1}^{m}\{g(\nabla_{b_\mu}\mathcal{F},b_\mu)+g(b_\mu,\nabla_{b_\mu}\mathcal{F})\}+\sum_{\mu=1}^{m}\tilde{\mathcal{S}}_{Ric}(b_\mu,b_\mu)+\sum_{\mu=1}^{m}\lambda g(b_\mu,b_\mu)=0, \tag{18}$$

where $\{b_1,\ldots,b_m\}$ is a local orthonormal tangent frame of the tangent bundle $T\Xi$ of Ξ.

Then, adopting (15), (13), (4) and (18) in (18) we turn up

$$\tilde{\mathcal{D}}iv(\mathcal{F})+\tilde{\mathcal{S}}_{Ric}(b_\mu,b_\mu)|_{T_p\Xi}(u,v)+\lambda\sum_{i=1}^{n}g(b_\mu,b_\mu)=0. \tag{19}$$

Adopting Gauss formula and (17) in (19), we obtain

$$\tilde{\mathcal{D}}iv(\mathcal{F})+2\widetilde{\mathcal{R}}_{scal}-\sum_{\mu,\nu=1}^{m}\{g(\Lambda(b_\mu,b_\mu),\Lambda(b_\nu,b_\nu))-g(\Lambda(b_\mu,b_\nu),\Lambda(b_\mu,b_\nu))\} \tag{20}$$

$$+\lambda\sum_{\mu=1}^{m}g(b_\mu,b_\mu)=0.$$

Then, the proof is completed. □

As a consequence of Equation (18), we articulate the following.

Theorem 6. *$(\mathcal{N}, g, \mathcal{F}, \lambda)$ be a RS and Ξ be a m-dimensional Θ-slant submanifold of a $(2m+1)$-dimensional TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm-connection $\tilde{\nabla}$ admits an RS with a potential vector field $\mathcal{F} \in T\Xi$ of Ricci soliton. Then the Ricci soliton on Ξ is expanding, steady, and shrinking according as*

1. $\frac{(m-2)(m-1)\mathrm{Cos}^2\Theta}{m} + m||\Pi||^2 > \frac{||\Lambda||^2}{m} + \frac{2m(\alpha^2-\beta^2)+\beta(m-1)[(m-2)\mathrm{Cos}^2\Theta+m]}{m} + \frac{\widetilde{\mathcal{D}iv(\mathcal{F})}}{m}$,
2. $\frac{(m-2)(m-1)\mathrm{Cos}^2\Theta}{m} + m||\Pi||^2 = \frac{||\Lambda||^2}{m} + \frac{2m(\alpha^2-\beta^2)+\beta(m-1)[(m-2)\mathrm{Cos}^2\Theta+m]}{m} + \frac{\widetilde{\mathcal{D}iv(\mathcal{F})}}{m}$,
3. $\frac{(m-2)(m-1)\mathrm{Cos}^2\Theta}{m} + m||\Pi||^2 < \frac{||\Lambda||^2}{m} + \frac{2m(\alpha^2-\beta^2)+\beta(m-1)[(m-2)\mathrm{Cos}^2\Theta+m]}{m} + \frac{\widetilde{\mathcal{D}iv(\mathcal{F})}}{m}$,

respectively.

At this stage, we remember the subsequent lemma from [33]

Lemma 2. *If $p_1, p_2 \cdots p_m$ for $m > 1$ are real numbers, then*

$$\frac{1}{m}\{\sum_{\mu=1}^{n} p_\mu\}^2 \leq \sum_{\mu=1}^{m} p_\mu^2$$

with equality holding if and only if $p_1 = p_2 = \cdots p_m$.

