

Article

The Existence and Uniqueness of Radial Solutions for Biharmonic Elliptic Equations in an Annulus

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Abstract: This paper concerns with the existence of radial solutions of the biharmonic elliptic equation $\Delta^2 u = f(|x|, u, |\nabla u|, \Delta u)$ in an annular domain $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\} (N \geq 2)$ with the boundary conditions $u|_{\partial\Omega} = 0$ and $\Delta u|_{\partial\Omega} = 0$, where $f : [r_1, r_2] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Under certain inequality conditions on f involving the principal eigenvalue λ_1 of the Laplace operator $-\Delta$ with boundary condition $u|_{\partial\Omega} = 0$, an existence result and a uniqueness result are obtained. The inequality conditions allow for $f(r, \xi, \zeta, \eta)$ to be a superlinear growth on ξ, ζ, η as $|(\xi, \zeta, \eta)| \rightarrow \infty$. Our discussion is based on the Leray–Schauder fixed point theorem, spectral theory of linear operators and technique of prior estimates.

Keywords: biharmonic elliptic equation; radial solution; annular domain; Leray–Schauder fixed point theorem; principal eigenvalue

MSC: 35J40; 35J60; 47H10

1. Introduction

In this paper, we discuss the existence and uniqueness of radially symmetric solutions of the biharmonic elliptic boundary value problem (BVP)

$$\begin{cases} \Delta^2 u = f(|x|, u, |\nabla u|, \Delta u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \quad (1)$$

in an annular domain $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$, where $N \geq 2, 0 < r_1 < r_2 < \infty$, $f : [r_1, r_2] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous nonlinear function.

Biharmonic elliptic equations arise in the study of traveling waves in suspension bridges [1,2], and the study of the static deflection of an elastic plate in a fluid [3,4]. BVP(1) is a general nonlinear biharmonic elliptic equation with the Navier boundary condition, and some of its special situations have been studied by many researchers; see [5–16] and the references therein. The authors of [5–9] considered the simple case where

$$\begin{cases} \Delta^2 u = f(x, u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0. \end{cases} \quad (2)$$

In [5–7], the authors applied the mountain pass lemma and critical point theory to obtain the existence and multiplicity results of solutions of BVP(2). Recently, the authors of [8,9], using the fixed point theorem on a cone, have obtained the existence and uniqueness results of positive solutions of BVP(2). Some researchers have also discussed the case where the right side of the equation with linear terms of Δu

$$\begin{cases} \Delta^2 u = c \Delta u + f(x, u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \quad (3)$$



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see [10–13]. The authors of [10–13] mainly applied variational methods and critical theory to discuss the existence of non trivial solutions of BVP(3).

For the case where the nonlinearity f contains Δu , but does not contain the gradient term ∇u , namely the boundary value problem

$$\begin{cases} \Delta^2 u = f(x, u, \Delta u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \quad (4)$$

the existence of solutions has also been discussed by several authors, see [14–16]. In [14], the existence results of solutions for BVP(4) are obtained by using the upper and lower solutions method, and in [15,16], the monotonic iterative programs for seeking the solution of BVP(4) are established.

The purpose of this paper is to study the general biharmonic elliptic boundary value problem BVP(1) and to obtain the existence results of radial symmetric solutions in the annular domain Ω . We have not yet seen any literature on this general case. Our main results are related to the principal eigenvalue λ_1 of the radially symmetric elliptic eigenvalue value problem (EVP),

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \\ u = u(|x|). \end{cases} \quad (5)$$

In next section, we will prove that EVP(5) has a minimum positive real eigenvalue λ_1 , and give the estimates of its upper and lower bounds (see Lemmas 2 and 3). Our main results are as follows:

Theorem 1. Let $f : [r_1, r_2] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfy the following condition:

(F1) There exist constants $a, b, c \geq 0$ with restriction $\frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c}{\lambda_1} < 1$ and $d > 0$, such that

$$-f(r, \xi, \zeta, \eta) \leq a\xi^2 + b\zeta^2 + c\eta^2 + d, \quad r \in [r_1, r_2], \quad (\xi, \zeta, \eta) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}.$$

Then, BVP(1) has at least one radial solution.

