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A Selberg Trace Formula for $GL_3(\mathbb{F}_p)\backslash GL_3(\mathbb{F}_q)/K$

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Abstract: In this paper, we prove a discrete analog of the Selberg Trace Formula for the group $GL_3(\mathbb{F}_q)$. By considering a cubic extension of the finite field $\mathbb{F}_q$, we define an analog of the upper half-space and an action of $GL_3(\mathbb{F}_q)$ on it. To compute the orbital sums, we explicitly identify the double coset spaces and fundamental domains in our upper half space. To understand the spectral side of the trace formula, we decompose the induced representation $\rho = \mathrm{Ind}_\Gamma^G 1$ for $G = GL_3(\mathbb{F}_q)$ and $\Gamma = GL_3(\mathbb{F}_p)$.

Keywords: general linear group over finite fields; finite upper half space; pre-trace formula; Selberg type trace formula; orbital sums

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1. Introduction

The Selberg trace formula [1] is one of the most celebrated mathematical results of the past century. In its original form, it relates the spectrum of the Laplacian on a hyperbolic surface to lengths of closed geodesics on the surface. It can be viewed as a non-Abelian generalization of the well-known Poisson summation formula in Fourier analysis, and has found numerous applications in many fields, for example, in number theory and the theory of automorphic forms (see Sarnak [2]).

The Selberg trace formula deals with surfaces which can be represented as double cosets of the form $\Gamma\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$, where Γ is a discrete subgroup of $SL_2(\mathbb{R})$. It is a natural question to work out the trace formula for higher-dimensional matrix groups. To quote Dennis Hejhal: "Of course, we won't really understand the trace formula until it is written down for $SL_4(\mathbb{Z})$." While Dorothy Wallace [3] has worked out the explicit details of the trace formula for $SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R})$, the four-dimensional case remains mysterious. A further generalization is the so-called Arthur–Selberg trace formula [4], which plays an important role in the Langlands program. In this regard, it should be mentioned that an extension of the Selberg trace formulae to higher-dimensional symmetric spaces $\Gamma\backslash G/K$ over real or complex numbers requires handling quite complicated analytic issues that do not appear in the original two-dimensional case. It is for this reason that study of Selberg trace formulae for higher-dimensional special linear groups over finite fields is a necessary step towards understanding a general trace formula. In this paper, we take a first step in this direction by proving a Selberg trace formula for a three-dimensional Poincaré upper half-plane over finite fields. At the same time, our results show that extending this formula to higher-dimensional upper half-planes for more general matrix groups G would involve complicated calculations.

Selberg-type trace formulae have been derived for discrete spaces such as graphs as well. By realizing a k-regular graph as a quotient of the infinite k-regular tree, Ahumada [5] found such a formula for k-regular graphs. Audrey Terras [6] introduced the concept

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of finite upper half-planes as a finite analogue of the Poincaré upper half-plane model in which the complex number field is replaced by a quadratic extension of a finite field $\mathbb{F}_q$, where $q = p^r$ is a prime power. She constructed a family of Ramanujan graphs [7] using these upper half-planes and developed a representation-theoretic trace formula on $\mathrm{GL}_2(\mathbb{F}_p)\backslash\mathrm{GL}_2(\mathbb{F}_q)/K$, where K is the finite analogue of $\mathrm{SO}_2(\mathbb{R})$.

The aim of this paper is to present an analogous trace formula for $\mathrm{GL}_3(\mathbb{F}_q)$. Our main result in this paper is as follows. Consider the pre-trace formula

$$\sum_{\pi\in\hat{G}} m(\pi,\rho)\,\mathrm{Tr}(\hat{f}(\pi)) = \sum_{\{\gamma\}\in C_\Gamma} \frac{|G_\gamma|}{|\Gamma_\gamma|} I_G(f,\gamma), \tag{1}$$

where G is a finite group, Γ is a subgroup of G, $\rho = \mathrm{Ind}_\Gamma^G 1$ is the induced representation of the trivial representation of Γ, and $I_G(f,\gamma)$ is the orbital sum of f at γ. These terms are defined in Section 3, and a proof of the pre-trace formula is recalled there as well. Note that (1) can be viewed as a representation-theoretic analogue of Selberg's trace formula [1,6]; the left hand side can be thought of as the spectral side, summing over the irreducible representations of G and their multiplicities in ρ, while the right hand side can be interpreted as the geometric side, summing over conjugacy classes in G. In this paper, to obtain a trace formula for the finite upper half-space we take $G = \mathrm{GL}_3(\mathbb{F}_q)$ and $\Gamma = \mathrm{GL}_3(\mathbb{F}_p)$; also, $f : G \to \mathbb{C}$ is chosen to be a $K-$bi-invariant function (K is the stabilizer of p_0 as above), i.e., $\forall k, h \in K$ and $\forall x \in G$ we have $f(kxh) = f(x)$. One can also think of f as a function on $G/K \simeq \mathbb{H}_q$ which is invariant under left-multiplication by elements in K.

The rest of this paper is organized as follows. In Section 2, we define an analogue, denoted $\mathbb{H}_q$, of the upper half-plane for the group $\mathrm{GL}_3(\mathbb{F}_q)$ by considering a cubic extension of the finite field $\mathbb{F}_q$. We define an action of $\mathrm{GL}_3(\mathbb{F}_q)$ on $\mathbb{H}_q$, which sets the stage for our discrete trace formula. In Section 3, we recall a standard pre-trace formula for finite groups. In Sections 4–6, we compute each term on the right-hand side of the pre-trace formula (5) (the so-called geometric side) using the explicit description for conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_q)$ provided in [8]. The conjugacy classes are separated into central, hyperbolic, parabolic, and elliptic terms. Each of these types of terms is explicitly identified. It would be helpful to have a more conceptual understanding of these computations. In Section 7, we provide some simple examples of the application of our formula in which the left-hand side of the pre-trace formula is known a priori. In Sections 8 and 9, we calculate the left-hand side of the pre-trace formula by computing the character of the induced representation $\rho = \mathrm{Ind}_\Gamma^G 1$, then using this to compute its decomposition into irreducible representations in the case that $2, 3 \nmid n$.

2. A Finite Upper Half-Space

To generalize the results of Terras to the $\mathrm{GL}_3(\mathbb{F}_q)$ case, we must first discuss the generalization of the finite upper half-space. Let p be a prime number and $q = p^n$ for a positive integer n. The definition of the upper half-space $\mathbb{H}_q$ comes from considering the cubic extension of $\mathbb{F}_q$. The construction was first provided by Martinez in [9]. Here, we describe it explicitly for our special case of $\mathrm{GL}_3(\mathbb{F}_q)$.

Lemma 1. *$\mathbb{F}_q$ has a cubic nonresidue if and only if $q \equiv 1 \pmod 3$.*

Proof. Let $q = p^n$. Consider the cube map $x \mapsto x^3 : \mathbb{F}_q^\times \to \mathbb{F}_q^\times$. This map has as a kernel the subgroup μ_3 consisting of the cube roots of unity in $\mathbb{F}_q^\times$. Let α be a generator of $\mathbb{F}_q^\times$. Then, if $q \equiv 1 \pmod 3$, we have $\mu_3 = \{1, \alpha^{(q-1)/3}, \alpha^{2(q-1)/3}\}$, in which case there are $(q-1)/3$ cubic residues. Otherwise, $\mu_3 = \{1\}$, meaning that each element of $\mathbb{F}_q^\times$ is a cubic residue. $\square$

From now on, we assume that $q \equiv 1 \pmod 3$. Take a cubic nonresidue $\delta \in \mathbb{F}_q$ and consider the cubic extension $\mathbb{F}_q\left(\sqrt[3]{\delta}\right) \simeq \mathbb{F}_{q^3}$. A basis for $\mathbb{F}_q\left(\sqrt[3]{\delta}\right)/\mathbb{F}_q$ is $\{1, \delta^{1/3}, \delta^{2/3}\}$. For an element α, we write $(\alpha_1, \alpha_2, \alpha_3)$ to denote its components, i.e.,

$$\alpha = \alpha_1 + \alpha_2\delta^{1/3} + \alpha_3\delta^{2/3}.$$

We define

$$\mathbb{H}_q = \left\{(\alpha, \beta) \in \mathbb{F}_q(\sqrt[3]{\delta}) \times \mathbb{F}_q(\sqrt[3]{\delta}) : \begin{vmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{vmatrix} \neq 0\right\}.$$

Lemma 2. *The following map defines an action of* $\mathrm{GL}_3(\mathbb{F}_q)$ *on* $\mathbb{H}_q$*:*

$$\begin{bmatrix} a & b & c \\ d & e & f \\ r & s & t \end{bmatrix}(\alpha, \beta) = \left(\frac{a\alpha + b\beta + c}{r\alpha + s\beta + t}, \frac{d\alpha + e\beta + f}{r\alpha + s\beta + t}\right).$$

Proof. The compatibility of the action with multiplication in $\mathrm{GL}_3(\mathbb{F}_q)$ follows from the fact that $\mathrm{GL}_3(\mathbb{F}_q)$ has an action on the projective space $\mathbb{P}^2(\mathbb{F}_{q^3})$. Thus, we need to check that the action is closed in $\mathbb{H}_q$. This is a routine verification, and can be found in Appendix A. □

Our distinguished point will be $p_0 = (\delta^{2/3}, \delta^{1/3}) \in \mathbb{H}_q$. The action of $\mathrm{Aff}_3(\mathbb{F}_q) \leq \mathrm{GL}_3(\mathbb{F}_q)$ is transitive on $\mathbb{H}_q$, where

$$\mathrm{Aff}_3(\mathbb{F}_q) := \left\{\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}\right\}.$$

With this action, we calculate that

$$K := \mathrm{stab}(p_0) = \left\{\begin{bmatrix} a & c\delta & b\delta \\ b & a & c\delta \\ c & b & a \end{bmatrix}\right\}.$$

Indeed, we find that

$$K \xrightarrow{\sim} \mathbb{F}_q(\delta^{1/3})^\times : \begin{bmatrix} a & c\delta & b\delta \\ b & a & c\delta \\ c & b & a \end{bmatrix} \mapsto a + b\delta^{1/3} + c\delta^{2/3}.$$

In addition, we have the homogeneous space $\mathbb{H}_q \simeq \mathrm{GL}_3(\mathbb{F}_q)/K$. As a quick check, we can note that we have $|\mathrm{GL}_3(\mathbb{F}_q)| = (q^3-1)(q^3-q)(q^3-q^2)$ and $|K| = q^3 - 1$ (from the canonical isomorphism); thus, we should have $|\mathbb{H}_q|$ equal to

$$\frac{(q^3-1)(q^3-q)(q^3-q^2)}{q^3-1} = (q^2-1)(q-1)q^3.$$

Indeed, from the definition, we have

$$|\mathbb{H}_q| = |\mathrm{GL}_2(\mathbb{F}_q)| \cdot q^2 = (q^2-1)(q^2-q)q^2 = (q^2-1)(q-1)q^3.$$

Remark 1. *Observe that the above construction is a natural generalization of the finite upper half-plane defined by Terras [6] (here,* δ *is a nonsquare in the finite field* $\mathbb{F}_q$*):*

$$H_q = \left\{x + y\sqrt{\delta} : x, y \in \mathbb{F}_q, y \neq 0\right\}.$$

The condition $y \neq 0$ *is equivalent to* 1 *and* $x + y\sqrt{\delta}$ *being linearly independent elements of* $\mathbb{F}_q\left(\sqrt{\delta}\right)$ *over* $\mathbb{F}_q$*.*

3. Trace Formula

3.1. The Pre-Trace Formula

Our trace formula relies on the interpretation of a known result, namely, the pre-trace formula. We now derive the pre-trace formula and contextualize it in our setting of the group $G = \mathrm{GL}_3(\mathbb{F}_q)$. The idea behind the pre-trace formula is to compute the trace of a specific operator in two different bases. We may construct the relevant information for this operator as follows:

Suppose that G is a finite group and $\Gamma \leq G$ is a subgroup. Let $f : G \to \mathbb{C}$ be a complex-valued function on G. For a representation $\kappa : G \to \mathrm{GL}(V)$ of G (where V is a finite-dimensional complex vector space endowed with a linear G-action), define the Fourier transform of f at κ by

$$\hat{f}(\kappa) = \sum_{g \in G} f(g)\kappa\Big(g^{-1}\Big).$$

It can be observed that the Fourier transform $\hat{f}(\kappa)$ is End V-valued. The pre-trace formula is now obtained by computing the trace of $\hat{f}(\rho)$ in two different bases with $\rho = \mathrm{Ind}_\Gamma^G 1$. First, we can decompose ρ as a direct sum of irreducible representations of G to obtain

$$\mathrm{Tr}\Big(\hat{f}(\rho)\Big) = \sum_{\pi \in \hat{G}} m(\pi, \rho)\,\mathrm{Tr}\Big(\hat{f}(\pi)\Big). \tag{2}$$