Theorem 7. *Let $(\mathcal{N}, g, \mathcal{F}, \lambda)$ be a RS and let Ξ be an m-dimensional Θ-slant submanifold of a $(2m+1)$-dimensional TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm connection $\tilde{\nabla}$. Then, we have*

$$\begin{aligned} \widetilde{\mathcal{D}iv(\mathcal{F})} \quad \leq \quad & m(m-1)||\Pi||^2 - m\lambda \\ & +(m-2)(m-1)\mathrm{Cos}^2\Theta \\ & -2m(\alpha^2-\beta^2) - \beta(m-1)[(m-2)\mathrm{Cos}^2\Theta + m]. \end{aligned}$$

Proof. In light of Equation (18), we show that if

$$\begin{aligned} &(m-2)(m-1)\mathrm{Cos}^2\Theta - 2m(\alpha^2-\beta^2) - \beta(m-1)[(m-2)\mathrm{Coss}^2\Theta + m] \\ &\qquad = -\widetilde{\mathcal{D}iv(\mathcal{F})} + m^2||\Pi||^2 - \|\Lambda\|^2 - m\lambda \end{aligned}$$

$$\begin{aligned} &(m-2)(m-1)\mathrm{Cos}^2\Theta - 2m(\alpha^2-\beta^2) - \beta(m-1)[(m-2)\mathrm{Cos}^2\Theta + m] \\ &= -\widetilde{\mathcal{D}iv(\mathcal{F})} + m^2||\Pi||^2 - m\lambda - \textstyle\sum_{\alpha=m+1}^{m}\sum_{\mu,\nu=1}^{m}(\Lambda_{\mu\nu}^{\alpha})^2 - \sum_{a=m+1}^{2m+1}\sum_{\mu,\mu\neq1}^{m}(\Lambda_{\mu\nu}^{a})^2 \end{aligned}$$

$$\begin{aligned} &(m-2)(m-1)\mathrm{Cos}^2\Theta - 2m(\alpha^2-\beta^2) - \beta(m-1)[(m-2)\mathrm{Cos}^2\Theta + m] \\ &\qquad \leq -\widetilde{\mathcal{D}iv(\mathcal{F})} + m^2||\Pi||^2 - m\lambda - \frac{m^2||\Pi||^2}{m} - \textstyle\sum_{a=m+1}^{2m+1}\sum_{\mu,\mu\neq1}^{m}(\Lambda_{\mu\nu}^{a})^2 \end{aligned}$$

$$\leq -\widetilde{\mathcal{D}iv(\mathcal{F})} - m\lambda - m(m-1)||\Pi||^2 - \sum_{\alpha=m+1}^{2m+1}\sum_{\mu,\mu\neq1}^{m}(\Lambda_{\mu\nu}^{a})^2$$

is found, then

$$\begin{aligned} &(m-2)(m-1)\mathrm{Cos}^2\Theta - 2m(\alpha^2-\beta^2) - \beta(m-1)[(m-2)\mathrm{Cos}^2\Theta + m] \\ &\qquad \leq -\widetilde{\mathcal{D}iv(\mathcal{F})} - m\lambda - m(m-1)||\Pi||^2 \end{aligned}$$

is obtained, which gives us (21). If the equality of (21) is satisfied, then M is totally umbilical. □

Now, let the soliton vector field $\mathcal{F}$ be of gradient type i.e., $\mathcal{F} = \nabla\psi$. Thus, in view of (3) and (18), we articulate the following.

Theorem 8. *Let $(\mathcal{N}, g, \mathcal{F} = \nabla\psi, \lambda)$ be a GRs with a soliton vector field $\mathcal{F} \in T\Xi$ is of gradient type, and let Ξ be an m-dimensional Θ-slant submanifold of a $(2m+1)$-dimensional TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm connection $\tilde{\nabla}$. Then, we have*

$$\begin{aligned} ||\widetilde{\mathcal{S}_{Ric}}||_g^2 \geq\ & ||\widetilde{\mathrm{Hess}}||_g^2 - \tfrac{(m-2)^2(m-1)^2\mathrm{Cos}^4\Theta}{m} - m^3||\Pi||^4 - \tfrac{||\Lambda||^4}{m} \\ & +4m(\alpha^2-\beta^2)^2 + \tfrac{\beta^2(m-1)^2}{m}[(m-2)^2\mathrm{Cos}^4\Theta + m^2] + m\lambda^2 \end{aligned}$$