Theorem 2. Let $f : [r_1, r_2] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfy the following condition:

(F2) There exist constants $a, b, c \geq 0$ satisfied $\frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c}{\lambda_1} < 1$ such that

$$\begin{aligned} -(f(r, \xi_2, \zeta_2, \eta_2) - f(r, \xi_1, \zeta_1, \eta_1))(\eta_2 - \eta_1) &\leq a(\xi_2 - \xi_1)^2 + b(\zeta_2 - \zeta_1)^2 + c(\eta_2 - \eta_1)^2, \\ r &\in [r_1, r_2], \quad (\xi_1, \zeta_1, \eta_1), (\xi_2, \zeta_2, \eta_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}. \end{aligned}$$

Then, BVP(1) has a uniqueness radial solution.

The proofs of Theorems 1 and 2 are based on the Leray–Schauder fixed point theorem of completely continuous operators, which will be given in Section 3. Some preliminaries to prove the main results are presented in Section 2. To illustrate the applicability of Theorems 1 and 2, three application examples are presented in Section 4.

2. Preliminaries

Let $u(|x|)$ be a radially symmetric solution of BVP(1), writing $r = |x| = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_N|^2}$ and setting $v = -\Delta u$, since

$$\begin{aligned} |\nabla u| &= \left| \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N} \right) \right| = |u'(r)|, \\ \Delta u &= \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2} = u''(r) + \frac{N-1}{r} u'(r) = \frac{1}{r^{N-1}} (r^{N-1} u'(r))', \\ \Delta v &= v''(u) + \frac{N-1}{r} v'(r) = \frac{1}{r^{N-1}} (r^{N-1} v'(r))', \end{aligned} \quad (6)$$

it follows that $(u(r), v(r))$ satisfies the boundary value problem of the ordinary differential equations system

$$\begin{cases} -\frac{1}{r^{N-1}} (r^{N-1} u'(r))' = v(r), & r \in [r_1, r_2], \\ -\frac{1}{r^{N-1}} (r^{N-1} v'(r))' = f(r, u(r), |u'(r)|, -v(r)), & r \in [r_1, r_2], \\ u(r_1) = u(r_2) = 0, & u(r_1) = u(r_2) = 0. \end{cases} \quad (7)$$

Conversely, if $(u(r), v(r))$ is a solution of BVP(7), by (6), $u(|x|)$ is a radially symmetric solution of BVP(1). We discuss BVP(7) to obtain radial solutions of BVP(1).

Let $I = [r_1, r_2]$. Let $C(I)$ denote the Banach space of all continuous functions $u(r)$ on I with norm $\|u\|_C = \max_{r \in I} |u(r)|$, and $C^1(I)$ denote the Banach space of all continuous differentiable functions on I with the norm $\|u\|_{C^1} = \max\{\|u\|_C, \|u'\|_C\}$. Let $L^2(I)$ be the Hilbert of all square integrable functions on I with the interior product $(u, v) = \int_{r_1}^{r_2} u(t)v(t)dt$ and the norm $\|u\|_2 = \left(\int_{r_1}^{r_2} |u(t)|^2 dt \right)^{1/2}$. For $n \in \mathbb{N}$, $H^n(I)$ denotes the usual Sobolev space. $u \in H^n(I)$ means that $u \in C^{n-1}(I)$, $u^{(n-1)}(t)$ is absolutely continuous on I and $u^{(n)} \in L^2(I)$. $H^n(I)$ is with the norm $\|u\|_{n,2} = \left(\|u\|_2^2 + \|u'\|_2^2 + \dots + \|u^{(n)}\|_2^2 \right)^{1/2}$. Let X and Y be two Banach spaces, then the product space $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is Banach space with norm $\|(x, y)\| = \|x\| + \|y\|$.

To discuss BVP(7), for the given $h \in L^2(I)$, we first consider the linear boundary value problem (LBVP)

$$\begin{cases} -\frac{1}{r^{N-1}} (r^{N-1} u'(r))' = h(r), & r \in [r_1, r_2], \\ u(r_1) = u(r_2) = 0. \end{cases} \quad (8)$$

Lemma 1. For every $h \in C(I)$, BVP(8) has a unique solution $u := Sh \in C^2(I)$. Moreover, the operator $S : C(I) \rightarrow C^1(I)$ is a completely continuous linear operator.

Lemma 1 is known, see Lemma 2.1 of [17].

Lemma 2. The radially symmetric elliptic eigenvalue problem EVP(5) has a minimum positive real eigenvalue λ_1 .