Here, $\hat{G}$ is the set of irreducible representations of G and $m(\pi, \rho)$ is the multiplicity of the irreducible representation $\pi \in \hat{G}$ in ρ. Let

$$V = \{f : G \to \mathbb{C} \,|\, f(\gamma g) = f(g) \quad \forall \gamma \in \Gamma, g \in G\} = L^2(\Gamma\backslash G).$$

As G acts linearly on V via right-multiplication, the corresponding representation is precisely $\rho = \mathrm{Ind}_\Gamma^G 1$. More concretely, for $\phi \in V$, $g, x \in G$, we have

$$[\rho(g)\phi](x) = \phi(xg).$$

Thus,

$$\Big[\hat{f}(\rho)\phi\Big](x) = \sum_{y \in G} f(y)\phi\Big(xy^{-1}\Big) = \sum_{u \in G} f\Big(u^{-1}x\Big)\phi(u).$$

Further, as ϕ is Γ-invariant,

$$\Big[\hat{f}(\rho)\phi\Big](x) = \sum_{y \in \Gamma\backslash G} \sum_{\gamma \in \Gamma} f\Big(y^{-1}\gamma x\Big)\phi(y).$$

Now, we are ready to compute the trace of $\hat{f}(\rho)$ in another way. Consider the indicator basis $\{\delta_y\}_{y \in \Gamma\backslash G}$ of V (for $y \in \Gamma\backslash G$, δ_y is the function on $\Gamma\backslash G$ which outputs 1 if the input is y and 0 otherwise). With respect to this basis, $\hat{f}(\rho)$ is represented by a matrix where the entry corresponding to $x, y \in \Gamma\backslash G$ is $\sum_{\gamma \in \Gamma} f\big(y^{-1}\gamma x\big)$. Therefore,

$$\mathrm{Tr}\Big[\hat{f}(\rho)\Big] = \sum_{x \in \Gamma\backslash G} \sum_{\gamma \in \Gamma} f\Big(x^{-1}\gamma x\Big). \tag{3}$$

We can rewrite the right-hand side as a sum over conjugacy classes in G. Let:

$$\Gamma_\gamma = \Big\{x \in \Gamma \,|\, x^{-1}\gamma x = \gamma\Big\} = \text{centralizer of } \gamma \text{ in } \Gamma$$

$$G_\gamma = \Big\{x \in G \,|\, x^{-1}\gamma x = \gamma\Big\} = \text{centralizer of } \gamma \text{ in } G$$

$$\{\gamma\} = \Big\{x^{-1}\gamma x \,|\, x \in G\Big\} = \text{conjugacy class of } \gamma \text{ in } G$$

$$C_\Gamma = \text{set of conjugacy classes in } \Gamma.$$

Note that there is a one-to-one correspondence between the right cosets $\Gamma_\gamma\backslash\Gamma$ and elements of $\{\gamma\}$ provided by $\Gamma_\gamma x \mapsto x^{-1}\gamma x$. Because conjugacy classes partition Γ, we can rewrite the inner sum as

$$\sum_{x\in\Gamma\backslash G}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma x) = \sum_{x\in\Gamma\backslash G}\sum_{\{\gamma\}\in C_\Gamma}\sum_{u\in\Gamma_\gamma\backslash\Gamma} f\left(x^{-1}u^{-1}\gamma u x\right).$$

Setting $y = ux$, we can then rewrite this as a sum over right cosets $y \in \Gamma_\gamma\backslash G$:

$$\sum_{x\in\Gamma\backslash G}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma x) = \sum_{y\in\Gamma_\gamma\backslash G}\sum_{\{\gamma\}\in C_\Gamma} f(y^{-1}\gamma y).$$

Observing that $\Gamma_\gamma \leq G_\gamma$, we can write $y = ts$ for $t \in \Gamma_\gamma\backslash G_\gamma$ and $s \in G_\gamma\backslash G$:

$$\sum_{x\in\Gamma\backslash G}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma x) = \sum_{t\in\Gamma_\gamma\backslash G_\gamma}\sum_{s\in G_\gamma\backslash G}\sum_{\{\gamma\}\in C_\Gamma} f\left(s^{-1}t^{-1}\gamma t s\right) = \sum_{\{\gamma\}\in C_\Gamma}\frac{|G_\gamma|}{|\Gamma_\gamma|}\sum_{s\in G_\gamma\backslash G} f(s^{-1}\gamma s). \tag{4}$$

We define the orbital sum $I_G(f,\gamma)$ of f at γ to be

$$I_G(f,\gamma) := \sum_{s\in G_\gamma\backslash G} f(s^{-1}\gamma s).$$

The pre-trace formula now follows from combining (2)–(4):

$$\sum_{\pi\in\hat{G}} m(\pi,\rho)\,\mathrm{Tr}(\hat{f}(\pi)) = \sum_{\{\gamma\}\in C_\Gamma}\frac{|G_\gamma|}{|\Gamma_\gamma|} I_G(f,\gamma) \tag{5}$$

where $\rho = \mathrm{Ind}_\Gamma^G 1$ and $I_G(f,\gamma)$ is the orbital sum of f at γ as defined above. Note that (5) can be viewed as a representation-theoretic analogue of Selberg's trace formula [1,6]; the left-hand side can be thought of as the spectral side, summing over the irreducible representations of G and their multiplicities in ρ, while the right-hand side can be interpreted as the geometric side, summing over conjugacy classes in G. For further references, see [6] (Ch. 22).

To obtain the trace formula for the finite upper half-space, we take $G = \mathrm{GL}_3(\mathbb{F}_q)$ and $\Gamma = \mathrm{GL}_3(\mathbb{F}_p)$. In addition, $f : G \to \mathbb{C}$ is chosen to be a $K-$bi-invariant function (K is the stabilizer of p_0, as above), i.e., $\forall k,h \in K$ and $\forall x \in G$ we have $f(kxh) = f(x)$. One can also think of f as a function on $G/K \simeq \mathbb{H}_q$ which is invariant under left-multiplication by elements in K.

Remark 2. *The left-hand side of* (5) *can be expressed solely in terms of the characters χ_π of the group G:*

$$\mathrm{Tr}\left[\hat{f}(\pi)\right] = \mathrm{Tr}\left[\sum_{g\in G} f(g)\pi\left(g^{-1}\right)\right] = \sum_{g\in G} f(g)\,\mathrm{Tr}\left[\pi\left(g^{-1}\right)\right] =$$
$$= \sum_{g\in G} f(g)\chi_\pi\left(g^{-1}\right) = \sum_{g\in G} f(g)\overline{\chi_\pi(g)} = \langle f,\chi_\pi\rangle.$$

Here, $\langle x,y\rangle$ denotes the usual Hermitian inner product on the complex vector space $L^2(G)$:

$$\langle x,y\rangle = \sum_{g\in G} x(g)\overline{y(g)}$$

.

In Sections 4–6, we compute each term in the sum on the right-hand side of the pre-trace formula (5) using the explicit description of conjugacy classes of $\mathrm{GL}_3(\mathbb{F}_p)$ provided in [8]. The conjugacy classes are separated into the cases of central, hyperbolic, parabolic, and elliptic terms. It is helpful to introduce one more tool that will help us simplify these computations.

3.2. Double Cosets and Fundamental Domains

To calculate the orbital sums, it is convenient to identify $G_\gamma\backslash G$ with $G_\gamma\backslash G/K \times K/Z$ (when possible), where Z is the center of G (the subgroup of diagonal matrices $\{aI : a \in \mathbb{F}_q^\times\}$). Let $\{t_i\}_{i=1}^l$ be representatives of double cosets in $G_\gamma\backslash G/K$, and consider the following map:

$$\begin{aligned} G_\gamma\backslash G/K \times K/Z &\to G_\gamma\backslash G \\ (G_\gamma t_i K, kZ) &\mapsto G_\gamma t_i k. \end{aligned}$$

We first show that this map is well-defined and onto. If $z \in Z$, then $G_\gamma t_i kz = G_\gamma z t_i k = G_\gamma t_i k$, because $z \in G_\gamma$ (as z commutes with every element of G). Now, let $G_\gamma x \in G\backslash G_\gamma$; then, because double cosets $G_\gamma t_i K$ partition G and are unions of right cosets $G_\gamma y$, we can find a representative $t_i \in G$ and $k \in K$ such that $G_\gamma x = G_\gamma t_i k$.

For injectivity, let t_i, t_j be representatives of two double cosets in $G_\gamma\backslash G/K$ and let k, k' be representatives of two left cosets in K/Z. Suppose that $G_\gamma t_i k = G_\gamma t_j k'$. Because double cosets are either disjoint or identical, this is only possible if $t_i = t_j$ while letting $t = t_i$. Thus, we have

$$G_\gamma = G_\gamma t k' k^{-1} t^{-1}, \tag{6}$$

which happens if and only if $tk'k^{-1}t^{-1} \in G_\gamma$, or equivalently, $k'k^{-1} \in tG_\gamma t^{-1} \cap K$. This can also be stated as $kH = k'H$, where $H = tG_\gamma t^{-1} \cap K$. Note that $Z \leq K$ and $Z \leq tG_\gamma t^{-1} = G_{t\gamma t^{-1}}$, as Z is the intersection of all centralizers in G; hence, we always have $Z \leq K \cap tG_\gamma t^{-1} = H$. If $Z = H$, then (6) is equivalent to $kZ = k'Z$, which implies that the above map is injective. In our specific example, the centralizers G_γ contain no non-central elements conjugate to some matrix in K unless γ itself is conjugate to a member of K (this can be seen by, e.g., taking simple representatives γ of conjugacy classes in G and looking at the characteristic polynomials and eigenvalues of elements in G_γ, which are preserved by conjugation).

Thus, we have two cases:

Case 1: γ is not conjugate to an element of K. Then, we have

$$I_G(f,\gamma) = \sum_{s \in G_\gamma\backslash G} f(s^{-1}\gamma s) = \sum_{t \in G_\gamma\backslash G/K} \sum_{k \in K/Z} f((tk)^{-1}\gamma tk) = |K/Z| \sum_{t \in G_\gamma\backslash G/K} f(t^{-1}\gamma t).$$

The last equality follows from the fact that f is $K-$bi-invariant. We can identify $G_\gamma\backslash G/K$ with $G_\gamma\backslash\mathbb{H}_q$. In order to calculate orbital sums, we find fundamental domains for the action of G_γ on $\mathbb{H}_q$, i.e., subsets of $\mathbb{H}_q$ which contain exactly one element from each G_γ-orbit.

Case 2: γ is conjugate to an element of K. In this case, the above map is not necessarily injective; however, $G_\gamma = G$ or K, depending on whether γ is conjugate to a central element of K or not. Then, $G_\gamma\backslash G$ is either trivial or can be identified with $\mathbb{H}_q$.

4. Central and Hyperbolic Terms

4.1. Central Terms

Let $\gamma = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ for $a \in \mathbb{F}_p^\times$. In this case, $G_\gamma = G$ and $\Gamma_\gamma = \Gamma$. Thus, $I_G(f,\gamma) = f(p_0)$, and the total contribution from such γ is

$$\frac{|G|}{|\Gamma|} f(p_0) \cdot (p-1) = \frac{(q^3-1)(q^2-1)(q-1)q^3}{(p^3-1)(p^2-1)p^3} f(p_0).$$

4.2. Hyperbolic Terms of the First Kind

Let $\gamma = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}, a \neq b \in \mathbb{F}_p^\times$. Here,

$$G_\gamma = \left\{ \begin{bmatrix} x & y & 0 \\ w & z & 0 \\ 0 & 0 & t \end{bmatrix} \right\} \simeq \mathrm{GL}_2(\mathbb{F}_q) \times \mathbb{F}_q^\times$$

and similarly for Γ_γ.

A fundamental domain for the action of G_γ is provided by

$$G_\gamma \backslash \mathbb{H}_q = \{(u + \delta^{1/3}, v + \delta^{2/3}) : u, v \in \mathbb{F}_q\}. \tag{7}$$

We verify this in the next proposition for the sake of illustration. In the next sections, the verification of the fundamental domains is relegated to Appendix A.

Proposition 1. *A fundamental domain for G_γ is provided by (7).*

Proof. First, to show the uniqueness of each representative, suppose that there exists $m \in G_\gamma$ such that

$$m \cdot (u_1 + \delta^{1/3}, v_1 + \delta^{2/3}) = (u_2 + \delta^{1/3}, v_2 + \delta^{2/3}).$$

Then, we deduce that $y = 0 = w$ and $x = t = z$; thus, $u_1 = u_2$ and $v_1 = v_2$.