Corollary 1. *Let $(\mathcal{N}, g, \mathcal{F} = \nabla\psi, \lambda)$ be a GRs with a soliton vector field $\mathcal{F} \in T\Xi$ of the gradient type, and let Ξ be an m-dimensional totally umbilical Θ-slant submanifold of a $(2m+1)$-dimensional TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm connection $\tilde{\nabla}$. Then, we have*

$$\begin{aligned} ||\widetilde{\mathcal{S}_{Ric}}||_g^2 \geq\ & ||\widetilde{\mathrm{Hess}}||_g^2 - \tfrac{(m-2)^2(m-1)^2\mathrm{Cos}^4\Theta}{m} + m\lambda^2 \\ & +4m(\alpha^2-\beta^2)^2 + \tfrac{\beta^2(m-1)^2}{m}[(m-2)^2\mathrm{Cos}^4\Theta + m^2]. \end{aligned}$$

Remark 1. *For anti-invariant submanifolds of trans-Sasakian manifolds using the quarter symmetric non-metric connection, a slant submanifold reduces to an anti-invariant submanifold when $\Theta = \frac{\pi}{2}$ and an invariant submanifold is the result of a slant submanifold when $\Theta = 0$*

Next, in light of Theorem 8 and Remark 1, we gain the following outcomes:

Corollary 2. *Let $(\mathcal{N}, g, \mathcal{F} = \nabla\psi, \lambda)$ be a GRs with a soliton vector field $\mathcal{F} \in T\Xi$ of the gradient type, and let Ξ be an m-dimensional anti-invariant submanifold of a $(2m+1)$-dimensional TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm connection $\tilde{\nabla}$. Then, we have*

$$||\widetilde{\mathcal{S}_{Ric}}||_g^2 \geq ||\widetilde{\mathrm{Hess}}||_g^2 - m^3||\Pi||^4 - \tfrac{||\Lambda||^4}{m} + m[4(\alpha^2-\beta^2)^2 + (m-1)^2\beta^2 + \lambda^2].$$

Corollary 3. *Let $(\mathcal{N}, g, \mathcal{F} = \nabla\psi, \lambda)$ be a GRs with a soliton vector field $\mathcal{F} \in T\Xi$ is of gradient type, and let Ξ be an m-dimensional invariant submanifold of a $(2m+1)$-dimensional TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm connection $\tilde{\nabla}$. Then, we have*

$$\begin{aligned} ||\widetilde{\mathcal{S}_{Ric}}||_g^2 \geq\ & ||\widetilde{\mathrm{Hess}}||_g^2 - \tfrac{(m-2)^2(m-1)^2}{m} - m^3||\Pi||^4 - \tfrac{||\Lambda||^4}{m} \\ & +4m(\alpha^2-\beta^2)^2 + \tfrac{\beta^2(m-1)^2}{m}[2m^2 - 4m + 4] + m\lambda^2 \end{aligned}$$

Remark 2. *Similarly, because of Corollary 1, we can easily obtain the parallel results for totally umbilical anti-invariant submanifold and a totally umbilical invariant submanifold [34] of a $(2m+1)$-dimensional TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm-connection $\tilde{\nabla}$.*

6. Application of Theorem 8

Corollary 4. *Let $(\mathcal{N}, g, \mathcal{F} = \nabla\psi, \lambda)$ be a GRs with a soliton vector field $\mathcal{F} \in T\Xi$ of the gradient type, and let Ξ be an m-dimensional Θ-slant submanifold of a $(2m+1)$-dimensional β-KM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm connection $\tilde{\nabla}$. Then, we have*

$$\begin{aligned} ||\widetilde{\mathcal{S}_{Ric}}||_g^2 \geq\ & ||\widetilde{\mathrm{Hess}}||_g^2 - \tfrac{(m-2)^2(m-1)^2\mathrm{Cos}^4\Theta}{m} - m^3||\Pi||^4 - \tfrac{||\Lambda||^4}{m} \\ & +\tfrac{\beta^2(m-1)^2}{m}[(m-2)^2\mathrm{Cos}^4\Theta + m^2] - 4m\beta^4 + m\lambda^2 \end{aligned}$$