Proof. For EVP(5), when setting $r = |x|$, by (6), it becomes the eigenvalue value problem of the ordinary differential equation

$$\begin{cases} -\frac{1}{r^{N-1}} (r^{N-1} u'(r))' = \lambda u(r), & r \in [r_1, r_2], \\ u(r_1) = u(r_2) = 0. \end{cases} \quad (9)$$

Define a linear operator $L : D(L) \subset L^2(I) \rightarrow L^2(I)$ by

$$D(L) = \{u \in H^2(I) \mid u(r_1) = u(r_2) = 0\}, \quad Lu = -\frac{1}{r^{N-1}}(r^{N-1}u'(r))', \quad (10)$$

then EVP(9) is rewritten as

$$Lu = \lambda u, \quad u \in D(L). \quad (11)$$

Hence, λ is an eigenvalue of EVP(5), if and only if λ is an eigenvalue of L .

In $L^2(I)$, we reintroduce the weighted inner product by

$$\langle u, v \rangle = \int_{r_1}^{r_2} r^{N-1} u(r) v(r) dr. \quad (12)$$

Clearly, the new inner product norm

$$\|u\| = \sqrt{\langle u, u \rangle} = \left(\int_{r_1}^{r_2} r^{N-1} |u(r)|^2 dr \right)^{1/2}. \quad (13)$$

is equivalent to the original norm $\|\cdot\|_2$. Hence, $L^2(I)$ forms a Hilbert space by this inner product, and we denote it by H . We can easily verify that L is a positive definite self-adjoint operator in H , and that it has a compact resolvent. Hence, by the spectral theory of positive definite operators, L has a minimum positive real eigenvalue λ_1 given by

$$\lambda_1 = \inf \left\{ \frac{\langle Lu, u \rangle}{\|u\|^2} \mid u \in D(L), \|u\| \neq 0 \right\} > 0. \quad (14)$$

λ_1 is also the minimum positive real eigenvalue of EVP(5). $\square$

Lemma 3. The minimum positive real eigenvalue λ_1 of EVP(5) satisfies

$$\frac{r_1^{N-1} \pi^2}{r_2^{N-1} (r_2 - r_1)^2} \leq \lambda_1 \leq \frac{r_2^{N-1} \pi^2}{r_1^{N-1} (r_2 - r_1)^2}. \quad (15)$$

Proof. Let $u \in D(L)$ and $\|u\| \neq 0$. By (10) and (12), we have

$$\langle Lu, u \rangle = - \int_{r_1}^{r_2} (r^{N-1} u'(r))' u(r) dr = \int_{r_1}^{r_2} r^{N-1} |u'(r)|^2 dr.$$

From this, it follows that

$$r_1^{N-1} \|u'\|_2^2 \leq \langle Lu, u \rangle \leq r_2^{N-1} \|u'\|_2^2.$$

Since

$$r_1^{N-1} \|u'\|_2^2 \leq \|u\|^2 \leq r_2^{N-1} \|u'\|_2^2,$$

we have

$$\frac{r_1^{N-1}}{r_2^{N-1}} \frac{\|u'\|_2^2}{\|u\|^2} \leq \frac{\langle Lu, u \rangle}{\|u\|^2} \leq \frac{r_2^{N-1}}{r_1^{N-1}} \frac{\|u'\|_2^2}{\|u\|^2}. \quad (16)$$

In $L^2(I)$ with original inner product (u, v) , we define another linear operator $L_0 : D(L_0) \subset L^2(I) \rightarrow L^2(I)$ by

$$D(L_0) = D(L), \quad L_0 u = -u''. \quad (17)$$

It is easy to verify that L_0 is a positive definite self-adjoint operator with compact resolvent. By direct calculation, it can be obtained that the minimum positive real eigenvalue of L_0 is

$$\lambda_1(L_0) = \inf \left\{ \frac{\|u'\|_2^2}{\|u\|_2^2} \mid u \in D(L), \|u\| \neq 0 \right\} = \frac{\pi^2}{(r_2 - r_1)^2},$$

and $\psi_1(r) = \sin \frac{\pi(r-r_1)}{r_2-r_1}$ is its eigenfunction. Hence, taking the infimum for (16), by (14), we determine that (15) holds. $\square$

Lemma 4. For every $h \in C(I)$, the unique solution of BVP(8) $u = Sh$ satisfies

$$\|u\| \leq \frac{1}{\sqrt{\lambda_1}} \|u'\|, \quad \|u'\| \leq \frac{1}{\sqrt{\lambda_1}} \|Lu\|. \quad (18)$$