Next, to show the completeness of this fundamental domain, we can take an arbitrary element $(\alpha, \beta) \in \mathbb{H}_q$. We wish to find $m \in G_\gamma, u, v \in \mathbb{F}_q$ such that $m \cdot (u + \delta^{1/3}, v + \delta^{2/3}) = (\alpha, \beta)$. Setting $x = \alpha_2, y = \alpha_3, w = \beta_2, z = \beta_3$, and $t = 1$, it can be seen that u, v is a solution to

$$\begin{bmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix},$$

which has a solution because the left matrix is invertible. □

Thus,

$$\begin{aligned} I_G(f, \gamma) &= \sum_{t \in G_\gamma \backslash G/K} \sum_{u \in K/\{aI\}} f((tu)^{-1} \gamma (tu)) \\ &= \sum_{t \in G_\gamma \backslash \mathbb{H}_q} \frac{|K|}{|\{aI\}|} f(t^{-1} \gamma t) \\ &= \frac{q^3 - 1}{q - 1} \sum_{x, y \in \mathbb{F}_q} f\left(\begin{bmatrix} 0 & 1 & -y \\ 1 & 0 & -x \\ 0 & 0 & 1 \end{bmatrix} \gamma \begin{bmatrix} 0 & 1 & x \\ 1 & 0 & y \\ 0 & 0 & 1 \end{bmatrix} p_0 \right) \\ &= (q^2 + q + 1) \sum_{x, y \in \mathbb{F}_q} f\left(\begin{bmatrix} a & 0 & (a-b)y \\ 0 & a & (a-b)x \\ 0 & 0 & b \end{bmatrix} p_0 \right) \\ &= (q^2 + q + 1) \sum_{x, y \in \mathbb{F}_q} f\Big((x + \frac{a}{b} \delta^{2/3}, y + \frac{a}{b} \delta^{1/3}) \Big). \end{aligned}$$

Let us define the *horocycle transform* $Hf : \mathrm{GL}_2(\mathbb{F}_q) \to \mathbb{C}$ of $f \in L^2(K\backslash G/K)$ by

$$Hf(\kappa) := \sum_{x,y\in\mathbb{F}_q} \sum_{\xi\in\{\kappa\}} f\left(\begin{bmatrix}\xi_{11} & \xi_{12} & x\\ \xi_{21} & \xi_{22} & y\\ 0 & 0 & 1\end{bmatrix} p_0\right),$$

where the inner sum runs over $\xi \in \mathrm{GL}_2(\mathbb{F}_q)$ in the conjugacy class of κ. Thus, this hyperbolic term is equal to

$$\begin{aligned}(q^2+q+1) &\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}} \frac{(q^2-1)(q^2-q)(q-1)}{(p^2-1)(p^2-p)(p-1)} \cdot Hf\left(\frac{a}{b}I\right)\\ &= \frac{(q^3-1)(q^2-1)(q^2-q)}{(p^2-1)(p^2-p)(p-1)} \sum_{\substack{a\in\mathbb{F}_p^\times\\ a\neq 1}} (p-1)\cdot Hf(aI)\\ &= \frac{(q^3-1)(q^2-1)(q^2-q)}{(p^2-1)(p^2-p)} \sum_{\substack{a\in\mathbb{F}_p^\times\\ a\neq 1}} Hf(aI).\end{aligned}$$

We have shown the full computation for the orbital sums for this case both to demonstrate the computational technique and to motivate and introduce the horocycle transform. For the next cases, we omit detailed computations for orbital sums; however, some are included in Appendix A for the curious reader.

4.3. Hyperbolic Terms of the Second Kind

Let $\gamma = \begin{bmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}$, with $a,b,c \in \mathbb{F}_p^\times$ being distinct. Here, G_γ is the set of diagonal matrices and

$$\begin{aligned}G_\gamma\backslash\mathbb{H}_q = &\{(x+\delta^{1/3}+y\delta^{2/3}, r+s\delta^{1/3}+\delta^{2/3}) : ys\neq 1\} \sqcup \{(x+\delta^{2/3}, r+\delta^{1/3}+s\delta^{2/3}) : s\neq 0\}\\ &\sqcup \{(x+y\delta^{1/3}+\delta^{2/3}, r+\delta^{1/3}) : y\neq 0\} \sqcup \{(x+\delta^{2/3}, y+\delta^{1/3})\}.\end{aligned}$$

We have

$$I_G(f,\gamma) = (q^2+q+1)\cdot Hf\left(\begin{bmatrix}a/c & 0\\ 0 & b/c\end{bmatrix}\right),$$

where $a,b,c \in \mathbb{F}_p^\times$ comes from a change of variables outlined in Appendix A.

The total contribution from the second hyperbolic term is

$$\begin{aligned}\frac{1}{6}(q^2+q+1)\frac{(q-1)^3}{(p-1)^3} \sum_{\substack{a,b,c\in\mathbb{F}_p^\times\\ a\neq b\neq c}} Hf\left(\begin{bmatrix}a/c & 0\\ 0 & b/c\end{bmatrix}\right) &= \frac{1}{6}(q^2+q+1)\frac{(q-1)^3}{(p-1)^2} \sum_{\substack{a,b\in\mathbb{F}_p^\times\backslash\{1\}\\ a\neq b}} Hf\left(\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\right) =\\ &= \frac{1}{3}(q^3-1)\frac{(q-1)^2}{(p-1)^2} \sum_{\{a,b\}\subseteq\mathbb{F}_p^\times\backslash\{1\}} Hf\left(\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}\right).\end{aligned}$$

5. Parabolic Terms

5.1. Parabolic Terms of the First Kind

Let $\gamma = \begin{bmatrix} a & 0 & a \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, a \in \mathbb{F}_p^{\times}$. In this case,

$$G_\gamma = \left\{ \begin{bmatrix} d & y & x \\ 0 & c & b \\ 0 & 0 & d \end{bmatrix} : c, d \in \mathbb{F}_q^{\times}, b, x, y \in \mathbb{F}_q \right\}.$$

Proposition 2. *A fundamental domain for G_γ is*

$$G_\gamma \backslash \mathbb{H}_q = \{(u\delta^{1/3}, v\delta^{1/3} + \delta^{2/3}) : u, v \in \mathbb{F}_q, u \neq 0\} \sqcup \{(\delta^{1/3} + u\delta^{2/3}, \delta^{1/3}) : u \in \mathbb{F}_q^{\times}\}.$$

Proof. See Appendix A. □

Following the computations outlined in Appendix A, the orbital sum in this case is

$$I_G(f, \gamma) = (q^2 + q + 1)(Hf(I) - f(p_0)).$$

In total, the contribution from the parabolic terms of the first kind is

$$\frac{q^3(q-1)^2}{p^3(p-1)^2}(q^2 + q + 1)(Hf(I) - f(p_0)) \cdot (p-1) = \frac{q^3(q^3-1)(q-1)}{p^3(p-1)}(Hf(I) - f(p_0)).$$

5.2. Parabolic Terms of the Second Kind

Let $\gamma = \begin{bmatrix} a & a & 0 \\ 0 & a & a \\ 0 & 0 & a \end{bmatrix}, a \in \mathbb{F}_p^{\times}$. We find that

$$G_\gamma = B = \left\{ \begin{bmatrix} z & y & x \\ 0 & c & b \\ 0 & 0 & d \end{bmatrix} : d, c, z \in \mathbb{F}_q^{\times}, b, x, y \in \mathbb{F}_q \right\}.$$

Here, B is the Borel subgroup of upper-triangular matrices.

Proposition 3. *A fundamental domain for B is*

$$B \backslash \mathbb{H}_q = \{(\delta^{1/3}, v\delta^{1/3} + \delta^{2/3}) : v \in \mathbb{F}_q\} \sqcup \{(\delta^{2/3}, \delta^{1/3})\}.$$

Proof. See Appendix A. □

We find that the orbital sum (see Appendix A) is

$$I_G(f, \gamma) = (q^2 + q + 1)\left(f\left(\delta^{2/3} + \delta^{1/3}, \delta^{1/3} + 1\right) + \sum_{v \in \mathbb{F}_q} f\left((1-v)\delta^{2/3} - v^2\delta^{1/3} + 1, \delta^{2/3} + (v+1)\delta^{1/3}\right)\right).$$

Note that the orbital sum is independent of a; therefore, we denote each $I_G(f, \gamma)$ by $\tilde{I}_G(f)$. Thus, in total, the contribution of these parabolic terms is

$$\frac{q^3(q-1)^2}{p^3(p-1)^2}\tilde{I}_G(f) \cdot (p-1) = \frac{q^3(q-1)^2}{p^3(p-1)}\tilde{I}_G(f).$$

5.3. Parabolic Terms of the Third Kind

Let $\gamma = \begin{bmatrix} a & 0 & 0 \\ a & a & 0 \\ 0 & 0 & b \end{bmatrix}$ with $a \neq b \in \mathbb{F}_p^\times$. Here,

$$G_\gamma = \left\{ \begin{bmatrix} x & 0 & 0 \\ y & x & 0 \\ 0 & 0 & z \end{bmatrix} : x, z \in \mathbb{F}_q^\times, y \in \mathbb{F}_q \right\}.$$

Again, by a change of variables, we find the following orbital sum (see Appendix A):

$$I_G(f, \gamma) = (q^2 + q + 1) \cdot Hf\left(\begin{bmatrix} a/b & 1 \\ 0 & a/b \end{bmatrix} \right).$$

The overall contribution of these terms is

$$\begin{aligned} \sum_{\substack{a,b \in \mathbb{F}_p^\times \\ a \neq b}} \frac{q(q-1)^2}{p(p-1)^2} (q^2 + q + 1) \cdot Hf\left(\begin{bmatrix} a/b & 1 \\ 0 & a/b \end{bmatrix} \right) &= \frac{q(q-1)^2}{p(p-1)^2} (q^2 + q + 1) \sum_{\substack{c \in \mathbb{F}_p^\times \\ c \neq 1}} (p-1) \cdot Hf\left(\begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} \right) \\ &= \frac{q(q-1)(q^3-1)}{p(p-1)} \sum_{\substack{c \in \mathbb{F}_p^\times \\ c \neq 1}} Hf\left(\begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix} \right). \end{aligned}$$

6. Elliptic Terms

6.1. Elliptic Terms of the First Kind

Irreducible γ. The characteristic polynomial of such a γ is irreducible in $\mathbb{F}_p[t]$. Recall that $q = p^n$. Depending on whether $n \equiv 0 \pmod 3$, there are two cases to consider.

- $n \equiv 0 \pmod 3$. In this case, we have the tower of field extensions $\mathbb{F}_q / \mathbb{F}_{p^3} / \mathbb{F}_p$. Thus, γ is similar to a diagonal matrix in $\mathrm{GL}_3(\mathbb{F}_q)$. However, the same is not the case in $\mathrm{GL}_3(\mathbb{F}_p)$, which is why this case is different from the second hyperbolic term. Therefore, G_γ is the subgroup of diagonal matrices over $\mathbb{F}_q$, while Γ_γ is K with entries in $\mathbb{F}_p$, i.e., $|\Gamma_\gamma| = p^3 - 1$. Because G_γ is the same, we can reuse the fundamental domain for G_γ from the earlier second hyperbolic case.

  Suppose that γ is similar to $\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}$ in $\mathrm{GL}_3(\mathbb{F}_q)$, where $\xi_1, \xi_2, \xi_3 \in \mathbb{F}_{p^3} \backslash \mathbb{F}_p$. Similar to the second hyperbolic term, the orbital sum will be

  $$I_G(f, \gamma) = |K/Z| \cdot Hf\left(\begin{bmatrix} \xi_1/\xi_3 & 0 \\ 0 & \xi_2/\xi_3 \end{bmatrix} \right).$$

  Because ξ_1, ξ_2, ξ_3 are Galois conjugates, without loss of generality, $\xi_1 = \xi_3^p$ and $\xi_2 = \xi_3^{p^2}$. Therefore, the total contribution of the first elliptic terms is

  $$\frac{1}{3} \cdot \frac{(q-1)^3}{p^3 - 1} \cdot \frac{q^3 - 1}{q - 1} \sum_{\xi \in \mathbb{F}_{p^3}^\times \backslash \mathbb{F}_p^\times} Hf\left(\begin{bmatrix} \xi^{p-1} & 0 \\ 0 & \xi^{p^2-1} \end{bmatrix} \right).$$

  Note that the factor of $\frac{1}{3}$ here comes from the fact that each γ is counted three times in the sum.