Corollary 5. *Let $(\mathcal{N}, g, \mathcal{F} = \nabla\psi, \lambda)$ be a GRs with a soliton vector field $\mathcal{F} \in T\Xi$ is of gradient type and Ξ be a m-dimensional Θ-slant submanifold of a $(2m+1)$-dimensional α-Sasakian manifold $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm-connection $\tilde{\nabla}$. Then we have*

$$\begin{aligned} ||\widetilde{\mathcal{S}_{Ric}}||_g^2 \quad \geq \quad & ||\widetilde{\mathrm{Hess}}||_g^2 - \tfrac{(m-2)^2(m-1)^2\mathrm{Cos}^4\Theta}{m} - m^3||\Pi||^4 - \tfrac{||\Lambda||^4}{m} \\ & + 4m\alpha^4 + m\lambda^2 \end{aligned}$$

Corollary 6. *Let $(\mathcal{N}, g, \mathcal{F} = \nabla\psi, \lambda)$ be a GRs with a soliton vector field $\mathcal{F} \in T\Xi$ of the gradient type, and let Ξ be an m-dimensional Θ-slant submanifold of a $(2m+1)$-dimensional cosymplectic manifold $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm connection $\tilde{\nabla}$. Then, we have*

$$||\widetilde{\mathcal{S}_{Ric}}||_g^2 \quad \geq \quad ||\widetilde{\mathrm{Hess}}||_g^2 - \tfrac{(m-2)^2(m-1)^2\mathrm{Cos}^4\Theta}{m} - m^3||\Pi||^4 - \tfrac{||\Lambda||^4}{m} + m\lambda^2$$

Remark 3. *Similarly, given Corollaries 4–6, we can easily obtain the parallel results for anti-invariant submanifold and invariant submanifold of a $(2m+1)$-dimensional β-Kenmotsu, α-Sasakian and cosympletic manifolds with a qsnm connection $\tilde{\nabla}$.*

Corollary 7. *Let $(\mathcal{N}, g, \mathcal{F} = \nabla\psi, \lambda)$ be a GRs with a soliton vector field $\mathcal{F} \in T\Xi$ of the gradient type, and let Ξ be an m-dimensional totally umbilical Θ-slant submanifold of a $(2m+1)$-dimensional β-KM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm-connection $\tilde{\nabla}$. Then, we have*

$$\begin{aligned} ||\widetilde{\mathcal{S}_{Ric}}||_g^2 \quad \geq \quad & ||\widetilde{Hess}||_g^2 - \tfrac{(m-2)^2(m-1)^2\mathrm{Cos}^4\Theta}{m} \\ & + \tfrac{\beta^2(m-1)^2}{m}[(m-2)^2\mathrm{Cos}^4\Theta + m^2] - 4m\beta^4 + m\lambda^2 \end{aligned}$$

Remark 4. *In addition, in light of Corollaries 4–7, we can easily obtain the parallel results for totally umbilical anti-invariant submanifolds and totally umbilical invariant submanifolds of a $(2m+1)$-dimensional β-Kenmotsu and α-Sasakian, and cosympletic manifolds with a qsnm connection $\tilde{\nabla}$.*

7. Solitonic Ineqiality with a Scalar Concircular Field

In this section, we deduced the above inequalities with a scalar concircular field in a $(2m+1)$-dimensional TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a $qsnm$-connection $\tilde{\nabla}$. Thus, we entail the following definition.