Proof. For $\forall h \in C(I)$, by the definitions of S and L , $u = Sh \in D(L)$ and $Lu = h$. Using integration by parts and (14), we have

$$\|u'\|^2 = \langle Lu, u \rangle \geq \lambda_1 \|u\|^2.$$

Hence, the first inequality of (18) holds. By this and the Schwarz inequality of inner products,

$$\|u'\|^2 = \langle Lu, u \rangle \leq \|Lu\| \cdot \|u\| \leq \|Lu\| \cdot \frac{1}{\sqrt{\lambda_1}} \|u'\|,$$

Hence, the second inequality holds. $\square$

We consider BVP(7). Let $f : [r_1, r_2] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Define a mapping $F : C^1(I) \times C(I) \rightarrow C(I)$ by

$$F(u, v)(r) = f(r, u(r), |u'(r)|, -v(r)), \quad (u, v) \in C^1(I) \times C(I). \quad (19)$$

Clearly, $F : C^1(I) \times C(I) \rightarrow C(I)$ is continuous, and it maps every bounded set in $C^1(I) \times C(I)$ into a bounded set in $C(I)$. Define an operator A by

$$A(u, v) = (Sv, S(F(u, v))), \quad (u, v) \in C^1(I) \times C(I). \quad (20)$$

By the complete continuity of $S : C(I) \rightarrow C^1(I)$ and the continuity of $F : C^1(I) \times C(I) \rightarrow C(I)$, $A : C^1(I) \times C(I) \rightarrow C^1(I) \times C(I)$ is completely continuous. If (u, v) is a fixed point of A then, by the definition of A ,

$$u = Sv, \quad v = S(F(u, v)). \quad (21)$$

By this and the definition of S and F , $(u, v) \in C^2(I) \times C^2(I)$ is a solution of BVP(7). Conversely, if $(u, v) \in C^2(I) \times C^2(I)$ is a solution of BVP(7), then u and v satisfy (21) and (u, v) is a fixed point of A . In next section, by finding the fixed point of A , we prove Theorems 1 and 2.

3. Proofs of the Main Results

Proof of Theorem 1. Let $A : C^1(I) \times C(I) \rightarrow C^1(I) \times C(I)$ be the completely continuous operator defined by (20). We use the Leray–Schauder fixed point theorem [18] to show that A has a fixed point. To do this, consider the family of the equations

$$(u, v) = \mu A(u, v), \quad 0 < \mu < 1. \quad (22)$$

We need to prove that the set of the solutions of (22) is bounded in $C^1(I) \times C(I)$. Let $(u, v) \in C^1(I) \times C(I)$ be a solution of (22) for $\mu \in (0, 1)$. By the definition of A , $(u, v) = \mu(Sv, S(F(u, v))) = (S(\mu v), S(\mu F(u, v)))$. Hence,

$$u = S(\mu v), \quad v = S(\mu F(u, v)). \quad (23)$$

By this and the definition of S , $u, v \in C^2(I)$ satisfy the differential equations system

$$\begin{cases} -\frac{1}{r^{N-1}}(r^{N-1}u'(r))' = \mu v(r), & r \in [r_1, r_2], \\ -\frac{1}{r^{N-1}}(r^{N-1}v'(r))' = \mu f(r, u(r), |u'(r)|, -v(r)), & r \in [r_1, r_2], \\ u(r_1) = u(r_2) = 0, & u(r_1) = u(r_2) = 0. \end{cases} \quad (24)$$

From the condition (F1), it follows that

$$\begin{aligned} f(r, u(r), |u'(r)|, -v(r)) v(r) &= -f(r, u(r), |u'(r)|, -v(r)) (-v(r)) \\ &\leq a|u(r)|^2 + b|u'(r)|^2 + c|v(r)|^2 + d. \end{aligned} \quad (25)$$

Multiplying the second equation of (24) by $r^{N-1}v(r)$ and integrating on I , then using integration by parts for the left side and using (25) for the right side, we have

$$\begin{aligned} \|v'\|^2 &= -\int_{r_1}^{r_2} (r^{N-1}v'(r))' v(r) dr \\ &= \mu \int_{r_1}^{r_2} r^{N-1} f(r, u(r), |u'(r)|, -v(r)) v(r) dr \\ &\leq \mu \int_{r_1}^{r_2} r^{N-1} (a|u(r)|^2 + b|u'(r)|^2 + c|v(r)|^2 + d) dr \\ &\leq \int_{r_1}^{r_2} r^{N-1} (a|u(r)|^2 + b|u'(r)|^2 + c|v(r)|^2 + d) dr \\ &\leq a\|u\|^2 + b\|u'\|^2 + c\|v\|^2 + r_2^{N-1}(r_2 - r_1)d. \end{aligned} \quad (26)$$