- $n \not\equiv 0 \pmod 3$. In this case, there is no cubic extension intermediate to $\mathbb{F}_q / \mathbb{F}_p$; thus, $\mathbb{F}_q(\delta^{1/3})$ is the minimal field (in the sense of containment, of course, not size) over which the characteristic polynomial of γ has a root. Suppose that an eigenvalue of γ is $\alpha_1 + \alpha_2 \delta^{1/3} + \alpha_3 \delta^{2/3}$ with eigenvector $x_1 + x_2 \delta^{1/3} + x_3 \delta^{2/3}$, where $\alpha_i \in \mathbb{F}_q$ and $x_i \in \mathbb{F}_q^3$. Then,

$$\gamma x_1 = \alpha_1 x_1 + \alpha_3 \delta x_2 + \alpha_2 \delta x_3,$$
$$\gamma x_2 = \alpha_2 x_1 + \alpha_1 x_2 + \alpha_3 \delta x_3,$$
$$\gamma x_3 = \alpha_3 x_1 + \alpha_2 x_2 + \alpha_1 x_3.$$

Thus, $\mathrm{GL}_3(\mathbb{F}_q)$ when viewed in γ is similar to a matrix in K. Conversely, non-diagonal matrices in K provide irreducible elements in $\mathrm{GL}_3(\mathbb{F}_p)$. Further, in this case G_γ is precisely K; thus, $G_\gamma \backslash G = \mathbb{H}_q$. After identifying elements $y \in \mathbb{H}_q$ with elements $y = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \in \mathrm{Aff}_3(\mathbb{F}_q)$ and writing $\gamma = \begin{bmatrix} \alpha & \varepsilon\delta & \beta\delta \\ \beta & \alpha & \varepsilon\delta \\ \varepsilon & \beta & \alpha \end{bmatrix}$, we can compute

$$\begin{aligned} I_G(f,\gamma) &= \sum_{y \in \mathbb{H}_q} f(y^{-1}\gamma y) \\ &= \sum_{a,b,c,d,e,f \in \mathbb{F}_q} f\left(\frac{1}{bd-ae} \begin{bmatrix} -e & b & ce-bf \\ d & -a & af-cd \\ 0 & 0 & bd-ae \end{bmatrix} \begin{bmatrix} \alpha & \varepsilon\delta & \beta\delta \\ \beta & \alpha & \varepsilon\delta \\ \varepsilon & \beta & \alpha \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} p_0 \right) \\ &= \sum_{a,b,c,d,e,f \in \mathbb{F}_q} f\left(\frac{1}{bd-ae} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} p_0 \right), \end{aligned}$$

where the entries of the matrix, by column, are as follows:

- ◇ $x_{11} = \alpha(bd-ae) + \beta(ab+cde-bdf) + \varepsilon(ace-abf-de\delta)$
- ◇ $x_{21} = \beta(adf-a^2-cd^2) + \varepsilon(a^2f-acd+d^2\delta)$
- ◇ $x_{31} = \beta d + \varepsilon a$
- ◇ $x_{12} = \beta(ce^2-bef+b^2) + \varepsilon(bce-b^2f-e^2\delta)$
- ◇ $x_{22} = \alpha(bd-ae) + \beta(aef-cde-ab) + \varepsilon(abf-bcd+de\delta)$
- ◇ $x_{32} = \beta e + \varepsilon b$
- ◇ $x_{13} = \beta(cef+bc-bf^2-e\delta)$
- ◇ $x_{23} = \beta(af^2-cdf-ac+d\delta) + \varepsilon(acf-ac^2+df\delta-a\delta)$
- ◇ $x_{33} = \alpha + \beta f + \varepsilon c$.

Unfortunately, there does not seem to be an obvious way to simplify this expression further. When computing this case for a specific field $\mathbb{F}_q$, it seems that applying this formula may not be more helpful than computing directly.

6.2. Elliptic Terms of the Second Kind

$\gamma = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$, where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is irreducible and $e \in \mathbb{F}_p^\times$. In addition,

$$\Gamma_\gamma = \left\{ \begin{bmatrix} a & b\xi & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix} : c(a^2 - b^2\xi) \neq 0 \right\},$$

where ξ is a nonsquare in $\mathbb{F}_p$.

Again, depending on whether $n \equiv 0 \pmod 2$, i.e., whether n is even or odd, there are two cases to consider.

- n is even. In this case, $\mathbb{F}_p \subset \mathbb{F}_{p^2} \subseteq \mathbb{F}_q$ and γ is diagonalizable in $\mathrm{GL}_3(\mathbb{F}_q)$. G_γ is a subgroup of diagonal matrices over $\mathbb{F}_q$, meaning that the fundamental domain $G_\gamma \backslash \mathbb{H}_q$ is the same as for the second hyperbolic case.

  Suppose that γ is similar to $\left(\begin{smallmatrix} \eta_1 & 0 & 0 \\ 0 & \eta_2 & 0 \\ 0 & 0 & e \end{smallmatrix}\right)$ in G, where $\eta_1, \eta_2 \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$ and $e \in \mathbb{F}_p^\times$. Then, (analogously to the second hyperbolic case) the orbital sum will be

$$I_G(f,\gamma) = |K/Z| \cdot Hf\left(\begin{bmatrix} \eta_1/e & 0 \\ 0 & \eta_2/e \end{bmatrix}\right).$$

As η_1, η_2 are Galois conjugates, without loss of generality we may assume that $\eta_1 = \eta_2^p$. Therefore, the total contribution of the second elliptic terms will be

$$\frac{1}{2} \cdot \frac{(q-1)^3}{(p-1)(p^2-1)} \cdot \frac{q^3-1}{q-1} \sum_{e\in\mathbb{F}_p^\times} \sum_{\eta\in\mathbb{F}_{p^2}^\times\backslash\mathbb{F}_p^\times} Hf\left(\begin{bmatrix} \eta/e & 0 \\ 0 & \eta^p/e \end{bmatrix}\right) =$$

$$= \frac{1}{2} \cdot \frac{(q-1)^3}{(p^2-1)} \cdot \frac{q^3-1}{q-1} \sum_{\eta\in\mathbb{F}_{p^2}^\times\backslash\mathbb{F}_p^\times} Hf\left(\begin{bmatrix} \eta & 0 \\ 0 & \eta^p \end{bmatrix}\right).$$

- n is odd. In this case, γ is not diagonalizable in $\mathrm{GL}_3(\mathbb{F}_q)$. In general, it will be conjugate to an element of the following form:

$$\begin{bmatrix} k & l\xi & 0 \\ l & k & 0 \\ 0 & 0 & m \end{bmatrix}$$

where $k \in \mathbb{F}_p$, $m, l, \xi \in \mathbb{F}_p^\times$ such that ξ is a nonsquare in $\mathbb{F}_p$ (this ensures that the determinant $m(k^2 - l^2\xi)$ is nonzero). In this case, we have

$$G_\gamma = \left\{ \begin{bmatrix} a & b\xi & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix} : a, b, c \in \mathbb{F}_q, c(a^2 - b^2\xi) \neq 0 \right\}.$$

We seek a fundamental domain for this specific G_γ.

Proposition 4. *A fundamental domain for G_γ is provided by*

$$G_\gamma\backslash\mathbb{H}_q = \left\{ \left(x + u\delta^{1/3} + v\delta^{2/3}, y + \delta^{1/3}\right) : v \in \mathbb{F}_q^\times, x, y, u \in \mathbb{F}_q \right\}.$$

The corresponding orbital sum is

$$I_G(f,\gamma) = \frac{(q^3-1)(q^2-1)}{(p-1)(p^2-1)} Hf\left(\begin{bmatrix} k/m & l\xi/m \\ l/m & k/m \end{bmatrix}\right).$$

The total contribution from the second elliptic terms in this case is

$$\frac{(q^3-1)(q^2-1)}{(p^2-1)} \sum_{\substack{s\in\mathbb{F}_p \\ t\in\mathbb{F}_p^\times}} Hf\left(\begin{bmatrix} s & t\xi \\ t & s \end{bmatrix}\right).$$

A detailed proof of both the proposition and the orbital sum calculation can be found in Appendix A.

7. Examples

Throughout this section, subscripts under multiplicities indicate which group is the domain of the corresponding representations. We set $\rho = \mathrm{Ind}_\Gamma^G 1$ and $\kappa = \mathrm{Ind}_K^G 1$.

Example 1. *Let $f : G \to \mathbb{C}$ be the constant one function. The right-hand side of the trace formula is $|G|$, while the left-hand side (by orthogonality of characters) becomes*

$$\sum_{\pi\in\hat{G}} m(\pi,\rho)_G \langle f, \chi_\pi \rangle = \sum_{\pi\in\hat{G}} m(\pi,\rho)_G \langle \chi_1, \chi_\pi \rangle = |G| \cdot m(1,\rho)_G.$$

Thus, the trace formula says $|G| \cdot m(1,\rho) = |G|$, which is a manifestation of Frobenius reciprocity:

$$m(1,\rho)_G = \frac{1}{|G|}\langle 1, \operatorname{Ind}_\Gamma^G 1\rangle_G = \frac{1}{|\Gamma|}\langle \operatorname{Res}_\Gamma^G 1, 1\rangle_\Gamma = m(1,1)_\Gamma = 1.$$

Example 2. *Let $f : \mathbb{H}_q \to \mathbb{C}$ be the indicator function δ_{p_0}, which is 1 on p_0 and zero otherwise. It is $K-$bi-invariant because $\operatorname{stab}(p_0) = K$. On the left-hand side of the trace formula, we have*

$$\begin{aligned}\sum_{\pi\in\hat{G}} m(\pi,\rho)_G\langle f,\chi_\pi\rangle &= \sum_{\pi\in\hat{G}} m(\pi,\rho)_G \sum_{g\in G} f(g\cdot p_0)\chi_\pi\left(g^{-1}\right) = \sum_{\pi\in\hat{G}} m(\pi,\rho)_G \sum_{k\in K}\chi_\pi\left(k^{-1}\right) = \\ &= \sum_{\pi\in\hat{G}} m(\pi,\rho)_G\langle 1,\chi_\pi\rangle_K = |K|\cdot\sum_{\pi\in\hat{G}} m(\pi,\rho)_G\cdot m(1,\operatorname{Res}_K^G\pi)_K = \\ &= |K|\cdot\sum_{\pi\in\hat{G}} m(\pi,\rho)_G\cdot m(\pi,\kappa)_G.\end{aligned}$$

On the right-hand side, the only nonzero terms are the central terms and the first elliptic terms when n is not divisible by 3. If that is the case, we claim that for a non-central γ in K we have $I_G(f,\gamma) = 3$.

Suppose that γ corresponds to multiplication by $\xi \in \mathbb{F}_q\left(\delta^{1/3}\right)^\times$ under the isomorphism $K \simeq \mathbb{F}_q\left(\delta^{1/3}\right)^\times$. When viewed as an element of $\mathrm{GL}_3(\mathbb{F}_q)$, the eigenvalues of γ are the Galois conjugates of ξ, which are ξ, ξ^q, and ξ^{q^2} in our setting. This means that the conjugates of γ lying in K are precisely γ^q and γ^{q^2}; hence, there will be exactly three non-zero terms in each orbital sum $I_G(f,\gamma)$.

The total contribution of the first elliptic terms is

$$\frac{1}{3}\cdot\frac{q^3-1}{p^3-1}\sum_{\substack{\gamma\in K\\ \gamma\notin Z}} I_G(f,\gamma) = \frac{(q^3-1)(q^3-q)}{p^3-1}.$$

Thus, we obtain the following cases:

Case 1: n is divisible by 3; then, we have

$$\sum_{\pi\in\hat{G}} m(\pi,\rho)_G\cdot m(\pi,\kappa)_G = \frac{q^3(q^2-1)(q-1)}{p^3(p^2-1)(p^3-1)}.$$

Case 2: n is not divisible by 3; then, we have

$$\sum_{\pi\in\hat{G}} m(\pi,\rho)_G\cdot m(\pi,\kappa)_G = \frac{q^3(q^2-1)(q-1)}{p^3(p^2-1)(p^3-1)} + \frac{q^3-q}{p^3-1}.$$

Remark 3. *Note that the sum on the left-hand side can also be written as*

$$\sum_{\pi\in\hat{G}} m(\pi,\rho)_G\cdot m(\pi,\kappa)_G = \frac{1}{|G|}\langle\chi_\rho,\chi_\kappa\rangle.$$

8. Character of the Representation $\rho = \operatorname{Ind}_\Gamma^G 1$

Having effectively computed the right-hand side of the pre-trace formula (5), we now turn to the left-hand side. For this, we require the character χ_ρ of ρ, which we now calculate. We use the usual formula for the character of an induced representation (see [6] (Ch. 16)):

$$\chi_\rho(\gamma) = \frac{1}{|\Gamma|}\sum_{x\in G}\mathbb{1}_\Gamma(x\gamma x^{-1}).$$

We treat each conjugacy class. By $H_{1,q}, H_{2,q}, P_{1,q}, \ldots, E_{2,q}$, we denote the size of various types of conjugacy classes in $\mathrm{GL}_3(\mathbb{F}_q)$. Given a set S, we use $\mathbb{1}_S$ to denote the indicator function for S.