Definition 3 ([35]). *If the scalar field $f \in C^\infty(\mathcal{M})$ fulfills the equation, it is considered a scalar concircular field*

$$Hessf = \pi g, \tag{21}$$

where the Riemannian metric is g and a scalar field is π.

Inserting (21) in (3), we gain an inequality for a lower bound of the geometry of g in terms of GRs with a scalar concircular field π in ambient space $\mathcal{N}$ such that

$$||\mathcal{S}_{Ric}||_g^2 \geq \pi^2||X||_g^2 - \frac{1}{n}(\Delta\psi)^2. \tag{22}$$

Next, in view of Equation (22) and Theorem 8 we obtain the following:

Corollary 8. *Let $(\mathcal{N}, g, \pi, \lambda)$ be a GRs with a scalar concircular field $\pi \in T\Xi$ and let Ξ be a m-dimensional submanifold of a $(2m+1)$-dimensional TSM $(\mathcal{N}, \Phi, \omega, \varsigma, g)$ equipped with a qsnm connection $\tilde{\nabla}$. Then we have*

S.No	***Submanifold Ξ is***	***Solitonical Inequality***
1	*Θ-slant*	$\begin{aligned} \Vert\widetilde{\mathcal{S}_{Ric}}\Vert_g^2 \geq\ & \pi^2\Vert X\Vert_g^2 - \frac{(m-2)^2(m-1)^2\mathrm{Cos}^4\Theta}{m} - m^3\Vert\Pi\Vert^4 \\ -\ & \frac{\Vert\Lambda\Vert^4}{m} + 4m(\alpha^2-\beta^2)^2 \\ +\ & \frac{\beta^2(m-1)^2}{m}[(m-2)^2\mathrm{Cos}^4\Theta + m^2] \\ +\ & m\lambda^2 \end{aligned}$
2	*totally umbilical Θ-slant*	$\begin{aligned} \Vert\widetilde{\mathcal{S}_{Ric}}\Vert_g^2 \geq\ & \pi^2\Vert X\Vert_g^2 - \frac{(m-2)^2(m-1)^2\mathrm{Cos}^4\Theta}{m} + m\lambda^2 \\ +\ & 4m(\alpha^2-\beta^2)^2 + \frac{\beta^2(m-1)^2}{m}[(m-2)^2\mathrm{Cos}^4\Theta + m^2] \end{aligned}$
3	*anti − ivariant*	$\begin{aligned} \Vert\widetilde{\mathcal{S}_{Ric}}\Vert_g^2 \geq\ & \pi^2\Vert X\Vert_g^2 - m^3\Vert\Pi\Vert^4 \\ -\ & \frac{\Vert\Lambda\Vert^4}{m} + m[4(\alpha^2-\beta^2)^2 + (m-1)^2\beta^2 + \lambda^2] \end{aligned}$
4	*invariant*	$\begin{aligned} \Vert\widetilde{\mathcal{S}_{Ric}}\Vert_g^2 \geq\ & \pi^2\Vert X\Vert_g^2 - \frac{(m-2)^2(m-1)^2}{m} - m^3\Vert\Pi\Vert^4 \\ -\ & \frac{\Vert\Lambda\Vert^4}{m} + 4m(\alpha^2-\beta^2)^2 \\ +\ & \frac{\beta^2(m-1)^2}{m[(m-2)^2+m^2]} \\ +\ & m\lambda^2 \end{aligned}$

Remark 5. *Moreover, in view of Corollary 8, we can easily obtain the parallel results for totally umbilical anti-invariant submanifold and totally umbilical invariant submanifold of $(2m+1)$-dimensional β-Kenmotsu, α-Sasakian and cosympletic manifolds with a qsnm connection $\tilde{\nabla}$.*

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Abbreviations

The following acronyms are used in this manuscript:

TSM	trans-Sasakian manifolds
qsnm	quarter symmetric non-metric connection
RS	Ricci soliton
GRs	gredient Ricci soliton

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