Since $u = S(\mu v) \in D(L)$, $Lu = \mu v$ and $v = S(\mu F(u, v)) \in D(L)$, by (18) of Lemma 4, we have

$$\begin{aligned} \|u\|^2 &\leq \frac{1}{\lambda_1} \|u'\|^2, \quad \|u'\|^2 \leq \frac{1}{\lambda_1} \|Lu\|^2 = \frac{1}{\lambda_1} \|\mu v\|^2 \leq \frac{1}{\lambda_1} \|v\|^2; \\ \|v\|^2 &\leq \frac{1}{\lambda_1} \|v'\|^2. \end{aligned}$$

From these inequalities, it follows that

$$\|u\|^2 \leq \frac{1}{\lambda_1^3} \|v'\|^2, \quad \|u'\|^2 \leq \frac{1}{\lambda_1^2} \|v'\|^2. \quad (27)$$

Hence, by (26), we obtain that

$$\|v'\|^2 \leq \left(\frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c}{\lambda_1} \right) \|v'\|^2 + r_2^{N-1}(r_2 - r_1)d.$$

So, we have

$$\|v'\| \leq \left(\frac{r_2^{N-1}(r_2 - r_1)d}{1 - \left(\frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c}{\lambda_1} \right)} \right)^{1/2} := C_0. \quad (28)$$

By the definition (13) of the norm $\|\cdot\|$,

$$\|v'\|_2 = \left(\int_{r_1}^{r_2} |v'(r)|^2 dr \right)^{1/2} \leq \left(\int_{r_1}^{r_2} \frac{r^{N-1}}{r_1^{N-1}} |v'(r)|^2 dr \right)^{1/2} = \frac{1}{r_1^{(N-1)/2}} \|v'\|.$$

Hence, by (12), we have

$$\|v'\|_2 \leq \frac{C_0}{r_1^{(N-1)/2}} := C_1. \quad (29)$$

For $\forall r \in I$, by the Holder inequality, we have

$$|v(r)| = \int_{r_1}^r v'(s) ds \leq \int_{r_1}^r |v'(s)| ds \leq \int_{r_1}^{r_2} |v'(s)| ds \leq (r_2 - r_1)^{1/2} \|v'\|_2.$$

Hence, by (29), we obtain that

$$\|v\|_C \leq (r_2 - r_1)^{1/2} C_1 := C_2. \quad (30)$$

Since $u = S(\mu v)$ and $S : C(I) \rightarrow C^1(I)$ is a linear bounded operator, we have

$$\begin{aligned} \|u\|_{C^1} &\leq \|S(\mu v)\|_C \leq \|S\|_{\mathfrak{B}(C(I), C^1(I))} \cdot \|\mu v\|_C \\ &\leq \|S\|_{\mathfrak{B}(C(I), C^1(I))} \|v\|_C \\ &\leq \|S\|_{\mathfrak{B}(C(I), C^1(I))} C_2 := C_3. \end{aligned} \quad (31)$$

Now, by (30) and (31), the norm of (u, v) in $C^1(I) \times C(I)$ satisfies

$$\|(u, v)\| = \|u\|_{C^1} + \|v\|_C \leq C_3 + C_2 := C_4. \quad (32)$$

Hence, the set of the solutions of (22) is bounded in $C^1(I) \times C(I)$. By the Leray–Schauder fixed point theorem, A has a fixed point (u_0, v_0) in $S : C(I) \rightarrow C^1(I)$. By the definition of A , (u_0, v_0) is a solution of BVP(7), and by (6), $u_0(|x|)$ is a radially symmetric solution of BVP(1).