- Central class: $\gamma = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, a \in \mathbb{F}_q^\times$. Because $\gamma \in Z(G)$, we have $x\gamma x^{-1} = \gamma$; thus,

$$\chi_\rho(\gamma) = \frac{|G|}{|\Gamma|} \mathbb{1}_{\mathbb{F}_p}(a).$$

- First hyperbolic class: $\gamma = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$ with $a \neq b \in \mathbb{F}_q^\times$. Here, γ is similar to an element of Γ only when a, b are roots of a polynomial over $\mathbb{F}_p$. This occurs only when $a, b \in \mathbb{F}_p^\times$; for instance, this can be seen by considering the determinant $\mathrm{Det}(\gamma) = a^2 b$. As this is the constant coefficient of $\mathrm{char}_\gamma \in \mathbb{F}_p[x]$, we have $\mathrm{Det}(\gamma) \in \mathbb{F}_p$, which implies that $\mathrm{Det}(\gamma)$ is invariant under the Frobenius automorphism (as $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is generated by the Frobenius). Because $\mathbb{F}_q/\mathbb{F}_p$ is Galois, if $a, b \notin \mathbb{F}_p$, then they must be Galois conjugates, i.e., $a^p = b$ and $b^p = a$. However, this forces $a^2 b = (a^2 b)^p = b^2 a$, i.e., $a = b$, which is a contradiction. Thus, by a simple application of the orbit-stabilizer theorem, it follows that
$$\chi_\rho(\gamma) = \frac{|G|}{|\Gamma|} \frac{H_{1,p}}{H_{1,q}} \mathbb{1}_\Gamma(\gamma).$$

- Second hyperbolic class: $\gamma = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ with $a, b, c \in \mathbb{F}_q^\times$ distinct. There are three cases which allow γ to be similar to an element of Γ: either all three are Galois conjugates in $\mathbb{F}_{p^3}/\mathbb{F}_p$ (only possible if $3 \mid n$), or a, b are Galois conjugates in $\mathbb{F}_{p^2}/\mathbb{F}_p$ and $c \in \mathbb{F}_p$ (only possible if $2 \mid n$), or all three belong to $\mathbb{F}_p$. These cases give rise to the three terms in the sum:

$$\chi_\rho(\gamma) = \frac{|G|}{|\Gamma| \cdot H_{2,q}} \Big(E_{2,p} \cdot \mathbb{1}_{\mathbb{F}_{p^3} \setminus \mathbb{F}_p}(a) \delta_b(a^p) \delta_c(a^{p^2}) + E_{1,p} \cdot \mathbb{1}_{\mathbb{F}_{p^2} \setminus \mathbb{F}_p}(a) \delta_b(a^p) \mathbb{1}_{\mathbb{F}_p}(c) + H_{2,p} \cdot \mathbb{1}_\Gamma(\gamma) \Big).$$

- First parabolic class: $\gamma = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}, a \in \mathbb{F}_q^\times$. If $a \notin \mathbb{F}_p$ were the root of a polynomial over $\mathbb{F}_p$, its Galois conjugate $a^p \neq a$ would also be a root, which is clearly not the case. Thus,
$$\chi_\rho(\gamma) = \frac{|G|}{|\Gamma|} \frac{P_{1,p}}{P_{1,q}} \mathbb{1}_{\mathbb{F}_p}(a).$$

- Second parabolic class: $\gamma = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}, a \in \mathbb{F}_q^\times$. Similar to the previous class,

$$\chi_\rho(\gamma) = \frac{|G|}{|\Gamma|} \frac{P_{2,p}}{P_{2,q}} \mathbb{1}_{\mathbb{F}_p}(a).$$

- Third parabolic class: $\gamma = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$ with $a \neq b \in \mathbb{F}_q^\times$. Following a similar argument as for the first hyperbolic class, we have

$$\chi_\rho(g) = \frac{|G|}{|\Gamma|} \frac{P_{3,p}}{P_{3,q}} \mathbb{1}_\Gamma(g).$$

- First elliptic class: $\gamma = \begin{bmatrix} w & 0 & 0 \\ 0 & w^q & 0 \\ 0 & 0 & r \end{bmatrix}$ with $w \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ and $r \in \mathbb{F}_q^\times$. Here, γ is similar to an element of Γ if and only if w, w^q are roots of a degree-2 irreducible polynomial over $\mathbb{F}_p$. However, if $2 \mid n$, then $\mathbb{F}_{p^2}$ is an intermediate extension in $\mathbb{F}_q/\mathbb{F}_p$, meaning that the roots of any degree-2 polynomial over $\mathbb{F}_p$ are included in $\mathbb{F}_q$. Thus, $\chi_\rho(\gamma) = 0$ if n is divisible by 2; otherwise,

$$\chi_\rho(\gamma) = \frac{|G|}{|\Gamma|} \frac{E_{1,p}}{E_{1,q}} \mathbb{1}_{\mathbb{F}_{p^2} \setminus \mathbb{F}_p}(w) \mathbb{1}_{\mathbb{F}_p}(r).$$

- Second elliptic class: $\gamma = \begin{bmatrix} w & 0 & 0 \\ 0 & w^q & 0 \\ 0 & 0 & w^{q^2} \end{bmatrix}, w \in \mathbb{F}_q(\delta^{1/3}) \setminus \mathbb{F}_q$. As for the previous class, $\chi_\rho(\gamma) = 0$ if n is divisible by 3; otherwise,

$$\chi_\rho(\gamma) = \frac{|G|}{|\Gamma|} \frac{E_{2,p}}{E_{2,q}} \mathbb{1}_{\mathbb{F}_{p^3} \setminus \mathbb{F}_p}(w).$$

9. Decomposition of ρ When $2, 3 \nmid N$

In this section, we compute the decomposition of ρ when neither 2 nor 3 divide n; thus, elliptic elements in $\mathrm{GL}_3(\mathbb{F}_p)$ remain elliptic in $\mathrm{GL}_3(\mathbb{F}_q)$.

We use the character table for $\mathrm{GL}_3(\mathbb{F}_q)$ in Terras [6] (pp. 383–384) and Steinberg [10]. As usual, to calculate the multiplicity $m(\pi, \rho)$ of an irreducible representation π present in ρ, we compute the inner product of their respective characters. By $H_1, H_2, P_1 \ldots, E_2$ we denote the size of different types of conjugacy classes in $\mathrm{GL}_3(\mathbb{F}_p)$, while by $\mathrm{Nm}_{E/F}$ we denote the norm of the field extension E/F.

1. α is a character of $\mathbb{F}_q^\times$. Then,

$$m(\alpha, \rho) = \frac{1}{|\Gamma|} \Bigg(\sum_{a \in \mathbb{F}_p^\times} \alpha(a^3) + H_1 \sum_{\substack{a,b \in \mathbb{F}_p^\times \\ a \neq b}} \alpha(a^2 b) + \frac{H_2}{6} \sum_{\substack{a,b,c \in \mathbb{F}_p^\times \\ a,b,c \text{ distinct}}} \alpha(abc) + P_1 \sum_{a \in \mathbb{F}_p^\times} \alpha(a^3) + P_2 \sum_{a \in \mathbb{F}_p^\times} \alpha(a^3)$$
$$+ P_3 \sum_{\substack{a,b \in \mathbb{F}_p^\times \\ a \neq b}} \alpha(a^2 b) + \frac{E_1}{2} \sum_{\substack{w \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p \\ r \in \mathbb{F}_p^\times}} \alpha(r\,\mathrm{Nm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(w)) + \frac{E_2}{3} \sum_{w \in \mathbb{F}_{p^3} \setminus \mathbb{F}_p} \alpha(\mathrm{Nm}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(w)) \Bigg).$$

We can simplify these sums on the basis of three cases:

- $\alpha^3 |_{\mathbb{F}_p^\times} \neq 1$. Each of the sums is zero; thus, $m(\alpha, \rho) = 0$.
- $\alpha^3 |_{\mathbb{F}_p^\times} = 1$ but $\alpha |_{\mathbb{F}_p^\times} \neq 1$. A short calculation yields

$$m(\alpha, \rho) = \frac{1}{|\Gamma|} \left(1 - H_1 + \frac{H_2}{3} + P_1 + P_2 - P_3 - \frac{E_2}{3} \right) (p-1) = 0.$$

- $\alpha |_{\mathbb{F}_p^\times} = 1$. Simply,

$$m(\alpha, \rho) = \frac{1}{|\Gamma|} \sum_{\{g\} \in C_\Gamma} |\{g\}| = 1.$$

2. π_α, where α is a character of $\mathbb{F}_q^\times$.

$$m(\pi_\alpha,\rho)=\frac{1}{|\Gamma|}\left((q^2+q)\sum_{a\in\mathbb{F}_p^\times}\alpha(a^3)+(q+1)H_1\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}\alpha(a^2b)+\frac{H_2}{3}\sum_{\substack{a,b,c\in\mathbb{F}_p^\times\\ a,b,c\text{ distinct}}}\alpha(abc)+qP_1\sum_{a\in\mathbb{F}_p^\times}\alpha(a^3)\right.$$
$$\left.+0+P_3\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}\alpha(a^2b)+0-\frac{E_2}{3}\sum_{w\in\mathbb{F}_{p^3}\setminus\mathbb{F}_p}\alpha(\mathrm{Nm}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(w))\right).$$

- $\alpha^3|_{\mathbb{F}_p^\times}\neq 1$. Here, all the sums are again zero, leading to $m(\pi_\alpha,\rho)=0$.
- $\alpha^3|_{\mathbb{F}_p^\times}=1$, but $\alpha|_{\mathbb{F}_p^\times}\neq 1$.

$$\begin{aligned}m(\pi_\alpha,\rho)&=\frac{1}{|\Gamma|}\left((q^2+q)-(q+1)H_1+\frac{2}{3}H_2+qP_1-P_3+\frac{E_2}{3}\right)(p-1)\\&=\frac{(q-p)(q-p^2)}{p^3(p-1)^2(p+1)(p^2+p+1)}.\end{aligned}$$

- $\alpha|_{\mathbb{F}_p^\times}=1$.

$$\begin{aligned}m(\pi_\alpha,\rho)=\frac{1}{|\Gamma|}&\Big((q^2+q)(p-1)+(q+1)H_1(p-1)(p-2)+(p-1)(p-2)(p-3)\frac{H_2}{3}\\&+qP_1(p-1)+(p-1)(p-2)P_3-\frac{E_2}{3}(p^3-p)\Big)\\=&\frac{(q-p)(q+p^5-2p^2)}{p^3(p-1)^2(p+1)(p^2+p+1)}.\end{aligned}$$

As a sanity check, showing that this is actually an integer is not very hard.

3. π'_α, where α is a character of $\mathbb{F}_q^\times$.

$$m(\pi'_\alpha,\rho)=\frac{1}{|\Gamma|}\left(q^3\sum_{a\in\mathbb{F}_p^\times}\alpha(a^3)+qH_1\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}\alpha(a^2b)+\frac{H_2}{6}\sum_{\substack{a,b,c\in\mathbb{F}_p^\times\\ a,b,c\text{ distinct}}}\alpha(abc)+0+0+0\right.$$
$$\left.-\frac{E_1}{2}\sum_{\substack{w\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p\\ r\in\mathbb{F}_p^\times}}\alpha(r\,\mathrm{Nm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(w))+\frac{E_2}{3}\sum_{w\in\mathbb{F}_{p^3}\setminus\mathbb{F}_p}\alpha(\mathrm{Nm}_{\mathbb{F}_{q^3}/\mathbb{F}_q}(w))\right).$$

- $\alpha^3|_{\mathbb{F}_p^\times}\neq 1$. $m(\pi_\alpha,\rho)=0$.
- $\alpha^3|_{\mathbb{F}_p^\times}=1$, but $\alpha|_{\mathbb{F}_p^\times}\neq 1$.

$$m(\pi'_\alpha,\rho)=\frac{1}{|\Gamma|}\left(q^3-qH_1+\frac{1}{3}H_2-\frac{E_2}{3}\right)(p-1)=\frac{(q-p)(q-p^2)(q+p^2+p)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

- $\alpha|_{\mathbb{F}_p^\times}=1$.

$$m(\pi'_\alpha,\rho) = \frac{1}{|\Gamma|}\Bigg(q^3(p-1) + q(p-1)(p-2)H_1 + (p-1)(p-2)(p-3)\frac{H_2}{6} - (p^2-p)(p-1)\frac{E_1}{2} + (p^3-p)\frac{E_2}{3}\Bigg)$$
$$= \frac{(q-p)(q^2+pq+p^5-p^4-p^3-p^2)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

4. $\pi_{\alpha,\beta}$ for distinct characters α and β of $\mathbb{F}_q^\times$.

$$m(\pi_{\alpha,\beta},\rho) = \frac{1}{|\Gamma|}\Bigg((q^2+q+1)\sum_{a\in\mathbb{F}_p^\times}\alpha(a^2)\beta(a) + H_1\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}((q+1)\alpha(ab)\beta(a) + \alpha(a^2)\beta(b))$$
$$+ \frac{H_2}{6}\sum_{\substack{a,b,c\in\mathbb{F}_p^\times\\ a,b,c \text{ distinct}}}(\beta(a)\alpha(bc) + \beta(b)\alpha(ac) + \beta(c)\alpha(ab)) + (q+1)P_1\sum_{a\in\mathbb{F}_p^\times}\alpha(a^2)\beta(a)$$
$$+ P_2\sum_{a\in\mathbb{F}_p^\times}\alpha(a^2)\beta(a) + P_3\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}(\alpha(ab)\beta(a) + \alpha(a^2)\beta(b))$$
$$+ \frac{E_1}{2}\sum_{\substack{w\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p\\ r\in\mathbb{F}_p^\times}}\alpha(\mathrm{Nm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(w))\beta(r) + 0\Bigg).$$

- $\alpha^2\beta\mid_{\mathbb{F}_p^\times}\neq 1$. $m(\pi_{\alpha,\beta},\rho) = 0$.
- $\alpha^2\beta\mid_{\mathbb{F}_p^\times} = 1$, but $\alpha^2\mid_{\mathbb{F}_p^\times}\neq 1$ and $\beta\mid_{\mathbb{F}_p^\times}\neq 1$.

$$m(\pi_{\alpha,\beta},\rho) = \frac{1}{|\Gamma|}\Bigg((q^2+q+1)(p-1) - (q+2)(p-1)H_1 + (p-1)H_2 + (q+1)(p-1)P_1 + (p-1)P_2 - 2(p-1)P_3\Bigg)$$
$$= \frac{(q-p)(q-p^2)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

- $\alpha^2\mid_{\mathbb{F}_p^\times} = 1$ and $\beta\mid_{\mathbb{F}_p^\times} = 1$, but $\alpha\mid_{\mathbb{F}_p^\times}\neq 1$.

$$m(\pi_{\alpha,\beta},\rho) = \frac{1}{|\Gamma|}\Bigg((q^2+q+1)(p-1) - (p-1)(q-p+3)H_1 - (p-1)(p-3)\frac{H_2}{2} + (q+1)(p-1)P_1 + (p-1)P_2 + (p-1)(p-3)P_3 - (p-1)^2\frac{E_1}{2}\Bigg)$$
$$= \frac{(q-p)(q-p^2)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

- $\alpha\mid_{\mathbb{F}_p^\times} = 1$ and $\beta\mid_{\mathbb{F}_p^\times} = 1$.

$$m(\pi_{\alpha,\beta},\rho)=\frac{1}{|\Gamma|}\Bigg((q^2+q+1)(p-1)+(q+2)(p-1)(p-2)H_1+(p-1)(p-2)(p-3)\frac{H_2}{2}$$
$$+(q+1)(p-1)P_1+(p-1)P_2+2(p-1)(p-2)P_3+(p-1)(p^2-p)\frac{E_1}{2}\Bigg)$$
$$=\frac{q(q+p^5-2p^2-p)+p^3(p^5-2p^3-p^2+3)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

5. $\pi'_{\alpha,\beta}$ for distinct characters α and β of $\mathbb{F}_q^\times$.

$$m(\pi'_{\alpha,\beta},\rho)=\frac{1}{|\Gamma|}\Bigg(q(q^2+q+1)\sum_{a\in\mathbb{F}_p^\times}\alpha(a^2)\beta(a)+H_1\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}((q+1)\alpha(ab)\beta(a)+q\alpha(a^2)\beta(b))$$
$$+\frac{H_2}{6}\sum_{\substack{a,b,c\in\mathbb{F}_p^\times\\ a,b,c\text{ distinct}}}(\beta(a)\alpha(bc)+\beta(b)\alpha(ac)+\beta(c)\alpha(ab))+qP_1\sum_{a\in\mathbb{F}_p^\times}\alpha(a^2)\beta(a)$$
$$+0+P_3\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}\alpha(ab)\beta(a)-\frac{E_1}{2}\sum_{\substack{w\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p\\ r\in\mathbb{F}_p^\times}}\alpha(\mathrm{Nm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(w))\beta(r)+0\Bigg).$$

- $\alpha^2\beta\mid_{\mathbb{F}_p^\times}\neq 1$. $m(\pi'_{\alpha,\beta},\rho)=0$.
- $\alpha^2\beta\mid_{\mathbb{F}_p^\times}=1$, but $\alpha^2\mid_{\mathbb{F}_p^\times}\neq 1$ and $\beta\mid_{\mathbb{F}_p^\times}\neq 1$.

$$m(\pi'_{\alpha,\beta},\rho)=\frac{1}{|\Gamma|}\Bigg(q(q^2+q+1)(p-1)-(2q+1)(p-1)H_1+(p-1)H_2$$
$$+q(p-1)P_1-(p-1)P_3\Bigg)$$
$$=\frac{(q-p)(q-p^2)(q+p^2+p+1)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

- $\alpha^2\mid_{\mathbb{F}_p^\times}=1$ and $\beta\mid_{\mathbb{F}_p^\times}=1$, but $\alpha\mid_{\mathbb{F}_p^\times}\neq 1$.

$$m(\pi_{\alpha,\beta},\rho)=\frac{1}{|\Gamma|}\Bigg(q(q^2+q+1)(p-1)+(p-1)(qp-3q-1)H_1-(p-1)(p-3)\frac{H_2}{2}$$
$$+q(p-1)P_1-(p-1)P_3+(p-1)^2\frac{E_1}{2}\Bigg)$$
$$=\frac{(q-p)(q(q+p+1)+p^2(p^3-p^2-p-2))}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

- $\alpha\mid_{\mathbb{F}_p^\times}=1$ and $\beta\mid_{\mathbb{F}_p^\times}=1$.

$$m(\pi_{\alpha,\beta},\rho) = \frac{1}{|\Gamma|}\Bigg(q(q^2+q+1)(p-1) + (2q+1)(p-1)(p-2)H_1 + (p-1)(p-2)(p-3)\frac{H_2}{2} + q(p-1)P_1 + (p-1)(p-2)P_3 - (p-1)(p^2-p)\frac{E_1}{2}\Bigg)$$
$$= \frac{(q-p)(q(q+p+1) + p^2(2p-3)(p^2+p+1))}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

6. $\pi_{\alpha,\beta,\gamma}$ for distinct characters α, β, γ of $\mathbb{F}_q^\times$.

$$m(\pi_{\alpha,\beta,\gamma},\rho) = \frac{1}{|\Gamma|}\Bigg((q+1)(q^2+q+1)\sum_{a\in\mathbb{F}_p^\times}\alpha(a)\beta(a)\gamma(a) + H_2\sum_{\substack{a,b,c\in\mathbb{F}_p^\times\\ a,b,c \text{ distinct}}}\alpha(a)\beta(b)\gamma(c) + (q+1)H_1\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}(\alpha(b)\beta(a)\gamma(a) + \beta(b)\gamma(a)\alpha(a) + \gamma(b)\alpha(a)\beta(a)) + (2q+1)P_1\sum_{a\in\mathbb{F}_p^\times}\alpha(a)\beta(a)\gamma(a) + P_2\sum_{a\in\mathbb{F}_p^\times}\alpha(a)\beta(a)\gamma(a) + P_3\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}(\alpha(b)\beta(a)\gamma(a) + \beta(b)\gamma(a)\alpha(a) + \gamma(b)\alpha(a)\beta(a)) + 0 + 0\Bigg).$$

- $\alpha\beta\gamma\mid_{\mathbb{F}_p^\times} \neq 1$. $m(\pi_{\alpha,\beta,\gamma},\rho) = 0$.
- $\alpha\beta\gamma\mid_{\mathbb{F}_p^\times} = 1$, but none of α, β, or γ are restricted to the trivial character on $\mathbb{F}_p^\times$.

$$m(\pi_{\alpha,\beta,\gamma},\rho) = \frac{1}{|\Gamma|}\Bigg((q+1)(q^2+q+1)(p-1) + 2(p-1)H_2 - 3(q+1)(p-1)H_1 + (2q+1)(p-1)P_1 + (p-1)P_2 - 3(p-1)P_3\Bigg)$$
$$= \frac{(q-p)(q-p^2)(q+p^2+p+2)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

- $\alpha\beta\mid_{\mathbb{F}_p^\times} = 1$ and $\gamma\mid_{\mathbb{F}_p^\times} = 1$, but α and β are nontrivial on $\mathbb{F}_p^\times$. Note that because $m(\pi_{\alpha,\beta,\gamma},\rho)$ is symmetric in α, β, γ, the calculation here also holds for the other two permutations of this case.

$$m(\pi_{\alpha,\beta,\gamma},\rho) = \frac{1}{|\Gamma|}\Bigg((q+1)(q^2+q+1)(p-1) - (p-1)(p-3)H_2 + (q+1)(p-1)(p-4)H_1 + (2q+1)(p-1)P_1 + (p-1)P_2 + (p-1)(p-4)P_3\Bigg)$$
$$= \frac{(q-p)(q(q+p+2) + p^2(p^3-p^2-p-3))}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

- All three of α, β, γ are restricted to the trivial character on $\mathbb{F}_p^\times$.

$$m(\pi_{\alpha,\beta,\gamma},\rho) = \frac{1}{|\Gamma|}\Bigg((q+1)(q^2+q+1)(p-1) + (p-1)(p-2)(p-3)H_2 + 3(q+1)(p-1)(p-2)H_1 + (2q+1)(p-1)P_1 + (p-1)P_2 + 3(p-1)(p-2)P_3\Bigg)$$
$$= \frac{(q(q^2+2q+3p^5-p^4-p^3-6p^2-2p)+p^3(p^5-4p^3+p+6))}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

7. $\rho_{\alpha,\nu}$ for α as a character of $\mathbb{F}_q^\times$ and ν as a character of $\mathbb{F}_{q^2}^\times$ such that $\nu^q \neq \nu$ (i.e., it is nondecomposable).

$$m(\rho_{\alpha,\nu},\rho) = \frac{1}{|\Gamma|}\Bigg((q-1)(q^2+q+1)\sum_{a\in\mathbb{F}_p^\times}\alpha(a)\nu(a) + (q-1)H_1\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}\alpha(b)\nu(a) + 0 - P_1\sum_{a\in\mathbb{F}_p^\times}\alpha(a)\nu(a) - P_2\sum_{a\in\mathbb{F}_p^\times}\alpha(a)\nu(a) - P_3\sum_{\substack{a,b\in\mathbb{F}_p^\times\\ a\neq b}}\alpha(b)\nu(a) - \frac{E_1}{2}\sum_{\substack{w\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p\\ r\in\mathbb{F}_p^\times}}\alpha(r)(\nu(w)+\nu(w^p)) + 0\Bigg).$$

Note that the last nonzero sum can be written as $-E_1\sum_{\substack{w\in\mathbb{F}_{p^2}\setminus\mathbb{F}_p\\ r\in\mathbb{F}_p^\times}}\alpha(r)\nu(w)$.

- $\alpha\nu\mid_{\mathbb{F}_p^\times}\neq 1$. $m(\rho_{\alpha,\nu},\rho)=0$.
- $\alpha\nu\mid_{\mathbb{F}_p^\times}=1$, but $\alpha\mid_{\mathbb{F}_p^\times}\neq 1$ and $\nu\mid_{\mathbb{F}_p^\times}\neq 1$.

$$m(\rho_{\alpha,\nu},\rho) = \frac{1}{|\Gamma|}\Bigg((q-1)(q^2+q+1)(p-1) - (q-1)(p-1)H_1 - (p-1)P_1 - (p-1)P_2 + (p-1)P_3\Bigg)$$
$$= \frac{(q-p)(q-p^2)(q+p^2+p)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

- $\alpha\mid_{\mathbb{F}_p^\times}=1$, $\nu\mid_{\mathbb{F}_{p^3}^\times}\neq 1$, but $\nu\mid_{\mathbb{F}_p^\times}=1$.

$$m(\rho_{\alpha,\nu},\rho) = \frac{1}{|\Gamma|}\Bigg((q-1)(q^2+q+1)(p-1) + (q-1)(p-1)(p-2)H_1 - (p-1)P_1 - (p-1)P_2 - (p-1)(p-2)P_3 + (p-1)^2E_1\Bigg)$$
$$= \frac{(q-p)(q(q+p)+p^2(p^3-p^2-p-1))}{(p-1)^2p^3(p+1)(p^2+p+1)}.$$

- $\alpha\mid_{\mathbb{F}_p^\times}=1$ and $\nu\mid_{\mathbb{F}_{p^3}^\times}=1$.

$$m(\rho_{\alpha,\nu},\rho) = \frac{1}{|\Gamma|}\Bigg((q-1)(q^2+q+1)(p-1) + (q-1)(p-1)(p-2)H_1$$
$$- (p-1)P_1 - (p-1)P_2 - (p-1)(p-2)P_3 - (p-1)(p^2-p)E_1\Bigg)$$
$$= \frac{q(q^2+p^5-p^4-p^3-2p^2)+p^4(-p^4+2p+1)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

8. σ_μ for μ as a character of $\mathbb{F}_{q^3}^\times$ such that $\mu^q \neq \mu$.

$$m(\sigma_\mu,\rho) = \frac{1}{|\Gamma|}\Bigg((q-1)^2(q+1)\sum_{a\in\mathbb{F}_p^\times}\mu(a) + 0 + 0 - (q-1)P_1\sum_{a\in\mathbb{F}_p^\times}\mu(a) + P_2\sum_{a\in\mathbb{F}_p^\times}\mu(a) + 0$$
$$+ 0 + \frac{E_2}{3}\sum_{w\in\mathbb{F}_{p^3}\setminus\mathbb{F}_p}(\mu(w)+\mu(w^p)+\mu(w^{p^2}))\Bigg).$$

- $\mu\mid_{\mathbb{F}_p^\times} \neq 1$. $m(\sigma_\mu,\rho) = 0$.
- $\mu\mid_{\mathbb{F}_{p^3}^\times} \neq 1$, but $\mu\mid_{\mathbb{F}_p^\times} = 1$.

$$m(\sigma_\mu,\rho) = \frac{1}{|\Gamma|}\Big((q-1)^2(q+1)(p-1) - (q-1)(p-1)P_1 + (p-1)P_2 - (p-1)E_2\Big)$$
$$= \frac{(q-p)(q-p^2)(q+p^2+p-1)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

- $\mu\mid_{\mathbb{F}_{p^3}^\times} = 1$.

$$m(\sigma_\mu,\rho) = \frac{1}{|\Gamma|}\Big((q-1)^2(q+1)(p-1) - (q-1)(p-1)P_1 + (p-1)P_2 + (p^3-p)E_2\Big)$$
$$= \frac{q(q^2-q-p^4-p^3+p)+p^4(p^4-p^2+1)}{p^3(p-1)^2(p+1)(p^2+p+1)}.$$

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Appendix A. Fundamental Domains and Orbital Sums

In this Appendix, we provide proofs of fundamental domains as well as some computations of orbital sums.