The proof of Theorem 1 is completed. $\square$

Proof of Theorem 2. We first prove that (F2) $\implies$ (F1). For the constants a, b, c in (F2), since $\frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c}{\lambda_1} < 1$, we can choose a positive $\delta > 0$, such that $\frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c+\delta}{\lambda_1} < 1$. Set $M = \max_{r \in I} |f(r, 0, 0, 0)| + 1$. For any $r \in I$ and $(\xi, \zeta, \eta) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, by Assumption (F2), we have

$$\begin{aligned} -f(r, \xi, \zeta, \eta) \eta &= -((f(r, \xi, \zeta, \eta) - f(r, 0, 0, 0))(\eta - 0) - f(r, 0, 0, 0)\eta) \\ &\leq a\xi^2 + b\zeta^2 + c\eta^2 + |f(r, 0, 0, 0)| \cdot |\eta| \\ &\leq a\xi^2 + b\zeta^2 + c\eta^2 + M|\eta| \\ &\leq a\xi^2 + b\zeta^2 + c\eta^2 + 2 \cdot \frac{M}{2\sqrt{\delta}} \cdot \sqrt{\delta}|\eta| \\ &\leq a\xi^2 + b\zeta^2 + (c + \delta)\eta^2 + \frac{M^2}{4\delta}. \end{aligned}$$

In the final step of the derivation above, the square inequality $2pq \leq p^2 + q^2$ is applied to $p = \frac{M}{2\sqrt{\delta}}$ and $q = \sqrt{\delta}|\eta|$. Hence, f satisfies Condition (F1). By the proof of Theorem 1, BVP(7) has at least one solution.

Let (u_1, v_1) and (u_2, v_2) be two solutions of BVP(7). Then, by the definition of L ,

$$Lu_1 = v_1, \quad Lv_1 = F(u_1, v_1); \quad Lu_2 = v_2, \quad Lv_2 = F(u_2, v_2). \quad (33)$$

Set $u = u_2 - u_1, v = v_2 - v_1$; then, $Lu = v$, and by Lemma 4,

$$\|u\|^2 \leq \frac{1}{\lambda_1^2} \|v\|^2, \quad \|u'\|^2 \leq \frac{1}{\lambda_1} \|v\|^2. \quad (34)$$

By (33), (F2) and (34), we have

$$\begin{aligned}
 \lambda_1 \|v\|^2 &\leq \langle Lv, v \rangle \\
 &= \langle Lv_2 - Lv_1, v_2 - v_1 \rangle \\
 &= \langle F(u_2, v_2) - F(u_1, v_1), v_2 - v_1 \rangle \\
 &= \int_{r_1}^{r_2} r^{N-1} (F(u_2, v_2)(r) - F(u_1, v_1)(r)) (v_2(r) - v_1(r)) dr \\
 &\leq \int_{r_1}^{r_2} r^{N-1} a(u_2(r) - v_2(r))^2 + b(|u_2'(r)| - |u_1'(r)|)^2 + c(v_2(r) - v_1(r))^2 dr \\
 &\leq a\|u\|^2 + b\|u'\|^2 + c\|v\|^2 \\
 &\leq \left(\frac{a}{\lambda_1^2} + \frac{b}{\lambda_1} + c \right) \|v\|^2.
 \end{aligned}$$

From this inequality, it follows that

$$\left[1 - \left(\frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c}{\lambda_1} \right) \right] \|v\|^2 \leq 0.$$

This implies that $\|v\| = 0$, so we have $v = 0$ and $u = Sv = 0$. Hence, $u_2 = u_1$ and $v_2 = v_1$. This means that BVP(7) has a unique solution. Hence, BVP(1) has a unique radially symmetric solution.

The proof of Theorem 2 is completed. $\square$

4. Examples

To illustrate the applicability of Theorems 1 and 2, we present three application examples.

Example 1. Let $\Omega = \{x \in \mathbb{R}^3 \mid 1 < |x| < 2\}$ be the annulus in $\mathbb{R}^3$. Consider the biharmonic elliptic boundary value problem on Ω with superlinear terms of u , $|\nabla u|$ and Δu

$$\begin{cases} \Delta^2 u = 2u + 3u^4(\Delta u)^3 + 5|\nabla u|^3 \Delta u + 2(\Delta u)^5 + |x|^2, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0. \end{cases} \quad (35)$$

Corresponding to BVP(1), the nonlinearity is

$$f(r, \xi, \zeta, \eta) = 2\xi + 3\xi^4\eta^3 + 5\xi^3\eta + 2\eta^5 + r^2, \quad r \in [1, 2], \xi, \eta \in \mathbb{R}, \zeta \in \mathbb{R}^+. \quad (36)$$