- **Proof of Lemma 2**

Proof. Observe that if $\alpha, \beta \in \mathbb{F}_q\left(\delta^{1/3}\right)$, then $(\alpha, \beta) \in \mathbb{H}_q$ if and only if $\{\alpha, \beta, 1\}$ form a basis of $\mathbb{F}_q\left(\delta^{1/3}\right)$ over $\mathbb{F}_q$.
Suppose that $(\alpha, \beta) \in \mathbb{H}_q$ and that $x, y, z \in \mathbb{F}_q$ are such that we have

$$x \cdot \frac{a\alpha + b\beta + c}{r\alpha + s\beta + t} + y \cdot \frac{d\alpha + e\beta + f}{r\alpha + s\beta + t} + z \cdot 1 = 0.$$

The denominator $r\alpha + s\beta + t$ can only be nonzero if $r = s = t = 0$ (as $\alpha, \beta, 1$ are linearly independent); however, that cannot happen, as $\begin{vmatrix} a & b & c \\ d & e & f \\ r & s & t \end{vmatrix} \neq 0$. Thus, we find

$$x(a\alpha + b\beta + c) + y(d\alpha + e\beta + f) + z(r\alpha + s\beta + t) = 0$$

.
In matrix form,

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ r & s & t \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix} = 0.$$

This implies that $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ r & s & t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, as $\alpha, \beta, 1$ are linearly independent over $\mathbb{F}_q$. Because our matrix is nonsingular, it follows that $x = y = z = 0$ and $\left\{ \frac{a\alpha+b\beta+c}{r\alpha+s\beta+t}, \frac{d\alpha+e\beta+f}{r\alpha+s\beta+t}, 1 \right\}$ is a basis of $\mathbb{F}_q\left(\delta^{1/3}\right)$ over $\mathbb{F}_q$, proving the claim that $\mathrm{GL}_3(\mathbb{F}_q)$ has an action on $\mathbb{H}_q$ of the above form. □

- **Second Hyperbolic Term.**

$$\begin{aligned}
I_G(f, \gamma) &= \sum_{t \in G_\gamma \backslash G/K} \sum_{u \in K/\{aI\}} f((tu)^{-1}\gamma(tu)) \\
&= \sum_{t \in G_\gamma \backslash \mathbb{H}_q} \frac{|K|}{|\{aI\}|} f(t^{-1}\gamma t) \\
&= \frac{q^3-1}{q-1} \sum_{\substack{x,y,r,s \in \mathbb{F}_q \\ ys \neq 1}} f\left(\begin{bmatrix} \frac{s}{sy-1} & \frac{1}{1-sy} & \frac{r-sx}{sy-1} \\ \frac{1}{1-sy} & \frac{y}{sy-1} & \frac{x-ry}{sy-1} \\ 0 & 0 & 1 \end{bmatrix} \gamma \begin{bmatrix} y & 1 & x \\ 1 & s & r \\ 0 & 0 & 1 \end{bmatrix} p_0 \right) + \\
&+ \frac{q^3-1}{q-1} \sum_{\substack{x,r,s \in \mathbb{F}_q \\ s \neq 0}} f\left(\begin{bmatrix} 1 & 0 & -x \\ -s & 1 & xs - r \\ 0 & 0 & 1 \end{bmatrix} \gamma \begin{bmatrix} 1 & 0 & x \\ s & 1 & r \\ 0 & 0 & 1 \end{bmatrix} p_0 \right) + \\
&+ \frac{q^3-1}{q-1} \sum_{\substack{x,y,r \in \mathbb{F}_q \\ y \neq 0}} f\left(\begin{bmatrix} 1 & -y & -x + yr \\ 0 & 1 & -r \\ 0 & 0 & 1 \end{bmatrix} \gamma \begin{bmatrix} 1 & y & x \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} p_0 \right) \\
&+ \frac{q^3-1}{q-1} \sum_{x,r \in \mathbb{F}_q} f\left(\begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -r \\ 0 & 0 & 1 \end{bmatrix} \gamma \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} p_0 \right)
\end{aligned}$$

$$= (q^2+q+1)\sum_{\substack{x,y,r,s\in\mathbb{F}_q\\ ys\neq 1}} f\left(\begin{bmatrix}\frac{b-asy}{1-sy} & \frac{s(a-b)}{sy-1} & \frac{asx-br+c(r-sx)}{sy-1}\\ \frac{y(b-a)}{sy-1} & \frac{a-bsy}{1-sy} & \frac{-ax+bry+c(x-ry)}{sy-1}\\ 0 & 0 & c\end{bmatrix}p_0\right)+$$

$$+ (q^2+q+1)\sum_{\substack{x,r,s\in\mathbb{F}_q\\ s\neq 0}} f\left(\begin{bmatrix}a & 0 & ax-cx\\ bs-as & b & -axs+br+c(xs-r)\\ 0 & 0 & c\end{bmatrix}p_0\right)+$$

$$+ (q^2+q+1)\sum_{\substack{x,y,r\in\mathbb{F}_q\\ y\neq 0}} f\left(\begin{bmatrix}a & ay-by & ax-byr+c(yr-x)\\ 0 & b & br-cr\\ 0 & 0 & c\end{bmatrix}p_0\right)+$$

$$+ (q^2+q+1)\sum_{x,y\in\mathbb{F}_q} f\left(\begin{bmatrix}a & 0 & ax-cx\\ 0 & b & br-cr\\ 0 & 0 & c\end{bmatrix}p_0\right) = (q^2+q+1)(S_1+S_2+S_3+S_4)$$

We simplify each of the four terms separately. We observe that the substitution

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} = \begin{bmatrix}\frac{asx-br+c(r-sx)}{sy-1}\\ \frac{-ax+bry+c(x-ry)}{sy-1}\end{bmatrix} = \frac{1}{sy-1}\begin{bmatrix}(a-c)s & c-b\\ c-a & (b-c)y\end{bmatrix}\begin{bmatrix}x\\ r\end{bmatrix}$$

is invertible, as $\begin{vmatrix}(a-c)s & c-b\\ c-a & (b-c)y\end{vmatrix} = (a-c)(b-c)(sy-1)\neq 0$. Therefore,

$$S_1 = \sum_{x,r\in\mathbb{F}_q}\sum_{\substack{y,s\in\mathbb{F}_q\\ ys\neq 1}} f\left(\begin{bmatrix}\frac{b-asy}{1-sy} & \frac{s(a-b)}{sy-1} & x\\ \frac{y(b-a)}{sy-1} & \frac{a-bsy}{1-sy} & r\\ 0 & 0 & c\end{bmatrix}p_0\right) = \sum_{x,r\in\mathbb{F}_q} f\left(\begin{bmatrix}b & 0 & x\\ 0 & a & r\\ 0 & 0 & c\end{bmatrix}p_0\right)+$$

$$+\sum_{\substack{x,y,r\in\mathbb{F}_q\\ y\neq 0}} f\left(\begin{bmatrix}b & 0 & x\\ y(a-b) & a & r\\ 0 & 0 & c\end{bmatrix}p_0\right)+\sum_{\substack{x,r,s\in\mathbb{F}_q\\ s\neq 0}} f\left(\begin{bmatrix}b & s(b-a) & x\\ 0 & a & r\\ 0 & 0 & c\end{bmatrix}p_0\right)+\sum_{x,r\in\mathbb{F}_q}\sum_{\substack{y,s\in\mathbb{F}_q^\times\\ ys\neq 1}} f\left(\begin{bmatrix}\frac{b-asy}{1-sy} & \frac{s(a-b)}{sy-1} & x\\ \frac{y(b-a)}{sy-1} & \frac{a-bsy}{1-sy} & r\\ 0 & 0 & c\end{bmatrix}p_0\right).$$

We can make a similar changes of variables $\begin{bmatrix}\tilde{x}\\ \tilde{r}\end{bmatrix}$ as above, finding that

$$S_2 = \sum_{\substack{x,r\in\mathbb{F}_q\\ s\in\mathbb{F}_q^\times}} f\left(\begin{bmatrix}a & 0 & x\\ (b-a)s & b & r\\ 0 & 0 & c\end{bmatrix}p_0\right)$$

$$S_3 = \sum_{\substack{x,r\in\mathbb{F}_q\\ y\in\mathbb{F}_q^\times}} f\left(\begin{bmatrix}a & (a-b)y & x\\ 0 & b & r\\ 0 & 0 & c\end{bmatrix}p_0\right)$$

$$S_4 = \sum_{x,r\in\mathbb{F}_q} f\left(\begin{bmatrix}a & 0 & x\\ 0 & b & r\\ 0 & 0 & c\end{bmatrix}p_0\right).$$

Putting together all the above, we find that $S_1+S_2+S_3+S_4$ is the following sum:

$$S_1+S_2+S_3+S_4 = Hf\left(\begin{bmatrix}a/c & 0\\ 0 & b/c\end{bmatrix}\right)$$

.

This is because each conjugate of $\left[\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right]$ appears exactly once in the upper 2×2 block in the sum $S_1+S_2+S_3+S_4$.

Thus, the total contribution is

$$\frac{1}{6}(q^2+q+1)\frac{(q-1)^3}{(p-1)^3}\sum_{\substack{a,b,c\in\mathbb{F}_p^\times\\ a\neq b\neq c}} Hf\left(\begin{bmatrix} a/c & 0\\ 0 & b/c\end{bmatrix}\right) = \frac{1}{6}(q^2+q+1)\frac{(q-1)^4}{(p-1)^3}\sum_{\substack{a,b\in\mathbb{F}_p^\times\setminus\{1\}\\ a\neq b}} Hf\left(\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}\right) =$$

$$= \frac{1}{3}(q^3-1)\frac{(q-1)^3}{(p-1)^3}\sum_{\{a,b\}\subseteq\mathbb{F}_p^\times\setminus\{1\}} Hf\left(\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}\right).$$

- **Proof of Proposition 2**

Proof. First, to show the uniqueness of each representative, suppose that there exists $m \in G_\gamma$ such that

$$m\cdot(u_1\delta^{1/3}, v_1\delta^{1/3}+\delta^{2/3}) = (u_2\delta^{1/3}, v_2\delta^{1/3}+\delta^{2/3}).$$

Then, we can deduce that $y = 0$, consequently, $u_1 = u_2$. In addition, it can be seen that $c = d$, implying that $v_1 = v_2$. Next, we suppose that

$$m\cdot(u_1\delta^{1/3}, v_1\delta^{1/3}+\delta^{2/3}) = (\delta^{1/3}+u_2\delta^{2/3}, \delta^{1/3}).$$

However, this immediately forces $c = 0$, which is a contradiction. Similarly,

$$m\cdot(\delta^{1/3}+u_1\delta^{2/3}, \delta^{1/3}) = (\delta^{1/3}+u_2\delta^{2/3}, \delta^{1/3}),$$

directly implies that $u_1 = u_2$.
To show the completeness of this fundamental domain, we can take an arbitrary element $(\alpha,\beta)\in\mathbb{H}_q$. If $\beta_3\neq 0$, then it is observed that

$$\begin{bmatrix} 1 & \alpha_3 & \alpha_1\\ 0 & \beta_3 & \beta_1\\ 0 & 0 & 1\end{bmatrix}\left((\alpha_2-\alpha_3\frac{\beta_2}{\beta_3})\delta^{1/3}, \frac{\beta_2}{\beta_3}\delta^{1/3}+\delta^{2/3}\right) = (\alpha,\beta);$$

otherwise, we must have $\beta_2 \neq 0$, in which case

$$\begin{bmatrix} 1 & \alpha_2-1 & \alpha_1\\ 0 & \beta_2 & \beta_1\\ 0 & 0 & 1\end{bmatrix}\left(\delta^{1/3}+\alpha_3\delta^{2/3}, \delta^{1/3}\right) = (\alpha,\beta).$$