For every $(r, \xi, \eta, \gamma) \in [1, 2] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, by (36),

$$\begin{aligned}
 -f(r, \xi, \zeta, \eta) \eta &= -2\xi\eta - 3\xi^4\eta^4 - 5\xi^3\eta^2 - 2\eta^6 - r^2\eta \\
 &\leq 2|\xi\eta| + 2|\eta| \\
 &\leq (4\xi^2 + \frac{1}{4}\eta^2) + (\frac{1}{4}\eta^2 + 4) \\
 &= 4\xi^2 + \frac{1}{2}\eta^2 + 4 \\
 &:= a\xi^2 + b\eta^2 + c\gamma^2 + d, \quad (37)
 \end{aligned}$$

where $a = 4$, $b = 0$, $c = \frac{1}{2}$, $d = 4$. By Lemma 3, $\lambda_1 \geq \frac{\pi^2}{4} > \frac{9}{4}$; hence,

$$\frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c}{\lambda_1} < \frac{4^3}{9^3} a + \frac{4}{9} c = \frac{256}{729} + \frac{2}{9} < 1.$$

By (37), f satisfies Condition (F1). Hence, by Theorem 1, BVP(35) has at least one radially symmetric solution.

Example 2. Let $N \geq 3$, $\Omega = \{x \in \mathbb{R}^N \mid 1 < |x| < 2\}$ be the annulus in $\mathbb{R}^N$. Consider the biharmonic elliptic boundary value problem

$$\begin{cases} \Delta^2 u = \alpha u - \beta |\nabla u| + \gamma (\Delta u)^3 + 1, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \quad (38)$$

where α, β, γ are positive constants. We show that BVP(38) has a unique radially symmetric solution when $2^{3N-11}\alpha + 2^{2N-7}\beta < 1$.

For BVP(38), the nonlinearity corresponding to BVP(1) is

$$f(r, \xi, \zeta, \eta) = \alpha \xi - \beta |\zeta| + \gamma \eta^3 + 1, \quad r \in [1, 2], \quad \xi, \eta \in \mathbb{R}, \quad \zeta \in \mathbb{R}^+. \quad (39)$$

For any $\forall r \in [1, 2]$, $(\xi_1, \eta_1, \gamma_1), (\xi_2, \eta_2, \gamma_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, by (39) and the differential mean value theorem, there exist $\theta \in (0, 1)$ and $\xi' = \xi_1 + \theta(\xi_2 - \xi_1)$, $\zeta' = \zeta_1 + \theta(\zeta_2 - \zeta_1)$, $\eta' = \eta_1 + \theta(\eta_2 - \eta_1)$, such that

$$\begin{aligned} & -(f(r, \xi_2, \zeta_2, \eta_2) - f(r, \xi_1, \zeta_1, \eta_1))(\eta_2 - \eta_1) = -f_\xi(r, \xi', \zeta', \eta')(\xi_2 - \xi_1)(\eta_2 - \eta_1) \\ & \quad - f_\zeta(r, \xi', \zeta', \eta')(\zeta_2 - \zeta_1)(\eta_2 - \eta_1) - f_\eta(r, \xi', \zeta', \eta')(\eta_2 - \eta_1)^2 \\ & = -\alpha(\xi_2 - \xi_1)(\eta_2 - \eta_1) + \beta(\zeta_2 - \zeta_1)(\eta_2 - \eta_1) - 3\gamma^2(\eta_2 - \eta_1)^2 \\ & \leq \alpha|(\xi_2 - \xi_1)(\eta_2 - \eta_1)| + \beta|(\zeta_2 - \zeta_1)(\eta_2 - \eta_1)| \\ & \leq \alpha(\xi_2 - \xi_1)^2 + \frac{\alpha}{4}(\eta_2 - \eta_1)^2 + \beta(\zeta_2 - \zeta_1)^2 + \frac{\beta}{4}(\eta_2 - \eta_1)^2 \\ & \leq \alpha(\xi_2 - \xi_1)^2 + \beta(\zeta_2 - \zeta_1)^2 + \frac{\alpha+\beta}{4}(\eta_2 - \eta_1)^2 \\ & := a(\xi_2 - \xi_1)^2 + b(\zeta_2 - \zeta_1)^2 + c(\eta_2 - \eta_1)^2, \end{aligned} \quad (40)$$

where $a = \alpha$, $b = \beta$, $c = \frac{\alpha+\beta}{4}$. By Lemma 3, $\lambda_1 \geq \frac{\pi^2}{2^{N-1}} > \frac{9}{2^{N-1}}$. Hence, we have