□

- **First Parabolic Term.**

$$\begin{aligned} I_G(f,\gamma) &= \sum_{t\in G_\gamma\setminus G/K}\ \sum_{u\in K/\{aI\}} f((tu)^{-1}\gamma(tu))\\ &= \sum_{t\in G_\gamma\setminus\mathbb{H}_q}\frac{|K|}{|\{aI\}|}f(t^{-1}\gamma t)\\ &= \frac{q^3-1}{q-1}\left(\sum_{\substack{u,v\in\mathbb{F}_q\\ u\neq 0}} f\left(\begin{bmatrix} -\frac{v}{u} & 1 & 0\\ \frac{1}{u} & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\gamma\begin{bmatrix} 0 & u & 0\\ 1 & v & 0\\ 0 & 0 & 1\end{bmatrix}p_0\right) + \sum_{u\in\mathbb{F}_q^\times} f\left(\begin{bmatrix} \frac{1}{u} & -\frac{1}{u} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\gamma\begin{bmatrix} u & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}p_0\right)\right)\end{aligned}$$

$$= (q^2+q+1)\left(\sum_{\substack{u,v\in\mathbb{F}_q\\ u\neq 0}} f\left(\begin{bmatrix} a & 0 & -\frac{av}{u} \\ 0 & a & \frac{a}{u} \\ 0 & 0 & a \end{bmatrix} p_0\right) + \sum_{u\in\mathbb{F}_q^\times} f\left(\begin{bmatrix} a & 0 & \frac{a}{u} \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} p_0\right)\right)$$

$$= (q^2+q+1)\left(\sum_{\substack{u,v\in\mathbb{F}_q\\ u\neq 0}} f\left(-\frac{v}{u}+\delta^{2/3}, \frac{1}{u}+\delta^{1/3}\right) + \sum_{u\in\mathbb{F}_q^\times} f\left(\delta^{2/3}+\frac{1}{u}, \delta^{1/3}\right)\right)$$

$$= (q^2+q+1)(Hf(I) - f(p_0))$$

- **Proof of Proposition 3**

Proof. First , we check that if $m \in B$ such that $mx = y$ for $x, y \in B\backslash\mathbb{H}_q$, then $x = y$. Suppose that there exists $m \in B$ such that

$$m(\delta^{1/3}, v_1\delta^{1/3} + \delta^{2/3}) = (\delta^{1/3}, v_2\delta^{1/3} + \delta^{2/3}).$$

We can immediately see see that $c/a = 1$; thus, $v_1 = v_2$. Finally, it is easy to see that we cannot use B to move elements between the two sets in the disjoint union of $B\backslash\mathbb{H}_q$. Next, we check that for any arbitrary element $(\alpha, \beta) \in \mathbb{H}_q$ (where $\alpha = \alpha_1 + \alpha_2\delta^{1/3} + \alpha_3\delta^{2/3}$ and $\beta = \beta_1 + \beta_2\delta^{1/3} + \beta_3\delta^{2/3}$) there exists $m \in B$ and $x \in B\backslash\mathbb{H}_q$ such that $mx = (\alpha, \beta)$. First, we suppose that $\beta_3 \neq 0$.

$$\begin{bmatrix} \alpha_2 - \alpha_3\frac{\beta_2}{\beta_3} & \alpha_3 & \alpha_1 \\ 0 & \beta_3 & \beta_1 \\ 0 & 0 & 1 \end{bmatrix}\left(\delta^{1/3}, \frac{\beta_2}{\beta_3}\delta^{1/3} + \delta^{2/3}\right) = (\alpha, \beta)$$

If $\beta_3 = 0$, then

$$\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ 0 & \beta_2 & \beta_1 \\ 0 & 0 & 1 \end{bmatrix}\left(\delta^{2/3}, \delta^{1/3}\right) = (\alpha, \beta).$$

□

- **Second Parabolic Term.**

$$I_G(f,\gamma) = \sum_{t\in G_\gamma\backslash G/K}\sum_{u\in K/\{aI\}} f((tu)^{-1}\gamma(tu))$$

$$= \sum_{t\in G_\gamma\backslash\mathbb{H}_q} \frac{|K|}{|aI|} f(t^{-1}\gamma t)$$

$$= \frac{q^3-1}{q-1}\left(f\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\gamma\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} p_0\right) + \sum_{v\in\mathbb{F}_q} f\left(\begin{bmatrix} -v & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\gamma\begin{bmatrix} 0 & 1 & 0 \\ 1 & v & 0 \\ 0 & 0 & 1 \end{bmatrix} p_0\right)\right)$$

$$= (q^2+q+1)\left(f\left(\begin{bmatrix} a & a & 0 \\ 0 & a & a \\ 0 & 0 & a \end{bmatrix} p_0\right) + \sum_{v\in\mathbb{F}_q} f\left(\begin{bmatrix} -av+a & -av^2 & a \\ a & av+a & 0 \\ 0 & 0 & a \end{bmatrix} p_0\right)\right)$$

$$= (q^2+q+1)\left(f\left(\delta^{2/3}+\delta^{1/3}, \delta^{1/3}+1\right) + \sum_{v\in\mathbb{F}_q} f\left((1-v)\delta^{2/3} - v^2\delta^{1/3} + 1, \delta^{2/3} + (v+1)\delta^{1/3}\right)\right)$$

- **Third Parabolic Term.**

$$G_\gamma\backslash\mathbb{H}_q = \{(\delta^{1/3} + u\delta^{2/3} + v, r\delta^{2/3} + s) : r \neq 0\} \sqcup \{(\delta^{2/3} + v, r\delta^{1/3} + s) : r \neq 0\}.$$

$$
\begin{aligned}
I_G(f,\gamma) &= \sum_{t\in G_\gamma\backslash G/K}\sum_{u\in K/\{aI\}} f((tu)^{-1}\gamma(tu)) \\
&= \sum_{t\in G_\gamma\backslash \mathbb{H}_q} \frac{|K|}{|\{aI\}|} f(t^{-1}\gamma t) \\
&= \frac{q^3-1}{q-1}\left(\sum_{\substack{u,v,r,s\in\mathbb{F}_q\\ r\neq 0}} f\left(\begin{bmatrix} 0 & \frac{1}{r} & -\frac{s}{r} \\ 1 & -\frac{u}{r} & \frac{su}{r}-v \\ 0 & 0 & 1\end{bmatrix}\gamma\begin{bmatrix} u & 1 & v\\ r & 0 & s\\ 0 & 0 & 1\end{bmatrix}p_0\right) + \sum_{\substack{v,r,s\in\mathbb{F}_q\\ r\neq 0}} f\left(\begin{bmatrix} 1 & 0 & -v\\ 0 & \frac{1}{r} & -\frac{s}{r}\\ 0 & 0 & 1\end{bmatrix}\gamma\begin{bmatrix} 1 & 0 & v\\ 0 & r & s\\ 0 & 0 & 1\end{bmatrix}p_0\right)\right) \\
&= (q^2+q+1)\left(\sum_{\substack{u,v,r,s\in\mathbb{F}_q\\ r\neq 0}} f\left(\begin{bmatrix} \frac{a(r+u)}{r} & \frac{a}{r} & \frac{a(s+v)-bs}{r} \\ -\frac{au^2}{r} & \frac{a(r-u)}{r} & -\frac{-arv+asu+auv+brv-bsu}{r} \\ 0 & 0 & b\end{bmatrix}p_0\right)\right. \\
&\quad \left. + \sum_{\substack{v,r,s\in\mathbb{F}_q\\ r\neq 0}} f\left(\begin{bmatrix} a & 0 & v(a-b)\\ \frac{a}{r} & a & \frac{a(s+v)-bs}{r}\\ 0 & 0 & b\end{bmatrix}p_0\right)\right).
\end{aligned}
$$

We employ the following change of variables in the two sums.

$$
\begin{bmatrix}\tilde{v}\\ \tilde{s}\end{bmatrix} = \frac{1}{r}\begin{bmatrix} a & (a-b)\\ (a-b)r-au & (b-a)u\end{bmatrix}\begin{bmatrix} v\\ s\end{bmatrix}
$$

$$
\begin{bmatrix} v^*\\ s^*\end{bmatrix} = \begin{bmatrix} a-b & 0\\ a/r & (a-b)/r\end{bmatrix}\begin{bmatrix} v\\ s\end{bmatrix}
$$

The determinants are both $(a-b)^2/r \neq 0$; hence, the transformations are invertible. It follows that the above sums reduce to horocycle transforms of parabolic conjugacy classes in $\mathrm{GL}_2(\mathbb{F}_q)$.

$$
I_G(f,\gamma) = (q^2+q+1)Hf\left(\begin{bmatrix} a/b & 1\\ 0 & a/b\end{bmatrix}\right)
$$

- **Proof of Proposition 4.**

Proof. We first show that every G_γ-orbit on $\mathbb{H}_q$ contains at least one element of the above form. We let $z \in \mathbb{H}_q$, and write

$$
z = \begin{bmatrix} a & b & c\\ d & e & f\\ 0 & 0 & 1\end{bmatrix} p_0
$$

.

We want to find $r, s \in \mathbb{F}_q$ that are not both zero and $t \in \mathbb{F}_q^\times$ such that

$$
\begin{bmatrix} r & s\xi & 0\\ s & r & 0\\ 0 & 0 & t\end{bmatrix}\begin{bmatrix} a & b & c\\ d & e & f\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix} v & u & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}
$$

for some $v \in \mathbb{F}_q^\times$ and $u, x, y \in \mathbb{F}_q$. This amounts to solving the following set of equations for s, r, t over $\mathbb{F}_q$.

$$
\begin{aligned}
sa + rd &= 0\\
sb + re &= 1\\
t &= 1
\end{aligned}
$$

By assumption, the determinant $\begin{vmatrix} a & b \\ d & e \end{vmatrix}$ is nonzero; therefore, the above system admits a unique solution for s, r, t over $\mathbb{F}_q$. Moreover, s and r cannot simultaneously be zero. We now need to show that the domain does not contain any orbit repetitions. Suppose that

$$\begin{bmatrix} r & s\xi & 0 \\ s & r & 0 \\ 0 & 0 & t \end{bmatrix} \begin{bmatrix} v & u & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} v' & u' & x' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{bmatrix}$$

.

Writing up the above equations, we again obtain the following.

$$\begin{aligned} sv &= 0 \\ su + r &= 1 \\ t &= 1 \end{aligned}$$

As $v \neq 0$, we find that $s = 0$, $r = 1$, and $t = 1$; thus, the first matrix on the left-hand side is the identity, which implies that $x = x'$, $y = y'$, $u = u'$, and $v = v'$, concluding the proof. $\square$

- **Second Elliptic Term.**

$$I_G(f,\gamma) = \frac{(q-1)(q^2-1)}{(p-1)(p^2-1)} \cdot \frac{q^3-1}{q-1} \sum_{\substack{x,y,u \in \mathbb{F}_q \\ v \in \mathbb{F}_q^\times}} f\left(\begin{bmatrix} v & u & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} k & l\xi & 0 \\ l & k & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} v & u & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} p_0 \right) =$$

$$= \frac{(q^3-1)(q^2-1)}{(p-1)(p^2-1)} \sum_{\substack{u \in \mathbb{F}_q \\ v \in \mathbb{F}_q^\times}} \sum_{x,y \in \mathbb{F}_q} f\left(\begin{bmatrix} -lu+k & \frac{l\xi - lu^2}{v} & \frac{x(-lu+k)+y(l\xi-ku)+m(uy-x)}{v} \\ vl & lu+k & lx+ky-my \\ 0 & 0 & m \end{bmatrix} p_0 \right)$$

We set

$$\begin{aligned} \tilde{x} &= \frac{(k-m-lu)}{v} \cdot x + \frac{(m-k)u + l\xi}{v} \cdot y \\ \tilde{y} &= lx + (k-m)y. \end{aligned}$$

The determinant corresponding to this change of variables is $\big((k-m)^2 - l^2\xi\big)/v$. It can only be zero if $k - m = 0$ and $l = 0$, as ξ is not a square; however, γ would then be similar to a diagonal matrix, contradicting our assumption. Thus, the orbital sum can be simplified as follows:

$$I_G(f,\gamma) = \frac{(q^3-1)(q^2-1)}{(p-1)(p^2-1)} \sum_{\substack{u \in \mathbb{F}_q \\ v \in \mathbb{F}_q^\times}} \sum_{\tilde{x},\tilde{y} \in \mathbb{F}_q} f\left(\begin{bmatrix} -lu+k & \frac{l\xi - lu^2}{v} & \tilde{x} \\ vl & lu+k & \tilde{y} \\ 0 & 0 & m \end{bmatrix} p_0 \right)$$

.

Finally, notice that the upper-left 2×2 block is similar to the elliptic matrix $\left[\begin{smallmatrix} k & l\xi \\ l & k \end{smallmatrix}\right]$ over $\mathrm{GL}_2(\mathbb{F}_q)$. Hence, we obtain

$$I_G(f,\gamma) = \frac{(q^3-1)(q^2-1)}{(p-1)(p^2-1)} Hf\left(\begin{bmatrix} k/m & l\xi/m \\ l/m & k/m \end{bmatrix} \right).$$

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