$$\begin{aligned} \frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c}{\lambda_1} & \leq \frac{2^{3(N-1)}}{9^3}\alpha + \frac{2^{2(N-1)}}{9^2}\beta + \frac{2^{N-1}}{9} \frac{(\alpha + \beta)}{4} \\ & < \frac{2^{3(N-1)}}{2^9}\alpha + \frac{2^{2(N-1)}}{2^6} + \frac{2^{N-1}}{2^5}(\alpha + \beta) \\ & = (2^{3N-12} + 2^{N-6})\alpha + (2^{2N-8} + 2^{N-6})\beta \\ & < 2^{3N-11}\alpha + 2^{2N-7}\beta. \end{aligned}$$

Hence, when $2^{3N-11}\alpha + 2^{2N-7}\beta < 1$, by (40), the nonlinearity f satisfies Condition (F2). By Theorem 2, BVP(38) has a unique radially symmetric solution.

Example 3. Let $\Omega = \{x \in \mathbb{R}^N \mid r_1 < |x| < r_2\}$ ($N \geq 2$, $0 < r_1 < r_2$) be an annulus in $\mathbb{R}^N$. Consider biharmonic elliptic boundary value problem with the asymptotically linear nonlinearity at 0

$$\begin{cases} \Delta^2 u = \frac{\alpha u + \beta \Delta u + 1}{\sqrt{1 + |\nabla u|^2}}, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \quad (41)$$

where $0 < \alpha < \lambda_1^2$ and $\beta > 0$ are constants. We show that BVP(41) has at least one radially symmetric solution.

Corresponding to BVP(1), the nonlinearity of BVP(41) is

$$f(r, \xi, \zeta, \eta) = \frac{\alpha \xi + \beta \eta + 1}{\sqrt{1 + \zeta^2}}, \quad r \in [0, 1], \quad \xi, \eta \in \mathbb{R}, \quad \zeta \in \mathbb{R}^+. \quad (42)$$

Since $\alpha^{\frac{1}{2}} < \lambda_1$, we can choose a positive constant ε , such that $\alpha^{\frac{1}{2}} < \lambda_1 - \varepsilon$. For every $(r, \xi, \eta, \gamma) \in [0, 2] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, by (42),

$$\begin{aligned} -f(r, \xi, \zeta, \eta) \eta &= \frac{-\alpha \xi \eta - \beta \eta^2 - \eta}{\sqrt{1 + \zeta^2}} \\ &\leq \alpha |\xi \eta| + |\eta| \\ &= 2 \cdot \frac{\alpha^{\frac{3}{4}}}{\sqrt{2}} \zeta \cdot \frac{\alpha^{\frac{1}{4}}}{\sqrt{2}} \eta + 2 \cdot \frac{\sqrt{\varepsilon} |\eta|}{\sqrt{2}} \cdot \frac{1}{\sqrt{2\varepsilon}} \\ &\leq \frac{1}{2} \alpha^{\frac{3}{2}} \zeta^2 + \frac{1}{2} \alpha^{\frac{1}{2}} \eta^2 + \frac{\varepsilon}{2} \eta^2 + \frac{1}{2\varepsilon} \\ &< \frac{1}{2} (\lambda_1 - \varepsilon)^3 \zeta^2 + \frac{1}{2} (\lambda_1 - \varepsilon) \eta^2 + \frac{\varepsilon}{2} \eta^2 + \frac{1}{2\varepsilon} \\ &= \frac{1}{2} (\lambda_1 - \varepsilon)^3 \zeta^2 + \frac{1}{2} \lambda_1 \eta^2 + \frac{1}{2\varepsilon} \\ &:= a \zeta^2 + b \eta^2 + c \gamma^2 + d, \end{aligned}$$

where $a = \frac{1}{2} (\lambda_1 - \varepsilon)^3$, $b = 0$, $c = \frac{1}{2} \lambda_1$ and $d = \frac{1}{2\varepsilon}$. Since

$$\frac{a}{\lambda_1^3} + \frac{b}{\lambda_1^2} + \frac{c}{\lambda_1} = \frac{1}{2} \frac{(\lambda_1 - \varepsilon)^3}{\lambda_1^3} + \frac{1}{2} < 1,$$

f satisfies Condition (F1). By Theorem 1, BVP(41) has at least one radially symmetric solution.

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